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Multivalued Equations on a Bounded Domain via Minimization on Orlicz-Sobolev Spaces*

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We exploit minimization of locally Lipschitz functionals defined on Orlicz-Sobolev spaces along with convexity techniques, to investigate existence of solution of the multivalued equation $-\Delta_\Phi u \in \partial j(.,u) + h$ in Ω, where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, $\Phi : \mathbf{R} \longrightarrow [0,\infty)$ is an N-function, Δ_Φ is the corresponding Φ-Laplacian, h is a measure on Ω and $\partial j(.,u)$ stands for the Clarke generalized gradient of a function j linked with critical growth. Regularity of the solutions is addressed, as well.

Keywords: Minimization, convexity, Orlicz-Sobolev space

1. Introduction

We study existence of solution of the multivalued equation

$$-\Delta_\Phi u \in \partial j(x,u) + h \quad \text{in } \Omega \tag{1}$$

where $\Omega \subset \mathbf{R}^\mathbf{N}$, $\mathbf{N} \geq \mathbf{3}$, is a bounded domain with smooth boundary $\partial\Omega$, $j : \Omega \times \mathbf{R} \to \mathbf{R}$ and $h : \Omega \to \mathbf{R}$ are measdurable, Δ_Φ is the Φ-Laplace operator,

$$\Delta_\Phi := \operatorname{div}(\phi(|\nabla u|)\nabla u),$$

where $\phi : (0,\infty) \to (0,\infty)$ is a continuous function satisfying,

(ϕ_1) (i) $\lim_{s\to 0} s\phi(s) = 0$, (ii) $\lim_{s\to\infty} s\phi(s) = \infty$,
(ϕ_2) $s \mapsto s\phi(s)$ is nondecreasing in $(0,\infty)$,
(ϕ_3) there exist $\ell, m \in (1,N)$ such that $\ell \leq \frac{t^2\phi(t)}{\Phi(t)} \leq m$.

We extend $s \mapsto s\phi(s)$ to $\mathbf{R}$ as an odd function and define Φ, as an even function, by

$$\Phi(t) = \int_0^t s\phi(s)ds, \quad t \in \mathbf{R}.$$

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The complementary function $\widetilde{\Phi}$ associated to Φ is defined as

$$\widetilde{\Phi}(t) = \max_{s \geq 0}\{st - \Phi(s)\}, \quad t \geq 0.$$

We shall assume in this work that j is measurable, $j(x,.) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R},\mathbf{R})$ and $j(x,0) \in L^1(\Omega)$. Clarke's generalized gradient of the map $s \mapsto j(x,s)$ is defined by

$$\partial j(x,t) = \{\mu \in \mathbf{R} \mid j^0(x,t;r) \geq \mu r, \ r \in \mathbf{R}\}$$

in the direction of r, that is

$$j^0(x,t;r) = \limsup_{y \to t, \ \lambda \downarrow 0} \frac{j(x,y+\lambda r) - j(x,y)}{\lambda}.$$

In this work $L_\Phi(\Omega)$ denotes the Orlicz space and the expression

$$\|u\|_\Phi = \inf\left\{\lambda > 0 \mid \int_\Omega \Phi\left(\frac{u(x)}{\lambda}\right) \leq 1\right\}$$

defines the Luxemburg norm, (cf. [1]). By [6, Lemma D2],

$$L_\Phi(\Omega) \overset{\hookrightarrow}{\text{cont}} L^\ell(\Omega) \tag{2}$$

and we denote by $C_\ell > 0$ the best constant such that

$$C_\ell |u|_\ell^\ell \leq \|u\|_\Phi^\ell, \quad u \in L_\Phi(\Omega). \tag{3}$$

The Orlicz-Sobolev space associated to Φ (also denoted $W^1L_\Phi(\Omega)$) is defined by

$$W^{1,\Phi}(\Omega) = \left\{u \in L_\Phi(\Omega) \mid \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), \ i = 1, ..., N\right\},$$

while $W_0^{1,\Phi}(\Omega)$ is the closure of $\mathbf{C}_0^\infty(\Omega)$ with respect to the norm of $W^{1,\Phi}(\Omega)$.

The critical growth function asociated to Φ, denoted by Φ_*, is defined as the inverse of

$$t \in (0,\infty) \mapsto \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds.$$

We extend Φ_* to $\mathbf{R}$ by setting $\Phi_*(t) = \Phi_*(-t)$ for $t \leq 0$.

Remark 1.1. If $N \geq 3$ and $\phi(t) = 2$ then, by computing, we obtain for $t > 0$: $\Phi_*^{-1}(t) = t^{\frac{N-2}{2N}}$ and $\Phi_*(t) = t^{\frac{2N}{N-2}}$. The function Φ_* plays the role of the critical Sobolev exponent in the case of Sobolev spaces.

Definition 1.2. A function $u \in W_0^{1,\Phi}(\Omega)$ is a solution of (1) if there is an element $\rho = \rho_u$ in $L_{\Phi_*}(\Omega)' = L_{\widetilde{\Phi}_*}(\Omega)$ such that

$$\rho(x) \in \partial j(x,u(x)) \quad \text{a.e. } x \in \Omega, \tag{4}$$

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla v dx = \int_\Omega \rho v dx + \int_\Omega hv, \quad v \in W_0^{1,\Phi}(\Omega). \tag{5}$$

The limit function below will be needed:

$$A_\infty(x) = \limsup_{|s|\to\infty} \frac{j(x,s)}{|s|^\ell}.$$

Theorem 1.3. *Assume* (ϕ_1)–(ϕ_3). *Suppose there are a numbers* $c_1, c_2 \geq 0$ *and a nonnegative function* $b \in L_{\Phi_*}(\Omega)'$ *such that*

$$|\eta| \leq c_1 \widetilde{\Phi}_*^{-1} \circ \Phi_*(c_2 s) + b(x), \quad s \in \mathbf{R},\ x \in \Omega,\ \eta \in \partial j(x,s). \tag{6}$$

Assume that there exist a constant $A \geq 0$ *and a nonnegative function* $B \in L^1(\Omega)$ *such that*

$$j(x,s) \leq A|s|^\ell + B(x), \quad s \in \mathbf{R},\ \mathbf{x} \in \boldsymbol{\Omega}. \tag{7}$$

If in addition,

$$\inf_{v \in W_0^{1,\Phi}(\Omega),\ \|v\|_\Phi = 1} \left\{ \int_\Omega \Phi(|\nabla v|)dx - \int_{\{v \neq 0\}} A_\infty(x)|v(x)|^\ell dx \right\} > 0, \tag{8}$$

then for each $h \in L_\Phi(\Omega)'$, *there is a solution* $u \in W_0^{1,\Phi}(\Omega)$ *of* (1).

Condition (8) is motivated by the work of Brézis & Oswald [2].

Set

$$\Delta_{\Phi_\gamma} u = \operatorname{div}\left(\gamma \frac{(\sqrt{1+|\nabla u|^2} - 1)^{\gamma-1}}{\sqrt{1+|\nabla u|^2}} \nabla u \right), \tag{9}$$

where $1 < \gamma < \infty$ and consider the equation

$$-\Delta_{\Phi_\gamma} u \in \partial j(x,u) + h \ \text{ in } \Omega, \tag{10}$$

which is an example of (1). The result below is an application Theorem 1.3.

Theorem 1.4. *Assume* $1 < \gamma,\ 2\gamma < N$. *Assume also that*

$$|\eta| \leq a|s|^{\gamma^\star - 1} + b(x), \quad s \in \mathbf{R},\ x \in \Omega,\ \eta \in \partial j(x,t), \tag{11}$$

where $a \geq 0$ *is some constant,* $b \in L^{\gamma^*}(\Omega)$ *is nonnegative,* $\gamma^\star = N\gamma/(N-\gamma)$ *and*

$$j(x,s) \leq A|s|^\gamma + B(x), \quad s \in \mathbf{R},\ x \in \Omega, \tag{12}$$

for some constant $A \geq 0$ *and some nonnegative function* $B \in L^1(\Omega)$. *If in addition,*

$$\inf_{v \in W_0^{1,\gamma},\ |v|_\gamma = 1} \left\{ \int_\Omega \left(\sqrt{1+|\nabla v|^2} - 1\right)^\gamma dx - \int_{\{v \neq 0\}} A_\infty(x)|v(x)|^\gamma dx \right\} > 0, \tag{13}$$

then for each $h \in L^{\frac{\gamma}{\gamma-1}}(\Omega)$ *problem* (10) *admits a solution* $u \in W_0^{1,\gamma}(\Omega)$.

Theorems 1.3 and 1.4 extend to the case of multivalued equations involving the operator $-\Delta_\Phi$ the main result of [12] where the operator $-\Delta$ is treated. There is a broad literature on multivalued equations and we refer the reader to [10, 17] and their references.

In the proposition below we give conditions for (8) to hold. At first, we recall that by a result of Garcia-Huidobro et al. in [11], the expression

$$\lambda_1 = \inf_{v\in W_0^1 L_\Phi,\ \|v\|_\Phi=1} \int_\Omega \Phi(|\nabla v|)dx$$

is attained at some positive function $v \in W_0^{1,\Phi}(\Omega)$ in such a way that λ_1 is an eigenvalue and v is an eigenfunction of $(-\Delta_\Phi, W_0^{1,\Phi}(\Omega))$.

Proposition 1.5. *Assume (ϕ_1)–(ϕ_3). Then (8) holds if there is a function $\alpha \in L^\infty(\Omega)$ and a subset Ω_0 of Ω with $|\Omega_0| > 0$ such that*

$$A_\infty \le C_\ell\ \alpha \quad a.e.\ in\ \Omega, \qquad \alpha \le \lambda_1 \quad a.e.\ in\ \ \Omega, \qquad \alpha < \lambda_1 \quad a.e.\ in\ \ \Omega_0. \tag{14}$$

For the reader's convenience a proof is provided in Section 5.

Remark 1.6. If $\phi(t) \equiv 2$ one has

(i) $\Delta_\Phi = \Delta,\ W_0^{1,\Phi}(\Omega) = H_0^1(\Omega)$ and equation (1) becomes

$$-\Delta u \in \partial j(x,u) + h \quad \text{in } \Omega, \tag{15}$$

(ii) condition (8) reads as

$$\inf_{v\in H_0^1(\Omega),\ |v|_2=1} \left\{ \int_\Omega |\nabla v|^2 dx - \int_{\{v\neq 0\}} A_\infty(x) v^2 dx \right\} > 0 \tag{16}$$

and by Proposition 1.5, (16) holds when (14) holds.

To close this section we state some results on the regularity of solutions. In this regard let $f : \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ be a function satisfying

$$\begin{gathered} f \text{ is Baire measurable,} \\ f(x,.) \in \mathbf{C}(\mathbf{R} - \{a\}), \\ f(x,a-0) < f(x,a+0), \quad x \in \Omega, \end{gathered} \tag{17}$$

where

$$f(x,t-0) := \lim_{\delta\to t^-} f(x,\delta), \qquad f(x,t+0) := \lim_{\delta\to t^+} f(x,\delta).$$

Let

$$j(x,s) := \int_0^s f(x,t)dt.$$

If u is a solution of (1), consider the level set

$$\Gamma_a(u) = \{x \in \Omega \mid u(x) = a\}.$$

Next we state a result which establishes that under suitable conditions,

$$|\Gamma_a(u)| = 0, \qquad -\Delta_\Phi u = f(x,u) + h \ \text{ a.e. in } \Omega. \tag{18}$$

Theorem 1.7. *Let ϕ satisfy (ϕ_1)–(ϕ_3) and let $f : \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ be nondecreasing in the real variable. Assume also that f satisfies* (1) *and*

$$|f(x,s)| \leq c_1 \widetilde{\Phi}_*^{-1} \circ \Phi_*(c_2 s) + b(x). \tag{19}$$

where $c_1, c_2 \geq 0$ are constants and $b \in L_{\Phi_*}(\Omega)'$ is nonnegative. If in addition, (7)–(8) *hold, then* (1) *admits a solution u satisfying* (18).

Remark 1.8. The growth condition

$$|f(x,s)| \leq a|s|^{\gamma^\star - 1} + b(x), \tag{20}$$

where $a \geq 0$ is a constant, $b \in L^{\gamma^*}(\Omega)$ is nonnegative and $\gamma^\star = N\gamma/(N-\gamma)$ will be assumed in the theorem below. It clarifies the technical condition (19) as well as the more general condition (6), which was employed earlier, for instance, in [16].

Theorem 1.9. *Assume $1 < \gamma,\ 2\gamma < N$. Assume also* (12)*,* (13)*,* (1) *and* (20)*. Then for each $h \in L^{\frac{\gamma}{\gamma-1}}(\Omega)$, problem* (10) *admits a solution $u \in W_0^{1,\gamma}(\Omega)$ satisfying* (18).

An Application. Consider the equation

$$-\Delta_\Phi u = \chi_{u>a} + h \ \text{ a.e. in } \Omega, \tag{21}$$

where $a > 0$, $h \in L_{\widetilde{\Phi}}(\Omega)$ and $(\phi_1) - (\phi_3)$ hold.

We claim that (21) admits a weak solution in $W_0^{1,\Phi}(\Omega)$. Indeed, let $f(x,t) = \chi_{t>a}$ and set

$$j(u) := \int_0^u f(x,t)dt = (u-a)^+.$$

Next consider the associated multivalued equation

$$-\Delta_\Phi u \in \partial j(u) + h. \tag{22}$$

We will show that the assumptions of Theorem 1.7 hold true. Indeed, since Ω is bounded, it is straightforward to check that conditions (8), (16) and (19) hold. Thus by Theorem 1.7, (22) admits a solution $u \in W_0^{1,\Phi}(\Omega)$. Notice also that (1) holds because

$$\lim_{t \to a_-} \chi_{t>a} = 0 < 1 = \lim_{t \to a_+} \chi_{t>a}.$$

Finally by Theorem 1.7, $|\Gamma_a(u)| = 0$ and so $u \in W_0^{1,\Phi}(\Omega)$ is a solution of (21). $\square$

Remark 1.10. In the case $\Phi = \Phi_\gamma$ as in (9), with $1 < \gamma,\ 2\gamma < N$, then by Theorem 1.9, the solution u of (21) belongs to $W_0^{1,\gamma}(\Omega)$.

2. Preliminary Results and Notations

We recall a few notations and results on the Critical Point Theory for locally Lipschitz functionals defined on a real Banach space X with norm $\|.\|$ and refer the reader to Clarke [5], Chang [4] and Carl, Le & Motreanu [3] and references therein.

Let $I : X \to \mathbf{R}$ be a locally Lipschitz functional ($I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbf{R})$ for short) that is: given $u \in X$ there are an open neighborhood $V = V_u \subset X$ and a constant $K = K_V > 0$ such that

$$|I(v_2) - I(v_1)| \leq K\|v_2 - v_1\|, \quad v_1, v_2 \in V.$$

The directional derivative of I at u in the direction of $v \in X$ is defined by

$$I^0(u; v) = \limsup_{h \to 0 \ \lambda \to 0^+} \frac{I(u + h + \lambda v) - I(u + h)}{\lambda}.$$

It follows that $I^0(u; .)$ is subadditive and positively homogeneous, that is

$$I^0(u; v_1 + v_2) \leq I^0(u; v_1) + I^0(u; v_2),$$
$$I^0(u; \lambda v) = \lambda I^0(u; v),$$

where $u, v, v_1, v_2 \in X$ and $\lambda > 0$. As a byproduct

$$\begin{aligned}\left|I^0(u; v_1) - I^0(u; v_2)\right| &\leq |I^0(u; v_1 - v_2)| \\ &\leq K \parallel v_1 - v_2 \parallel_X,\end{aligned}$$

for some $K = K_u > 0$ and in addition $I^0(u; .)$ is continuous and convex. Its subdifferential at $z \in X$ is given by

$$\partial I^0(u; z) = \left\{\mu \in X^* \mid I^0(u; v) \geq I^0(u; z) + \langle \mu, v - z\rangle, \ v \in X\right\}.$$

The generalized gradient of I at u is the set

$$\partial I(u) = \left\{\mu \in X^* \mid \langle \mu, v\rangle \leq I^0(u; v), \ v \in X\right\},$$

Since $I^0(u; 0) = 0$, $\partial I(u)$ is the subdifferential of $I^0(u; 0)$. Further properties, definitions and remarks are given below (cf. [3], [4] for the proofs):

1. $\partial I(u) = \left\{I'(u)\right\}$ if $I \in C^1(X, \mathbf{R})$,
2. *a point* $u_0 \in X$ *is a critical point of* I *if* $0 \in \partial I(u_0)$,
3. *if* u_0 is a local minimum of I then it is a critical point of I,
4. *if* $J \in C^1(X, \mathbf{R})$ *then* $\partial(I + J)(u) = \partial I(u) + \partial J(u)$.

Next we will introduce some notations concerning Orlicz and Orliccz-Sobolev spaces (cf. [1, 15]). The function Φ is said to satisfy the Δ_2-condition, ($\Phi \in \Delta_2$ for short), if

$$\text{there is a constant } K > 0 \text{ such that } \ \Phi(2t) \leq K\Phi(t), \quad t \geq 0.$$

The usual Orlicz-Sobolev norm of $W^{1,\Phi}(\Omega)$ is

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_\Phi .$$

It follows by $(\phi_1)-(\phi_3)$ that Φ and $\widetilde{\Phi}$ are N-functions and satisfy the Δ_2-condition (cf. [15, Remark 3.4.6]) and [8, Lemma 2.5]. Moreover, $L_\Phi(\Omega)$ and $W^{1,\Phi}(\Omega)$ are separable, reflexive, Banach spaces, see e.g. [1]. By the Poincaré inequality, (cf. [1]),

$$\int_\Omega \Phi(u)dx \leq \int_\Omega \Phi(2d|\nabla u|)dx \tag{23}$$

where $d = \text{diam}(\Omega)$, and it follows that

$$\|u\|_\Phi \leq 2d\|\nabla u\|_\Phi, \quad u \in W_0^{1,\Phi}(\Omega). \tag{24}$$

Remark 2.1. We claim that $\Delta_\Phi u \in L_{\widetilde{\Phi}_*}(\Omega\ )$ for each $u \in W_0^{1,\Phi}(\Omega)$. Indeed, set

$$\langle -\Delta_\Phi u, v\rangle = \int_\Omega \phi(|\nabla u|)\nabla u \nabla v dx \ \text{ for } u, v \in W_0^{1,\Phi}(\Omega).$$

By [8, p. 263],

$$\int_\Omega \widetilde{\Phi}(\phi(|\nabla u|)|\nabla u|)dx \leq \int_\Omega \Phi(2|\nabla u|)dx < \infty,$$

which shows that $\phi(|\nabla u|)|\nabla u| \in L_{\widetilde{\Phi}}(\Omega)$. By the Hölder inequality,

$$\int_\Omega |\phi(|\nabla u|)\nabla u \nabla v| dx \leq 2\|\phi(|\nabla u|)|\nabla u|\|_{\widetilde{\Phi}} \|\nabla v\|_\Phi.$$

As a consequence of the inequality above and the embedding $L_{\Phi_*}(\Omega) \overset{cont}{\hookrightarrow} L_\Phi(\Omega)$ we have,

$$\Delta_\Phi u \in L_\Phi(\Omega\ )' = L_{\widetilde{\Phi}}(\Omega\ ) \subseteq L_{\widetilde{\Phi}_*}(\Omega\ )$$

Since $L_{\Phi_*}(\Omega)' = L_{\widetilde{\Phi_*}}(\Omega)$, it follows that

$$\langle -\Delta_\Phi u, \varphi\rangle = \int_\Omega (-\Delta_\Phi u)\varphi dx, \quad \varphi \in L_{\Phi_*}(\Omega).$$

The energy functional associated to equation (1) is

$$I(u) = \int_\Omega \Phi(|\nabla u|)dx - \int_\Omega j(x,u)dx - \int_\Omega hudx, \quad u \in W_0^{1,\Phi}(\Omega). \tag{25}$$

By Lebourg's Theorem (cf. [5, Theorem 2.3.7]) there are $\tau \in (0,t)$ and $\eta_\tau(x) \in \partial j(x,\tau)$ such that

$$j(x,t) = j(x,0) + \eta_\tau(x)t \ \text{ a.e. } x \in \Omega.$$

Since $s \mapsto \widetilde{\Phi}_*^{-1} \circ \Phi_*(c_2 s)$ is increasing, it follows by (6) that

$$\begin{aligned} |j(x,t)| &\le |j(x,0)| + |\eta_\tau(x)t| \\ &\le |j(x,0)| + [c_1 \widetilde{\Phi}_*^{-1} \circ \Phi_*(c_2 t) + b(x)]|t|. \end{aligned}$$

As a consequence for each $u \in W_0^{1,\Phi}(\Omega)$, $j(x,u) \in L^1(\Omega)$ and by the Hölder inequality $I : W_0^{1,\Phi}(\Omega) \longrightarrow \mathbf{R}$ is defined.

Setting

$$Q(u) = \int_\Omega \Phi(|\nabla u|)dx - \int_\Omega uhdx \quad \text{and} \quad J(u) = \int_\Omega j(x,u)dx, \tag{26}$$

it follows that

$$I(u) = Q(u) - J(u).$$

It can also be shown that,

$$\begin{gathered} Q \in \mathbf{C}^1(W_0^{1,\Phi}(\Omega), \mathbf{R}) \quad \text{and} \\ \langle Q'(u), v\rangle = \int_\Omega \phi(|\nabla u|)\nabla u \nabla v dx - \int_\Omega hv dx, \end{gathered} \tag{27}$$

$$\begin{gathered} J \in \mathrm{Lip}_{\mathrm{loc}}(W_0^{1,\Phi}(\Omega), \mathbf{R}) \quad \text{and} \\ \partial J(u) \subseteq \{\rho \in L_{\widetilde{\Phi}_*} \mid \rho(x) \in \partial j(x, u(x)) \text{ a.e. } x \in \Omega\}. \end{gathered} \tag{28}$$

Thus,

$$I \in \mathrm{Lip}_{\mathrm{loc}}(W_0^{1,\Phi}(\Omega), \mathbf{R}) \quad \text{and} \quad \partial I(u) = Q'(u) - \partial J(u). \tag{29}$$

So if u is a critical point of I in the sense that $0 \in \partial I(u)$, there is some $\rho \in L_{\widetilde{\Phi}_*}(\Omega)$ such that

$$\langle Q'(u), v\rangle - \langle \rho, v\rangle = 0 \ \text{ for } v \in W_0^{1,\Phi}(\Omega).$$

Remark 2.2. In our arguments below, C will denote a positive (*cumulative*) constant.

3. Proof of Theorem 1.3

At first we state and prove some technical results.

Lemma 3.1. *Assume* (ϕ_1)–(ϕ_3) *and* (7)*. Then* I *is weakly lower semicontinuous,* wlsc *for short.*

Proof. Let $(u_n) \subseteq W_0^{1,\Phi}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$. Then $u_n \to u$ in $L_\Phi(\Omega)$ and, by eventually passing to subsequences, $u_n \to u$ a.e. in Ω and there is $\theta_2 \in L^\ell(\Omega)$ such that $|u_n| \le \theta_2$ a.e. in Ω. By (7) we have

$$j(x, u_n) \le A|\theta_2|^\ell + B(x).$$

Since j is a Carathéodory function (remember that j is measurable and $j(x,.) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R},\mathbf{R})$),

$$j(x,u_n(x)) \longrightarrow j(x,u(x)) \ \text{ a.e. } x \in \Omega.$$

By Fatou's Lemma,

$$\limsup \int_\Omega j(x,u_n)dx \leq \int_\Omega j(x,u)dx.$$

Hence

$$I(u) \leq \liminf \left\{ \int_\Omega \Phi(|\nabla u_n|)dx - \int_\Omega j(x,u_n)dx - \int_\Omega h u_n dx \right\} = \liminf I(u_n)$$

showing that I is wlsc. □

The result below due to Le, Motreanu & Motreanu [16], is a variant for the case of Orlicz spaces, of the Aubin-Clarke Theorem.

Theorem 3.2 (Aubin-Clarke). *Assume* $(\phi_1)-(\phi_2)$. *If* $j:\Omega\times\mathbf{R}\longrightarrow\mathbf{R}$ *is a Carathéodory function such that* $j(x,.)\in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R},\mathbf{R})$, $j(x,0)\in L^1(\Omega)$ *and* (6) *holds. Then*
$J \in \mathrm{Lip}_{\mathrm{loc}}(L_{\Phi_*}(\Omega),\mathbf{R})$ *and*

$$\partial J(u) \subseteq \{w \in L_{\widetilde{\Phi}_*}(\Omega) \mid w(x) \in \partial j(x,u(x)) \ \textit{a.e. } x\in\Omega\}. \tag{30}$$

Corollary 3.3. *Under the assumptions of Theorem* 3.2*,*

$$H := J\Big|_{W_0^{1,\Phi}(\Omega)} \in \mathrm{Lip}_{\mathrm{loc}}(W_0^{1,\Phi}(\Omega),\mathbf{R}),$$

$$\partial H(u) \subseteq \{w \in L_{\widetilde{\Phi}_*}(\Omega) \mid w(x) \in \partial j(x,u(x)) \ \textit{a.e. } x\in\Omega\}.$$

Proof. Since $W_0^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi_*}(\Omega)$ and $\mathbf{C}_0^\infty(\Omega) \subseteq W_0^{1,\Phi}(\Omega)$ then $\overline{W_0^{1,\Phi}(\Omega)}^{L_{\Phi_*}} = L_{\Phi_*}(\Omega)$. By [4, Theorem 2.2], H is locally Lipschitz continuous. Again by both [4, Theorem 2.2] and Theorem 3.2 above we get to

$$\partial H(u) \subseteq \partial J(u) \subseteq \{w \in L_{\widetilde{\Phi}_*}(\Omega) \mid w(x) \in \partial j(x,u(x)) \text{ a.e. } x\in\Omega\},$$

endding the proof. □

Proof of Theorem 1.3 (finished). At first we show that I is coercive. Indeed, assume by the way of contradiction, that there is $(u_n) \subseteq W_0^{1,\Phi}(\Omega)$ such that

$$\|\nabla u_n\|_\Phi \to \infty \quad \text{and} \quad I(u_n) \leq C.$$

Using (7) and the Hölder Inequality we have

$$\int_\Omega \Phi(|\nabla u_n|)dx \leq A \int_\Omega |u_n|^\ell dx + 2\|h\|_{\widetilde{\Phi}} \|u_n\|_\Phi + C \tag{31}$$

We claim that $\int_\Omega |u_n|^\ell dx \to \infty$. Indeed, assume on the contrary, that

$$\int_\Omega |u_n|^\ell dx \leq C.$$

By (31) and Poincaré's Inequality (cf. [13]),

$$\int_\Omega \Phi(|\nabla u_n|)dx \leq C(1 + \|\nabla u_n\|_\Phi),$$

which is impossible because by [13, Lemma 3.14],

$$\frac{\int_\Omega \Phi(|\nabla u_n|)dx}{\|\nabla u_n\|_\Phi} \to \infty,$$

showing that $\int_\Omega |u_n|^\ell dx \to \infty$. We infer, using (2) that $\|u_n\|_\Phi \to \infty$.

By (31) and Lemma A.1 in the Appendix, we have

$$\|\nabla u_n\|_\Phi^\ell \leq \int_\Omega \Phi(|\nabla u_n|)dx \leq C\|u_n\|_\Phi^\ell + 2\|h\|_{\widetilde{\Phi}}\|u_n\|_\Phi + C. \tag{32}$$

Dividing in (32) by $\|u_n\|_\Phi^\ell$ we get

$$\|\nabla v_n\|_\Phi^\ell \leq C + \frac{2\|h\|_{\widetilde{\Phi}}}{\|u_n\|_\Phi^{\ell-1}} + \frac{C}{\|u_n\|_\Phi^\ell},$$

where $v_n = \frac{u_n}{\|u_n\|_\Phi}$. It follows that $(\|\nabla v_n\|_\Phi)$ is bounded. Passing to a subsequence, we have,

– $v_n \rightharpoonup v$ in $W_0^{1,\Phi}(\Omega)$ and $\int_\Omega \Phi(|\nabla v|)dx \leq \liminf \int_\Omega \Phi(|\nabla v_n|)dx$,
– $v_n \to v$ in $L_\Phi(\Omega)$ and there is $\theta_3 \in L^\ell(\Omega)$ such that $|v_n| \leq \theta_3$ a.e. in Ω.

We claim that

$$\limsup \int_\Omega \frac{j(x, \|u_n\|_\Phi v_n)}{\|u_n\|_\Phi^\ell} dx \leq \int_{\{v\neq 0\}} A_\infty(x)|v(x)|^\ell dx. \tag{33}$$

Indeed, it follows using (7) that

$$\frac{j(x, \|u_n\|_\Phi v_n(x))}{\|u_n\|_\Phi^\ell} \leq A\theta_3^\ell(x) + B(x). \tag{34}$$

In addition,

$$\limsup \frac{j(x, \|u_n\|_\Phi v_n)}{\|u_n\|_\Phi^\ell} \leq \limsup \frac{j(x, \|u_n\|_\Phi v_n)}{\|u_n\|_\Phi^\ell |v_n(x)|^\ell} |v_n(x)|^\ell \chi_{\{v_n \neq 0\}}$$

and hence

$$\limsup \frac{j(x, \|u_n\|_\Phi v_n)}{(\|u_n\|_\Phi v_n(x))^\ell} |v_n(x)|^\ell \chi_{\{v_n\neq 0\}} \leq A_\infty(x)|v(x)|^\ell, \quad v(x) \neq 0. \tag{35}$$

By (34), (35) and Fatou's Lemma, (33) follows. Using again the fact that Φ is convex and continuous, Lemma A.1, $I(u_n) \leq C$ and (33) it follows that

$$\begin{aligned}
\int_\Omega \Phi(|\nabla v|)dx &\leq \liminf \int_\Omega \Phi(|\nabla v_n|)dx \\
&\leq \liminf \frac{1}{\| u_n\|_\Phi^\ell} \int_\Omega \Phi(|\nabla u_n|)dx \\
&\leq \liminf \left\{ \int_\Omega \left[\frac{j(x, \|u_n\|_\Phi v_n)}{\|u_n\|_\Phi^\ell} + \frac{2\|h\|_{\tilde{\Phi}}}{\|u_n\|_\Phi^{\ell-1}} \right] dx + \frac{C}{\|u_n\|_\Phi^\ell} \right\} \\
&\leq \limsup \int_\Omega \frac{j(x, \|u_n\|_\Phi v_n)}{\|u_n\|_\Phi^\ell} dx \\
&\leq \int_{\{v \neq 0\}} A_\infty(x)|v(x)|^\ell dx,
\end{aligned}$$

which contradicts (8). So, I is coercive. By Lemma 3.1, there is $u \in W_0^{1,\Phi}(\Omega)$ such that

$$I(u) = \min_{v \in W_0^{1,\Phi}(\Omega)} I(v).$$

As a byproduct u is a critical point, that is, $0 \in \partial I(u)$. Hence, there is $\rho \in \partial J(u)$ such that

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla v dx = \int_\Omega \rho v dx + \int_\Omega h v dx, \quad v \in W_0^{1,\Phi}(\Omega)$$

and by Theorem 3.2

$$\rho(x) \in \partial j(x, u(x)) \quad \text{a.e. } x \in \Omega,$$

showing that u is a solution of (1). $\square$

4. Proof of Theorem 1.7

At first we establish some technical results, some of which are variants for the Orlicz spaces framework, of results due to Chang [4].

Proposition 4.1 (cf. Chang [4]). *Let $f : \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ be a Baire measurable function, assume that f is monotone non-decreasing in the second variable, and set*

$$j(x,t) := \int_0^t f(x,s)ds. \tag{36}$$

Then $j(x,.) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}, \mathbf{R})$ and

$$\partial j(x, u(x)) = [f(x, u(x) - 0), f(x, u(x) + 0)]. \tag{37}$$

Corollary 4.2. *Assume $(\phi_1) - (\phi_3)$. Let $f : \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ be a Baire measurable function, non-decreasing in the second variable, satisfying (19). Then J is convex and*

$$\partial J(u)(x) = [f(x, u(x) - 0), f(x, u(x) + 0)].$$

Proof. Proceeding as in [15, Lemma 3.2.2] we find that J is convex. Now let $\eta \in \partial j(x, u(x))$. By Proposition 4.1,

$$\eta(x) \in \partial j(x, u(x)) = [f(x, u(x) - 0), f(x, u(x) + 0)].$$

Using (19) we get

$$\begin{aligned} |\eta(x)| &\leq \max\{|f(x, u(x) - 0)|, |f(x, u(x) + 0)|\} \\ &\leq c_1 \widetilde{\Phi}_*^{-1} \circ \Phi_*(c_2 u(x)) + b(x), \end{aligned}$$

which is condition (6). Employing both Theorem 3.2 and Corollary 3.3, we have

$$\partial J(u)(x) \subseteq [f(x, u(x) - 0), f(x, u(x) + 0)].$$

On the other hand, take $w \in L_{\widetilde{\Phi}*}(\Omega)$ with $w(x) \in [f(x, u(x) - 0), f(x, u(x) + 0)]$. Then

$$\begin{aligned} w(x)(v(x) - u(x)) &\leq \int_{u(x)}^{v(x)} f(x, t)dt \\ &= \left(\int_0^{v(x)} - \int_0^{u(x)}\right) f(x, t)dt \\ &= j(x, v(x)) - j(x, u(x)). \end{aligned}$$

Integrating we have,

$$\int_\Omega w(x)(v(x) - u(x))dx \leq \int_\Omega j(x, v(x))dx - \int_\Omega j(x, u(x))dx = J(v) - J(u),$$

that is,

$$\langle w, v - u\rangle \leq J(v) - J(u).$$

Since J is convex it follows (cf. [3, p. 25]) that

$$\partial J(u) = \{w \in L_{\widetilde{\Phi}_*}(\Omega) \mid \langle w, v - u\rangle \leq J(v) - J(u),\ v \in L_{\Phi_*}(\Omega)\}.$$

We infer that $w \in \partial J(u)$, ending the proof. $\square$

Proof of Theorem 1.7 (finished). Since f is Baire-measurable and non-decreasing in the real variable, it follows by Corollary 4.2 that

$$\partial J(u)(x) = \partial j(x, v(x)) = [f(x, v(x) - 0), f(x, v(x) + 0)], \quad v \in W_0^{1,\Phi}(\Omega).$$

Let $u \in W_0^{1,\Phi}(\Omega)$ be a solution of (1) which minimizes the energy functional I, given for instance by Theorem 1.3. Then

$$-\Delta_\Phi u \in \partial j(x, u) + h \quad \text{and} \quad I(u) \leq I(u + \epsilon\varphi) \ \text{ for } \epsilon > 0 \text{ and } \varphi \in W_0^{1,\Phi}(\Omega).$$

It follows that

$$\int_\Omega \frac{j(x,u+\epsilon\varphi)-j(x,u)}{\epsilon}dx \le \int_\Omega \frac{\Phi(|\nabla(u+\epsilon\varphi)|)-\Phi(|\nabla u|)}{\epsilon}dx - \int_\Omega h\varphi dx. \quad (38)$$

We distinguish between two cases:

Case 1. $\varphi \in \mathbf{C}_0^\infty(\Omega)$ and $\varphi \ge 0$. By Lebourg's Theorem there is ρ_ϵ such that

$$\frac{j(x,u+\epsilon\varphi)-j(x,u)}{\epsilon} = \xi_\epsilon\varphi,$$

where

$$u < \rho_\epsilon < u+\epsilon\varphi, \quad \xi_\epsilon \in \partial j(x,\rho_\epsilon) = [f(x,\rho_\epsilon - 0), \quad f(x,\rho_\epsilon+0)].$$

By (19) we get

$$\begin{aligned} |\xi_\epsilon| &\le |f(x,\rho_\epsilon-0)| + |f(x,\rho_\epsilon+0)| \\ &\le 2c_1\widetilde{\Phi}_*^{-1}\circ\Phi_*(c_2\rho_\epsilon) + 2b(x). \end{aligned}$$

We infer that

$$\left|\frac{j(x,u+\epsilon\varphi)-j(x,u)}{\epsilon}\right| \le [2c_1\widetilde{\Phi}_*^{-1}\circ\Phi_*(c_2|u|+c_2|\varphi|)+2b(x)]\varphi \in L^1(\Omega).$$

By easy arguments,

$$\lim_{\epsilon\to 0_+} \frac{j(x,u+\epsilon\varphi)-j(x,u)}{\epsilon} = f(x,u(x)+0), \quad \text{qtp } x\in\Omega,$$

and applying Lebesgue's Theorem in (38) we get

$$\begin{aligned} \int_\Omega f(x,u(x)+0)\varphi dx &\le \int_\Omega \phi(|\nabla u|)\nabla u\nabla\varphi dx - \int_\Omega h\varphi dx \\ &= \int_\Omega (-\Delta_\Phi u - h)\varphi dx, \quad \varphi\in W_0^{1,\Phi}(\Omega), \end{aligned}$$

Hence

$$-\Delta_\Phi u(x) \ge f(x,u(x)+0)+h(x) \ \text{ a.e. } x\in\Omega,$$

which gives

$$-\Delta_\Phi u = f(x,u(x)+0)+h(x) \ \text{ a.e. } x\in\Omega. \quad (39)$$

Case 2. $\varphi \in \mathbf{C}_0^\infty(\Omega)$ and $\varphi \le 0$. By arguments similar to the ones in Case 1 we get to

$$-\Delta_\Phi u = f(x,u(x)-0)+h(x) \ \text{ a.e. } x\in\Omega. \quad (40)$$

Assume, by the way of contradiction, that $|\Gamma_a(u)| \ne 0$. By (39) and (40) we have

$$f(x,a-0) = f(x,a+0) \ \text{ a.e. } x\in\Gamma_a(u),$$

imposible. Hence $|\Gamma_a(u)| = 0$.

Since u is a solution of (1) there is $\rho \in \partial J(u)$ such that

$$\int_\Omega [-\Delta_\Phi u - \rho - h]\varphi dx = 0, \quad \varphi \in W_0^{1,\Phi}(\Omega).$$

Hence

$$-\Delta_\Phi u(x) = \rho(x) + h(x) \text{ a.e. } x \in \Omega,$$

and since $f \in \mathbf{C}(\mathbf{R} - \{a\})$ and $|\Gamma_a(u)| = 0$ we infer that

$$-\Delta_\Phi u(x) = f(x, u(x)) + h(x) \text{ a.e. } x \in \Omega. \qquad \square$$

5. Proofs of Theorems 1.4 and 1.9

In order to prove Theorem 1.4 some technical results are needed. Set

$$\phi(t) = \gamma \frac{(\sqrt{1+t^2} - 1)^{\gamma-1}}{\sqrt{1+t^2}}, \quad t \geq 0. \tag{41}$$

Lemma 5.1. *The function ϕ satisfies $(\phi_1), (\phi_2)$, and*

$$\gamma \leq \frac{t^2\phi(t)}{\Phi(t)} \leq 2\gamma, \quad t > 0. \tag{42}$$

In addition, when $1 < \gamma < N$,

$$L_\Phi(\Omega) = L^\gamma(\Omega),$$

$$|u|_\Phi \leq |u|_\gamma, \ u \in L_\Phi(\Omega), \tag{43}$$

and

$$W_0^{1,\Phi}(\Omega) = W_0^{1,\gamma}(\Omega).$$

Proof. Of course $\phi \in C(0, \infty)$. By a direct computation we infer that for $t > 0$,

$$\lim_{t\to 0} t\phi(t) = 0, \quad \lim_{t\to\infty} t\phi(t) = \infty, \quad (t\phi(t))' > 0 \text{ and } \gamma \leq \frac{t^2\phi(t)}{\Phi(t)} \leq 2\gamma,$$

showing (ϕ_1), (ϕ_2) and (42). To prove that $L_\Phi(\Omega) = L^\gamma(\Omega)$, we point out that

$$\Phi(t) \leq t^\gamma, \quad t \geq 0 \quad \text{and} \quad \Phi(t) \geq \frac{1}{2^\gamma} t^\gamma, \quad t \geq 2.$$

By [15, Theorem 3.17.3], $L_\Phi(\Omega) = L^\gamma(\Omega)$. As a consequence, $W_0^{1,\Phi}(\Omega) = W_0^{1,\gamma}(\Omega)$, In order to show (43), take $u \in L^\gamma(\Omega)$ and $k > 0$ and notice that

$$\int_\Omega \frac{|u(x)|^\gamma}{k^\gamma} = 1 \text{ iff } |u|_\gamma = k.$$

Since $\Phi(t) \leq t^\gamma$ for $t \geq 0$ we have

$$\int_\Omega \Phi\left(\frac{u}{k}\right) dx \leq \frac{|u|_\gamma^\gamma}{k^\gamma}.$$

Setting $k = |u|_\gamma$ we get (43). This proves Lemma 5.1. □

Lemma 5.2. *Assume ϕ as in* (41). *Assume in addition* (13). *Then* (8) *holds.*

Proof. By computing we get $\Phi(t) = (\sqrt{1+t^2} - 1)^\gamma$. Let $u \in W_0^{1,\Phi}(\Omega)$ with $\|u\|_\Phi = 1$. Set $v = u/|u|_\gamma$. By (13) there is $\delta > 0$ such that

$$\delta \leq \int_\Omega \Phi\left(\frac{|\nabla u|}{|u|_\gamma}\right) dx - \frac{1}{|u|_\gamma^\gamma} \int_{u\neq 0} A_\infty(x)|u|^\gamma dx.$$

Using the inequality $\|u\|_\Phi \leq |u|_\gamma$ given by Lemma 5.1 we have, applying Lemma A.1,

$$\begin{aligned}
\delta \leq\ & \max\left\{\frac{1}{|u|_\gamma^\gamma}, \frac{1}{|u|_\gamma^{2\gamma}}\right\} \int_\Omega \Phi\left(|\nabla u|\right) dx - \frac{1}{|u|_\gamma^\gamma} \int_{u\neq 0} A_\infty(x)|u|^\gamma dx \\
=\ & \frac{1}{|u|_\gamma^\gamma} \left\{ \int_\Omega \Phi\left(|\nabla u|\right) dx - \int_{u\neq 0} A_\infty(x)|u|^\gamma dx \right\} \\
\leq\ & \int_\Omega \Phi\left(|\nabla u|\right) dx - \int_{u\neq 0} A_\infty(x)|u|^\gamma dx
\end{aligned}$$

Hence

$$\inf_{v\in W_0^{1,\Phi}(\Omega),\ \|v\|_\Phi = 1} \left\{ \int_\Omega \left(\sqrt{1+|\nabla v|^2} - 1\right)^\gamma dx - \int_{\{v\neq 0\}} A_\infty(x)|v(x)|^\gamma dx \right\} > 0.$$

This proves Lemma 5.2. □

Lemma 5.3. *Assume* $1 < \gamma, 2\gamma < N$ *and* (11). *Then* (6) *holds.*

Proof. Set $\Phi(t) = t^\gamma$. By Lemma A.2 (in the Appendix), we have for $t \geq 1$,

$$\Phi_*(1)\ t^{\gamma^*} \leq \Phi_*(t).$$

Using the inequality above and (11) we get (6). This proves Lemma 5.3. □

Proof of Theorem 1.4. As a consequence of Lemmas 5.1, 5.2 and 5.3, Theorem 1.3 applies ending the proof of Theorem 1.4. □

Proof of Theorem 1.9. It is enough to observe that condition (20) is equivalent to condition (19) of Theorem 1.7, since $L_{\Phi_\star}(\Omega) = L^{\gamma^\star}(\Omega)$. Now it is enough to apply Theorem 1.7.

Proof of Proposition 1.5. Based on [11] choose $v \in W_0^{1,\Phi}(\Omega)$ such that

$$\|v\|_\Phi = 1 \quad \text{and} \quad \lambda_1 = \int_\Omega \Phi(|\nabla v|)dx.$$

We have

$$\begin{aligned}\int_\Omega \Phi(|\nabla v|)dx - \int_{v\neq 0} C_\ell \alpha(x)|v|^\ell dx &= \lambda_1 - \left(\int_{\Omega_0} + \int_{\Omega-\Omega_0}\right) C_\ell \alpha(x)|v|^\ell dx \\ &> \lambda_1 - \lambda_1 C_\ell \int_\Omega |v|^\ell dx \\ &\geq \lambda_1 - \lambda_1 \|v\|_\Phi^\ell = 0.\end{aligned} \tag{44}$$

If on the other hand if $v \in W_0^{1,\Phi}(\Omega)$ satisfies

$$\|v\|_\Phi = 1 \quad \text{and} \quad \lambda_1 < \int_\Omega \Phi(|\nabla v|)dx,$$

then

$$\int_\Omega \Phi(|\nabla v|)dx - \int_{v\neq 0} C_\ell \alpha(x)|v|^\ell dx > \lambda_1 - \lambda_1 \|v\|_\Phi^\ell = 0. \tag{45}$$

By (44)–(45),

$$\int_\Omega \Phi(|\nabla v|)dx - \int_{v\neq 0} C_\ell \alpha(x)|v|^\ell dx > 0, \quad v \in W_0^{1,\Phi}(\Omega),\ \|v\|_\Phi = 1.$$

Now let

$$i(C_\ell\alpha) = \inf_{v\in W_0^{1,\Phi}(\Omega),\ \|v\|_\Phi = 1} \left\{\int_\Omega \Phi(|\nabla v|)dx - \int_{v\neq 0} C_\ell \alpha(x)|v|^\ell dx\right\}.$$

Let (v_n) be a minimizing sequence for $i(\alpha)$, that is $v_n \in W_0^{1,\Phi}(\Omega)$, $\|v_n\|_\Phi = 1$ and

$$\int_\Omega \Phi(|\nabla v_n|)dx - \int_{\{v_n\neq 0\}} C_\ell \alpha(x)|v_n|^\ell dx \longrightarrow i(C_\ell\alpha).$$

We claim that $(\|\nabla v_n\|_\Phi)$ is bounded. Indeed, assume that $\|\nabla v_n\|_\Phi \to \infty$ so that $\|\nabla v_n\|_\Phi \geq 1$.

Let $\varepsilon > 0$. Then for n large,

$$\int_\Omega \Phi(|\nabla v_n|)dx - \int_{\{v_n\neq 0\}} C_\ell \alpha(x)|v_n|^\ell dx \leq i(C_\ell\alpha) + \varepsilon. \tag{46}$$

Since $\alpha(x) \leq \lambda_1$ and $\|v_n\|_\Phi = 1$ we get

$$\begin{aligned}\int_\Omega \Phi(|\nabla v_n|)dx &\leq i(C_\ell\alpha) + \varepsilon + \int_{\{v_n\neq 0\}} C_\ell \alpha(x)|v_n|^\ell dx \\ &\leq i(C_\ell\alpha) + \varepsilon + \lambda_1.\end{aligned} \tag{47}$$

We infer that

$$\|\nabla v_n\|_\Phi \le \int_\Omega \Phi(|\nabla v_n|)dx \le i(C_\ell \alpha) + \varepsilon + \lambda_1,$$

a contradiction. Thus $(\|\nabla v_n\|_\Phi)$ is bounded.

Passing to a subsequence we find some $v \in W_0^{1,\Phi}(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{and} \quad \int_\Omega \Phi(|\nabla v|)dx \le \liminf_n \int_\Omega \Phi(|\nabla v_n|)dx.$$

Employing the compactness of the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L_\Phi(\Omega)$ we get

$$v_n \to v \ \text{ in } L_\Phi(\Omega) \quad \text{and} \quad \|v\|_\Phi = 1, \quad v_n \to v \text{ a.e. in } \Omega,$$

$$\text{there is } \theta_1 \in L_\Phi(\Omega) \ \text{ such that } |v_n| \le \theta_1 \text{ a.e. in } \Omega.$$

By Lebesgue's Theorem and (46) we get to

$$\begin{aligned} 0 &< \int_\Omega \Phi(|\nabla v|)dx - \int_{\{v_n \neq 0\}} C_\ell \alpha(x)|v|^\ell dx \\ &\le \liminf_n \left\{ \int_\Omega \Phi(|\nabla v_n|)dx - \int_{\{v_n \neq 0\}} C_\ell \alpha |v_n|^\ell dx \right\} \\ &\le i(C_\ell \alpha) + \varepsilon \end{aligned}$$

for each $\epsilon > 0$. It follows that $i(A_\infty) \ge i(C_\ell \alpha) > 0$. It follows that (8) holds. This ends the proof of Proposition 1.5. □

A. Appendix

We refer the reader to [8] for the lemmas below whose proofs are elementary.

Lemma A.1. *Assume that ϕ satisfies (ϕ_1)–(ϕ_3). Set*

$$\zeta_0(t) = \min\{t^\ell, t^m\}, \quad \zeta_1(t) = \max\{t^\ell, t^m\}, \quad t \ge 0.$$

Then Φ satisfies

$$\zeta_0(t)\Phi(\rho) \le \Phi(\rho t) \le \zeta_1(t)\Phi(\rho), \quad \rho, t > 0,$$

$$\zeta_0(\|u\|_\Phi) \le \int_\Omega \Phi(u)dx \le \zeta_1(\|u\|_\Phi), \quad u \in L_\Phi(\Omega).$$

Lemma A.2. *Assume that ϕ satisfies (ϕ_1)–(ϕ_3). Set*

$$\zeta_2(t) = \min\{t^{\ell^*}, t^{m^*}\}, \quad \zeta_3(t) = \max\{t^{\ell^*}, t^{m^*}\}, \quad t \ge 0$$

where $1 < \ell, m < N$ and $m^* = \frac{mN}{N-m}$, $\ell^* = \frac{\ell N}{N-\ell}$. Then

$$\ell^* \le \frac{t^2 \Phi_*'(t)}{\Phi_*(t)} \le m^*, \quad t > 0,$$

$$\zeta_2(t)\Phi_*(\rho) \le \Phi_*(\rho t) \le \zeta_3(t)\Phi_*(\rho), \quad \rho, t > 0,$$

$$\zeta_2(\|u\|_{\Phi_*}) \le \int_\Omega \Phi_*(u)dx \le \zeta_3(\|u\|_{\Phi_*}), \quad u \in L_{\Phi_*}(\Omega).$$

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