Journal of Convex Analysis
Volume 21 (2014), No. 2, 307–316

Relative Chebyshev Centers in $L_\infty(\mu, X)^*$

Tijani Pakhrou
Department of Mathematical Analysis, Faculty of Sciences,
University of Alicante, 03080-Alicante, Spain
tijani.pakhrou@ua.es

Received: June 22, 2012
Revised manuscript received: April 16, 2013
Accepted: April 17, 2013

Let X be a Banach space and Y a weakly $\mathcal{K}$-analytic subspace of X. In this paper we study the simultaneous proximinality in the space $L_\infty(\mu, X)$. In this sense we have proved that $L_\infty(\mu, Y)$ is simultaneously proximinal in $L_\infty(\mu, X)$ if, and only if, Y is simultaneously proximinal in X.

1. Introduction and Notation

The definition of the relative Chebyshev center and the initial study thereof are due to Garkavi [6, 7]. In the work of Amir and Ziegler [1] there are some characterizations of these centers. These centers, also known as *best simultaneous approximation*, continue to generate much interest, for the enormous applications that they have in economy and telecommunications industry, for example.

Throughout this paper, $(X, \|\cdot\|)$ is a real Banach space, (Ω, Σ, μ) a complete probability space and $L_\infty(\mu, X)$ the Banach space of all μ-measurable and essentially bounded functions defined on Ω with values in X endowed with the usual norm

$$\|f\|_\infty = \sup_{s \in \Omega} \text{ess}\, \|f(s)\|$$

for every $f \in L_\infty(\mu, X)$ (see [4]).

Let Y be a subset of X and n be any positive integer. Let $A = \{a_1, \ldots, a_n\} \subset X$, we say that $y_0 \in Y$ is a *relative Chebyshev center of A in Y* if

$$\max_{1\le i\le n} \|a_i - y_0\| \le \max_{1\le i\le n} \|a_i - y\|, \quad \forall\, y \in Y.$$

If every n-tuple of vectors $a_1, \ldots, a_n \in X$ admits a relative Chebyshev center in Y, then Y is said to be *simultaneously proximinal in* X. Of course, for $n = 1$ the notion of relative Chebyshev center is just that of *best approximation* and the concept of simultaneously proximinal coincides with the *proximinality* (see [11]).

*Research partially supported by MTM 2012-31286 (Spanish Ministry of Economy and Competitiveness) and GRE11-23 (University of Alicante, Spain).

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

Various examples of finite sets lacking Chebyshev centers are known. The first such example was given by Garkavi [6] who defined a closed hyperplane H in $\mathcal{C}([-1,1])$, the vector space of all real-valued continuous functions on $[-1,1]$, and a three-points set in H without relative Chebyshev center in H.

Konyagin in [10] proved that: *let $(X, \|\cdot\|)$ be a nonreflexive Banach space, then there exists a three-point set $A = \{a_1, a_2, a_3\} \subset X$ and an equivalent norm $\|\cdot\|_0$ on X such that A has no relative Chebyshev center in $(X, \|\cdot\|_0)$.*

As far as we know, there are no characterizations of the spaces X with the property consisting of every subset $A = \{a_1, \ldots, a_n\} \subset X$ to have a relative Chebyshev center in X. However, if X be a Banach space that is norm-one complemented in its bidual (i.e. there exists a bounded linear projection of the bidual X^{**} onto X with norm one; to see the definition and properties of that concept we suggest [5, p. 137–138]). Then the existence of relative Chebyshev center of every set $A = \{a_1, \ldots, a_n\} \subset X$ follows straightforward from lower-semicontinuity and compactness arguments (see [18, Proposition 2.2]).

One of the questions that are studied within the relative Chebyshev center problems, is to find the type of subspaces that are simultaneously proximinal in a Banach space.

In this paper we will focus on the simultaneous proximinality in the space $L_\infty(\mu, X)$. Natural subspaces of $L_\infty(\mu, X)$ are either of the kind $L_\infty(\mu, Y)$, where Y is a closed subspace of X, or of the kind $L_\infty(\mu_0, X)$, where μ_0 is the restriction of μ to a certain sub-σ-algebra Σ_0 of Σ.

When Y is a reflexive subspace, it was proved in [13] that $L_\infty(\mu, Y)$ is simultaneously proximinal in $L_\infty(\mu, X)$. Furthermore, it was proved that $L_\infty(\mu_0, X)$ is simultaneously proximinal in $L_\infty(\mu, X)$ when X is reflexive. Indeed, the existence of relative Chebyshev center follows from Dunford's theorem [4, p. 101]. We have also obtained similar results in the space $L_1(\mu, X)$ (the Banach space of X-valued Bochner μ-integrable functions defined on Ω, see [12], [14]).

As it is shown in [12], the simultaneously proximinality is strictly stronger than the proximinality. In fact, in [12, Example 2] it was proved that there exist a Banach space X and a sub-σ-algebra Σ_0 of Σ such that $L_1(\mu_0, X)$ is proximinal but not simultaneously proximal in $L_1(\mu_0, X)$.

We denote by $\ell_\infty^n(X)$ the Banach space consisting of all n-tuple

$$(x_1, \ldots, x_n) \in X^n = \overbrace{X \times X \times \cdots \times X}^{n \text{ times}}$$

with the norm

$$|||(x_1, \ldots, x_n)||| := \max_{1 \le i \le n} \|x_i\|.$$

Let $A = \{a_1, \ldots, a_n\} \subset X$. Then it is obvious that y_0 is a relative Chebyshev center of A in Y if and only if $(y_0, \ldots, y_0)$ is a best approximation to $(a_1, \ldots, a_n)$

from the subspace

$$\operatorname{diag}(\ell_\infty^n(Y)) := \{(y, \ldots, y) : y \in Y\}.$$

2. Selectors for the metric projection

We will start by recalling some definitions from descriptive set theory. Denote by $\mathbb{N}^{\mathbb{N}}$ the set of all infinite sequences $(n_i)_{i\geq 1}$ with natural components.

Definition 2.1 ([2, Definition 1.10.1]). Let X be a nonempty set and let $\mathcal{A}$ be some class of its subsets. We say that we are given a Souslin scheme $\{A_{n_1,\ldots,n_k}\}$ with values in $\mathcal{A}$ if, to every finite sequence $(n_1, \ldots, n_k)$ of natural numbers, there corresponds a set $A_{n_1,\ldots,n_k} \in \mathcal{A}$. The Souslin operation over the class $\mathcal{A}$ is a mapping that to every Souslin scheme $\{A_{n_1,\ldots,n_k}\}$ with values in $\mathcal{A}$ associates the set

$$A = \bigcup_{(n_i)_{i\geq 1}\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} A_{n_1,\ldots,n_k}.$$

The sets $A \subset X$ of this form are called $\mathcal{A}$-Souslin.

We recall that a multivalued mapping Ψ from a topological space X to the set of nonempty subsets of a topological space Y is called upper semi-continuous if for every $x \in X$ and every open set V in Y containing the set $\Psi(x)$, there exists a neighborhood U of the point x such that

$$\Psi(U) := \bigcup_{u\in U} \Psi(u) \subset V.$$

Definition 2.2 ([2, Definition 6.10.12]). Let X be a Hausdorff space and let $\mathbb{N}^{\mathbb{N}}$ be the space of sequences of positive integers endowed with its product topology. A subset A of X is called $\mathcal{K}$-analytic if there exists an upper semi-continuous multivalued mapping Ψ on $\mathbb{N}^{\mathbb{N}}$ with values in the set of nonempty compact sets in X such that the equality

$$A = \bigcup_{(n_i)_{i\geq 1}\in\mathbb{N}^{\mathbb{N}}} \Psi\left((n_i)_{i\geq 1}\right)$$

holds.

Good references for $\mathcal{A}$-Souslin and $\mathcal{K}$-analytic notions are [2, 9].

The metric projection on a subset Y of a Banach space $(X, \|\cdot\|)$ is the multivalued mapping given by

$$P_Y(x) = \{y \in Y : \|x - y\| = \inf_{z\in Y} \|x - z\|\}$$

for every $x \in X$. Note that $P_Y(x)$ may be the empty set for some $x \in X$. A convex subset Y of X is said to be proximinal in X provided that for every $x \in X$

the set $P_Y(x)$ is non empty (see [16]). In this section we recall a useful result that it was proved in [3] of measurable selectors for the metric projection when Y is proximal but not necessarily reflexive. To state the result we need two definitions.

Definition 2.3. Let Y and Z be two topological spaces. A selector for a multivalued mapping $\phi : Y \longrightarrow 2^Z$ is a singlevalued map $f : Y \longrightarrow Z$ such that $f(y) \in \phi(y)$ for every $y \in Y$.

Definition 2.4. Let Y and Z be two topological spaces. We will say that a map $f : Y \longrightarrow Z$ is analytic measurable if the preimage of any Borel subset of Z belongs to the smallest σ-algebra containing the analytic subsets of Y.

Theorem 2.5 ([3, Theorem 3.3]). *Let X be a Banach space and Y a proximinal and weakly $\mathcal{K}$-analytic convex subset of X. Then for every separable closed subset $M \subset X$ the metric projection $P_Y|_M : M \longrightarrow 2^Y$ has an analytic measurable selector with separable range.*

To really measure the scope of the preceding theorem it suffices to say that the class of proximinal vector subspaces Y which are $\mathcal{K}$-analytic for the weak topology includes the classes of reflexive subspaces, proximinal separable subspaces, proximinal weakly compactly generated subspaces and proximinal quasi-reflexive subspaces, among others, see [17].

We use Theorem 2.5 to prove a previous result which is necessary to establish a relationship between a relative Chebyshev center point in X and a relative Chebyshev center in the corresponding function space $L_\infty(\mu, X)$.

Proposition 2.6. *Let X be a Banach space and let Y be a weakly $\mathcal{K}$-analytic subspace of X. Let (Ω, Σ, μ) be a complete probability space. Let us suppose that Y is simultaneously proximinal in X and let $f_1, \dots, f_n : \Omega \longrightarrow X$ be μ-measurable functions. Then there exists a μ-measurable function $g_0 : \Omega \longrightarrow Y$ such that $g_0(s)$ is a relative Chebyshev center of $\{f_1(s), \dots, f_n(s)\} \subset X$ in Y, for almost all $s \in \Omega$.*

Proof. Let $f_1, \dots, f_n : \Omega \longrightarrow X$ be μ-measurable functions. It is clear that the function defined as

$$\begin{array}{rccl} f: & \Omega & \longrightarrow & \ell_\infty^n(X) \\ & s & \longrightarrow & f(s) = (f_1(s), \dots, f_n(s)) \end{array}$$

is μ-measurable. Then there exists $\Omega_0 \in \Sigma$ such that $\mu(\Omega \backslash \Omega_0) = 0$ and $f(\Omega_0)$ is a (norm) separable subset of $\ell_\infty^n(X)$ (see [4, p. 42]).

On the other hand we know that [8, Proposition 3.3] the countable products of $\mathcal{K}$-analytic sets is $\mathcal{K}$-analytic. Since Y is weakly $\mathcal{K}$-analytic, then $\operatorname{diag}(\ell_\infty^n(Y))$ is weakly $\mathcal{K}$-analytic.

Now let M be the closure of $f(\Omega_0)$ in $\ell_\infty^n(X)$ for the norm topology and let

$$h : M \longrightarrow \operatorname{diag}(\ell_\infty^n(Y))$$

be the analytic selector with separable range for the norm topology of the metric projection

$$P_{\mathrm{diag}(\ell_\infty^n(Y))|_M} : M \longrightarrow 2^{\mathrm{diag}(\ell_\infty^n(Y))},$$

ensured by Theorem 2.5. The function

$$\begin{array}{rccl} g: & \Omega & \longrightarrow & \mathrm{diag}(\ell_\infty^n(Y)) \\ & s & \longrightarrow & g(s) = \begin{cases} h\left(f(s)\right) & \text{if } s \in \Omega_0 \\ 0 & \text{if } s \in \Omega \backslash \Omega_0 \end{cases} \end{array}$$

obviously satisfies by construction that

$$g(s) = h\left(f(s)\right) \in P_{\mathrm{diag}(\ell_\infty^n(Y))|_M}\left(f(s)\right) \quad \text{if } s \in \Omega_0.$$

Therefore

$$|||f(s) - g(s)||| = \inf_{u \in \mathrm{diag}(\ell_\infty^n(Y))} |||f(s) - u||| \quad \text{if } s \in \Omega_0$$

and so $g(s)$ is a best approximation of $f(s)$ for almost all $s \in \Omega$.

On the other hand g is μ-measurable where the measurability follows from the fact that $f|_{\Omega_0}$ is $\Sigma|_{\Omega_0}$-analytic measurable because a complete probability space is stable by the Souslin operation, [9, Theorem 29.16] and h is analytic measurable.

Finally, we define the function $g_0 : \Omega \longrightarrow Y$ by $g_0(s) = y_s$ where y_s is the component of $g(s) = (y_s, \ldots, y_s) \in \mathrm{diag}(\ell_\infty^n(Y))$ if $s \in \Omega_0$ and $g_0(s) = 0$ otherwise. It is clear that g_0 is μ-measurable.

Thus, taking $y \in Y$ we have

$$|||f(s) - g(s)||| \leq |||f(s) - u|||$$

for almost all $s \in \Omega$, where $u = (y, \ldots, y) \in \mathrm{diag}(\ell_\infty^n(Y))$. This implies that

$$\max_{1 \leq i \leq n} \|f_i(s) - g_0(s)\| \leq \max_{1 \leq i \leq n} \|f_i(s) - y\|$$

for almost all $s \in \Omega$.

Therefore, $g_0(s)$ is a relative Chebyshev center of $\{f_1(s), \ldots, f_n(s)\} \subset X$ in Y, for almost all $s \in \Omega$. □

3. Relative Chebyshev centers in $L_\infty(\mu, X)$

This section is devoted to construct the relative Chebyshev center in the space $L_\infty(\mu, X)$ using the relative Chebyshev center in X.

Proposition 3.1. *Let X be a Banach space, Y be a closed subspace of X. Let (Ω, Σ, μ) be a complete probability space. Let $f_1, \ldots, f_n \in L_\infty(\mu, X)$ and $g \in L_\infty(\mu, Y)$. If $g(s)$ is a relative Chebyshev center of $\{f_1(s), \ldots, f_n(s)\} \subset X$ in Y, for almost all $s \in \Omega$, then g is a relative Chebyshev center of $\{f_1, \ldots, f_n\} \subset L_\infty(\mu, X)$ in $L_\infty(\mu, Y)$.*

Proof. Let $h \in L_\infty(\mu, Y)$. Since $g(s)$ is a relative Chebyshev center of $\{f_1(s), \dots, f_n(s)\} \subset X$ in Y, for almost all $s \in \Omega$, then

$$\max_{1\le i\le n} \|f_i(s) - g(s)\| \le \max_{1\le i\le n} \|f_i(s) - h(s)\|$$

for almost all $s \in \Omega$. On the other hand,

$$\|f_i(s) - h(s)\| \le \|f_i - h\|_\infty$$

for almost all $s \in \Omega$ and each $i \in \{1, \dots, n\}$. Therefore, we have

$$\|f_i(s) - g(s)\| \le \max_{1\le i\le n} \|f_i(s) - g(s)\| \le \max_{1\le i\le n} \|f_i(s) - h(s)\| \le \max_{1\le i\le n} \|f_i - h\|_\infty$$

for almost all $s \in \Omega$ and each $i \in \{1, \dots, n\}$. Then

$$\|f_i(s) - g(s)\| \le \inf_{h\in L_\infty(\mu,Y)} \max_{1\le i\le n} \|f_i - h\|_\infty.$$

for almost all $s \in \Omega$ and each $i \in \{1, \dots, n\}$. This implies that

$$\|f_i - g\|_\infty \le \inf_{h\in L_\infty(\mu,Y)} \max_{1\le i\le n} \|f_i - h\|_\infty$$

for each $i \in \{1, \dots, n\}$. We conclude

$$\max_{1\le i\le n} \|f_i - g\|_\infty \le \inf_{h\in L_\infty(\mu,Y)} \max_{1\le i\le n} \|f_i - h\|_\infty \le \max_{1\le i\le n} \|f_i - h\|_\infty, \quad \forall\, h \in L_\infty(\mu, Y),$$

this means that g is is a relative Chebyshev center of $\{f_1, \dots, f_n\} \subset L_\infty(\mu, X)$ in $L_\infty(\mu, Y)$. $\square$

Lemma 3.2. *Let X be a Banach space, Y be a closed subspace of X. Let (Ω, Σ, μ) be a complete probability space. Let $f_1, \dots, f_n \in L_\infty(\mu, X)$ and $g : \Omega \longrightarrow Y$ be a μ-measurable function such that $g(s)$ is a relative Chebyshev center of $\{f_1(s), \dots, f_n(s)\} \subset X$ in Y, for almost all $s \in \Omega$. Then g is a relative Chebyshev center of $\{f_1, \dots, f_n\} \subset L_\infty(\mu, X)$ in $L_\infty(\mu, Y)$.*

Proof. Since $g(s)$ is a relative Chebyshev center of $\{f_1(s), \dots, f_n(s)\} \subset X$ in Y, for almost all $s \in \Omega$, we have

$$\max_{1\le i\le n} \|f_i(s) - g(s)\| \le \max_{1\le i\le n} \|f_i(s) - y\|$$

for almost all $s \in \Omega$ and each $y \in Y$.

On the other hand, for $i_0 \in \{1, \dots, n\}$ we have

$$\begin{aligned}\|g(s)\| \le \|f_{i_0}(s) - g(s)\| + \|f_{i_0}(s)\| &\le \max_{1\le i\le n} \|f_i(s) - g(s)\| + \|f_{i_0}(s)\| \\ &\le \max_{1\le i\le n} \|f_i(s) - y\| + \|f_{i_0}(s)\|\end{aligned}$$

for almost all $s \in \Omega$ and each $y \in Y$. By taking $y = 0$, we get

$$\|g(s)\| \le \max_{1\le i\le n} \|f_i(s)\| + \|f_{i_0}(s)\| \le \max_{1\le i\le n} \|f_i\|_\infty + \|f_{i_0}\|_\infty$$

for almost all $s \in \Omega$. Thus

$$\|g\|_\infty \le \max_{1\le i\le n} \|f_i\|_\infty + \|f_{i_0}\|_\infty.$$

Consequently $g \in L_\infty(\mu, Y)$, and from Proposition 3.1, is a relative Chebyshev center of $\{f_1, \dots, f_n\} \subset L_\infty(\mu, X)$ in $L_\infty(\mu, Y)$. □

Theorem 3.3. *Let X be a Banach space and let Y be a weakly $\mathcal{K}$-analytic subspace of X. Let us suppose that Y is simultaneously proximinal in X. Then $L_\infty(\mu, Y)$ is simultaneously proximinal in $L_\infty(\mu, X)$.*

Proof. Let $f_1, \dots, f_n$ be functions in $L_\infty(\mu, X)$. Proposition 2.6 guarantees the existence a μ-measurable function $g_0 : \Omega \longrightarrow Y$ such that $g_0(s)$ is a relative Chebyshev center of $\{f_1(s), \dots, f_n(s)\} \subset X$ in Y, for almost all $s \in \Omega$. By Lemma 3.2 it follows that g_0 is a relative Chebyshev center of $\{f_1, \dots, f_n\} \subset L_\infty(\mu, X)$ in $L_\infty(\mu, Y)$. □

Remark 3.4. The above result is not true if Y is not separable, as it was proved in [11, Example 3.1].

4. Relative Chebyshev centers in X

In this section we construct the relative Chebyshev center in the space X using the relative Chebyshev center in $L_\infty(\mu, X)$.

Lemma 4.1. *Let X be a Banach space, Y a closed subspace of X. Let (Ω, Σ, μ) be a complete probability space, $A \in \Sigma$ a set with positive measure, and let $f_1, \dots, f_n \in L_\infty(\mu, X)$ be such that*

$$\inf_{y\in Y} \max_{1\le i\le n} \|f_i(s) - y\| = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \in \Omega \setminus A. \end{cases}$$

Then

$$\inf_{h\in L_\infty(\mu,Y)} \max_{1\le i\le n} \|f_i - h\|_\infty = 1.$$

Proof. It is clear that

$$1 \le \inf_{h\in L_\infty(\mu,Y)} \max_{1\le i\le n} \|f_i - h\|_\infty.$$

For the converse inequality, given $\varepsilon > 0$ we choose measurable countably valued function $f_1', \dots, f_n' \in L_\infty(\mu, X)$ such that

$$f_i'(\Omega) \subset f_i(\Omega) \quad \text{and} \quad \|f_i - f_i'\|_\infty < \frac{\varepsilon}{2}$$

for each $i \in \{1, \ldots, n\}$. The functions $f'_1, \ldots, f'_n$ have the form

$$f'_i(\cdot) = \sum_{k=1}^{\infty} \chi_{B_k}(\cdot) x_{k,i}$$

for each $i \in \{1, \ldots, n\}$, with the B_k's disjoint and measurable, $\chi_{B_k}(\cdot)$ the characteristic function of B_k and $x_{k,i} \in f_i(\Omega)$. For each $k \geq 1$ take $y_k \in Y$ such that

$$\max_{1 \leq i \leq n} \|x_{k,i} - y_k\| < \inf_{y \in Y} \max_{1 \leq i \leq n} \|x_{k,i} - y\| + \frac{\varepsilon}{2} \leq 1 + \frac{\varepsilon}{2}.$$

It is clear that g defined by

$$g(\cdot) = \sum_{k=1}^{\infty} \chi_{B_k}(\cdot) y_k$$

belongs to $L_\infty(\mu, Y)$ and satisfies

$$\|f_i - g\|_\infty \leq \|f_i - f'_i\|_\infty + \|f'_i - g\|_\infty \leq 1 + \varepsilon$$

for each $i \in \{1, \ldots, n\}$. This implies that

$$\max_{1 \leq i \leq n} \|f_i - g\|_\infty \leq 1 + \varepsilon.$$

Therefore,

$$\inf_{h \in L_\infty(\mu, Y)} \max_{1 \leq i \leq n} \|f_i - h\|_\infty \leq 1 + \varepsilon. \qquad \square$$

Proposition 4.2. *Let X be a Banach space and let Y be a closed subspace of X. Let (Ω, Σ, μ) be a complete probability space. Then $L_\infty(\mu, Y)$ is simultaneously proximinal in $L_\infty(\mu, X)$ if and only if for each $f_1, \ldots, f_n \in L_\infty(\mu, X)$ there exists $g \in L_\infty(\mu, Y)$ such that $g(s)$ is a relative Chebyshev center of $\{f_1(s), \ldots, f_n(s)\} \subset X$ in Y for almost all $s \in \Omega$.*

Proof. Assume that $L_\infty(\mu, Y)$ is simultaneously proximinal in $L_\infty(\mu, X)$ and take $f_1, \ldots, f_n \in L_\infty(\mu, X)$. Consider the non-negative measurable function

$$\begin{array}{rccl} h: & \Omega & \longrightarrow & [0, +\infty) \\ & s & \longrightarrow & \inf_{y \in Y} \max_{1 \leq i \leq n} \|f_i(s) - y\|. \end{array}$$

Case 1. Let $s \in \Omega$ such that $h(s) = 0$, then as $\operatorname{diag}(\ell^n_\infty(Y))$ is a closed subspace of the Banach space $(\ell^n_\infty(X), |||\cdot|||)$, it follows that $(f_1(s), \ldots, f_n(s)) \in \operatorname{diag}(\ell^n_\infty(Y))$, so

$$f_1(s) = \cdots = f_n(s) \in Y.$$

Define $\Omega_0 := \{s \in \Omega : f_1(s) = f_2(s) = \cdots = f_n(s) \in Y\}$.

Case 2. Let $s \in \Omega$ such that $h(s) \neq 0$, then for each $k \geq 1$, define

$$\Omega_k := \{s \in \Omega : k - 1 < h(s) \leq k\}.$$

Of course, we may forget those Ω_k which are μ-null sets, so, without loss of generality, we will assume $\mu(\Omega_k) > 0$ for all $k \geq 1$. Now, for each $k \geq 1$ and each $i \in \{1, \ldots, n\}$ we define $f_{k,i} : \Omega \longrightarrow X$ by

$$f_{k,i}(s) = \begin{cases} \frac{1}{h(s)} f_i(s) & \text{if } s \in \Omega_k \\ 0 & \text{if } s \in \Omega \setminus \Omega_k. \end{cases}$$

It is clear that $f_{(k,i)}$ belongs to $L_\infty(\mu, X)$ and also that

$$\begin{aligned} \inf_{y \in Y} \max_{1 \leq i \leq n} \|f_{k,i}(s) - y\| &= \inf_{y \in Y} \max_{1 \leq i \leq n} \left\| \frac{1}{h(s)} f_i(s) - y \right\| \\ &= \frac{1}{h(s)} \inf_{y \in Y} \max_{1 \leq i \leq n} \|f_i(s) - y\| = 1 \end{aligned}$$

for all $s \in \Omega_k$. So, it follows from the preceding lemma that

$$\inf_{g \in L_\infty(\mu, Y)} \max_{1 \leq i \leq n} \|f_{k,i} - g\|_\infty = 1.$$

On the other hand, using that $L_\infty(\mu, Y)$ is simultaneously proximinal in $L_\infty(\mu, X)$, we deduce that there exists $g_k \in L_\infty(\mu, Y)$ such that

$$\max_{1 \leq i \leq n} \|f_{k,i} - g_k\|_\infty = \inf_{g \in L_\infty(\mu, Y)} \max_{1 \leq i \leq n} \|f_{k,i} - g\|_\infty = 1.$$

Therefore, we have

$$1 = \inf_{y \in Y} \max_{1 \leq i \leq n} \|f_{k,i}(s) - y\| \leq \max_{1 \leq i \leq n} \|f_{k,i}(s) - g_k(s)\| \leq \max_{1 \leq i \leq n} \|f_{k,i} - g_k\|_\infty = 1$$

for almost all $s \in \Omega_k$, and so

$$\max_{1 \leq i \leq n} \|f_{k,i}(s) - g_k(s)\| = 1$$

for almost all $s \in \Omega_k$. This implies that

$$\begin{aligned} \inf_{y \in Y} \max_{1 \leq i \leq n} \|f_i(s) - y\| = h(s) &= h(s) \max_{1 \leq i \leq n} \|f_{k,i}(s) - g_k(s)\| \\ &= \max_{1 \leq i \leq n} \|f_i(s) - h(s) g_k(s)\| \end{aligned}$$

for almost all $s \in \Omega_k$. Now it is clear that g defined by

$$g(s) = \chi_{\Omega_0}(s) f_1(s) + \sum_{k=1}^{\infty} \chi_{\Omega_k}(s) h(s) g_k(s)$$

for all $s \in \Omega$ enjoys the required property. □

Corollary 4.3. *Let X be a Banach space and Y a closed subspace of X. Let us suppose that $L_\infty(\mu, Y)$ is simultaneously proximinal in $L_\infty(\mu, X)$. Then Y is simultaneously proximinal in X.*

References

[1] D. Amir, Z. Ziegler: Relative Chebyshev centers in normed linear spaces. I, J. Approx. Theory 29 (1980) 235–252.

[2] V. I. Bogachev: Measure Theory. Vol. I, II, Springer, Berlin (2007).

[3] B. Cascales, M. Raja: Measurable selectors for the metric projection, Math. Nachr. 254 (2003) 27–34.

[4] J. Diestel, J. R. Uhl: Vector Measures, Math. Surveys 15, Amer. Math. Soc, Providence (1977).

[5] M. Fabian, P. Habala, P. Hájek, J. Pelant, V. Montesinos, V. Zizler: Functional Analysis and Infinite Dimensional Geometry, Springer, New York (2001).

[6] A. L. Garkavi: The best possible net and the best possible cross–section of a set in a normed space, Izv. Akad. Nauk SSSR, Ser. Mat. 26 (1962) 87–106 (in Russian); Amer. Math. Soc., Transl., II. Ser. 39 (1964) 111–132 (in English).

[7] A. L. Garkavi: On the Chebyshev center and the convex hull of a set, Usp. Mat. Nauk 19 (1964) 139–145.

[8] J. Kakol, W. Kubiś, M. López-Pellicer: Descriptive Topology in Selected Topics of Functional Analysis, Developments in Mathematics 24, Springer, Berlin (2011).

[9] A. S. Kechris: Classical Descriptive Set Theory, Springer, Berlin (1995).

[10] S. V. Konyagin: A remark on renormings of nonreflexive spaces and the existence of a Chebyshev center, Mosc. Univ. Math. Bull. 43(2) (1988) 55-56.

[11] J. Mendoza: Proximinality in $L^p(\mu, X)$, J. Approx. Theory 93 (1998) 369–379.

[12] J. Mendoza, T. Pakhrou: Best simultaneous approximation in $L_1(\mu, X)$, J. Approx. Theory 145 (2007) 212–220.

[13] T. Pakhrou: Best simultaneous approximation in $L_\infty(\mu, X)$, Math. Nachr. 281 (2008) 396–401.

[14] T. Pakhrou: On nonlinear simultaneous approximation problems, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM 105 (2011) 305–313.

[15] T. Pakhrou: Some subspaces simultaneously proximinal, Kybernetes 41 (2012) 108–115.

[16] I. Singer: Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer, New York (1970).

[17] M. Talagrand: Espaces de Banach faiblement $\mathcal{K}$-analytiques, Ann. Math. (2) 110 (1979) 407–438.

[18] L. Veselý: Generalized centers of finite sets in Banach spaces, Acta Math. Univ. Comen., New Ser. 66 (1997) 83–115.