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Existence and Uniqueness of Renormalized Solutions to Some Nonlinear Elliptic Equations with Variable Exponents and Measure Data *

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In this paper we prove the existence and uniqueness of renormalized solutions to a class of nonlinear elliptic equations with variable exponents and measure data in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$.

Keywords: Nonlinear elliptic equations, variable exponents, measure data, renormalized solution, existence, uniqueness

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1. Introduction

Our main goal is to prove the existence and uniqueness of renormalized solutions to the following nonlinear elliptic equation

$$(P) \quad \begin{cases} -\operatorname{div}(a(x,u,\nabla u)) - \operatorname{div}\phi(u) + g(x,u) = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N (N \geq 2)$ with Lipschitz boundary $\partial\Omega$, μ is a signed measure in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$, and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

We make the following assumptions on a, g and ϕ:

(H_1) The function $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and there exist a continuous function $p : \bar{\Omega} \to (1, +\infty)$ and a positive constant α such that

$$a(x,s,\xi)\xi \geq \alpha|\xi|^{p(x)}, \quad \text{a.e. } x \in \Omega, \ \forall s \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N.$$

(H_2) There exist a continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ and a nonnegative function $c \in L^{p'(\cdot)}(\Omega)$ such that

$$|a(x,s,\xi)| \leq b(|s|)[|\xi|^{p(x)-1} + c(x)], \quad \text{a.e. } x \in \Omega, \ \forall s \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N.$$

(H_3) $(a(x,s,\xi) - a(x,s,\zeta)) \cdot (\xi - \zeta) \geq 0$ holds for almost every $x \in \Omega$ and for every $\xi, \zeta \in \mathbb{R}^N$.

(H_4) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$\sup_{|s| \leq n} |g(\cdot, s)| = h_n(\cdot) \in L^1(\Omega), \quad g(x,s)s \geq 0, \text{ a.e. } x \in \Omega, \ \forall s \in \mathbb{R}.$$

(H_5) The function $\phi : \mathbb{R} \to \mathbb{R}^N$ is assumed to be continuous.

To deal with this kind of problem, it is natural to use Lebesgue and Sobolev spaces with variable exponents which are denoted by $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ respectively. The study of differential equations with variable exponents has been a very active field in recent years, since it has important and extensive application background such as the applications to electro-rheological fluids (see [31, 37]) and image processing (see [16]). There are already numerous results for such problems, see [1–5, 14–15, 18–20, 22–23, 25, 28–29, 32–33, 35–36] and their references.

Under our assumptions, problem (P) does not admit, in general, a weak solution since the terms $a(x,u,\nabla u)$ and $\phi(u)$ may not belong to $(L^1_{loc}(\Omega))^N$. In order to overcome this difficulty, we work with the framework of renormalized solutions. This notion was first introduced by DiPerna and Lions [18] for the study of Boltzmann equation. It was then used by many authors to study the elliptic equations (see [7–10, 13, 17, 26]) and the parabolic equations (see [11–12, 30]) in case of a constant exponent $p(\cdot) \equiv p$. We point out that in [7], the authors have proved that there exists a renormalized solution when right hand side belongs to $W^{-1,p'}(\Omega, \omega)$ with the non-uniformly elliptic condition but not degenerate with respect u, where ω is the weighted function and $\operatorname{esssup}_{|s| \leq k} b(|s|) < \infty$ for any

$k > 0$ (see Assumption (3.1) in [7]). However, the proof in [7] seems to be valid only for $\text{esssup}_{s\in\mathbb{R}} b(|s|) < \infty$, i.e., the method which is used to prove the existence results of renormalized solutions in [7] can not apply to our case.

Recently, while $a(x, s, \xi) = |\xi|^{p(x)-2}\xi$, $g \equiv 0$ and $\phi \equiv 0$, Bendahmane and Wittbold (see [4]) have proved the existence and uniqueness of renormalized solutions to problem (P) with $\mu \in L^1(\Omega)$. Then, Zhang and Zhou (see [35]) have obtained the above results for measure data $\mu \in L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$. In [33], Wittbold and Zimmermann have extended the results of [4] to the case $a(x, s, \xi) = a(x, \xi)$, $g(x, u) = \beta(u)$ and $\mu \in L^1(\Omega)$.

Concerning the notion of entropy solution (introduced by Bénilan et al. in [6]), Sanchón and Urbano (see [32]) have proved the existence and uniqueness of entropy solution to the $p(x)$-Laplace equation with L^1 data. For more papers about the entropy solutions for nonlinear elliptic equations with variable exponents, please see [29], [14] and the references therein. Finally, we mention that there are also some papers [28, 15] which are interesting in the nontrivial solutions for elliptic equations with variable exponents.

In this paper, we will extend the results of [4, 35, 33, 13, 7] to the general case, and our main ideas and methods come from [35, 33, 11]. The outline of this paper is as follows. In Section 2, we give some preliminaries and notations. In Section 3, the existence of renormalized solutions to problem (P) is obtained. In Section 4, we prove the uniqueness of renormalized solution to problem (P).

2. Some preliminaries and notations

In what follows, we recall some definitions and basic properties of Lebesgue and Sobolev spaces with variable exponents.

Set $C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : \inf_{x\in\bar{\Omega}} h(x) > 1\}$. For any $h \in C_+(\bar{\Omega})$, we define $h^+ = \sup_{x\in\bar{\Omega}} h(x)$ and $h^- = \inf_{x\in\bar{\Omega}} h(x)$. For any $p \in C_+(\bar{\Omega})$, the Lebesgue space with variable exponents, denoted by $L^{p(\cdot)}(\Omega)$, consists of all measurable functions such that the modular $\rho_p(f) := \int_\Omega |f(x)|^{p(x)}\mathrm{d}x$ is finite, endowed with the Luxemburg norm $\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_p(\frac{u}{\lambda}) \le 1\right\}$. For $u_n, u \in L^{p(\cdot)}(\Omega), n = 1, 2, \cdots$, we say u_n strongly converges to u in $L^{p(\cdot)}(\Omega)$ if $\lim_{n\to\infty} \|u_n - u\|_{p(\cdot)} = 0$.

Lemma 2.1 (See [20]).

(*1*) *The space $L^{p(\cdot)}(\Omega)$ is a separable and reflexive Banach space, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$,*

$$\left|\int_\Omega uv\mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

(*2*) *If $p_1, p_2 \in C_+(\bar{\Omega})$ with $p_1(x) \le p_2(x)$ for any $x \in \Omega$, then there exists a continuous embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.*

(3) $C_0^\infty(\Omega)$ *is dense in* $L^{p(\cdot)}(\Omega)$.

(4) *For any* $u \in L^{p(\cdot)}(\Omega)$, *there holds*

$$\min\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\} \le \int_\Omega |u(x)|^{p(x)}\mathrm{d}x \le \max\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\}. \quad (1)$$

(5) *Let* $u_n, u \in L^{p(\cdot)}(\Omega), n = 1, 2, \cdots$, *then the following statements are equivalent:*
  (i) $\lim_{n\to\infty} \|u_n - u\|_{p(\cdot)} = 0$;
  (ii) $\lim_{n\to\infty} \rho_p(u_n - u) = 0$;
  (iii) u_n *converges to* u *in* Ω *in measure and* $\lim_{n\to\infty} \rho_p(u_n) = \rho_p(u)$.

Set $W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}$, where the norm is defined by $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$, and denote $W_0^{1,p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$.

Lemma 2.2 (See [20]). *$W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are reflexive Banach spaces. For any $u \in W_0^{1,p(\cdot)}(\Omega)$, the Poincaré inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad (2)$$

holds true, where C is a constant depending on Ω.

Lemma 2.3 (See [20]). *Let $p, d \in C_+(\bar\Omega)$ with $p^+ < N$ and $d(x) < p^*(x) := \frac{Np(x)}{N-p(x)}$ for any $x \in \Omega$, then there is a continuous and compact imbedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{d(\cdot)}(\Omega)$.*

We denote the dual space of $W_0^{1,p(\cdot)}(\Omega)$ by $W^{-1,p'(\cdot)}(\Omega)$, then

Lemma 2.4 (See [21]). *For any $H \in W^{-1,p'(\cdot)}(\Omega)$, there exists a unique vector of function $\tilde H \in (L^{p'(\cdot)}(\Omega))^N$ such that*

$$\langle H, v\rangle = \int_\Omega \tilde H \cdot \nabla v \mathrm{d}x, \quad \forall v \in W_0^{1,p(\cdot)}(\Omega).$$

$W^{-1,p'(\cdot)}(\Omega)$ can be endowed with the norm $\|H\|_{W^{-1,p'(\cdot)}(\Omega)} = \sup\left\{\frac{\langle H,v\rangle}{\|v\|_{W^{1,p(\cdot)}(\Omega)}} : v \in W_0^{1,p(\cdot)}(\Omega)\right\}$. Furthermore, it's easy to prove the following conclusion.

Lemma 2.5. *If $H \in (W_0^{1,p(\cdot)}(\Omega))^N$, then $\int_\Omega \operatorname{div} \mathrm{d}x = 0$.*

For any given $k > 0$, we define $T_k(s) = \min\{k, \max\{s, -k\}\}$, $S_k(r) = \min\{1, (k + 1 - |r|)_+\}$ and the function

$$\operatorname{sign}^+(s) = \begin{cases} 1 & s > 0, \\ 0 & s \le 0. \end{cases} \quad (3)$$

It is easy to find that S_n satisfies the following two conclusions: (S), $\operatorname{supp} S_n \subseteq [-n-1, n+1]$, and $\sup|S_n| = 1$; (S'), $\operatorname{supp} S'_n \subseteq [-n-1,-n] \cup [n, n+1]$, and $\sup\,|S'_n| = 1$. We use $C(\theta_1, \theta_2, \ldots, \theta_m)$ to denote positive constants depending only on specified quantities θ_1, θ_2,..., θ_m.

3. Existence of renormalized solutions to problem (P)

In this section, we study the existence of renormalized solutions to problem (P). Since μ is a signed measure in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$, we may write u again as

$$\mu = f - \operatorname{div} F, \tag{4}$$

where $f \in L^1(\Omega)$ and $F \in L^{p'(\cdot)}(\Omega)$.

Definition 3.1. A measurable function u is a renormalized solution to problem (P), if $g(x,u) \in L^1(\Omega)$, $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$ for any $k \geq 0$,

$$\lim_{n\to\infty} \int_{\{n\leq|u|\leq n+1\}} a(x,u,\nabla u)\nabla u \mathrm{d}x = 0, \tag{5}$$

$$\begin{aligned}
&\int_\Omega (a(x,u,\nabla u) + \phi(u))\nabla(h(u)\varphi)\mathrm{d}x + \int_\Omega g(x,u)h(u)\varphi\mathrm{d}x \\
&= \int_\Omega fh(u)\varphi\mathrm{d}x + \int_\Omega F\cdot\nabla(h(u)\varphi)\mathrm{d}x
\end{aligned} \tag{6}$$

holds for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support and $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Theorem 3.2. *Let $p \in C_+(\bar{\Omega}), p^+ < N$ and μ satisfies* (4). *Assume that the conditions (H_1)–(H_5) hold, then problem (P) admits a renormalized solution u in the sense of Definition* 3.1.

Proof of Theorem 3.2. The proof is divided into eight steps.

Step 1: We introduce a family of approximate problems and prove the existence of solutions to such problems.

For each $\varepsilon > 0$, we define the following approximations

$$g_\varepsilon(x,s) = T_{\frac{1}{\varepsilon}}(g(x,s)), \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R}, \tag{7}$$

$$a_\varepsilon(x,s,\xi) = a(x, T_{\frac{1}{\varepsilon}}(s), \xi), \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N, \tag{8}$$

$$\phi_\varepsilon(s) = \phi(T_{\frac{1}{\varepsilon}}(s)), \quad \forall s \in \mathbb{R}, \tag{9}$$

$$f_\varepsilon = T_{\frac{1}{\varepsilon}}(f). \tag{10}$$

In view of (8), the function a_ε satisfies (H_1), (H_3) with a replaced by a_ε. Moreover, due to (H_2), there exist $c_\varepsilon \in L^{p'(\cdot)}(\Omega)$ and a constant $b_\varepsilon > 0$ such that

$$|a_\varepsilon(x,s,\xi)| \leq b_\varepsilon|\xi|^{p(x)-1} + c_\varepsilon(x), \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N. \tag{11}$$

Now we introduce the approximate problems of (P)

$$(P_\varepsilon) \quad \begin{cases} -\operatorname{div} a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) - \operatorname{div}\phi_\varepsilon(u_\varepsilon) + g_\varepsilon(x,u_\varepsilon) = f_\varepsilon - \operatorname{div} F, & \text{in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

and prove the existence of weak solutions of problem (P_ε).

For any $v, w \in W_0^{1,p(\cdot)}(\Omega)$, we define

$$\langle \mathscr{A}_{\varepsilon 1}(v), w\rangle = \int_\Omega a_\varepsilon(x,v,\nabla v)\nabla w dx$$

and

$$\langle \mathscr{A}_{\varepsilon 2}(v), w\rangle = \int_\Omega \phi_\varepsilon(v)\nabla w dx + \int_\Omega g_\varepsilon(x,v) w dx,$$

it is easy to find that $\mathscr{A}_{\varepsilon 1},\ \mathscr{A}_{\varepsilon 2} : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$. First of all, following the same argument of [27] (see the proof of Theorem 2.8 in Chapter 2), we infer that $\mathscr{A}_{\varepsilon 1}$ is a variational operator and so $\mathscr{A}_{\varepsilon 1}$ is a pseudo-monotone operator. Secondly, it is easy to check that $\mathscr{A}_{\varepsilon 2}$ is a bounded and pseudo-monotone operator. Indeed, if $\{u_n\} \subseteq W_0^{1,p(\cdot)}(\Omega)$ such that

$$u_n \rightharpoonup u \ \text{ weakly in } W_0^{1,p(\cdot)}(\Omega), \tag{12}$$

we have by Lemma 2.3 that

$$u_n \to u \ \text{ strongly in } L^{d(\cdot)}(\Omega), \ \ d(x) < p^*(x), \ \text{and } u_n \to u \ \text{ a.e. in } \Omega. \tag{13}$$

Notice that g_ε and ϕ_ε are bounded and continuous, together with (13), it is easy to prove that

$$\phi_\varepsilon(u_n) \to \phi_\varepsilon(u) \ \text{ strongly in } L^{p'(\cdot)}(\Omega) \tag{14}$$

and

$$g_\varepsilon(x,u_n) \to g_\varepsilon(x,u) \ \text{ strongly in } L^{p'(\cdot)}(\Omega). \tag{15}$$

Furthermore, by the Hölder inequality, we have

$$\begin{aligned} &|\langle \mathscr{A}_{\varepsilon 2}u_n - \mathscr{A}_{\varepsilon 2}u, v\rangle| \\ \le &\left|\int_\Omega [\phi_\varepsilon(u_n) - \phi_\varepsilon(u)]\nabla v dx\right| + \left|\int_\Omega [g_\varepsilon(x,u_n) - g_\varepsilon(x,u)]\, v dx\right| \\ \le &\|\phi_\varepsilon(u_n) - \phi_\varepsilon(u)\|_{L^{p'(\cdot)}(\Omega)}\|\nabla v\|_{L^{p(\cdot)}(\Omega)} + \|g_\varepsilon(x,u_n) - g_\varepsilon(x,u)\|_{L^{p'(\cdot)}(\Omega)}\|v\|_{L^{p(\cdot)}(\Omega)} \end{aligned} \tag{16}$$

for any $v \in W_0^{1,p(\cdot)}(\Omega)$. Hence, we get from that (14)-(16) that

$$\|\mathscr{A}_{\varepsilon 2}u_n - \mathscr{A}_{\varepsilon 2}u\|_{W^{-1,p'(\cdot)}(\Omega)} \to 0,$$

i.e., $\mathscr{A}_{\varepsilon 2}$ is strongly continuous which implies that $\mathscr{A}_{\varepsilon 2}$ is a pseudo-monotone operator(see Proposition 27.6 in Chapter 27 of [34]).

Now let $\mathscr{A}_\varepsilon = \mathscr{A}_{\varepsilon 1} + \mathscr{A}_{\varepsilon 2}$, it is easy to see that

(i) $\mathscr{A}_\varepsilon$ is bounded.
(ii) $\mathscr{A}_\varepsilon$ is coercive.

Indeed, by (H_1), (H_4), (H_5), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}\frac{\langle \mathscr{A}_\varepsilon(v), v\rangle}{\|v\|_{W_0^{1,p(\cdot)}(\Omega)}} &\geq \frac{\alpha\int_\Omega |\nabla v|^{p(x)}\mathrm{d}x + \int_\Omega \phi_\varepsilon(v)\nabla v\mathrm{d}x}{C\|\nabla v\|_{L^{p(\cdot)}(\Omega)}}\\ &\geq \frac{\alpha}{C}\frac{\min\{\|\nabla v\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|\nabla v\|^{p^+}_{L^{p(\cdot)}(\Omega)}\}}{\|\nabla v\|_{L^{p(\cdot)}(\Omega)}} - C_\varepsilon,\end{aligned}$$

where $M_\varepsilon = \max_{|s|\leq \frac{1}{\varepsilon}} \phi(s)$, $C_\varepsilon = \frac{M_\varepsilon\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|1\|_{L^{p'(\cdot)}(\Omega)}}{C}$ is a constant depending on ε. Thus (ii) is true.

(iii) $\mathscr{A}_\varepsilon$ is a pseudo-monotone operator(see Proposition 27.6 in Chapter 27 of [34]).

Hence, (i)–(iii) together with Theorem 2.7 in [27] imply that there exist at least one weak solution u_ε to problem (P_ε) for any $0 < \varepsilon < 1$.

Step 2: We establish an uniform estimate on u_ε with respect to ε.

For any $k > 0$, taking $T_k(u_\varepsilon)$ as a test function in (P_ε), we have

$$\begin{aligned}&\int_\Omega a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\nabla T_k(u_\varepsilon)\mathrm{d}x + \int_\Omega \phi_\varepsilon(u_\varepsilon)\nabla T_k(u_\varepsilon)\mathrm{d}x + \int_\Omega g_\varepsilon(x, u_\varepsilon)T_k(u_\varepsilon)\mathrm{d}x\\ =& \int_\Omega f_\varepsilon T_k(u_\varepsilon)\mathrm{d}x + \int_\Omega F\nabla T_k(u_\varepsilon)\mathrm{d}x. \end{aligned}\tag{17}$$

Define $\tilde{\phi}_\varepsilon(t) = \int_0^t \phi_\varepsilon(s)\mathrm{d}s$. Applying the Divergence Theorem, we deduce that

$$\int_\Omega \phi_\varepsilon(u_\varepsilon)\nabla T_k(u_\varepsilon)\mathrm{d}x = \int_\Omega \operatorname{div}\tilde{\phi}_\varepsilon(T_k(u_\varepsilon))\mathrm{d}x = \int_{\partial\Omega} \tilde{\phi}_\varepsilon(T_k(0))\cdot\nu\mathrm{d}s = 0. \tag{18}$$

Combining (7)–(10), (17)–(18) with (H_1) and (H_4), by Young's inequality we obtain

$$\begin{aligned}&\int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)}\mathrm{d}x\\ \leq& \frac{kp^-}{\alpha(p^- - 1)}\int_\Omega |f(x)|\mathrm{d}x + \frac{p^-(p^+ - 1)}{(p^- - 1)p^+}\max\left\{1, \alpha^{\frac{p^-}{1-p^-}}\right\}\int_\Omega |F(x)|^{p'(x)}\mathrm{d}x. \end{aligned}\tag{19}$$

By Lemma 2.1 and (19), we have

$$\begin{aligned}\|\nabla T_k(u_\varepsilon)\|_{L^{p(\cdot)}(\Omega)} &\leq \max\left\{1, \left(\int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)}\mathrm{d}x\right)^{\frac{1}{p^-}}\right\}\\ &\leq C\left(\alpha, k, p, \|f\|_{L^1(\Omega)}, \|F\|_{L^{p'(\cdot)}(\Omega)}\right). \end{aligned}\tag{20}$$

For fixed $k>0$, we have in view of (8) that

$$\begin{aligned} &a_\varepsilon(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) \\ &= a(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) \ \text{ a.e. } x\in\Omega, \ \text{for any } 0<\varepsilon<\frac{1}{k}. \end{aligned} \tag{21}$$

From (H_2) and (20), $a(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))$ is bounded uniformly in $(L^{p'(\cdot)}(\Omega))^N$ with respect to ε. Hence, by (21), there exists a function $\sigma_k\in (L^{p'(\cdot)}(\Omega))^N$ and a subsequence (still indexed by ε) such that

$$a_\varepsilon(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) \rightharpoonup \sigma_k \ \text{ weakly in } (L^{p'(\cdot)}(\Omega))^N, \ \forall k>0. \tag{22}$$

Using (20), Lemma 2.2–2.3 and arguing as in the proof of Theorem 6.1 in [6], there exist a measurable function u and a subsequence of $\{T_k(u_\varepsilon)\}$ (still denoted by $\{T_k(u_\varepsilon)\}$) such that

$$u_\varepsilon\to u \ \text{ a.e. in } \Omega, \tag{23}$$

$$T_k(u_\varepsilon)\rightharpoonup T_k(u) \ \text{ weakly in } W_0^{1,p(\cdot)}(\Omega), \tag{24}$$

$$T_k(u_\varepsilon)\to T_k(u) \ \text{ strongly in } L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \tag{25}$$

Step 3: We prove in this step that

$$\overline{\lim}_{\varepsilon\to 0}\int_\Omega a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon)\mathrm{d}x \le \int_\Omega \sigma_k\nabla T_k(u)\mathrm{d}x. \tag{26}$$

Choosing $S_n(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))$ as a test function in (P_ε), we have

$$I_1+I_2+I_3+I_4+I_5=I_6+I_7+I_8, \tag{27}$$

where

$$\begin{aligned} I_1 &= \int_\Omega S_n(u_\varepsilon)a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)(\nabla T_k(u_\varepsilon)-\nabla T_k(u))\mathrm{d}x, \\ I_2 &= \int_\Omega S_n'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\mathrm{d}x, \\ I_3 &= \int_\Omega S_n(u_\varepsilon)\phi_\varepsilon(u_\varepsilon)(\nabla T_k(u_\varepsilon)-\nabla T_k(u))\mathrm{d}x, \\ I_4 &= \int_\Omega S_n'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\phi_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\mathrm{d}x, \\ I_5 &= \int_\Omega g_\varepsilon(x,u_\varepsilon)S_n(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\mathrm{d}x, \\ I_6 &= \int_\Omega f_\varepsilon S_n(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\mathrm{d}x, \\ I_7 &= \int_\Omega S_n(u_\varepsilon)F(\nabla T_k(u_\varepsilon)-\nabla T_k(u))\mathrm{d}x, \\ I_8 &= \int_\Omega S_n'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))F\nabla u_\varepsilon\mathrm{d}x. \end{aligned}$$

In the following, we will study the limit of I_i respectively.

The limit of I_2: By (S'), we have

$$\overline{\lim}_{\varepsilon\to 0}|I_2| \leq 2k\overline{\lim}_{\varepsilon\to 0}\int_{\{n\leq|u_\varepsilon|\leq n+1\}} a_\epsilon(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon \mathrm{d}x. \tag{28}$$

Using $T_{n+1}(u_\varepsilon)-T_n(u_\varepsilon)$ as a test function in (P_ε), we obtain

$$\begin{aligned}
&\int_\Omega a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\nabla(T_{n+1}(u_\varepsilon)-T_n(u_\varepsilon))\mathrm{d}x\\
&\qquad+\int_\Omega \phi_\epsilon(u_\varepsilon)\nabla(T_{n+1}(u_\varepsilon)-T_n(u_\varepsilon))\mathrm{d}x\\
&\qquad+\int_\Omega g_\varepsilon(x,u_\varepsilon)(T_{n+1}(u_\varepsilon)-T_n(u_\varepsilon))\mathrm{d}x\\
=&\int_\Omega f_\varepsilon(T_{n+1}(u_\varepsilon)-T_n(u_\varepsilon))\mathrm{d}x+\int_\Omega F\nabla(T_{n+1}(u_\varepsilon)-T_n(u_\varepsilon))\mathrm{d}x.
\end{aligned}\tag{29}$$

Similarly to (18), we have

$$\int_\Omega \phi_\varepsilon(u_\varepsilon)\nabla(T_{n+1}(u_\varepsilon)-T_n(u_\varepsilon))\mathrm{d}x=0. \tag{30}$$

Using (30) and applying Young's inequality in (29), we obtain that

$$\begin{aligned}
&\int_{\{n\leq|u_\varepsilon|\leq n+1\}} a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\mathrm{d}x+\frac{p^-}{p^--1}\int_\Omega g_\varepsilon(x,u_\varepsilon)(T_{n+1}(u_\varepsilon)-T_n(u_\varepsilon))\mathrm{d}x\\
&\leq \omega(n,\varepsilon),
\end{aligned}\tag{31}$$

where $\omega(n,\varepsilon):=\frac{p^-}{p^--1}\int_{\{|u_\varepsilon|\geq n\}}|f|\mathrm{d}x+\frac{p^-(p^+-1)}{(p^--1)p^+}\max\left\{\alpha,\alpha^{\frac{1}{1-p^-}}\right\}\int_{\{n\leq|u_\varepsilon|\leq n+1\}}|F|^{p'(x)}\mathrm{d}x.$

On the other hand, by Poincaré inequality and (19), we have

$$\begin{aligned}
&|\{|u_\varepsilon|\geq n\}|\\
\leq&\int_\Omega\frac{|T_n(u_\varepsilon)|^{p^-}}{n^{p^-}}\mathrm{d}x\leq\frac{C(p^-,N,\Omega)}{n^{p^-}}\int_\Omega|\nabla T_n(u_\varepsilon)|^{p^-}\mathrm{d}x\\
\leq&\frac{C(p^-,N,\Omega)}{n^{p^-}}\left[\int_{\{|\nabla T_n(u_\varepsilon)|<1\}}|\nabla T_n(u_\varepsilon)|^{p^-}\mathrm{d}x+\int_{\{|\nabla T_n(u_\varepsilon)|\geq 1\}}|\nabla T_n(u_\varepsilon)|^{p(x)}\mathrm{d}x\right]\\
\leq&\frac{C(p^-,N,\Omega)}{n^{p^-}}\left[|\Omega|+\int_\Omega|\nabla T_n(u_\varepsilon)|^{p(x)}\mathrm{d}x\right]\\
\leq&\frac{C\left(\alpha,p^-,N,\Omega,p^+,\|f\|_{L^1(\Omega)},\|F\|_{L^{p'(\cdot)}(\Omega)}\right)}{n^{p^--1}},\quad\forall n>1.
\end{aligned}\tag{32}$$

Let $\varepsilon \to 0$ in (32), then it follows from (23) and Lebesgue's dominated convergence theorem that

$$|\{|u| \geq n\}| \leq \frac{C\left(\alpha, p^-, N, \Omega, p^+, \|f\|_{L^1(\Omega)}, \|F\|_{L^{p'(\cdot)}(\Omega)}\right)}{n^{p^- -1}}, \quad \forall n > 1. \tag{33}$$

Combining (H_4), (23) with (31) and (33), by the absolute continuity of integral and Lebesgue's dominated convergence theorem, we find that

$$\overline{\lim_{n\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \int_{\{n\leq |u_\varepsilon|\leq n+1\}} a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\nabla u_\varepsilon \mathrm{d}x \leq \lim_{n\to\infty}\lim_{\varepsilon\to 0}\omega(n,\varepsilon) = 0. \tag{34}$$

Hence, according to (28) and (34), we have

$$\overline{\lim_{n\to\infty}}\ \overline{\lim_{\varepsilon\to 0}}\ |I_2| = 0. \tag{35}$$

The limit of I_3, I_4, I_7 and I_8: By (S), for enough small $\varepsilon > 0$, we have

$$S_n(u_\varepsilon)\phi_\varepsilon(u_\varepsilon) = S_n(u_\varepsilon)\phi(T_{n+1}(u_\varepsilon)).$$

Note that S_n is continuous and bounded, the above equality and (23) yield that

$$\begin{aligned} S_n(u_\varepsilon)\phi_\varepsilon(u_\varepsilon) &\to S_n(u)\phi(T_{n+1}(u)) \\ &= S_n(u)\phi(u) \ \text{ strongly in } (L^{p'(\cdot)}(\Omega))^N, \text{ as } \varepsilon \to 0. \end{aligned} \tag{36}$$

The convergence result (36) together with (24) imply that

$$\lim_{\varepsilon\to 0} I_3 = \lim_{\varepsilon\to 0}\int_\Omega S_n(u_\varepsilon)\phi_\varepsilon(u_\varepsilon)(\nabla T_k(u_\varepsilon) - \nabla T_k(u))\mathrm{d}x = 0. \tag{37}$$

Similarly to the limit of I_3, we have

$$\lim_{\varepsilon\to 0} I_4 = \lim_{\varepsilon\to 0}\int_\Omega S_n'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))\phi_\varepsilon(u_\varepsilon)\nabla T_{n+1}(u_\varepsilon)\mathrm{d}x = 0, \tag{38}$$

$$\lim_{\varepsilon\to 0} I_7 = \lim_{\varepsilon\to 0}\int_\Omega S_n(u_\varepsilon)F(\nabla T_k(u_\varepsilon) - \nabla T_k(u))\mathrm{d}x = 0, \tag{39}$$

$$\lim_{\varepsilon\to 0} I_8 = \lim_{\varepsilon\to 0}\int_\Omega S_n'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))F\nabla T_{n+1}(u_\varepsilon)\mathrm{d}x = 0, \tag{40}$$

where (S') is used in (38) and (40).

The limit of I_5 and I_6: From (S), (23) and Lebesgue's dominated convergence theorem, we deduce that

$$\lim_{\varepsilon\to 0} I_5 = \lim_{\varepsilon\to 0}\int_\Omega g_\varepsilon(x, u_\varepsilon)S_n(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))\mathrm{d}x = 0, \tag{41}$$

$$\lim_{\varepsilon\to 0} I_6 = \lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon S_n(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))\mathrm{d}x = 0. \tag{42}$$

Passage to the limit in (27): Let $\varepsilon \to 0$ and then $n \to \infty$ in (27), it follows from (35) and (37)–(40) that

$$\overline{\lim_{n\to\infty}}\ \overline{\lim_{\varepsilon\to 0}}\ I_1 = \overline{\lim_{n\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \int_\Omega S_n(u_\varepsilon) a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)(\nabla T_k(u_\varepsilon) - \nabla T_k(u))\mathrm{d}x \le 0. \quad (43)$$

By (8), we infer that

$$\begin{aligned} &S_n(u_\varepsilon) a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\nabla T_k(u_\varepsilon) \\ &= a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \ \text{ a.e. } x \in \Omega, \ \text{for } 0 < k < \max\left\{\frac{1}{\varepsilon}, n\right\}, \end{aligned} \quad (44)$$

$$\begin{aligned} &S_n(u_\varepsilon) a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \\ &= S_n(u_\varepsilon) a(x, T_{n+1}(u_\varepsilon), \nabla T_{n+1}(u_\varepsilon)) \ \text{ a.e. } x \in \Omega, \ \text{for } 0 < \varepsilon < \frac{1}{n+1}. \end{aligned} \quad (45)$$

Combing (22) with (23) and (43)–(45), we get

$$\begin{aligned} &\overline{\lim_{n\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \int_\Omega a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\nabla T_k(u_\varepsilon)\mathrm{d}x \\ &\le \overline{\lim_{n\to\infty}} \int_\Omega S_n(u)\sigma_{n+1}\nabla T_k(u)\mathrm{d}x, \quad \text{for } 0 < k < n. \end{aligned} \quad (46)$$

Moreover, we have for $0 < k < n$ that

$$\begin{aligned} &a_\varepsilon(x, T_{n+1}(u_\varepsilon), \nabla T_{n+1}(u_\varepsilon))\chi_{\{|u_\varepsilon|\le k\}} \\ &= a_\varepsilon(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon))\chi_{\{|u_\varepsilon|\le k\}} \ \text{ a.e. } x \in \Omega. \end{aligned} \quad (47)$$

Using (22), (23) and (47), we find that

$$\sigma_{n+1}\ \chi_{\{|u|\le k\}} = \sigma_k\ \chi_{\{|u|\le k\}} \ \text{ a.e. } x \in \Omega. \quad (48)$$

From (46) and (48), we get the desired result (26).

Step 4: The aim of this step is to obtain

$$\lim_{\varepsilon\to 0} \int_\Omega [a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), \nabla T_k(u))](\nabla T_k(u_\varepsilon) - \nabla T_k(u))\mathrm{d}x = 0. \quad (49)$$

Indeed, by (H_2), (23) and Lebesgue's dominated convergence theorem, we have

$$a(x, T_k(u_\varepsilon), \nabla T_k(u)) \to a(x, T_k(u), \nabla T_k(u)) \ \text{ strongly in } (L^{p'(\cdot)}(\Omega))^N, \ \text{as } \varepsilon \to 0. \quad (50)$$

Thus, using (24), we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u))(\nabla T_k(u_\varepsilon) - \nabla T_k(u))\mathrm{d}x = 0. \quad (51)$$

Note that

$$a_\varepsilon(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) = a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)), \quad \text{for } 0 < \varepsilon < \frac{1}{k}, \tag{52}$$

then (22), (26) and (51) yield

$$\overline{\lim_{\varepsilon \to 0}} \int_\Omega [a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), \nabla T_k(u))](\nabla T_k(u_\varepsilon) - \nabla T_k(u))\mathrm{d}x \le 0.$$

Hence, conclusion (49) follows from the above inequality and (H_3).

Step 5: In this step, we prove that $\sigma_k = a(x, T_k(u), \nabla T_k(u))$ and

$$\begin{aligned} &a_\varepsilon(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \\ &\to a(x, T_k(u), \nabla T_k(u))\nabla T_k(u) \ \text{ weakly in } L^1(\Omega), \ \text{as } \varepsilon \to 0. \end{aligned} \tag{53}$$

In fact, by (22), (49)–(52), we have

$$\lim_{\varepsilon \to 0} \int_\Omega a_\varepsilon(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon)\mathrm{d}x = \int_\Omega \sigma_k \cdot \nabla T_k(u)\mathrm{d}x. \tag{54}$$

Similarly to the argument in [11], in view of (H_3), (22), (24), (52) and (54), the Minty's Lemma (Lemma 1.5 in Chapter III of [24]) implies that

$$\sigma_k = a(x, T_k(u), \nabla T_k(u)). \tag{55}$$

Using (22), (55), we obtain that

$$\begin{aligned} &a_\varepsilon(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon))\nabla T_k(u) \\ &\to a(x, T_k(u), \nabla T_k(u))\nabla T_k(u) \ \text{ weakly in } L^1(\Omega), \ \text{as } \varepsilon \to 0. \end{aligned}$$

Since $a_\varepsilon(x, T_k(u_\varepsilon), \nabla T_k(u)) = a(x, T_k(u_\varepsilon), \nabla T_k(u))$ for $0 < \varepsilon < \frac{1}{k}$, by (50) we have

$$\begin{aligned} &a_\varepsilon(x, T_k(u_\varepsilon), \nabla T_k(u))\nabla T_k(u_\varepsilon) \\ &\to a(x, T_k(u), \nabla T_k(u))\nabla T_k(u) \ \text{ weakly in } L^1(\Omega), \ \text{as } \varepsilon \to 0, \end{aligned}$$

$$\begin{aligned} &a_\varepsilon(x, T_k(u_\varepsilon), \nabla T_k(u))\nabla T_k(u) \\ &\to a(x, T_k(u), \nabla T_k(u))\nabla T_k(u) \ \text{ strongly in } L^1(\Omega), \ \text{as } \varepsilon \to 0. \end{aligned}$$

The above three convergence results and (49) imply that the desired result (53) holds.

Step 6: We want to get the following limit

$$\lim_{n \to \infty} \int_{\{n \le |u| \le n+1\}} a(x, u, \nabla u)\nabla u\mathrm{d}x = 0. \tag{56}$$

Indeed, replacing k by $n+1$ in (53), from (23) we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\{n\le|u_\varepsilon|\le n+1\}} a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon \mathrm{d}x\\
&=\lim_{\varepsilon\to 0}\int_\Omega a_\varepsilon(x,T_{n+1}(u_\varepsilon),\nabla T_{n+1}(u_\varepsilon))\nabla T_{n+1}(u_\varepsilon)\mathcal{X}_{\{n\le|u_\varepsilon|\le n+1\}}\mathrm{d}x\\
&=\int_\Omega a(x,T_{n+1}(u),\nabla T_{n+1}(u))\nabla T_{n+1}(u)\mathcal{X}_{\{n\le|u|\le n+1\}}\mathrm{d}x\\
&=\int_{\{n\le|u|\le n+1\}} a(x,u,\nabla u)\nabla u\mathrm{d}x. \qquad (57)
\end{aligned}$$

Conclusion (56) follows immediately from (34) and (57).

Step 7: Our aim is to prove that $g(x,u)\in L^1(\Omega)$.

Indeed, in view of (31), we obtain

$$\int_\Omega g_\varepsilon(x,u_\varepsilon)(T_{n+1}(u_\varepsilon)-T_n(u_\varepsilon))\mathrm{d}x\le\frac{p^--1}{p^-}\omega(n,\varepsilon).$$

By (H_4), (23) and Fatou's Lemma, we deduce from the above inequality that

$$g(x,u)(T_{n+1}(u)-T_n(u))\in L^1(\Omega)$$
$$\text{and}\quad \|g(x,u)(T_{n+1}(u)-T_n(u))\|_{L^1(\Omega)}\le\frac{p^--1}{p^-}\omega_n,$$

where $\omega_n=\lim_{\varepsilon\to 0}\omega(n,\varepsilon)=\frac{p^-}{p^--1}\int_{\{|u|\ge n\}}|f|\mathrm{d}x+\frac{p^-(p^+-1)}{(p^--1)p^+}\max\left\{\alpha,\alpha^{\frac{1}{1-p^-}}\right\}\int_{\{n\le|u|\le n+1\}}$ $|F|^{p'(x)}\mathrm{d}x$. Thus we have

$$\begin{aligned}
&\int_\Omega|g(x,u)|\mathrm{d}x\\
&=\int_{\Omega\cap\{|u|\le n+1\}}|g(x,u)|\mathrm{d}x+\int_{\Omega\cap\{|u|>n+1\}}|g(x,u)(T_{n+1}(u)-T_n(u))|\mathrm{d}x\\
&\le\int_\Omega h_{n+1}(x)\mathrm{d}x+\frac{p^--1}{p^-}\omega_n.
\end{aligned}$$

Since $\lim_{n\to\infty}\omega(n)=0$ (see (34)), we conclude from the above inequality that $\int_\Omega|g(x,u)|\mathrm{d}x\le C$, i.e., $g(x,u)\in L^1(\Omega)$.

Step 8: We prove that u satisfies (6).

Let $\varphi\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ and $h\in W^{1,\infty}(\mathbb{R})$ with compact support belonging to $[-k,k]$. Taking $S_n(u_\varepsilon)h(u)\varphi$ as a test function in (P_ε), we get

$$J_1+J_2+J_3+J_4+J_5=J_6+J_7+J_8, \qquad (58)$$

where

$$J_1 = \int_\Omega S_n(u_\varepsilon)a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\nabla(h(u)\varphi)\mathrm{d}x,$$
$$J_2 = \int_\Omega S_n'(u_\varepsilon)a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\nabla u_\varepsilon h(u)\varphi\mathrm{d}x,$$
$$J_3 = \int_\Omega S_n(u_\varepsilon)\phi_\varepsilon(u_\varepsilon)\nabla(h(u)\varphi)\mathrm{d}x,$$
$$J_4 = \int_\Omega S_n'(u_\varepsilon)\phi_\varepsilon(u_\varepsilon)\nabla u_\varepsilon h(u)\varphi\mathrm{d}x,$$
$$J_5 = \int_\Omega g_\varepsilon(x, u_\varepsilon)S_n(u_\varepsilon)h(u)\varphi\mathrm{d}x,$$
$$J_6 = \int_\Omega f_\varepsilon(x)S_n(u_\varepsilon)h(u)\varphi\mathrm{d}x,$$
$$J_7 = \int_\Omega S_n(u_\varepsilon)F\nabla(h(u)\varphi)\mathrm{d}x,$$
$$J_8 = \int_\Omega S_n'(u_\varepsilon)F\nabla u_\varepsilon h(u)\varphi\mathrm{d}x.$$

In the following, we will examine the limit of J_i respectively.

The limit of J_1: By (S), the term J_1 can be rewritten as follows,

$$J_1 = \int_\Omega S_n(u_\varepsilon)a(x, T_{n+1}(u_\varepsilon), \nabla T_{n+1}(u_\varepsilon))\nabla(h(u)\varphi)\mathrm{d}x. \tag{59}$$

Lebesgue's dominated convergence theorem and (23) imply that

$$S_n(u_\varepsilon)\nabla(h(u)\varphi) \to S_n(u)\nabla(h(u)\varphi) \text{ strongly in } (L^{p(\cdot)}(\Omega))^N, \text{ as } \varepsilon \to 0. \tag{60}$$

Replacing k by $n+1$ in (22) and (55), it follows from (59) and (60) that

$$\begin{aligned}\lim_{n\to\infty}\lim_{\varepsilon\to 0} J_1 &= \lim_{n\to\infty}\int_\Omega S_n(u)a(x, T_{n+1}(u), \nabla T_{n+1}(u))\nabla(h(u)\varphi)\\ &= \int_\Omega a(x, u, \nabla u)\nabla(h(u)\varphi)\mathrm{d}x.\end{aligned} \tag{61}$$

The limit of J_2: In view of (S'), we have

$$|J_2| \le \|h(u)\varphi\|_{L^\infty(\Omega)}\int_{\{n\le|u_\varepsilon|\le n+1\}} a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\nabla u_\varepsilon\mathrm{d}x. \tag{62}$$

Combining (34) with (62), we obtain

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} J_2 = 0. \tag{63}$$

The limit of J_3: Recall that supp $h \subset [-k,k]$, similarly to (61), we have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\varepsilon\to 0} J_3\\
&= \lim_{n\to\infty}\int_\Omega S_n(u)\phi(T_{n+1}(u))\nabla(h(u)\varphi)\mathrm{d}x = \lim_{n\to\infty}\int_\Omega S_n(u)\phi(u)\nabla(h(u)\varphi)\mathrm{d}x,\\
&= \lim_{n\to\infty}\int_\Omega S_n(u)\phi(T_k(u))\nabla(h(u)\varphi)\mathrm{d}x = \int_\Omega \phi(u)\nabla(h(u)\varphi)\mathrm{d}x. \qquad (64)
\end{aligned}$$

The limit of J_4: Similarly to (62), using Young's inequality, we have

$$\begin{aligned}
|J_4| &\le \|h(u)\varphi\|_{L^\infty(\Omega)}\int_{\{n\le|u_\varepsilon|\le n+1\}}|\phi_\varepsilon(u_\varepsilon)\nabla u_\varepsilon|\mathrm{d}x\\
&\le \|h(u)\varphi\|_{L^\infty(\Omega)}\Bigg[\frac{1}{p^-}\int_{\{n\le|u_\varepsilon|\le n+1\}}|\nabla u_\varepsilon|^{p(x)}\mathrm{d}x\\
&\qquad + \frac{p^+-1}{p^+}\int_{\{n\le|u_\varepsilon|\le n+1\}\cap\{|u|\le k\}}|\phi_\varepsilon(u_\varepsilon)|^{p'(x)}\mathrm{d}x\Bigg]. \qquad (65)
\end{aligned}$$

By (H_1), (8), (33) and (34), we obtain

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} J_4 = 0. \qquad (66)$$

The limit of J_5: Notice that supp $h \subset [-k,k]$ and (S), we may assume that $n+1>k$ and obtain

$$\begin{aligned}
\lim_{n\to\infty}\lim_{\varepsilon\to 0} J_5 &= \lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega g_\varepsilon(x,T_{n+1}(u_\varepsilon))S_n(u_\varepsilon)h(u)\varphi\mathrm{d}x\\
&= \int_\Omega g(x,u)h(u)\varphi\mathrm{d}x. \qquad (67)
\end{aligned}$$

The limit of J_6 and J_7: It is easy to see that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} J_6 = \int_\Omega f(x)h(u)\varphi\mathrm{d}x, \qquad \lim_{n\to\infty}\lim_{\varepsilon\to 0} J_7 = \int_\Omega F\nabla(h(u)\varphi)\mathrm{d}x. \qquad (68)$$

The limit of J_8: Similarly to (66), we obtain

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} J_8 = 0. \qquad (69)$$

Passing to the limit in (58): Let $\varepsilon\to 0$ and then $n\to\infty$ in (58), it follows from (61), (63), (64), and (66)–(69) that

$$\begin{aligned}
&\int_\Omega (a(x,u,\nabla u)+\phi(u))\nabla(h(u)\varphi)\mathrm{d}x + \int_\Omega g(x,u)h(u)\varphi\mathrm{d}x\\
&= \int_\Omega fh(u)\varphi\mathrm{d}x + \int_\Omega F\cdot\nabla(h(u)\varphi)\mathrm{d}x, \qquad (70)
\end{aligned}$$

holds for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support and $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

In view of (56) and (70), we conclude that u is a renormalized solution to problem (P) and the proof of Theorem 3.2 is finished. □

4. Uniqueness of renormalized solution to problem (P)

In order to get the uniqueness of renormalized solution to problem (P), we make the following assumptions:

(H_6) ϕ is locally Lipschitz continuous function.

(H_7) For a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^N$ and $\forall k > 0$, there exist $\bar{c}_k \in L^{p'(\cdot)}(\Omega)$ and a constant $\beta_k > 0$ such that

$$|a(x, s_1, \xi) - a(x, s_2, \xi)| \leq |s_1 - s_2|[\beta_k |\xi|^{p(x)-1} + \bar{c}_k(x)], \tag{71}$$

holds for any $|s_1| \leq k,\ |s_2| \leq k$.

(H_8) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is strictly increasing with respect to the second variable.

Theorem 4.1. *Let $p \in C_+(\bar{\Omega}), p^+ < N$ and μ satisfies (4). Assume that the conditions (H_1)–(H_8) hold, then problem (P) admits a unique renormalized solutions u in the sense of Definition 3.1.*

Proof of Theorem 4.1. By Theorem 3.2, we need only to prove the uniqueness result.

Defined $H_\varepsilon \in W^{1,\infty}(\mathbb{R})$ as $H_\varepsilon(r) = \min\{1, \frac{r_+}{\varepsilon}\}$. Let u_1, u_2 be two renormalized solutions to problem (P). According to Definition 3.1, we have

$$\begin{aligned}
&\int_\Omega (a(x, u_i, \nabla u_i) + \phi(u_i))\nabla(S_n(u_i) H_\varepsilon(T_{n+1}(u_1) - T_{n+1}(u_2)))\mathrm{d}x \\
&\qquad + \int_\Omega g(x, u_i) S_n(u_i) H_\varepsilon(T_{n+1}(u_1) - T_{n+1}(u_2))\mathrm{d}x \\
&= \int_\Omega f S_n(u_i) H_\varepsilon(T_{n+1}(u_1) - T_{n+1}(u_2))\mathrm{d}x \\
&\qquad + \int_\Omega F \cdot \nabla(S_n(u_i) H_\varepsilon(T_{n+1}(u_1) - T_{n+1}(u_2)))\mathrm{d}x,
\end{aligned} \tag{72}$$

where $i = 1, 2$. Denote $\Omega_\varepsilon = \{x \in \Omega : 0 < T_{n+1}(u_1) - T_{n+1}(u_2) < \varepsilon\}$, we obtain from (72) that

$$\begin{aligned}
&I_1(n, \varepsilon) + I_2(n, \varepsilon) + I_3(n, \varepsilon) + I_4(n, \varepsilon) + I_5(n, \varepsilon) \\
&= I_6(n, \varepsilon) + I_7(n, \varepsilon) + I_8(n, \varepsilon),
\end{aligned} \tag{73}$$

where

$$\begin{aligned}
I_1(n,\varepsilon) = &\int_\Omega [S_n'(u_1)a(x,u_1,\nabla u_1)\nabla u_1 - S_n'(u_2)a(x,u_2,\nabla u_2)\nabla u_2]H_\varepsilon(T_{n+1}(u_1) \\
&\quad - T_{n+1}(u_2))\mathrm{d}x, \\
I_2(n,\varepsilon) = &\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} [S_n(u_1)a(x,u_1,\nabla u_1) - S_n(u_2)a(x,u_2,\nabla u_2)]\nabla(T_{n+1}(u_1) \\
&\quad - T_{n+1}(u_2))\mathrm{d}x, \\
I_3(n,\varepsilon) = &\int_\Omega [S_n'(u_1)\phi(u_1)\nabla u_1 - S_n'(u_2)\phi(u_2)\nabla u_2]H_\varepsilon(T_{n+1}(u_1) - T_{n+1}(u_2))\mathrm{d}x, \\
I_4(n,\varepsilon) = &\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} [S_n(u_1)\phi(u_1) - S_n(u_2)\phi(u_2)]\nabla(T_{n+1}(u_1) - T_{n+1}(u_2))\mathrm{d}x, \\
I_5(n,\varepsilon) = &\int_\Omega [g(x,u_1)S_n(u_1) - g(x,u_2)S_n(u_2)]H_\varepsilon(T_{n+1}(u_1) - T_{n+1}(u_2))\mathrm{d}x, \\
I_6(n,\varepsilon) = &\int_\Omega f[S_n(u_1) - S_n(u_2)]H_\varepsilon(T_{n+1}(u_1) - T_{n+1}(u_2))\mathrm{d}x, \\
I_7(n,\varepsilon) = &\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} [S_n(u_1) - S_n(u_2)]F\cdot\nabla(T_{n+1}(u_1) - T_{n+1}(u_2))\mathrm{d}x, \\
I_8(n,\varepsilon) = &\int_\Omega [S_n'(u_1)F\cdot\nabla u_1 - S_n'(u_2)F\cdot\nabla u_2]H_\varepsilon(T_{n+1}(u_1) - T_{n+1}(u_2))\mathrm{d}x.
\end{aligned}$$

In the following, we will examine the limit of each term in (73) as $\varepsilon \to 0$ and then $n \to \infty$.

The limit of $I_1(n,\varepsilon)$: On one hand, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} I_1(n,\varepsilon) \\
= &\int_\Omega [S_n'(u_1)a(x,u_1,\nabla u_1)\nabla u_1 - S_n'(u_2)a(x,u_2,\nabla u_2)\nabla u_2]\mathrm{sign}^+(T_{n+1}(u_1) \\
&\quad - T_{n+1}(u_2))\mathrm{d}x, \qquad (74)
\end{aligned}$$

where the function sign^+ is defined by (3). On the other hand, notice that (S'), we deduce from (5) and (74) that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} I_1(n,\varepsilon) = 0. \qquad (75)$$

The limit of $I_2(n,\varepsilon)$: Rewritten $I_2(n,\varepsilon)$ as $I_2(n,\varepsilon) = I_{21}(n,\varepsilon) + I_{22}(n,\varepsilon) + I_{23}(n,\varepsilon)$, where

$$\begin{aligned}
I_{21}(n,\varepsilon) = &\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} S_n(u_1)[a(x,T_{n+1}(u_1),\nabla T_{n+1}(u_1)) \\
&\quad - a(x,T_{n+1}(u_1),\nabla T_{n+1}(u_2))]\nabla(T_{n+1}(u_1) - T_{n+1}(u_2))\mathrm{d}x,
\end{aligned}$$

$$I_{22}(n,\varepsilon)=\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[S_n(u_1)-S_n(u_2)]a(x,T_{n+1}(u_1),\nabla T_{n+1}(u_2))\nabla(T_{n+1}(u_1)\\ -T_{n+1}(u_2))\mathrm{d}x,$$

$$I_{23}(n,\varepsilon)=\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}S_n(u_2)[a(x,T_{n+1}(u_1),\nabla T_{n+1}(u_2))\\ -a(x,T_{n+1}(u_2),\nabla T_{n+1}(u_2))]\nabla(T_{n+1}(u_1)-T_{n+1}(u_2))\mathrm{d}x.$$

In view of (H_3), we have

$$I_{21}(n,\varepsilon)\geq 0. \tag{76}$$

Notice that (S) and (S'), by (H_2) we have

$$\begin{aligned}&|I_{22}(n,\varepsilon)|\\ \leq&\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\sup|S'_n|\cdot|T_{n+1}(u_1)-T_{n+1}(u_2)|\\ &\quad\cdot|a(x,T_{n+1}(u_1),\nabla T_{n+1}(u_2))\nabla(T_{n+1}(u_1)-T_{n+1}(u_2))|\mathrm{d}x\\ \leq&\int_{\Omega_\varepsilon}b_{n+1}[|\nabla T_{n+1}(u_2)|^{p(x)-1}+c_{n+1}(x)]\cdot|\nabla(T_{n+1}(u_1)-\nabla T_{n+1}(u_2))|\mathrm{d}x,\end{aligned}$$

where the constant b_{n+1} and the function C_{n+1} is independent of ε, which yields

$$\lim_{\varepsilon\to 0}I_{22}(n,\varepsilon)=0. \tag{77}$$

As far as $I_{23}(n,\varepsilon)$ is concerned, we have in view of (H_7) (replacing k by $n+1$) that

$$\begin{aligned}&|I_{23}(n,\varepsilon)|\\ \leq&\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}|T_{n+1}(u_1)-T_{n+1}(u_2)|\cdot[\bar{c}_{n+1}(x)+\beta_{n+1}|\nabla T_{n+1}(u_2)|^{p(x)-1}]\\ &\quad\cdot|\nabla(T_{n+1}(u_1)-T_{n+1}(u_2))|\mathrm{d}x\\ \leq&\int_{\Omega_\varepsilon}[\bar{c}_{n+1}(x)+\beta_{n+1}|\nabla T_{n+1}(u_2)|^{p(x)-1}]\cdot|\nabla(T_{n+1}(u_1)-T_{n+1}(u_2))|\mathrm{d}x,\end{aligned}$$

and thus

$$\lim_{\varepsilon\to 0}I_{23}(n,\varepsilon)=0. \tag{78}$$

Recall that $I_2(n,\varepsilon)=I_{21}(n,\varepsilon)+I_{22}(n,\varepsilon)+I_{23}(n,\varepsilon)$, it follows from (76)–(78) that

$$\overline{\lim_{\varepsilon\to 0}}\,I_2(n,\varepsilon)\geq 0. \tag{79}$$

The limit of $I_3(n,\varepsilon)$: For the term $I_3(n,\varepsilon)$, we have following calculations

$$\begin{aligned}
|I_3(n,\varepsilon)| &= \left|\int_\Omega \left[\operatorname{div}\int_{T_{n+1}(u_2)}^{T_{n+1}(u_1)} S_n'(r)\phi(r)\mathrm{d}r\right] H_\varepsilon(T_{n+1}(u_1)-T_{n+1}(u_2))\mathrm{d}x\right| \\
&\le \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\left|\int_{T_{n+1}(u_2)}^{T_{n+1}(u_1)} S_n'(r)\phi(r)\mathrm{d}r\right|\cdot|\nabla(T_{n+1}(u_1)-T_{n+1}(u_2))|\mathrm{d}x \\
&\le \sup_{r\in[-n-1,\ n+1]}|\phi(r)|\int_{\Omega_\varepsilon}|\nabla T_{n+1}(u_1)-\nabla T_{n+1}(u_2)|\mathrm{d}x,
\end{aligned}$$

which implies that

$$\lim_{\varepsilon\to 0} I_3(n,\varepsilon)=0. \tag{80}$$

The limit of $I_4(n,\varepsilon)$: The term $I_4(n,\varepsilon)$ can be written as follows

$$\begin{aligned}
&|I_4(n,\varepsilon)| \\
=&\frac{1}{\varepsilon}\bigg|\int_{\Omega_\varepsilon} S_n(u_1)[\phi(T_{n+1}(u_1))-\phi(T_{n+1}(u_2))]\cdot\nabla(T_{n+1}(u_1)-T_{n+1}(u_2))\mathrm{d}x \\
&\qquad +\int_{\Omega_\varepsilon}[S_n(u_1)-S_n(u_2)]\phi(T_{n+1}(u_2))\cdot\nabla(T_{n+1}(u_1)-T_{n+1}(u_2))\mathrm{d}x\bigg| \\
\le&\|\phi'\|_{L^\infty[-n-1,\ n+1]}\int_{\Omega_\varepsilon}|\nabla T_{n+1}(u_1)-\nabla T_{n+1}(u_2)|\mathrm{d}x \\
&\qquad +\sup_{r\in[-n-1,\ n+1]}|\phi(r)|\int_{\Omega_\varepsilon}|\nabla T_{n+1}(u_1)-\nabla T_{n+1}(u_2)|\mathrm{d}x,
\end{aligned}$$

where (H_6) is used. Therefore,

$$\lim_{\varepsilon\to 0} I_4(n,\varepsilon)=0. \tag{81}$$

The limit of $I_5(n,\varepsilon)$: Notice that $g(x,u_1), g(x,u_2)\in L^1(\Omega)$, it is easy to see that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\varepsilon\to 0} I_5(n,\varepsilon) \\
=&\lim_{n\to\infty}\int_\Omega[g(x,u_1)S_n(u_1)-g(x,u_2)S_n(u_2)]\operatorname{sign}^+(T_{n+1}(u_1)-T_{n+1}(u_2))\mathrm{d}x \\
=&\int_\Omega[g(x,u_1)-g(x,u_2)]\chi_{\{u_1-u_2>0\}}\mathrm{d}x.
\end{aligned} \tag{82}$$

The limit of $I_6(n,\varepsilon)$: It is easy to calculate that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\varepsilon\to 0} I_6(n,\varepsilon) \\
=&\lim_{n\to\infty}\int_\Omega f[S_n(u_1)-S_n(u_2)]\operatorname{sign}^+(T_{n+1}(u_1)-T_{n+1}(u_2))\mathrm{d}x=0.
\end{aligned} \tag{83}$$

The limit of $I_7(n,\varepsilon)$: Similarly to (77), we have

$$\lim_{\varepsilon\to 0} I_7(n,\varepsilon) = 0. \tag{84}$$

The limit of $I_8(n,\varepsilon)$: We observe that

$$\begin{aligned} &\lim_{\varepsilon\to 0} I_8(n,\varepsilon) \qquad (85)\\ = &\int_\Omega [S_n'(u_1)F\cdot\nabla u_1 - S_n'(u_2)F\cdot\nabla u_2]\mathrm{sign}^+(T_{n+1}(u_1)-T_{n+1}(u_2))\mathrm{d}x = I_{8n}. \end{aligned}$$

By Hölder's inequality, we deduce that

$$\begin{aligned} |I_{8n}| \le \left(\frac{1}{p-}+\frac{1}{(p')^-}\right)\Big[&\|\chi_{\{n\le|u_1|\le n+1\}}\nabla u_1\|_{L^{p(\cdot)}(\Omega)} \\ &+ \|\chi_{\{n\le|u_2|\le n+1\}}\nabla u_2\|_{L^{p(\cdot)}(\Omega)}\Big]\|F\|_{L^{p'(\cdot)}(\Omega)}. \end{aligned} \tag{86}$$

Since $\lim_{n\to\infty}\int_{\{n\le|u_1|\le n+1\}} a(x,u,\nabla u_1)\nabla u_1\mathrm{d}x = 0$, there exists a $N_1>0$ such that $\int_{\{n\le|u_1|\le n+1\}}|\nabla u_1|^{p(x)}\mathrm{d}x<1$ for $n\ge N_1$. Thus, by Lemma 2.1, we have for $n>N_1$ that

$$\begin{aligned} \|\chi_{\{n\le|u_1|\le n+1\}}\nabla u_1\|_{L^{p(\cdot)}(\Omega)} &\le \left(\int_{\{n\le|u_1|\le n+1\}}|\nabla u_1|^{p(x)}\mathrm{d}x\right)^{\frac{1}{p^+}} \\ &\le \left(\frac{1}{\alpha}\int_{\{n\le|u_1|\le n+1\}} a(x,u_1,\nabla u_1)\nabla u_1\mathrm{d}x\right)^{\frac{1}{p^+}}. \end{aligned} \tag{87}$$

Similarly, there exists a $N_2\ge N_1$ such that for $n>N_2$,

$$\|\chi_{\{n\le|u_2|\le n+1\}}\nabla u_2\|_{L^{p(\cdot)}(\Omega)} \le \left(\frac{1}{\alpha}\int_{\{n\le|u_2|\le n+1\}} a(x,u_2,\nabla u_2))\nabla u_2\mathrm{d}x\right)^{\frac{1}{p^+}}. \tag{88}$$

It follows from (5) (replace u by u_1 and u_2 respectively) and (85)–(88) that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} I_8(n,\varepsilon) = \lim_{n\to\infty} I_{8n} = 0. \tag{89}$$

Passage to the limit as $\varepsilon\to 0$ and then $n\to\infty$ in (73): It follows from (75), (79)–(81), (82)–(84) and (89) that

$$\int_\Omega [g(x,u_1)-g(x,u_2)]\chi_{\{u_1-u_2>0\}}\mathrm{d}x \le 0. \tag{90}$$

Combining (90) with (H_8), we infer that $u_1\le u_2$ a.e. in Ω.

Interchanging the role of u_1 and u_2, we obtain $u_2\le u_1$ a.e. in Ω. Therefore, $u_1\equiv u_2$ a.e. in Ω. Thus, the proof is finished. □

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