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# Right Convexity*

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A convex set is $\mathcal{F}$-convex if every pair of points in the set lie in a right triangle included in the set.

We characterize $\mathcal{F}$-convex sets, find some classes of $\mathcal{F}$-convex sets, investigate $\mathcal{F}$-convexity for cones and cylinders, and find out that most convex bodies are $\mathcal{F}$-convex.

On our way, we also describe the curvature at the endpoints of diameters of most convex bodies.

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## Introduction

At the 1974 Convexity meeting in Oberwolfach, the author proposed the investigation of various cases of the following convexity type concept: Given a family $\mathcal{F}$ of sets in a certain space $X$, a set $M \subset X$ is called *$\mathcal{F}$-convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$. This appeared explicitly as a problem.

Obviously, affine linearity, arc-wise connectedness, usual convexity are all examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the author [1], and later Böröczky Jr. [2], investigated the rectangular convexity, the case when $\mathcal{F}$ contains all non-degenerate rectangles, but a conjectured characterization remained unproved.

By taking as $\mathcal{F}$ the family of all convex hulls of surfaces obtained by rotating a circular arc (smaller than a half-circle) around the line through its endpoints, we obtain the usual strict convexity. This family $\mathcal{F}$ has been used for a characterization of convexity in [7].

Among the more recent investigations, let us mention Magazanik and Perles' staircase connectedness, where $\mathcal{F}$ consists of a special kind of polygonal arcs [6].

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Here we take the family $\mathcal{F}$ to be that of all right triangles in the Hilbert space $H$ of dimension at least 2. A *right triangle* is the convex hull of 3 distinct points $x, y, z \in H$ with $\angle xyz = \pi/2$. Our $\mathcal{F}$-convex sets are all convex, because so are the elements of $\mathcal{F}$. In fact this $\mathcal{F}$-convexity is a special case of $\mathcal{F}'$-convexity, where $\mathcal{F}'$ is the family of all triples $\{x, y, z\} \subset H$ such that $\angle xyz = \pi/2$. The (interesting) study of $\mathcal{F}'$-convexity includes a more discrete geometric research, while $\mathcal{F}$-convexity, called *right convexity*, is fully embedded in convex geometry.

The convex sets considered here shall always be closed.

A *convex body* is a closed bounded convex set of dimension at least 2.

For a convex body $K$, we denote by $S_K$ the circumsphere of $K$, i.e. the boundary of the smallest (unique) ball in $H$ including $K$, and by $\operatorname{bd} K$ the relative boundary of $K$.

Many convex bodies are $\mathcal{F}$-convex, i.e., rightly convex, many are not. So, in $\mathbb{R}^d$, all (right bounded) cylinders are $\mathcal{F}$-convex, while all ellipsoids with distinct axis lengths are not. Of course, rectangular convexity implies right convexity (but not conversely).

We prove here that further classes of convex bodies, including all convex bodies of constant width, are $\mathcal{F}$-convex, but leave the pleasure of finding more such classes to the reader. Moreover, we show that, in the sense of Baire categories, most convex bodies in $\mathbb{R}^d$ are $\mathcal{F}$-convex.

A line-segment of maximal length in a convex set $K$ is called a *diameter* of $K$. Let $\mathcal{B}$ be the set of all convex bodies in $H$ admitting a diameter. If $H = \mathbb{R}^d$, $\mathcal{B}$ coincides with the space of all convex bodies.

## Characterizations

We start with a characterization of $\mathcal{F}$-convex sets among the convex bodies from $\mathcal{B}$.

**Theorem 1.** *A convex body $K \in \mathcal{B}$ is rightly convex if and only if some diameter $\sigma$ of $K$ is seen from some point in $K \setminus \sigma$ under a right angle.*

**Proof.** Assume the condition in the statement is verified. Let $x, y \in K$. The condition in the definition of $\mathcal{F}$-convexity is obviously satisfied at $x, y$ if $xy = \sigma$.

If $xy \neq \sigma$, let $\sigma = uv$. Let $w \in K$ be the point satisfying $\angle uwv = \pi/2$. Either $uv$ is another diameter of the hypersphere $S$ of diameter $xy$, or at least one of the two endpoints of $uv$, say $v$, lies outside $S$.

In the first case $\angle xuy = \pi/2$. In the second case, if $x$, $y$, $v$ are not collinear, $xv \cup yv$ meets $S$ in at least three points, $x$, $y$, and a third point $z$. Thus, $\angle xzy = \pi/2$. If $x$, $y$, $v$ are collinear, for example $y$ lies between $x$ and $v$, then the hyperplane through $y$ orthogonal onto $xy$ meets $\operatorname{conv}\{x, u, v, w\} \subset K$ along a line-segment or a triangle, where we can choose a point $z \neq y$. Now, $\angle xyz = \pi/2$, and $K$ is $\mathcal{F}$-convex.

Conversely, if $K$ is $\mathcal{F}$-convex and $\sigma$ is any diameter of $K$, the existence of a right triangle in $K$ including $\sigma$ is only possible if $\sigma$ is its hypothenuse.

The following simple but useful sufficient condition for $\mathcal{F}$-convexity follows from Theorem 1.

**Corollary 2.** *Let $K$ be a convex body. If there exists a diameter $\sigma$ of $K$ and a rightly convex set $L$ such that $\sigma \subset L \subset K$, then $K$ is rightly convex too.*

**Theorem 3.** *Let* $\dim H < \infty$. *A compact convex set $K \subset H$ is a rightly convex body if and only if* $\operatorname{card}(K \cap S_K) \geq 3$.

**Proof.** It is well-known that $\dim K \geq 1$ implies $\operatorname{card}(K \cap S_K) \geq 2$.

Suppose $\operatorname{card}(K \cap S_K) = 2$. Put $K \cap S_K = \{x, y\}$. Then $xy$ is the single common diameter of $K$ and $S_K$, and for no third point $z \in K$, $\angle xzy = \pi/2$. Thus, $K$ is not $\mathcal{F}$-convex.

Conversely, first remark that $K$ is a convex body if $\dim K \geq 2$, and this is indeed guaranteed by the assumption that $\operatorname{card}(K \cap S_K) \geq 3$. Now suppose $K$ is not $\mathcal{F}$-convex. By Theorem 1, for any diameter $xy$ of $K$ and any point $z \in K \setminus xy$, $\angle xzy \neq \pi/2$. This implies that $K \cap S = \{x, y\}$, where $S$ is the sphere of diameter $xy$. Then $S = S_K$ and $\operatorname{card}(K \cap S_K) = 2$.

## Convex bodies of constant width and other classes of rightly convex bodies

In [12] we described the curvature aspect at diametrally opposite points of a planar convex body $C$ of constant width $w$. Let $\rho_i(x)$ and $\rho_s(x)$ denote the lower and upper curvature radii of $\operatorname{bd} C$ at the point $x \in \operatorname{bd} C$ in counterclockwise direction on $\operatorname{bd} C$ (for a definition of the lower and upper curvature radii, see [3]). What happens is that, for any pair of diametrally opposite points $x, y \in \operatorname{bd} C$,

$$\rho_i(x) + \rho_s(y) = w.$$

This yields the existence of a third point $z \in C$ at distance $w/2$ from the midpoint of $xy$, but a proof requires several lines. Once proven, this in turn would suffice for $C$ to be rightly convex. But that this is indeed the case follows more elegantly from the next result, valid in any dimension.

**Theorem 4.** *Every convex body with more than one diameter is rightly convex.*

**Proof.** Let $xy$ be a diameter of the convex body $K$. Let $uv$ be a diameter of $K$ distinct from $xy$. Then either $uv$ is another diameter of the hypersphere $S$ of diameter $xy$, or at least one of the two endpoints of $uv$, say $v$, lies outside $S$. In both cases $xv \cup yv$ meets $S$ in at least three points, $x$, $y$, and another point $z$. Thus, $\angle xzy = \pi/2$. By Theorem 1, $K$ is $\mathcal{F}$-convex.

Since any pair of boundary points of a convex body $C$ of constant width admitting opposite unit normal vectors determine a diameter of $C$, we get the following.

**Corollary 5.** *Every convex body of constant width is rightly convex.*

**Theorem 6.** *Each convex body from $\mathcal{B}$, symmetric with respect to a linear subspace $A \subset H$, with no diameter in $A$, is rightly convex.*

**Proof.** The image of any diameter of the convex body through the existing symmetry is another diameter. Now, the $\mathcal{F}$-convexity follows from Theorem 4.

### Rightly convex cones and cylinders

A *cone* $P \subset H$ is the convex hull of the union of a convex body $B$ of codimension 1, called *base*, with a point $p \notin \operatorname{aff} B$ called *apex*.

$P$ is called a *right cone* if the orthogonal projection $p'$ of $p$ onto $\operatorname{aff} B$ belongs to $B$.

The *distance* $d(x, M)$ from a point $x$ to a set $M \subset H$ equals $\inf_{y \in M} \|x - y\|$. Set $\operatorname{diam} M = \sup_{y,z \in M} \|y - z\|$. (For $M \in \mathcal{B}$, $\operatorname{diam} M$ is the length of a diameter of $M$.)

The *height* of a cone of apex $p$ and base $B$ is $d(p, \operatorname{aff} B)$.

**Theorem 7.** *A right cone is rightly convex if it has a rightly convex base or its height is at least* $\operatorname{diam} B/2$.

**Proof.** Let $P$ be a right cone with apex $p$, base $B$ and height $h$.

Choose arbitrarily $x, y \in P$.

Assume first that $B$ is $\mathcal{F}$-convex.

*Case I.* $x, y \in B$. The existence of a right triangle containing $x, y$ is now assumed.

*Case II.* $x \in B$, $y \in P \setminus B$. We show that there exists a right triangle in $P$ containing both $x$ and $y$. Indeed, if $x \neq y'$, where $y'$ is the orthogonal projection of $y$ onto $\operatorname{aff} B$, such a triangle is $xy'y$. If $x = y'$, a suitable triangle is $yxz$, where $z$ can be arbitrarily chosen in $B \setminus \{x\}$.

*Case III.* $x, y \in P \setminus B$. Assume without loss of generality that $d(x, B) \leq d(y, B)$. The hyperplane through $x$ parallel to $\operatorname{aff} B$ intersects $P$ along a convex body $B'$. Since $B'$ is similar to $B$ and $P' = \operatorname{conv}(\{p\} \cup B')$ is similar to $P$, we arrive at Case I or II with respect to $P'$ and $B'$ in place of $P$ and $B$.

Assume now that $h \geq \operatorname{diam} B/2$.

We have again the same three cases as before. Only Case I needs a different treatment. We have to show that, for any pair of points $x, y \in B$, there exists a right triangle in $P$ containing both $x$ and $y$.

Let $S \subset H$ be the hypersphere of diameter $xy$. Since

$$\left\| p - \frac{x+y}{2} \right\| \geq d(p, B) = h \geq \operatorname{diam} B/2 \geq \|x - y\|/2,$$

$p$ lies outside $S$ or on it. Then $px \cup py$ intersects $S \setminus \{x, y\}$, and for $z$ in this intersection we have $\angle xzy = \pi/2$.

**Theorem 8.** *Let the cone $P \subset H$ have apex $p$, base $B$, and height $h$. Assume that $B$ is smooth at $a \in \operatorname{bd} B$ and $pa$ is a diameter of $P$. If the upper radius of curvature $\rho_s^\tau(a)$ of $\operatorname{bd} B$ at $a$ in some tangent direction $\tau$ satisfies*

$$\rho_s^\tau(a) > \frac{1}{2}\sqrt{(\operatorname{diam} P)^2 - h^2},$$

*then $P$ is rightly convex.*

**Proof.** Let $p'$ be the orthogonal projection of $p$ on $\operatorname{aff} B$, and put

$$p'' = \frac{p' + a}{2}, \qquad r = \|p'' - a\|.$$

The normal line of $\operatorname{bd} B$ at $a$ passes through $p'$. Obviously, $r = \frac{1}{2}\sqrt{(\operatorname{diam} P)^2 - h^2}$. The condition $\rho_s^\tau(a) > r$ implies that in any neighbourhood of $a$ there are points of $\operatorname{bd} B$ outside the sphere of centre $p''$ and radius $r$. Let $x$ be any such point; then $xa$ meets the sphere in some point $y$ and $\angle p'ya = \pi/2$, whence $\angle pya = \pi/2$ too.

Now the conclusion follows from Theorem 1.

A *cylinder* is a set

$$Z = B + \mathbf{0}v = \{x + \lambda v : x \in B, \lambda \in [0, 1]\} \subset H,$$

where $B$ is a convex body of codimension 1 containing the origin $\mathbf{0}$ and $v \notin \operatorname{aff} B$. The sets $B$ and $B + v$ are called *bases* of $Z$, while the distance $d(v, \operatorname{aff} B)$ is called *height* of $Z$.

$Z$ is said to be a *right* cylinder if $\langle v, w\rangle = 0$ for all $w \in B$.

Every right cylinder is obviously rightly convex. Regarding the other cylinders, we have the following result similar to Theorem 8.

**Theorem 9.** *Let the cylinder $Z \subset H$ have bases $B$ and $B + v$, and height $h$. Assume that $Z$ has a diameter $ab$. Then its endpoints belong to the bases, say $a \in B$, $b \in B + v$. Also, assume that $B$ is smooth at $a$. If the upper radius of curvature $\rho_s^\tau(a)$ of $\operatorname{bd} B$ at $a$ in some tangent direction $\tau$ satisfies*

$$\rho_s^\tau(a) > \frac{1}{2}\sqrt{(\operatorname{diam} Z)^2 - h^2},$$

*then $Z$ is rightly convex.*

**Proof.** The cone $\operatorname{conv}(\{b\} \cup B)$ is $\mathcal{F}$-convex because the hypotheses of Theorem 8 are verified for that cone. Now, by Corollary 2, $Z$ is $\mathcal{F}$-convex too.

### Unbounded convex sets

So far, we could detect no great similarity between rectangular and right convexities. Also regarding unbounded convex sets, right triangular convexity and rectangular convexity behave very differently: Every $d$-dimensional unbounded closed convex set in $\mathbb{R}^d$ is rightly convex. Compare this to Theorems 1 and 6 in [1].

**Theorem 10.** *Every unbounded closed convex set in $H$, of dimension at least 2, is rightly convex.*

**Proof.** Let $C \subset H$ be an unbounded closed convex set with $\dim C \geq 2$. Consider the vector $v$ in its recession cone.

Let now $x, y \in C$.

If $v$ and $x - y$ are linearly dependent, for example $x - y = \alpha v$ with $\alpha > 0$, then the half-line through $x$ starting at $y$ lies in $C$, and $x + w \in C$ for some vector $w$ orthogonal to $v$, because $\dim C \geq 2$.

If $v$ and $x - y$ are not linearly dependent, some diagonal of the parallelogram of vertices $x, y, x + v, y + v \in C$, say $x(y + v)$, is not smaller than the other one. Then the orthogonal projection $y'$ of $y$ onto $x(y + v)$ lies in $C$, and $\angle xy'y = \pi/2$.

### Right convexity of most convex bodies

In this last section, $H = \mathbb{R}^d$. Consider the space of all continua in $\mathbb{R}^d$, and its subspace $\mathcal{K}$ of all $d$-dimensional convex bodies. Both spaces, endowed with the Pompeiu-Hausdorff distance, are Baire spaces. As usual, we say that *most* elements of a Baire space enjoy property **P** if those elements not enjoying **P** form a subset of first Baire category.

The generic curvature behaviour of convex surfaces, i.e., boundaries of convex bodies, was described in [8], [9], [11].

For a convex body $K$ to be rightly convex it suffices that $\mathrm{bd}\, K$ has upper radius of curvature larger than $\mathrm{diam}\, K/2$ at some endpoint of some diameter, in some tangent direction, by Theorem 1.

On most convex bodies, at most points, the lower curvature in any direction is 0. However, this cannot happen at the endpoints of a diameter. What it does happen is described in the next theorem.

**Theorem 11.** *Most convex bodies in $\mathbb{R}^d$ are smooth and have a single diameter. At both endpoints of that diameter, in some tangent direction, the upper curvature is $\infty$ and the lower one equals* $(\mathrm{diam}\, K)^{-1}$.

**Proof.** A direction or a line-segment will be called *horizontal*, respectively *vertical*, if it is parallel, respectively orthogonal, to a fixed (2-dimensional) plane.

Let $\mathcal{K}_n$ be the set of convex bodies admitting (at least) two diameters at Hausdorff distance at least $1/n$ from one another. This set is closed in $\mathcal{K}$.

A convex body $C$ can be approximated by a polytope $P$, i.e. any neighbourhood of $C$ in $\mathcal{K}$ contains a polytope $P$. Let $xy$ be a diameter of $P$. By gently extending $xy$ beyond $y$ to a line segment $xy'$, we obtain a polytope $P' = \operatorname{conv}(P \cup \{y'\})$ with the single diameter $xy'$ and also approximating $C$. As $P' \notin \mathcal{K}_n$, we obtain that $\mathcal{K}_n$ is nowhere dense, so most convex bodies have a single diameter.

Let now $xx^*$ be a diameter of $C \in \mathcal{K}$, and let the direction $\tau$ be orthogonal to $xx^*$. Consider the points $x_n, x_n^* \in xx^*$ such that $\|x - x_n\| = \|x^* - x_n^*\| = 1/n$, and the half-plane $\Pi$ with $xx^*$ on its boundary and $x + \tau \in \Pi$.

Let $A_n \subset \Pi$ be the arc of length $1/n$ of the circle of centre $x_n^*$ from $x$ to $y_n$. The radius is $\operatorname{diam} C - 1/n$. Let $A_n^* \subset \Pi$ be the analogous arc of same length $1/n$, of the circle of centre $x_n$ from $x^*$ to $y_n^*$. The two arcs are congruent.

Let's say that $C$ has the $(n)$-property if for every diameter $xx^*$ of $C$ and for any horizontal direction $\tau$ orthogonal to $xx^*$, either $A_n$ or $A_n^*$ does not meet $\operatorname{int} C$.

We prove that the set $\mathcal{K}_n'$ of those $C \in \mathcal{K}$ which enjoy the $(n)$-property is nowhere dense in $\mathcal{K}$.

First, it is easily seen that each $\mathcal{K}_n'$ is closed in $\mathcal{K}$. Then, let $C \in \mathcal{K}$. Approximate it by a polytope $P$ having a diameter $xx^*$. Choose a horizontal direction $\tau$ orthogonal to $xx^*$. Choose $\varepsilon > 0$ very small (compared with $1/n$), and the points $y \in x^*(x + \varepsilon\tau)$ and $y^* \in x(x^* + \varepsilon\tau)$ such that $\|y - x^*\| = \|y^* - x\| = \|x - x^*\|$. Then $P' = \operatorname{conv}(P \cup \{y\} \cup \{y^*\})$ has not the $(n)$-property, whence $\mathcal{K}_n'$ is nowhere dense.

It is immediately seen that those convex bodies which admit a vertical diameter form a nowhere dense subset of $\mathcal{K}$.

In conclusion, most $C \in \mathcal{K}$ are smooth (see [5], [4]), have a single diameter $xx^*$ which is not vertical, and also have the $(n)$-property for no natural number $n$. This means that for at least one of the two horizontal tangent directions $\tau, -\tau$ at $x$, say for $\tau$,

$$\rho_s^\tau(x) > \operatorname{diam} C - 1/n \quad \text{and} \quad \rho_s^\tau(x^*) > \operatorname{diam} C - 1/n$$

for infinitely many $n$'s, yielding $\rho_s^\tau(x) = \rho_s^\tau(x^*) = \operatorname{diam} C$.

By Theorem 1 in [8], for most $C \in \mathcal{K}$, at every point $z \in \operatorname{bd} C$ and for every tangent direction $\tau$, $\rho_i^\tau(z) = 0$ or $\rho_s^\tau(z) = \infty$. It follows that, at the endpoints $x, x^*$ of the unique diameter and for the determined direction $\tau$, $\rho_i^\tau(x) = \rho_i^\tau(x^*) = 0$.

Thus, for most convex bodies, the upper radius of curvature at both endpoints of the (unique) diameter equals $\operatorname{diam} K$ in some tangent direction, which implies, by Theorem 1, that they are rightly convex. But we are not satisfied with this. The following is namely true.

**Theorem 12.** *The set of all convex bodies in $\mathbb{R}^d$ which are not rightly convex is nowhere dense.*

**Proof.** We shall show that the set $\mathcal{C}$ of those $K \in \mathcal{K}$, for which $\operatorname{diam} K = \operatorname{diam} S_K$, is nowhere dense in $\mathcal{K}$.

First of all, it is a routine matter to verify that $\mathcal{C}$ is closed in $\mathcal{K}$. Now, let $C \in \mathcal{C}$. If $x, y$ are diametrally opposite in $C$, then the circumsphere $S_C$ of $C$ must admit $xy$ as a diameter. Let $\varepsilon > 0$. Choose a point $a$ outside $S_C$ but inside the sphere of centre $y$ and radius $\operatorname{diam} C$, at distance from $x$ less than $\varepsilon$. Clearly, $C' = \operatorname{conv}(\{a\} \cup C)$ has the same diameter $xy$ as $C$, but a different circumsphere. Also,

$$\rho(C, C') < \varepsilon.$$

Thus, $\mathcal{C}$ is nowhere dense in $\mathcal{K}$. Since, by Theorem 3, all convex bodies which are not $\mathcal{F}$-convex belong to $\mathcal{C}$ (but not conversely), the proof is finished.

The weaker assertion that most convex bodies are $\mathcal{F}$-convex also follows from Theorem 3 in [10] together with our Theorem 3.

What is the behaviour of continua and of compact starshaped sets? Since only few continua are starshaped sets, and even fewer are convex, we cannot expect many of them to be $\mathcal{F}$-convex. However, concerning $\mathcal{F}'$-convexity, things may become more interesting.

We leave the pleasure of investigating this to the reader. (For a while.)

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