Journal of Convex Analysis
Volume 21 (2014), No. 1, 237–252

From the Uniform Approximation of a Solution of the PDE to the L^2-Approximation of the Gradient of the Solution

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Received: November 23, 2012
Revised manuscript received: April 10, 2013
Accepted: April 11, 2013

We establish a new energy inequality for the difference of two semiconvex functions in a bounded open convex set D of R^n. This inequality is applied to the L^2-approximation problem of the gradient of the unknown solution of the nonlinear elliptic partial differential equation provided that the latter solution is a semiconvex function in D.

Keywords: Semiconvex and semiconcave functions, energy inequality, L^2-approximation, gradient of semiconvex function, convex envelope

2010 Mathematics Subject Classification: 26B25, 35J60, 49L25

1. Introduction

The semiconvexity, as well as semiconcavity, is a natural property of a large class of value functions of the optimization problems (see for instance Cannarsa and Sinestrari [3]).

Semiconvexity of the value function in the spatial variable was established in Krylov [13, Section 4.2] in the problem of maximization of the payoff function for stochastic control of diffusion processes on a fixed time interval $[t_0, t_1]$.

Lions [15] proved semiconcavity of the value function in the minimization problem of the cost function for stochastic control problems until exit from a bounded cylindrical region $Q = [t_0, t_1) \times D$ or until exit from D in the discounted infinite horizon problem.

For the optimal control of the reflected diffusion processes the semiconcavity of

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ISSN 0944-6532 / $ 2.50

the value function was shown in Lions [16, Section 3] and in Lions and Trudinger [17, Section 5].

It is well-known that for controlled Markov diffusion processes on n-dimensional Euclidean domain D the value function of the corresponding stochastic optimal control problem is the unique viscosity solution of the nonlinear partial differential equation of the second order, called a Hamilton–Jacobi–Bellman (HJB) equation with appropriate boundary conditions (see Fleming and Soner [6]).

For most optimal stochastic control models arising from applications, the HJB equation can only be solved approximately by numerical computations.

The finite difference numerical approximations for HJB equation are considered in Fleming and Soner [6, Chapter 9].

The convergence rates of finite difference approximations for second-order HJB equations were established by Krylov [14], Barles and Jakobsen [2] and many other authors. In case of fully nonlinear elliptic PDE and for monotone finite difference approximations the rate of convergence is given in Caffarelli and Souganidis [4].

In those papers the uniform convergence rate $\|v_\delta - v\|_{L^\infty(D)}$ is estimated, where v is the value function of the discounted optimal stochastic control problem on an infinite time horizon, v_δ is the solution of the discretized approximation scheme and δ is a small parameter measuring the mesh size.

However numerical computation of the value function v is insufficient for constructing the optimal Markov control policies, as the numerical computation of the gradient $\operatorname{grad} v$ and the Hessian matrix $\operatorname{Hess} v$ are equally needed.

In this paper we propose a method of L^2-approximation of the (Sobolev) gradient $\operatorname{grad} v$ under the basic assumption that the value function v be semiconvex (or semiconcave).

In case the value function is the subsolution of the linear second order uniformly elliptic partial differential equation, a similar approach has been already developed by Saleem and Shashiashvili [20].

It is a remarkable fact that semiconvexity of the solution is not restricted to the Hamilton–Jacobi–Bellman equation, but it holds also for more general class of fully nonlinear partial differential equations. Indeed, Alvarez, Lasry and Lions [1] considered viscosity solutions of the fully nonlinear second order elliptic partial differential equation

$$F(x, u, \operatorname{grad} u, \operatorname{Hess} u) = 0 \tag{1}$$

in a bounded open convex subset D of R^n.

Here $u \in C(\overline{D})$ is a real valued function, $\operatorname{grad} u$ is the gradient-vector of u, $\operatorname{Hess} u$ denotes the matrix of second order partial derivatives of u, $F \in C(\overline{D} \times R \times R^n \times S^n)$ is degenerate elliptic, i.e.

$$F(x, r, p, A) \leq F(x, r, p, B) \quad \text{provided } A \geq B. \tag{2}$$

S^n (S^n_{++}) are the sets of the symmetric $n \times n$ (respectively symmetric positive definite) matrices.

The authors [1] require the following concavity assumption, that the mapping

$$(x, r, A) \longrightarrow F(x, r, p, A^{-1}) \tag{3}$$

be concave for $(x, r, p, A) \in \overline{D} \times R \times R^n \times S^n_{++}$.

Given a continuous viscosity solution u, $u \in C(\overline{D})$, with the state constraints boundary conditions the authors establish [1, Section 3, Theorem 1] that the function $u(x)$, $x \in \overline{D}$ is convex in $\overline{D}$. This result is the generalization of the corresponding results of the previous authors Korevaar [12] and Kennington [11], who proved convexity of the solution of the quasilinear elliptic equation

$$-tr\big[a(\operatorname{grad} u) \cdot \operatorname{Hess} u\big] + b(x, u, \operatorname{grad} u) = 0 \quad \text{in } D \tag{4}$$

with the specific boundary conditions, where $a(p) \in S^n_{++}$ for every p and $b(x, r, p)$ is nondecreasing in r, where $b \in C(D \times R \times R^n)$.

The plan of this paper is as follows. In Section 2 we prove the new energy inequality for the difference of two bounded semiconvex functions in a nonempty bounded open convex set D of R^n. We note that no smoothness assumptions are required from the convex set D.

This inequality is the generalization of the corresponding one-dimensional inequality originally established by K. Shashiashvili and M. Shashiashvili [21] and subsequently applied to the hedging problems of Mathematical Finance in Hussain and Shashiashvili [9].

In Section 3 we consider the L^2-approximation problem of the gradient of the analytically unknown viscosity solution of the nonlinear second order elliptic partial differential equation.

Our method of approximation essentially uses the convex envelope $\Gamma(v)$ of a given bounded continuous function v in a domain D (see Alvarez, Lasry and Lions [1] or Oberman [18]).

2. Energy Inequality for the Difference of Two Bounded Semiconvex Functions

Throughout the paper we fix a nonempty bounded open convex subset D of Euclidean space R^n. For a continuous function $u : D \to R$ let us introduce the second order difference quotients

$$\Delta^2 u = \frac{1}{h^2}\,\left[u(x + h \cdot \gamma) + u(x - h \cdot \gamma) - 2u(x)\right], \tag{5}$$

where $h > 0$ is a scalar variable, $x \in D$, $\gamma \in R^n$ is a direction, $|\gamma| = 1$ (i.e. a unit vector in R^n) and it is assumed that $[x - h \cdot \gamma, x + h \cdot \gamma] \subset D$.

Definition 2.1 (see Cannarsa and Sinestrari [3, Definition 1.1.1]). We say that a continuous function $u : D \to R$ is semiconvex if there exists $c \geq 0$, such that

$$\Delta^2 u \geq -c \tag{6}$$

for any $h > 0$, $x \in D$, $\gamma \in R^n$ with $|\gamma| = 1$ for which $[x - h \cdot \gamma, x + h \cdot \gamma] \subset D$. The constant c is called the semiconvexity constant (obviously it is not unique).

A continuous function $u : D \to R$ is called semiconcave if $(-u)$ is semiconvex.

It is well-known that a continuous function $u : D \to R$ is convex if and only if it is semiconvex with $c = 0$.

The following equivalent characterization of the semiconvex functions will be used several times from now on (Cannarsa and Sinestrari [3, Proposition 1.1.3]):

the real valued function $u : D \to R$ is semiconvex in D with semiconvexity constant $c \geq 0$, if and only if the function

$$u_c(x) = u(x) + \frac{c}{2} \cdot |x|^2 \tag{7}$$

is convex in D.

Denote

$$v_0(x) = \frac{1}{2} \cdot |x|^2, \tag{8}$$

then we can write

$$u(x) = u_c(x) - c \cdot v_0(x), \quad x \in D,$$

from which we immediately get, that any semiconvex function is locally Lipschitz in D (see Evans and Gariepy [5, Section 6.3]). Locally Lipschitz functions belong to the Sobolev space $W^{1,\infty}_{loc}(D)$ and hence they are Sobolev functions in the terminology of Evans and Gariepy [5, Chapter 4]. This means, that the weak (Sobolev) gradient

$$\operatorname{grad} u(x) = \left(\frac{\partial u(x)}{\partial x_1}, \ldots, \frac{\partial u(x)}{\partial x_n} \right)$$

exists and is locally bounded in D. By Rademacher's remarkable theorem locally Lipschitz functions are differentiable a.e., thus the Sobolev gradient is equal to the classical gradient a.e. with respect to n-dimensional Lebesgue measure.

Let us introduce the distance function $d_{\partial D}(x)$ from the arbitrary point $x \in R^n$ to the boundary ∂D of the open convex set D.

We recall that the distance function from a given nonempty closed set $C \subset R^n$ is defined by

$$d_C(x) = \min_{y \in C} |y - x|, \quad x \in R^n.$$

We formulate now the main result of this section the origin of which is the corresponding one-dimensional inequality in K. Shashiashvili and M. Shashiashvili [21].

Theorem 2.2. *Let u and v be two bounded semiconvex functions in a nonempty bounded open convex subset D of R^n with the semiconvexity constants c_u and c_v, respectively.*

Then the following weighted energy inequality is valid for the difference $u - v$

$$\begin{aligned}
&\int\limits_D |\operatorname{grad} u - \operatorname{grad} v|^2 \cdot \frac{d^2_{\partial D}}{n}\, dx + \int\limits_D (u-v)^2\, dx \\
&\leq 5 \operatorname{meas} D \cdot \|u - v\|_{L^\infty(D)} \\
&\qquad \times \left(\|u\|_{L^\infty(D)} + \|v\|_{L^\infty(D)} + 2\max(c_u, c_v) \cdot \|v_0\|_{L^\infty(D)}\right). \qquad (9)
\end{aligned}$$

Proof. It is enough to establish the inequality (9) for convex functions. Indeed, from the characterization (7) of semiconvex functions we get that the following functions are convex in D

$$\begin{aligned}
\widetilde{u} &= u + \max(c_u, c_v) \cdot v_0, \\
\widetilde{v} &= v + \max(c_u, c_v) \cdot v_0.
\end{aligned} \qquad (10)$$

Suppose the energy inequality is valid for convex functions $\widetilde{u}$ and $\widetilde{v}$ (with $c_{\widetilde{u}} = c_{\widetilde{v}} = 0$). Then we have

$$\begin{aligned}
&\int\limits_D \left|\operatorname{grad} \widetilde{u} - \operatorname{grad} \widetilde{v}\right|^2 \cdot \frac{d^2_{\partial D}}{n}\, dx + \int\limits_D (\widetilde{u} - \widetilde{v})^2\, dx \\
&\leq 5 \operatorname{meas} D \cdot \|\widetilde{u} - \widetilde{v}\|_{L^\infty(D)} \left(\|\widetilde{u}\|_{L^\infty(D)} + \|\widetilde{v}\|_{L^\infty(D)}\right), \qquad (11)
\end{aligned}$$

from which (9) follows trivially.

To prove the estimate (9) for arbitrary nonsmooth bounded convex functions and nonsmooth convex domains D we need to approximate them by smooth ones. For this purpose we shall follow standard mollification arguments from Evans and Gariepy [5, Chapter 4].

Denote

$$\begin{aligned}
u_k(x) &= k^n \int\limits_D \eta(k(x-y)) \cdot u(y)\, dy, \quad k = 1, 2, \ldots, \\
v_k(x) &= k^n \int\limits_D \eta(k(x-y)) \cdot v(y)\, dy, \quad k = 1, 2, \ldots,
\end{aligned} \qquad (12)$$

where

$$\eta(x) = \begin{cases} c \cdot \exp\left(\dfrac{1}{|x|^2 - 1}\right), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases} \qquad (13)$$

and $c > 0$ is a constant, such that

$$\int\limits_{R^n} \eta(x)\, dx = 1,$$

$\eta(x)$, $x \in R^n$, is called a smoothing function. It is well-known that the functions u_k, v_k, $k = 1, 2, \ldots$ are infinitely differentiable in R^n.

Hammer [8] constructs smooth open convex subsets D_m, $m = 1, 2, \ldots$ of D such that

$$\overline{D}_m \subset D_{m+1} \subset \overline{D}_{m+1} \subset D \quad \text{and} \quad \bigcup_{m=1}^{\infty} D_m = D \tag{14}$$

with $\partial D_m \in C^\infty$, $m = 1, 2, \ldots$.

We fix arbitrary m, $m = 1, 2, \ldots$.

For fixed m we take k sufficiently large, in particular, we shall consider $k \geq k_m$, where

$$k_m = \inf\left\{k : \frac{1}{k} < \frac{1}{2} \cdot \operatorname{dist}(D_m, \partial D)\right\}. \tag{15}$$

Then we have

$$\begin{cases} u_k(x) = \displaystyle\int\limits_{B(0,1)} \eta(z) \cdot u\left(x - \frac{z}{k}\right) dz, \quad x \in \overline{D}_m, \\ v_k(x) = \displaystyle\int\limits_{B(0,1)} \eta(z) \cdot v\left(x - \frac{z}{k}\right) dz, \quad x \in \overline{D}_m, \end{cases} \tag{16}$$

if $k \geq k_m$, from which it is obvious that the functions

$$u_k, v_k : \overline{D}_m \to R, \quad k \geq k_m$$

are convex functions on $\overline{D}_m$.

From Evans and Gariepy [5, Chapter 4, Theorem 1] we get

$$\|u_k - u\|_{L^\infty(D_m)} \underset{k\to\infty}{\longrightarrow} 0, \qquad \|v_k - v\|_{L^\infty(D_m)} \underset{k\to\infty}{\longrightarrow} 0, \tag{17}$$

$$\frac{\partial u_k}{\partial x_i} \underset{k\to\infty}{\longrightarrow} \frac{\partial u}{\partial x_i}, \qquad \frac{\partial v_k}{\partial x_i} \underset{k\to\infty}{\longrightarrow} \frac{\partial v}{\partial x_i}, \quad i = 1, \ldots, n \tag{18}$$

in $L^p(D_m)$, $1 \leq p < \infty$ and in particular for $p = 2$.

For each fixed m define the weight function $h_m(x)$, $x \in \overline{D}_m$ as the unique solution of the following Dirichlet problem

$$\begin{cases} \Delta h_m(x) = -1 & \text{in } D_m, \\ h_m(x) = 0 & \text{on } \partial D_m, \end{cases} \tag{19}$$

where Δ is the Laplace operator.

As the domain D_m is smooth we can apply Theorem 6.14 from Gilbarg and Trudinger [7, Chapter 6], from which we know that the Dirichlet problem (19) has a unique solution h_m, which is smooth up to the boundary, i.e. $h_m \in C^2(\overline{D}_m)$. By the Hopf's strong maximum principle we have

$$h_m(x) > 0 \ \text{ in } D_m. \tag{20}$$

Take a ball $B(a,R)$ such that $\overline{D} \subset B(a,R)$ and consider the Dirichlet problem in $B(a,R)$

$$\begin{cases} \Delta h_B(x) = -1 & \text{in } B, \\ h_B(x) = 0 & \text{on } \partial B. \end{cases} \tag{21}$$

It is well known, that the unique solution of the latter problem is given by

$$h_B(x) = \frac{R^2 - |x-a|^2}{2n}\,, \quad x \in \overline{B}. \tag{22}$$

The function $h_B(x)$ satisfies the same equation in D_m

$$\begin{cases} \Delta h_B(x) = -1 & \text{in } D_m, \\ h_B(x) > 0 & \text{on } \partial D_m, \end{cases}$$

hence we get for the difference $h_B - h_m$

$$\begin{cases} \Delta (h_B - h_m) = 0 & \text{in } D_m, \\ (h_B - h_m) > 0 & \text{on } \partial D_m. \end{cases} \tag{23}$$

By the maximum principle we obtain

$$h_m(x) < h_B(x) \ \text{ in } \overline{D}_m,$$

hence the bound

$$0 \leq h_m(x) \leq \frac{R^2}{2n}\,, \quad x \in \overline{D}_m. \tag{24}$$

The similar argument shows that

$$h_m(x) \leq h_{m+1}(x), \quad x \in \overline{D}_m, \tag{25}$$

and the same inequality will hold if we trivially extend the functions $h_m(x)$ outside $\overline{D}_m$ to be equal to zero.

Thus we get that the sequence of nonnegative functions h_m, $m = 1, 2, \dots,$ is nondecreasing and bounded by constant, hence it has a finite limit

$$h_\infty(x) = \lim_{m\to\infty} h_m(x), \qquad 0 \leq h_\infty(x) \leq \frac{R^2}{2n}\,, \tag{26}$$

where $x \in \overline{D}$.

Let us prove now the following lower bound for $h_\infty(x)$

$$h_\infty(x) \geq \frac{d^2_{\partial D}(x)}{2n}, \quad x \in \overline{D}. \tag{27}$$

Take arbitrary point $x_0 \in D$ and choose $r > 0$ such that $\overline{B}(x_0, r) \subset D$. As $\bigcup_{m=1}^{\infty} D_m = D$, we conclude that there exists m_0, such that

$$\overline{B}(x_0, r) \subset D_m \ \text{ if } m \geq m_0.$$

Consider the Dirichlet problem in $B(x_0, r)$

$$\begin{cases} \Delta \widetilde{h}(x) = -1 & \text{in } B(x_0, r), \\ \widetilde{h}(x) = 0 & \text{on } \partial B(x_0, r). \end{cases} \tag{28}$$

As in (22) we have

$$\widetilde{h}(x) = \frac{r^2 - |x - x_0|^2}{2n}, \quad x \in \overline{B}(x_0, r). \tag{29}$$

Consider the difference $h_m - \widetilde{h}$ in the ball $B(x_0, r)$ for $m \geq m_0$, it satisfies the Laplace equation

$$\Delta(h_m - \widetilde{h}) = 0 \ \text{ in } B(x_0, r)$$

with the boundary condition

$$h_m - \widetilde{h} > 0 \ \text{ on } \partial B(x_0, r).$$

By the maximum principle we have

$$h_m(x) > \widetilde{h}(x) \ \text{ on } \overline{B}(x_0, r). \tag{30}$$

Take here $x = x_0$, then we get

$$h_m(x_0) > \frac{r^2}{2n} \ \text{ for } m \geq m_0. \tag{31}$$

Let us pass to the limit $m \to \infty$ in the latter inequality, we shall have

$$h_\infty(x) \geq \frac{r^2}{2n}.$$

Now letting r tend to the distance $d_{\partial D}(x_0)$, then we obtain ultimately

$$h_\infty(x_0) \geq \frac{d^2_{\partial D}(x_0)}{2n}, \quad x_0 \in D,$$

which is the desired inequality (27).

We fix m and take arbitrary k, $k \geq k_m$, where k_m is defined by (15). We know from the formulas (16) that the functions u_k, v_k are the infinitely differentiable convex functions on $\overline{D}_m$.

Introduce the notation

$$w_k = u_k - v_k, \quad k \geq k_m. \tag{32}$$

From now on we shall use several times the classical Green's second formula for smooth functions on $\overline{D}_m$.

We have

$$\int_{D_m} \Delta w_k^2 \cdot h_m \, dx = \int_{D_m} w_k^2 \cdot \Delta h_m \, dx - \int_{\partial D_m} w_k^2 \cdot (\operatorname{grad} h_m, n_m) \, d\sigma_m, \tag{33}$$

where $n_m(x) = (n_{i,m}(x))_{i=1,\dots,n}$ is the outward pointing unit normal vector at $x \in \partial D_m$, $d\sigma_m$ is the $(n-1)$-dimensional surface measure of the domain D_m.

We have

$$\Delta w_k^2 = 2 \sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i} \right)^2 + 2 w_k \cdot \Delta w_k. \tag{34}$$

Hence we get

$$\begin{aligned} &2 \int_{D_m} \sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i} \right)^2 h_m \, dx + 2 \int_{D_m} w_k \cdot \Delta w_k \cdot h_m \, dx \\ &= -\int_{D_m} w_k^2 \, dx - \int_{\partial D_m} w_k^2 (\operatorname{grad} h_m, n_m) \, d\sigma_m, \end{aligned} \tag{35}$$

but for any $x \in \partial D_m$ by the definition of the directional derivative we can write

$$(\operatorname{grad} h_m, n_m) = \lim_{t \downarrow 0} \frac{h_m(x) - h_m(x - t \cdot n_m)}{t} \leq 0. \tag{36}$$

Take $w_k = 1$ in the equality (35), we get

$$\int_{\partial D_m} (\operatorname{grad} h_m, n_m) \, d\sigma_m = -\operatorname{meas} D_m. \tag{37}$$

From the identity (35) we have the inequality

$$\begin{aligned} &2 \int_{D_m} |\operatorname{grad} w_k|^2 h_m \, dx + \int_{D_m} w_k^2 \, dx \\ &\leq 2 \|w_k\|_{L^\infty(D_m)} \int_{D_m} |\Delta u_k - \Delta v_k| h_m \, dx + \|w_k\|^2_{L^\infty(D_m)} \int_{\partial D_m} \big|(\operatorname{grad} h_m, n_m)\big| \, d\sigma_m. \end{aligned} \tag{38}$$

But the functions u_k, v_k are convex and hence

$$\Delta u_k \geq 0, \qquad \Delta v_k \geq 0. \tag{39}$$

We calculate from (36), (37)

$$\int_{\partial D_m} \big|(\operatorname{grad} h_m, n_m)\big| \, d\sigma_m = -\int_{\partial D_m} (\operatorname{grad} h_m, n_m) \, d\sigma_m = \operatorname{meas} D_m. \tag{40}$$

Hence we obtain the following estimate

$$\begin{aligned}
&\int_{D_m} |\operatorname{grad} w_k|^2 \cdot (2h_m) \, dx + \int_{D_m} w_k^2 \, dx \\
&\leq 2\|w_k\|_{L^\infty(D_m)} \int_{D_m} \Delta(u_k + v_k) h_m \, dx \\
&\qquad + \operatorname{meas} D_m \cdot \|w_k\|_{L^\infty(D_m)} \left(\|u_k\|_{L^\infty(D_m)} + \|v_k\|_{L^\infty(D_m)}\right).
\end{aligned} \tag{41}$$

We use once again the Green's second formula

$$\begin{aligned}
&\int_{D_m} \Delta(u_k + v_k) h_m \, dx \\
&= \int_{D_m} (u_k + v_k) \cdot \Delta h_m \, dx + \int_{\partial D_m} (u_k + v_k) \cdot (\operatorname{grad} h_m, -n_m) \, d\sigma_m \\
&\leq \left(\|u_k\|_{L^\infty(D_m)} + \|v_k\|_{L^\infty(D_m)}\right) \cdot \operatorname{meas} D_m \\
&\qquad + \left(\|u_k\|_{L^\infty(D_m)} + \|v_k\|_{L^\infty(D_m)}\right) \cdot \operatorname{meas} D_m \\
&= 2 \operatorname{meas} D_m \left(\|u_k\|_{L^\infty(D_m)} + \|v_k\|_{L^\infty(D_m)}\right),
\end{aligned}$$

thus we get

$$\int_{D_m} \Delta(u_k + v_k) h_m \, dx \leq 2 \operatorname{meas} D_m \left(\|u_k\|_{L^\infty(D_m)} + \|v_k\|_{L^\infty(D_m)}\right). \tag{42}$$

From the inequalities (41), (42) we obtain the estimate

$$\begin{aligned}
&\int_{D_m} |\operatorname{grad} w_k|^2 \cdot (2h_m) \, dx + \int_{D_m} w_k^2 \, dx \\
&\leq 5 \operatorname{meas} D_m \cdot \|w_k\|_{L^\infty(D_m)} \left(\|u_k\|_{L^\infty(D_m)} + \|v_k\|_{L^\infty(D_m)}\right).
\end{aligned} \tag{43}$$

We pass to the limit $k \to \infty$ in the latter estimate taking into account the convergence in norms (17), (18) and get

$$\int\limits_{D_m} \big|\operatorname{grad} u - \operatorname{grad} v\big|^2 \cdot (2h_m)\, dx + \int\limits_{D_m} (u-v)^2\, dx$$
$$\leq 5 \operatorname{meas} D_m \cdot \|u-v\|_{L^\infty(D_m)} \left(\|u\|_{L^\infty(D_m)} + \|v\|_{L^\infty(D_m)}\right). \qquad (44)$$

We change m to $m+l$, where $l = 0, 1, 2, \dots$ and restrict the left-hand side of (44) on D_m, we obtain

$$\int\limits_{D_m} \big|\operatorname{grad} u - \operatorname{grad} v\big|^2 \cdot (2h_{m+l})\, dx + \int\limits_{D_m} (u-v)^2\, dx$$
$$\leq 5 \operatorname{meas} D_{m+l} \cdot \|u-v\|_{L^\infty(D_{m+l})} \left(\|u\|_{L^\infty(D_{m+l})} + \|v\|_{L^\infty(D_{m+l})}\right). \qquad (45)$$

We let l tend to infinity and get

$$\int\limits_{D_m} \big|\operatorname{grad} u - \operatorname{grad} v\big|^2 \cdot (2h_\infty)\, dx + \int\limits_{D_m} (u-v)^2\, dx$$
$$\leq 5 \operatorname{meas} D \cdot \|u-v\|_{L^\infty(D)} \left(\|u\|_{L^\infty(D)} + \|v\|_{L^\infty(D)}\right). \qquad (46)$$

Taking into account the inequality (27) we derive from the latter inequality

$$\int\limits_{D_m} \big|\operatorname{grad} u - \operatorname{grad} v\big|^2 \cdot \frac{d^2_{\partial D}}{n}\, dx + \int\limits_{D_m} (u-v)^2\, dx$$
$$\leq 5 \operatorname{meas} D \cdot \|u-v\|_{L^\infty(D)} \left(\|u\|_{L^\infty(D)} + \|v\|_{L^\infty(D)}\right). \qquad (47)$$

Put $v = 0$ in the latter estimate, then we have the following bound

$$\int\limits_{D_m} |\operatorname{grad} u|^2 \cdot \frac{d^2_{\partial D}}{n}\, dx \leq 5 \operatorname{meas} D \cdot \|u\|^2_{L^\infty(D)}, \qquad (48)$$

which shows that the left-hand side of (48) is uniformly bounded with respect to m and hence passing to the limit $m \to \infty$ we get

$$\int\limits_{D} |\operatorname{grad} u|^2 \cdot \frac{d^2_{\partial D}}{n}\, dx \leq 5 \operatorname{meas} D \cdot \|u\|^2_{L^\infty(D)} \qquad (49)$$

for arbitrary bounded convex function $u : D \to R$ and arbitrary bounded open (nonsmooth in general) convex set D in R^n.

Based on the bound (49) we finally pass to limit in the inequality (47) and derive the desired estimate (9) for bounded convex functions. □

Corollary 2.3. *Let $u_k : D \to R$, $k = 1, 2, \dots$ be a sequence of semiconvex functions with the same semiconvexity constant c. Suppose this sequence converges in $L^\infty(D)$ to a continuous function u. Then we have the following estimate*

$$\int_D \left| \operatorname{grad} u_k - \operatorname{grad} u \right|^2 \cdot \frac{d^2_{\partial D}}{n}\, dx \tag{50}$$
$$\leq 5 \operatorname{meas} D \cdot \|u_k - u\|_{L^\infty(D)} \times \left(2\|u\|_{L^\infty(D)} + \|u_k - u\|_{L^\infty(D)} + 2c\|v_0\|_{L^\infty(D)}\right).$$

Indeed, recalling the definition of the semiconvex function (5) and passing to the limit $k \to \infty$ in the one sided estimate (6) we see that the limit function u is also semiconvex with the same semiconvexity constant c. We apply the weighted energy inequality (9) to the functions u_k and u and get the estimate (50).

3. The L^2-Approximation of the Gradient of the Semiconvex Function through the Convex Envelope

Let $u : D \to R$ be analytically unknown viscosity solution of the fully nonlinear second order elliptic partial differential equation

$$F(x, u, \operatorname{grad} u, \operatorname{Hess} u) = 0 \tag{51}$$

in a bounded open convex subset D of R^n.

As pointed out in the introduction the solution of the equation (51) turns out to be semiconvex (or semiconcave) function if the latter equation is related to different kind of optimization problems.

Suppose the bounded viscosity solution u of equation (51) is a semiconvex function and we are given its uniform continuous numerical approximation $u_\delta : D \to R$, where δ is a small parameter, which typically measures the mesh size. The objective consists in constructing interior L^2-approximation of the unknown Sobolev gradient $\operatorname{grad} u$ based on the uniform approximation u_δ. Moreover, it is desirable to estimate the gradient's L^2-error through the $L^\infty(D)$-uniform error of approximation.

We shall see in this section that such a construction is possible and it uses two ingredients: the energy inequality (9) and the notion of the convex envelope.

The convex envelope $\Gamma(u)$ of a bounded continuous function u in D is defined as the supremum of all convex functions which are majorized by the function u (see, for instance, Alvarez, Lasry and Lions [1], or Oberman [18])

$$\Gamma(u) = \sup\{v(x) :\ v(x) \text{ convex in } D,\ v(x) \leq u(x) \text{ for all } x \in D\}. \tag{52}$$

The mapping $u \to \Gamma(u)$ possesses some nice properties which we prove below

Lemma 3.1. *The mapping $u \to \Gamma(u)$ has the Lipschitz property*

$$\|\Gamma(u) - \Gamma(v)\|_{L^\infty(D)} \leq \|u - v\|_{L^\infty(D)} \tag{53}$$

if only u, v belong to $C(D) \cap L^\infty(D)$.

Proof. Denote

$$d = \|u - v\|_{L^\infty(D)},$$

then we have $-d \le u(x) - v(x) \le d$, i.e. $v(x) - d \le u(x)$, $u(x) - d \le v(x)$.

Hence we have

$$\Gamma(v) - d \le u, \qquad \Gamma(u) - d \le v.$$

This means that the convex functions $\Gamma(v) - d$ and $\Gamma(u) - d$ are majorized respectively by u, v.

By the definition of the convex envelope we obtain

$$\Gamma(v) - d \le \Gamma(u), \qquad \Gamma(u) - d \le \Gamma(v),$$

i.e. $-d \le \Gamma(u) - \Gamma(v) \le d$, thus we derive the inequality (53). □

Taking successively $u = 0$ and $v = 0$ in (53) we get

$$\begin{cases} \|\Gamma(v)\|_{L^\infty(D)} \le \|v\|_{L^\infty(D)}, \\ \|\Gamma(u)\|_{L^\infty(D)} \le \|u\|_{L^\infty(D)}. \end{cases} \tag{54}$$

Proposition 3.2. *On the space $C(D) \cap L^\infty(D)$ the mapping $u \to \Gamma(u)$ possesses the following important property*

$$\int_D \left| \operatorname{grad} \Gamma(u) - \operatorname{grad} \Gamma(v) \right|^2 \cdot \frac{d^2_{\partial D}}{n}\, dx$$
$$\le 5 \operatorname{meas} D \cdot \|u - v\|_{L^\infty(D)} \left(\|u\|_{L^\infty(D)} + \|v\|_{L^\infty(D)} \right). \tag{55}$$

Proof. We have from the bound (54) that the convex functions $\Gamma(u)$ and $\Gamma(v)$ are bounded, thus we can apply the energy inequality (9) for the latter convex functions and get

$$\int_D \left| \operatorname{grad} \Gamma(u) - \operatorname{grad} \Gamma(v) \right|^2 \cdot \frac{d^2_{\partial D}}{n}\, dx$$
$$\le 5 \operatorname{meas} D \cdot \|\Gamma(u) - \Gamma(v)\|_{L^\infty(D)} \left(\|\Gamma(u)\|_{L^\infty(D)} + \|\Gamma(v)\|_{L^\infty(D)} \right). \tag{56}$$

The assertion follows after application of Lemma 3.1 and the bound (54). □

Consider now the bounded viscosity solution u of the equation (51) which is assumed to be semiconvex with semiconvexity constant $c \ge 0$ and its uniform continuous numerical approximation u_δ, i.e.

$$\|u_\delta - u\|_{L^\infty(D)} \underset{\delta \to 0}{\longrightarrow} 0. \tag{57}$$

Further consider the bounded continuous functions

$$u + c \cdot v_0 \quad \text{and} \quad u_\delta + c \cdot v_0 \tag{58}$$

and their convex envelopes

$$\Gamma(u + c \cdot v_0) \quad \text{and} \quad \Gamma(u_\delta + c \cdot v_0). \tag{59}$$

The next proposition is the main result of Section 3.

Theorem 3.3. *The following weighted L^2-estimate is valid for the unknown* grad u *through the function* grad$(\Gamma(u_\delta + c \cdot v_0) - c \cdot v_0)$

$$\int\limits_D |\operatorname{grad}(\Gamma(u_\delta + c \cdot v_0) - c \cdot v_0) - \operatorname{grad} u|^2 \cdot \frac{d^2_{\partial D}}{n}\, dx \tag{60}$$
$$\leq 5 \operatorname{meas} D \cdot \|u_\delta - u\|_{L^\infty(D)} \times \left(2\|u\|_{L^\infty(D)} + \|u_\delta - u\|_{L^\infty(D)} + 2c\|v_0\|_{L^\infty(D)}\right).$$

Proof. Let us apply Proposition 3.2 to the functions $u_\delta + c \cdot v_0$ and $u + c \cdot u_0$, we shall have

$$\int\limits_D |\operatorname{grad} \Gamma(u_\delta + c \cdot v_0) - \operatorname{grad} \Gamma(u + c \cdot v_0)|^2 \cdot \frac{d^2_{\partial D}}{n}\, dx$$
$$\leq 5 \operatorname{meas} D \cdot \|u_\delta - u\|_{L^\infty(D)} \left(\|u_\delta + c \cdot v_0\|_{L^\infty(D)} + \|u + c \cdot v_0\|_{L^\infty(D)}\right). \tag{61}$$

By the semiconvexity criteria (7) we have that the function $u + c \cdot v_0$ is convex and therefore coincides with its convex envelope $\Gamma(u + c \cdot v_0)$, hence we get

$$\operatorname{grad} \Gamma(u + c \cdot v_0) = \operatorname{grad}(u + c \cdot v_0) = \operatorname{grad} u + \operatorname{grad}(c \cdot v_0),$$

the rest is obvious. $\square$

Thus the L^2-approximation problem of the unknown grad u is reduced to the efficient numerical computation of convex envelope $\Gamma(u_\delta + c \cdot v_0)$ and its gradient (for this item see Kadhi and Trad [10] and Oberman [19]).

We note here that if the solution of PDE (51) is convex (as is the case in Alvarez, Lasry and Lions [1]) the unknown grad u is approximated by the grad $\Gamma(u_\delta)$.

At the end of our paper we give an application of the energy inequality (9) to the distance function from a set.

It is well known (see Cannarsa and Sinestrari [3, Chapter 2, Proposition 2.2.2] that the square of the distance function d_C^2 is semiconcave in R^n with the semiconcavity constant 2.

Consider two bounded nonempty closed sets C and $\widetilde{C}$ in R^n (in particular the boundaries of the open sets) and arbitrary ball B in R^n. If we put the semiconvex functions $(-d_C^2)$ and $(-d_{\widetilde{C}}^2)$ in the energy inequality (9) we get the following interesting (and impressive) estimate

$$\int\limits_B \left|\operatorname{grad} d_C^2 - \operatorname{grad} d_{\widetilde{C}}^2\right|^2 \cdot \frac{d^2_{\partial B}}{n}\, dx$$
$$\leq 5 \operatorname{meas} B \cdot \|d_C^2 - d_{\widetilde{C}}^2\|_{L^\infty(B)} \times \left(\|d_C^2\|_{L^\infty(B)} + \|d_{\widetilde{C}}^2\|_{L^\infty(B)} + 4\|v_0\|_{L^\infty(B)}\right). \tag{62}$$

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