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Closedness of the Set of Extreme Points in Calderón-Lozanovskiĭ Spaces

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It is known (see [1]) that a compact linear operator from a Banach space X into the space of continuous functions $C(Z,\mathbb{R})$ is extreme provided it is nice, i.e. $T^*(Z) \subset \operatorname{Ext} B(X^*)$, where Z is a compact Hausdorff space and $T^* : Z \to X^*$ is a continuous function defined by $T^*(z)(x) = T(x)(z)$. The nice operator condition can be weakened as long as the set of extreme points $\operatorname{Ext} B(X^*)$ is closed, namely it suffices to assume than $T^*(Z_0) \subset \operatorname{Ext} B(X^*)$ for some dense subset $Z_0 \subset Z$ in that case. The aim of this paper is to characterize the closedness of the set of extreme points of the unit ball of Calderón-Lozanovskiĭ spaces E_φ generated by the Köthe space E and the Orlicz function φ. The main theorem of the paper (Theorem 2.14) gives conditions under which the closedness of the set $\operatorname{Ext} B(E_\varphi)$ is equivalent to the closedness of the set of extreme points of the unit ball of the corresponding Köthe space E.

Keywords: Calderón-Lozanovskiĭ spaces, extreme points, compact operators, Orlicz spaces, Köthe spaces

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1. Introduction

Recall that for a Banach space $(X, ||\cdot||)$ a point $x \in S(X)$ is said to be an extreme point of $B(X)$ ($x \in \operatorname{Ext} B(X)$) if for any $y, z \in B(X)$ such that $x = \frac{y+z}{2}$ we have $y = z$.

For a Banach space X and a compact Hausdorff space Z, it is known ([7]) that the space $K(X, C(Z,\mathbb{R}))$ of linear, bounded compact operators $T : X \to C(Z,\mathbb{R})$, is isometric to the space $C(Z, X^*)$ of continuous functions on Z with values in the dual Banach space X^*. In fact, for every linear compact operator $T : X \to C(Z,\mathbb{R})$ we can assign a continuous mapping $T^* : Z \to X^*$ defined by the formula $T^*(z)(x) = T(x)(z)$. The operator T is called nice if $T^*(Z) \subset \operatorname{Ext} B(X^*)$. It is

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easy to see that every nice linear compact operator is an extreme point of the unit ball $B(K(X, C(Z, \mathbb{R}))$, but the inverse implication is not true even in finite dimensional Banach spaces ([1]).

Let us note that if the set $\operatorname{Ext} B(X)$ is closed then for T to be nice it suffices to assume that $T^*(Z_0) \subset \operatorname{Ext} B(X^*)$ on a dense subset $Z_0 \subset Z$.

In the following by (Ω, Σ, μ) we will denote a σ-finite and complete measure space and $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ will stand for the space of all (equivalence classes of) Σ-measurable real valued functions defined on Ω.

Evidently, in the case of Lebesgue function spaces $L^p(\Omega)$, the set $\operatorname{Ext} B(L^p(\Omega))$ is closed for all $1 \leq p \leq \infty$. Indeed, for $p = 1$, $\operatorname{Ext} B(L^1(\Omega)) = \emptyset$, for $1 < p < \infty$ the space $L^p(\Omega)$ is rotund and for $p = \infty$ the set $\operatorname{Ext} B(L^\infty(\Omega))$ consists of isolated points. In the case of Orlicz spaces $L^\varphi(\Omega)$ (see [3]) either with the Luxemburg or the Orlicz norm the closedness of $\operatorname{Ext} B(L^\varphi(\Omega))$ depends on the function φ (see [29], [30], [31]). In this paper we will investigate the closedness of the set of extreme points of Calderón-Lozanovskĭi spaces E_φ. This class of spaces includes the class of Orlicz spaces with Luxemburg norm – it suffices to put $E = L^1(\Omega)$. We will give conditions under which the following equivalence holds true: the set $\operatorname{Ext} B(E_\varphi)$ is closed if and only if the set $\operatorname{Ext} B(E)$ is closed. We start with basic notation and definitions that will be used in the paper.

By a Köthe space $E \subset L^0(\Omega)$ we mean a Banach space $(E, ||\cdot||_E)$ satisfying the following conditions:

(i) for every $x \in L^0(\Omega)$ and $y \in E$ such that $|x(t)| \leq |y(t)|$ for μ-a.e. $t \in \Omega$ we have $x \in E$ and $||x||_E \leq ||y||_E$,

(ii) there is a function $x \in E$ such that $x(t) > 0$ for μ-a.e. $t \in \Omega$.

We say that a Köthe space E has the Fatou property ($E \in (FP)$ for short) if for any $x \in L^0$ and (x_n) in E^+ such that $x_n \uparrow |x|$ μ-a.e. and $\sup_{n\in\mathbb{N}} ||x_n||_E < \infty$ we have $x \in E$ and $||x_n||_E \to ||x||_E$.

By E^+ we will denote the positive cone of E, i.e. $E^+ = \{x \in E : x \geq 0\}$. It is known that for every Köthe space there exists a sequence of disjoint measurable sets (A_n) such that $\bigcup_{n=1}^\infty A_n = \Omega$ and $\chi_{A_n} \in E$ for all $n \in \mathbb{N}$ ([32], Theorem 86.2).

A point $x \in E^+$ is called a point of upper monotonicity (UM-point for short) if for any $y \in E^+ \setminus \{0\}$ we have $||x||_E < ||x + y||_E$. A point $x \in E^+ \setminus \{0\}$ is called a point of lower monotonicity (LM-point for short) if for every $y \in E^+$ such that $y \leq x$ and $y \neq x$ we have $||y||_E < ||x||_E$. If every point of $S(E^+)$ is a UM-point (alternatively an LM-point), then we say that the space E is strictly monotone.

The relation between UM-points and extreme points in Köthe spaces describes the following lemma.

Lemma 1.1 ([12], Lemma 1.4). *Let E be an arbitrary Köthe space. A point $x \in S(E)$ is an extreme point of $B(E)$ if and only if $|x|$ is an UM-point and $|x| \in \operatorname{Ext} B(E^+)$.*

For a nonnegative measurable function x define the functions

$$n_x(\tau) = \mu(\{t \in \Omega : x(t) > \tau\}), \qquad x^*(t) = \inf\{\tau : n_x(\tau) < t\}.$$

We say that two nonnegative functions x and y are equimeasurable whenever $n_x(\tau) = n_y(\tau)$ for every $\tau > 0$. A Köthe space E is called symmetric if for every equimeasurable functions $x \in L^0(\Omega)$ and $y \in E$ we have $x \in E$ and $||x||_E = ||y||_E$ or equivalently

$$x \in L^0(\Omega) \wedge y \in E \wedge x^*(t) \leq y^*(t)\ \mu - a.e. \quad \Rightarrow \quad x \in E \wedge ||x||_E \leq ||y||_E.$$

For more information about symmetric Köthe spaces we refer to [25] and [26].

A function $\varphi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if $\varphi(0) = 0$, φ is not identically equal to 0, it is even and convex on the interval $(-b_\varphi, b_\varphi)$ and left continuous at b_φ, i.e., $\lim_{u \to b_\varphi^-} \varphi(u) = \varphi(b_\varphi)$, where $b_\varphi = \sup\{u > 0 : \varphi(u) < \infty\}$.

Moreover, for any Orlicz function φ we define $a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\}$. If $a_\varphi = 0$ we will write $\varphi > 0$. If φ takes only finite values (i.e. $b_\varphi = \infty$), then we will write $\varphi < \infty$. Let G_φ denotes the set $[a_\varphi, b_\varphi]$ if $\varphi(b_\varphi) < \infty$ or the set $[a_\varphi, b_\varphi)$ if $\varphi(b_\varphi) = \infty$. Note that the function $\varphi|_{G_\varphi}$ is strictly monotone, so it posesses the inverse function.

An interval $[a, b]$ is called a structurally affine interval for an Orlicz function φ provided that φ is affine on $[a, b]$ and it is not affine either on $[a-\varepsilon, b]$ or $[a, b+\varepsilon]$ for any $\varepsilon > 0$. For an Orlicz function φ we define the set of its strictly convex points by $SC(\varphi) = G_\varphi \setminus \bigcup_{n \in N_\varphi} (a_n, b_n)$, where $N_\varphi \subset \mathbb{N}$ denotes the set of all indices of structurally affine intervals for φ.

We say that an Orlicz function φ satisfies the $\Delta_2(0)$-condition ($\varphi \in \Delta_2(0)$ in short) if there exist numbers $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) > 0$ and $\varphi(2u) \leq K\varphi(u)$ for all $u \leq u_0$. We say that an Orlicz function φ satisfies the $\Delta_2(\infty)$-condition ($\varphi \in \Delta_2(\infty)$ in short) if there exist numbers $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) < \infty$ and $\varphi(2u) \leq K\varphi(u)$ for all $u \geq u_0$. Finally, we say that an Orlicz function φ satisfies the $\Delta_2(\mathbb{R}^+)$-condition ($\varphi \in \Delta_2(\mathbb{R}^+)$ in short) if there exists a number $K > 0$ such that $\varphi(2u) \leq K\varphi(u)$ for all $u \geq 0$.

For any Köthe space E and any Orlicz function φ, on the space $L^0(\Omega)$ we define the convex semimodular ρ_φ by the formula

$$\rho_\varphi(x) = \begin{cases} ||\varphi \circ |x|\,||_E & \text{if } \varphi \circ |x| \in E, \\ \infty & \text{if } \varphi \circ |x| \notin E. \end{cases}$$

The Calderón-Lozanovskĭi space E_φ generated by the couple (E, φ) is defined as the set $E_\varphi = \{x \in L^0(\Omega) : \exists_{\lambda > 0}\ \ \rho_\varphi(\lambda x) < \infty\}$. In the Calderón-Lozanovskĭi space E_φ we define a norm by the formula $||x||_\varphi = \inf\{\lambda > 0 : \rho_\varphi(\frac{x}{\lambda}) \leq 1\}$. It is known that if E has the Fatou property then E_φ has this property as well. It is also known ([25], Lemma 4.3) that if the Köthe space E is symmetric then the Calderón-Lozanovskĭi space E_φ is symmetric as well.

Let us note that every Calderón-Lozanovskiĭ space over $E = L^1(\Omega)$ becomes an Orlicz space $L^\varphi(\Omega)$ and the norm $\|\cdot\|_\varphi$ is called the Luxemburg norm in that case. Moreover, Orlicz spaces with the Luxemburg norm are examples of symmetric Köthe spaces. In the sequel we will use the following theorem characterizing the UM-points of the unit sphere of Orlicz spaces equipped with the Luxemburg norm, that is an immediate consequence of Theorem 1 presented in [11].

Theorem 1.2 ([11]). *A point $x \in S(L^\varphi(\Omega)^+)$ is upper monotone with respect to the Luxemburg norm if and only if the following conditions are satisfied:*

(*i*) $x(t) \geq a_\varphi$ *μ-a.e.,*
(*ii*) $x(t) = b_\varphi$ *μ-a.e. or* $\int_\Omega \varphi(x)d\mu = 1$.

If E is a Köthe space and φ is an Orlicz function then we say that φ satisfies the Δ_2^E-condition ($\varphi \in \Delta_2^E$ for short, [14]) if:

(1) $\varphi \in \Delta_2(0)$ whenever $E \hookrightarrow L^\infty(\Omega)$ (i.e. E is continuously included in $L^\infty(\Omega)$)
(2) $\varphi \in \Delta_2(\infty)$ whenever $L^\infty(\Omega) \hookrightarrow E$
(3) $\varphi \in \Delta_2(\mathbb{R}^+)$ whenever neither $E \hookrightarrow L^\infty(\Omega)$ nor $L^\infty(\Omega) \hookrightarrow E$.

Lemma 1.3 ([2], [9]). *Let E be a Köthe space and let φ be an Orlicz function. For every $x \in E_\varphi$ and (x_n) in E_φ we have:*

(*i*) *if* $\rho_\varphi(x) = 1$, *then* $||x||_\varphi = 1$,
(*ii*) *if* $\rho_\varphi(x_n) \to 1$, *then* $||x_n||_\varphi \to 1$,
(*iii*) *if* $||x_n||_\varphi \to 0$, *then* $\rho_\varphi(x_n) \to 0$.

Lemma 1.4 ([2], [9]). *Let E be a Köthe space and let φ be an Orlicz function with $\varphi > 0$ and $\varphi \in \Delta_2^E$. For every sequence (x_n) in E_φ we have $||x_n||_\varphi \to 0$ whenever $\rho_\varphi(x_n) \to 0$.*

Lemma 1.5 ([2], [9]). *Let E be a Köthe space and let φ be any Orlicz function such that $\varphi < \infty$ and $\varphi \in \Delta_2^E$. For every $x \in E_\varphi$ and (x_n) in E_φ we have:*

(*i*) $\rho_\varphi(x) = 1$ *whenever* $||x||_\varphi = 1$,
(*ii*) $\rho_\varphi(x_n) \to 1$ *whenever* $||x_n||_\varphi \to 1$.

The condition (i) of Lemma 1.5 appears to be the crucial one while investigating several types of geometric properties of Calderón-Lozanovskiĭ spaces. In the paper [16] P. Kolwicz defined a weakened property that implies the condition (i), namely we say that the Calderón-Lozanowskii space E_φ satisfies the norm-modular property ($E_\varphi \in (nm)$) if for every $x \in E_\varphi$ the implication $||x||_\varphi = 1 \Rightarrow \rho_\varphi(x) = 1$ holds. It is obvious that if $\varphi < \infty$, the condition $\varphi \in \Delta_2^E$ implies $E_\varphi \in (nm)$. From the following lemma it is easy to conclude that the inverse implication is not true.

Lemma 1.6 ([16]). *Assume that E is a symmetric space over the space of finite measure and let $\varphi < \infty$. We have $E_\varphi \in (nm)$ if and only if $\varphi \in \Delta_2(\infty)$ or $E_a = \{0\}$, where E_a denotes the ideal of elements of absolutely continuous norm* *in* E*, i.e.* $E_a = \{x \in E : ||x\chi_{A_n}||_E \to 0 \ \ as \ \ A_n \searrow_\mu \emptyset\}$.

The following theorem gives the complete characterization of extreme points in Calderón-Lozanovskĭi spaces.

Theorem 1.7 ([13], Theorem 1). *Let* E_φ *be a Calderón-Lozanovskĭi space and* $x \in S(E_\varphi)$*. Then* $x \in \operatorname{Ext} B(E_\varphi)$ *if and only if one of the following conditions is satisfied:*

(i) $|x| = b_\varphi \chi_\Omega$ *or*

(ii) $\rho_\varphi(x) = 1$, $|x| \geq a_\varphi \chi_\Omega$, $\varphi \circ |x|$ *is* UM*-point in* E *and the following condition holds:*

(o) *for every* $u, v \in S(E)$ *such that* $0 \leq u, v \leq \varphi(b_\varphi)\chi_\Omega$ *and* $\frac{u+v}{2} = \varphi \circ |x|$ *we have either* $u = v$ *or* $\varphi \circ (\frac{y+z}{2}) < \frac{1}{2}(\varphi \circ y + \varphi \circ z)$*, where* $y = g^{-1} \circ u$*,* $z = g^{-1} \circ v$*, and* g *is the restriction of* φ *to the set* G_φ*.*

It is obvious that if $x \in S(E_\varphi)$ and $\varphi \circ |x| \in \operatorname{Ext} B(E)$, then the condition (o) is satisfied. Moreover, by Lemma 1.1, $\varphi \circ |x| \in \operatorname{Ext} B(E)$ implies that $\varphi \circ |x|$ is an UM-point of E. Thus, by the condition (ii) of Theorem 1.7, for $x \in S(E_\varphi)$ to be an extreme point of $B(E_\varphi)$ it suffices to prove that $\rho_\varphi(x) = 1$, $|x| \geq a_\varphi \chi_\Omega$ and $\varphi \circ |x| \in \operatorname{Ext} B(E)$.

But extreme points x of $B(E_\varphi)$ need not to satisfy the condition $\varphi \circ |x| \in \operatorname{Ext} B(E)$ even if $\varphi < \infty$. For example, take the Lebesgue space $E = L^1[0,1]$ and put $\varphi(u) = u^2$. Then $E_\varphi = L^2[0,1]$, so $\operatorname{Ext}(L^2[0,1]) \neq \emptyset$. But $\varphi \circ |x| \notin \operatorname{Ext} B(L^1[0,1])$ since the unit ball $B(L^1[0,1])$ does not possess extreme points.

More information about Orlicz spaces can be found in [3] and [24]. For information about extreme points and geometric properties of Calderón-Lozanovskĭi spaces we refer to: [2], [8], [9], [10], [12], [13], [14], [16], [17], [18], [19], [20], [21], [22], [23].

2. Main Results

In the paper [29] the necessary and sufficient conditions for the closedness of the set of extreme points in Orlicz spaces $L^\varphi(\Omega)$ equipped with Luxemburg norm were presented. Recall that an Orlicz function φ is said to satisfy the Γ_a-condition ($\varphi \in \Gamma_a$ for short) if there exist constants $c, K > 1$ and $d \geq 0$ such that $d\mu(\Omega) < \infty$ and $\varphi(t) \leq K \cdot \varphi(s) + d$ for every $t \in SC(\varphi) \cap [0, cs]$ and $s \in SC(\varphi)$.

Theorem 2.1 ([29]). *The set* $\operatorname{Ext} B(L^\varphi(\Omega))$ *is closed if and only if* $\varphi \in \Gamma_a$*.*

We start with an auxiliary lemma that will be used later.

Lemma 2.2. *If the composition of two Orlicz functions* $\psi \circ \varphi$ *satisfies the* Γ_a*-condition, then the function* ψ *satisfies the* Γ_a*-condition as well.*

Proof. Assume that $\psi \circ \varphi \in \Gamma_a$. Then there exist constants $c, K > 1$ and $d \geq 0$ such that $d\mu(\Omega) < \infty$ and $(\psi \circ \varphi)(t) \leq K(\psi \circ \varphi)(s) + d$ for every $t \in$

$SC(\psi \circ \varphi) \cap [0, cs]$ and $s \in SC(\psi \circ \varphi)$. Put $s = \varphi(u)$. We will show that if $s \in SC(\psi)$, then $u \in SC(\psi \circ \varphi)$. Suppose that $u \notin SC(\psi \circ \varphi)$. Then we can find $x \neq y$ and $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ such that $u = \alpha x + \beta y$ and

$$\begin{aligned}\alpha\psi(\varphi(x)) + \beta\psi(\varphi(y)) &= \alpha(\psi \circ \varphi)(x) + \beta(\psi \circ \varphi)(y) \\ &= (\psi \circ \varphi)(\alpha x + \beta y) = \psi(\varphi(u)) = \psi(s).\end{aligned}$$

Since $s = \varphi(u) \in SC(\psi)$, we have $\varphi(u) = \varphi(x) = \varphi(y) > 0$, whence $u = x = y$, a contradiction. By convexity of φ and by the Γ_a-condition for $\psi \circ \varphi$ we get

$$\psi(t) \leq \psi(cs) = \psi(c\varphi(u)) \leq \psi(\varphi(cu)) \leq K\psi(\varphi(u)) + d = K\psi(s) + d$$

for every $t \in SC(\psi) \cap [0, cs]$ and $s \in SC(\psi)$, i.e., the Γ_a-condition for the function ψ holds true. □

Since the condition $\Delta_2(\mathbb{R}^+)$ implies the condition Γ_a, the set $\operatorname{Ext} B(L^{\psi\circ\varphi})$ is closed provided both functions φ and ψ satisfy the condition $\Delta_2(\mathbb{R}^+)$. Indeed, it suffices to note that the composition $\psi \circ \varphi$ satisfies the condition $\Delta_2(\mathbb{R}^+)$ in this case. Let us note that the condition $\psi \in \Delta_2(\mathbb{R}^+)$ is far from being necessary for the closedness of the set $\operatorname{Ext} B(L^{\psi\circ\varphi})$.

In the sequel we will use the closedness of the set of UM-points in Köthe spaces. Below we present such theorem for Orlicz spaces.

Theorem 2.3. *For any Orlicz function ψ the set of UM-points of the unit sphere $S(L^\psi(\Omega)^+)$ is closed under the Luxemburg norm topology.*

Proof. Let (x_n) be a sequence of UM-points of the unit sphere $S(L^\psi(\Omega)^+)$ and let $\|x_n - x\| \to 0$ for some $x \in L^\psi(\Omega)^+$. Evidently $\|x\| = 1$. By Theorem 1.2, for every $n \in \mathbb{N}$ we have $x_n \geq a_\psi \chi_\Omega$ and $x_n = b_\psi \chi_\Omega$ or $\int_\Omega \psi(x_n) d\mu = 1$. Thus, evidently, $x \geq a_\psi \chi_\Omega$ as well.

Suppose that $\int_\Omega \psi(x_{n_0}) d\mu < 1$ for some $n_0 \in \mathbb{N}$. Then $x_{n_0} = b_\psi \chi_\Omega$, so

$$\int_\Omega \psi(z) d\mu \leq \int_\Omega \psi(b_\psi) d\mu = \int_\Omega \psi(x_{n_0}) d\mu < 1$$

for every function $z \in L^\psi(\Omega)^+$ with $\int_\Omega \psi(z) d\mu < \infty$, whence $x_n = b_\psi \chi_\Omega$ for all $n \in \mathbb{N}$, i.e., the sequence (x_n) is constant. Thus $x = b_\psi \chi_\Omega$ as well, so x is an UM-point of $S(L^\psi(\Omega)^+)$.

Now assume that $\int_\Omega \psi(x_n) d\mu = 1$ for all $n \in \mathbb{N}$. By Lemma 2.3 in [5] we can find a subsequence (x_{n_m}) such that

$$1 = \int_\Omega \psi(x_{n_m}) d\mu \to \int_\Omega \psi(x) d\mu,$$

whence x is an UM-point of $S(L^\psi(\Omega)^+)$. □

In order to prove our main theorem we need to recall a few more properties of Köthe spaces. We say that a Köthe space E has the (H_μ)-property (we write $E \in (H_\mu)$) if for an arbitrary sequence (x_n) in $S(E)$ and an arbitrary $x \in S(E)$ we have

$$x_n \xrightarrow{\mu} x \quad \Rightarrow \quad ||x_n - x||_E \to 0.$$

Analogously, a Köthe space E has a locally (H_μ)-property ($E \in (H^l_\mu)$) if for an arbitrary sequence (x_n) in $S(E)$ and an arbitrary $x \in S(E)$, if $x_n\chi_A \xrightarrow{\mu} x\chi_A$ for every set A of finite measure then $||x_n - x||_E \to 0$. The (H^l_μ)-property implies the order continuity property but the (H_μ)-property does not have to imply order continuity.

Example 2.4. There exists a Köthe space E, which is not strictly monotone, but the set of UM-points of $S(E^+)$ is closed and $E \in (H_\mu)$.

Indeed, take the Orlicz space $E = L^\psi(\Omega)$ with the Luxemburg norm, $\mu(\Omega) < \infty$, $a_\psi > 0$ and $\psi \in \Delta_2(\infty)$. Since $a_\psi > 0$, the space $L^\psi(\Omega)$ is not strictly monotone (see Corollary 1 in [11]). Moreover, since $\mu(\Omega) < \infty$ and $\psi \in \Delta_2(\infty)$, we have $L^\psi(\Omega) \in (H_\mu)$, Finally, the set of UM-points of $S(E^+)$ is closed by Theorem 2.3.

In some Köthe spaces E it is possible that $||x_n - x||_E \to 0$ but $x_n \overset{\mu}{\nrightarrow} x$, i.e., the norm convergence does not always imply the convergence in the measure μ.

Definition 2.5. We will say that the Calderón-Lozanovskĭi space E_φ over the measure space (Ω, Σ, μ) satisfies the $(N\mu)$-property ($E_\varphi \in (N\mu)$ in short) if for every sequence (x_n) in $S(E_\varphi)$ and $x \in S(E_\varphi)$,

$$\|x_n - x\|_\varphi \to 0 \quad \Rightarrow \quad \varphi \circ x_n \xrightarrow{\mu} \varphi \circ x.$$

It is known that if the Köthe space E is symmetric then the norm convergence in E implies the convergence in the measure μ, so (see Lemma 3.9 in [17]) E_φ satisfies the $(N\mu)$-property as well. In the following lemma we give examples of Calderón-Lozanovski spaces with the $(N\mu)$-property, that are defined over a Köthe space that need not to be symmetric.

Lemma 2.6. *Assume that, in the Köthe space E over the measure space (Ω, Σ, μ), the norm convergence implies the convergence in the measure μ. If the function φ admits an asymptote at infinity then the Calderón-Lozanowskii space E_φ satisfies the $(N\mu)$-property.*

Proof. Let x_n be a sequence of elements of $S(E_\varphi)$ such that $\|x_n - x\|_\varphi \to 0$ for some $x \in E_\varphi$. Then, for every $\lambda > 0$,

$$\|\lambda(x_n - x)\|_\varphi \geq \|\varphi \circ (\lambda(x_n - x))\|_E \to 0$$

whence, by assumption, $\varphi \circ (\lambda(x_n - x)) \xrightarrow{\mu} 0$. In consequence, $x_n \xrightarrow{\mu} x$. Indeed, otherwise, passing to a subsequence if necessary,

$$\mu(\{t \in \Omega : |x_n(t) - x(t)| \geq \varepsilon\}) \geq \eta > 0$$

for some $\varepsilon > 0$, $\eta > 0$ and all $n \in \mathbb{N}$. Take $\lambda > 0$ such that $\lambda\varepsilon > a_\varphi$. Then

$$\mu(\{t \in \Omega : \varphi(\lambda(x_n(t) - x(t))) \geq \varphi(\lambda\varepsilon) > 0\}) \geq \eta > 0$$

for all $n \in \mathbb{N}$, whence $\varphi \circ (\lambda(x_n - x)) \overset{\mu}{\not\to} 0$, a contradiction.

Let $\varepsilon > 0$. Since φ admits an asymptote at infinity, we can find constants $b, c, u_\varepsilon > 0$ such that $a_\varphi < u_\varepsilon < \infty$ and

$$cu - b \leq \varphi(u) \leq cu - b + \varepsilon/2$$

for all $u \in [u_\varepsilon, \infty)$. Therefore

$$|\varphi(u) - \varphi(v)| \leq c(\max\{|u|, |v|\} - \min\{|u|, |v|\}) + \varepsilon/2 \leq c|u - v| + \varepsilon/2 \tag{1}$$

for every $|u|, |v| \geq u_\varepsilon$.

In order to simplify notation, put $F_n = \{t \in \Omega : |\varphi(x_n(t)) - \varphi(x(t)| \geq \varepsilon\}$ for $n \in \mathbb{N}$. In order to prove that $\mu(F_n) \to 0$ as $n \to \infty$ take η as an arbitrary positive number.

Define $A_n = \{t \in \Omega : |x_n(t) - x(t)| \geq u_\varepsilon\}$ for $n \in \mathbb{N}$. Since $x_n \xrightarrow{\mu} x$, we can find $n_0 \in \mathbb{N}$ such that $\mu(A_n) < \eta/3$, whence $\mu(F_n \cap A_n) \leq \mu(A_n) < \eta/3$ for all $n \geq n_0$.

Put $B_n = \{t \in \Omega \setminus A_n : \max\{|x_n(t)|, |x(t)|\} \geq 2u_\varepsilon\}$. Since $|x_n(t) - x(t)| < u_\varepsilon$ for $t \notin A_n$, we have $\min\{|x_n(t)|, |x(t)|\} \geq u_\varepsilon$ for all $t \in B_n$. Since $x_n \xrightarrow{\mu} x$ and by 1, we can find $n_1 \geq n_0$ such that

$$\begin{aligned}\mu(F_n \cap B_n) &\leq \mu(\{t \in \Omega : |\varphi(x_n(t)) - \varphi(x(t))| \geq \varepsilon,\ |x_n(t)| \geq u_\varepsilon,\ |x(t)| \geq u_\varepsilon\}) \\ &\leq \mu(\{t \in \Omega : c(\max\{|x_n(t)|, |x(t)|\} - \min\{|x_n(t)|, |x(t)|\}) \geq \varepsilon/2\}) \\ &\leq \mu(\{t \in \Omega : |x_n(t) - x(t)| \geq \varepsilon/(2c)\}) < \eta/3\end{aligned}$$

for all $n \geq n_1$.

Put $C_n = \Omega \setminus (A_n \cup B_n)$. We have $\max\{|x_n(t)|, |x(t)|\} \leq 2u_\varepsilon$ for all $t \in C_n$. Since the function φ is uniformly continuous on the interval $[-2u_\varepsilon, 2u_\varepsilon]$, we can find $\delta > 0$ such that $|u - v| < \delta$ implies $|\varphi(u) - \varphi(v)| < \varepsilon$ for all $u, v \in [-2u_\varepsilon, 2u_\varepsilon]$. Since $x_n \xrightarrow{\mu} x$ we can find $n_2 \geq n_1$ such that

$$\begin{aligned}\mu(F_n \cap C_n) &\leq \mu(\{t \in \Omega : |\varphi(x_n(t)) - \varphi(x(t))| \geq \varepsilon,\ |x_n(t)| \leq 2u_\varepsilon,\ |x(t)| \leq 2u_\varepsilon\}) \\ &\leq \mu(\{t \in \Omega : |x_n(t) - x(t)| \geq \delta\}) < \eta/3\end{aligned}$$

for all $n \geq n_2$. Therefore,

$$\mu(F_n) = \mu(F_n \cap A_n) + \mu(F_n \cap B_n) + \mu(F_n \cap C_n) < \eta$$

for all $n \geq n_2$, i.e. $\varphi \circ x_n \xrightarrow{\mu} \varphi \circ x$. □

The Orlicz space $L^\psi(\Omega)$ can also be equipped with the Orlicz norm defined by

$$||x||_o = \inf_{k>0} \frac{1}{k}\left(1 + \int_\Omega \psi(k|x(t)|)d\mu\right).$$

Since, in the Orlicz spaces, the norm convergence with respect to both, Orlicz and Luxemburg norm, implies the convergence in the measure μ (in fact, the Orlicz and the Luxemburg norms are equivalent), we get the following corollary.

Corollary 2.7. *If E is an Orlicz space with either the Luxemburg or the Orlicz norm and the function φ admits an asymptote at infinity, then the Calderón-Lozanowskii space E_φ satisfies the $(N\mu)$-property.*

We will use yet another property of Calderón-Lozanovskĭi spaces.

Definition 2.8. The Calderón-Lozanovskĭi space E_φ satisfies the condition (e) (we will write $E_\varphi \in (e)$) if for every point $x \in S(E_\varphi)$ the following implication holds true

$$x \in \operatorname{Ext} B(E_\varphi) \quad \Rightarrow \quad \varphi \circ |x| \in \operatorname{Ext} B(E).$$

Evidently, for every rotund Köthe space E and every Orlicz function φ the Calderón-Lozanovskĭi space E_φ satisfies the condition (e). Indeed, it suffices to note that, for every $x \in \operatorname{Ext} B(E_\varphi)$, $\|\varphi \circ |x|\|_E = \|x\|_\varphi = 1$, whence, by rotundity of E, $\varphi \circ |x| \in \operatorname{Ext} B(E)$. It is also obvious that the condition (e) is satisfied whenever $\operatorname{Ext} B(E_\varphi) = \emptyset$.

There exists an Orlicz function such that for any Köthe space E with $\operatorname{Ext} B(E) \neq \emptyset$ the condition (e) is satisfied. In order to prove that result, we will use the Orlicz function φ defined by the formula $\varphi(u) = \max\{0, |u| - 1\}$. Let us note that the Orlicz space $L^\varphi(\Omega)$ generated by this function and equipped with the Orlicz norm coincides with the classical Banach space $L^1(\Omega) + L^\infty(\Omega)$ (see [3], [15], [27], [28]). Since the Orlicz norm and the Luxemburg norm are equivalent, the Calderón-Lozanovskĭi space $(L^1(\Omega))_{\max\{0,|u|-1\}}$ is linearly isomorphic to the Banach space $L^1(\Omega) + L^\infty(\Omega)$.

Theorem 2.9. *Let $\varphi(u) = \max\{0, |u| - 1\}$ for $u \geq 0$. For every Köthe space E with $\operatorname{Ext} B(E) \neq \emptyset$ we have $E_\varphi \in (e)$.*

Proof. Let E be an arbitrary Köthe space with $\operatorname{Ext} B(E) \neq \emptyset$. If $\operatorname{Ext} B(E_\varphi) = \emptyset$, then the condition (e) is satisfied.

Assume that $\operatorname{Ext} B(E_\varphi) \neq \emptyset$ and let $x \in \operatorname{Ext} B(E_\varphi)$. Since $\varphi < \infty$, the condition (ii) of Theorem 1.7 holds true, whence $\rho_\varphi(x) = 1$, $|x| \geq \chi_\Omega$, $\varphi \circ |x|$ is an UM-point in E and the condition (o) is satisfied. We will show that $\varphi \circ |x| \in \operatorname{Ext} B(E)$. By Lemma 1.1 it is sufficient to prove that $\varphi \circ |x| \in \operatorname{Ext} B(E^+)$.

Let $\varphi \circ |x| = \frac{y+z}{2}$, where $y, z \in S(E^+)$. Since the function φ is monotone and affine on the interval $I = [1, \infty)$, the inverse function $\varphi_I^{-1} : [0, \infty) \to [1, \infty)$ exists and it is affine, where φ_I denotes the restriction of the function φ to the interval I. We have

$$|x| = \varphi_I^{-1} \circ \frac{y+z}{2} = \frac{1}{2}(\varphi_I^{-1} \circ y + \varphi_I^{-1} \circ z).$$

Moreover,

$$\rho_\varphi(\varphi_I^{-1} \circ y) = ||\varphi \circ (\varphi_I^{-1} \circ y)||_E = ||y||_E = 1$$

and

$$\rho_\varphi(\varphi_I^{-1} \circ z) = ||\varphi \circ (\varphi_I^{-1} \circ z)||_E = ||z||_E = 1.$$

By Lemma 1.3 we get $||\varphi_I^{-1} \circ y||_\varphi = 1$ and $||\varphi_I^{-1} \circ z||_\varphi = 1$. Since $|x| \in \operatorname{Ext} B(E_\varphi^+)$, we have $\varphi_I^{-1} \circ y = \varphi_I^{-1} \circ z = |x|$. Thus $y = z = \varphi \circ |x|$, whence $\varphi \circ |x| \in \operatorname{Ext} B(E^+)$. $\square$

Example 2.10. There exist an Orlicz function φ and a non-rotund Köthe space E such that $E_\varphi \in (e)$.

Proof. Indeed, let $\varphi(u) = \max\{0, |u| - 1\}$, $\psi(u) = \max\{0, u^2 - 1\}$. Take $E = L^\psi(\Omega)$ with the Luxemburg norm. Since ψ is not strictly convex on $(0, \infty)$, the Orlicz space $L^\psi(\Omega)$ is not rotund (see, e.g., Theorem 6.2 in [4]). Moreover, since ψ is strictly convex on $(1, \infty)$, we have $ExtB(L^\psi(\Omega)) \neq \emptyset$ (see Corollary 5.5 in [4]). Thus, by Theorem 2.9, E_φ satisfies the condition (e). $\square$

Theorem 2.11. *Assume that the Calderón-Lozanovskĭi space E_φ posseses the properties (nm), $(N\mu)$ and (e). Moreover, assume that E is a Köthe space with the (H_μ)-property and let the set of UM-points of $S(E^+)$ be closed. If the set* $\operatorname{Ext} B(E)$ *is closed then the set* $\operatorname{Ext} B(E_\varphi)$ *is closed as well.*

Proof. Without loss of generality we can assume that $\operatorname{Ext} B(E_\varphi) \neq \emptyset$. Assume that the set $\operatorname{Ext} B(E)$ is closed and let (e_n) be a sequence of extreme points in $B(E_\varphi)$ such that $||e_n - y||_\varphi \to 0$ for some $y \in S(E_\varphi)$. Evidently

$$1 = ||e_n||_\varphi = ||\varphi \circ e_n||_E \to ||\varphi \circ y||_E = \|y\|_\varphi = 1$$

By the $(N\mu)$-property, $\varphi \circ e_n \xrightarrow{\mu} \varphi \circ y$, so, since $E \in (H_\mu)$, we get $||\varphi \circ e_n - \varphi \circ y||_E \to 0$. By the condition (e) we know that $\varphi \circ e_n$ are extreme points of $B(E)$. The set $\operatorname{Ext} B(E)$ is closed, so $\varphi \circ y$ is an extreme point in $B(E)$. Thus the condition (o) from Theorem 1.7 is satisfied for the function y.

Since $y \in S(E_\varphi)$ and $E_\varphi \in (mn)$, we have

$$\rho_\varphi(y) = ||y||_\varphi = 1.$$

Moreover, from the inequality $|e_n| \geq a_\varphi \chi_\Omega$ μ-a.e. we have $|y| \geq a_\varphi \chi_\Omega$ μ-a.e. By Theorem 1.7, in order to prove that $y \in \operatorname{Ext} B(E_\varphi)$ it remains to show that $|y|$ is an UM-point of $S(E^+)$ only. Evidently

$$||\varphi \circ |e_n| - \varphi \circ |y|\, ||_E = ||\varphi \circ e_n - \varphi \circ y||_E \to 0$$

Since $\varphi \circ |e_n|$ are UM-points and, by assumption, the set of UM-points of $S(E^+)$ is closed, we infer that $\varphi \circ |y|$ is an UM-point in $S(E^+)$. Thus $y \in \operatorname{Ext} B(E_\varphi)$. $\square$

Since it is possible that $x \in \operatorname{Ext} B(E_\varphi)$, but $\varphi \circ |x| \notin \operatorname{Ext} B(E)$, the inverse to Theorem 2.5 is not true. But under additional assumptions on the function φ or the space E we get the equivalence. Let us note that the equivalence is true in case $\operatorname{Ext} B(E) = \emptyset$ or when the space E is rotund.

Corollary 2.12. *Let $\varphi(u) = \max\{0, |u| - 1\}$, $\mu(\Omega) < \infty$ and consider the Orlicz space $L^\psi(\Omega)$ equipped with the Luxemburg norm. Moreover, assume that $\psi \in \Delta_2(\infty)$, $a_\psi > 0$ and $\operatorname{Ext} B(L^\psi(\Omega)) \neq \emptyset$. The set $\operatorname{Ext} B(L^{\psi\circ\varphi}(\Omega))$ is closed if and only if the set $\operatorname{Ext} B(L^\psi(\Omega))$ is closed, i.e. $\psi \in \Gamma_a$.*

Proof. Assume that the set $\operatorname{Ext} B(L^{\psi\circ\varphi}(\Omega))$ is closed. By Theorem 2.1 we have $\psi \circ \varphi \in \Gamma_a$ and by Lemma 2.2, $\psi \in \Gamma_a$. Hence, the set $\operatorname{Ext} B(L^\psi(\Omega))$ is closed as well.

In order to prove the converse implication we will verify the assumptions of Theorem 2.11. Assume that the set $\operatorname{Ext} B(L^\psi(\Omega))$ is closed. Since, by assumption, $\operatorname{Ext} B(L^\psi(\Omega)) \neq \emptyset$, by Theorem 2.9 we get $L^{\psi\circ\varphi} \in (e)$. By Corollary 2.7, $L^{\psi\circ\varphi} \in (N\mu)$. Since $a_\varphi = 1$, we have $L^\infty(\Omega) \hookrightarrow L^\psi(\Omega)$. Further, since $\varphi \in \Delta_2(\infty)$, we have $\varphi \in \Delta_2^{L^\psi(\Omega)}$, whence, by Lemma 1.4, $L^{\psi\circ\varphi} \in (nm)$.

Moreover, since $\mu(\Omega) < \infty$, $a_\psi > 0$ and $\psi \in \Delta_2(\infty)$, we have $L^\psi(\Omega) \in (H_\mu)$. By Theorem 2.3 the set of UM-points of the unit sphere $S(L^\psi(\Omega)^+)$ is closed. Since all the assumptions of Theorem 2.11 are satisfied, we infer that the set $\operatorname{Ext} B(L^{\psi\circ\varphi}(\Omega))$ is closed. □

Theorem 2.13. *Let E_φ be a Calderón-Lozanovskĭi space with the property (e). Moreover, assume that the set of UM-points of $S(E^+)$ is closed in the Köthe space E. If φ is a strictly convex function on the interval $[0, \infty)$ with $\varphi \in \Delta_2^E$, then the closedness of the set $\operatorname{Ext} B(E_\varphi)$ implies that the set $\operatorname{Ext} B(E)$ is closed as well.*

Proof. Without loss of generality we can assume that $\operatorname{Ext} B(E) \neq \emptyset$. Now, assume that the set $\operatorname{Ext} B(E_\varphi)$ is closed and let (b_n) be a sequence of extreme points of $B(E)$ such that $||b_n - x||_E \to 0$ for some $x \in S(E)$. Since φ is strictly convex, the converse function φ^{-1} exists on $[0, \infty)$. Put $e_n = \varphi^{-1} \circ |b_n|$. Then

$$\rho_\varphi(e_n) = ||\varphi \circ (\varphi^{-1} \circ |b_n|)||_E = ||b_n||_E = 1,$$

whence $||e_n||_\varphi = 1$. Since $b_n \in \operatorname{Ext} B(E)$, from Lemma 1.1 it follows that the functions $|b_n| = \varphi \circ e_n$ are UM-points in $S(E^+)$. Since $a_\varphi = 0$, it is evident that $e_n \geq a_\varphi \chi_\Omega$. From the strict convexity of the function φ it follows that the condition (o) and thus the condition (ii) of Theorem 1.7 is satisfied. Therefore (e_n) is a sequence of extreme points in $B(E_\varphi)$.

We will show that the sequence (e_n) is convergent to $y = \varphi^{-1} \circ |x| \in S(E_\varphi)$. Since the function φ^{-1} is concave on $(0, \infty)$, it is subadditive. Hence we have

$$\varphi^{-1} \circ |b_n| \leq \varphi^{-1} \circ |b_n - x| + \varphi^{-1} \circ |x|,$$

and

$$\varphi^{-1} \circ |x| \leq \varphi^{-1} \circ |b_n - x| + \varphi^{-1} \circ |b_n|$$

Thus

$$\left|\varphi^{-1} \circ |b_n| - \varphi^{-1} \circ |x|\right| \leq \varphi^{-1} \circ |b_n - x|,$$

so
$$||e_n - y||_\varphi = ||\varphi^{-1} \circ |b_n| - \varphi^{-1} \circ |x|\,||_\varphi \leq ||\varphi^{-1} \circ |b_n - x|\,||_\varphi.$$
Since $\varphi \in \Delta_2^E$ and
$$\rho_\varphi(\varphi^{-1} \circ |b_n - x|) = ||\varphi \circ (\varphi^{-1} \circ |b_n - x|)||_E = ||b_n - x||_E \to 0,$$
we get $||\varphi^{-1} \circ |b_n - x|\,||_\varphi \to 0$, whence $||e_n - y||_\varphi \to 0$. From the closedness of the set $\operatorname{Ext} B(E_\varphi)$ it follows that $y \in \operatorname{Ext} B(E_\varphi)$. By assumption $E_\varphi \in (e)$, so $|x| = \varphi \circ y$ is an extreme point of the unit ball $B(E)$. Therefore, by Lemma 1.1, $|x|$ is an UM-point of $S(E^+)$ whence $x \in \operatorname{Ext} B(E)$ and the proof is completed. □

Since the condition $\varphi \in \Delta_2^E$ implies that $E_\varphi \in (nm)$, combining Theorem 2.11 and Theorem 2.13 we get the following conclusion.

Theorem 2.14. *Let E_φ be a Calderón-Lozanovskĭi space with the properties $(N\mu)$ and (e). Moreover, assume that E is a Köthe space with the (H_μ)-property and the set of UM-points of $S(E^+)$ is closed. If φ is a strictly convex function with $\varphi \in \Delta_2^E$, then the set $\operatorname{Ext} B(E_\varphi)$ is closed if and only if the set $\operatorname{Ext} B(E)$ is closed.*

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