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Mean-Value Inequalities for Convex Functions and the Chebysev-Vietoris Inequality

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It is shown that if $B = [-b_1, b_1] \times \cdots \times [-b_n, b_n] \subset \mathbb{R}^n$, where $b_i > 0$ for $i = 1, ..., n$, and if A is a convex and compact subset of B of positive Lebesgue measure, which is preserved by reflections with respect to all coordinate hyperplanes $x_i = 0$ for $i = 1, ..., n$, then A is convexly majorized by B, i.e., for every continuous convex function v defined over B, the mean of v over A is not exceeding the mean of v over B. In the proof an n-dimensional extension of the integral form of the Chebysev inequality, which was given by L. Vietoris in [8], is used.

1. Introduction

In this article, we continue the study of convex majorization of sets, i.e., the study of mean-value inequalities for convex functions, which has been initiated in [4] and continued in [5]. This type of majorization played an essential role in our study of an n-dimensional extension of the the right-hand side of classical Hadamard inequality [6]. It allowed us to construct boundary measures of some surfaces. This approach showed a natural way to apply the concept of majorization of sets. We will prove new mean-value inequalities in this paper.

The proof of our new mean-value inequality relies heavily on the integral form of the Chebysev inequality, obtained essentially by Vietoris in [8]. In this little known gem of a paper, Vietoris proves a discrete version of the n-dimensional Chebysev inequality, and then states its integral version. He comments that the proof of the latter is analogous to that of the former, with sums replaced by integrals, but he gives no indications what kind of integrals he has in mind, nor why are the functions under consideration integrable. We felt more comfortable to sketch a proof of this integral inequality, by noticing that order-monotone functions on closed and bounded boxes in $\mathbb{R}^n$ are Lebesgue measurable and even Riemann integrable, but not necessarily Borel measurable. These facts yield the

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integral version of the Chebysev inequality by application of the discrete one to suitable Riemann sums. The same integral inequality under different assumptions was stated in [7].

First, some notation will be discussed, and then some definitions and results will be recalled from [4] and [5]. Some additional definitions will be also discussed.

If A_i are sets for $1 \leq i \leq n$, the Cartesian product $A_1 \times \cdots \times A_n$ will be also denoted by $X_{i=1}^n A_i$. Let A be a bounded Lebesgue measurable subset of $\mathbb{R}^n$ of positive Lebesgue measure and let $v : A \to \mathbb{R}$ be continuous. The integral mean of v over A, denoted by v_A, is defined as

$$v_A := \frac{1}{\mathcal{L}^n(A)} \int_A v(x) d\mathcal{L}^n(x), \tag{1}$$

where $\mathcal{L}^n(A)$ denotes the (n-dimensional) Lebesgue measure of A.

If $A \subset \mathbb{R}^n$, then $\overline{A}$ denotes its closure, $\mathrm{int}(A)$ its interior, χ_A its characteristic function, $\mathrm{co}(A)$ (resp. $\overline{\mathrm{co}}(A)$) denotes the convex hull (resp. the closed convex hull) of A.

The following partial order on $\mathbb{R}^n$ will be used in this paper.

Definition 1.1. Let $x = (x_1, ..., x_n) \in \mathbb{R}^n, y = (y_1, ..., y_n) \in \mathbb{R}^n$. We will say that $x \leq y$ if $x_i \leq y_i$ for $i = 1, ..., n$.

Corresponding to this partial order, a notion of (order) monotonicity can be defined in $\mathbb{R}^n$.

Definition 1.2. Let $B \subset \mathbb{R}^n$, and let $f : B \to \mathbb{R}$. The function f is said to be (order) nondecreasing on B, if for any $x, y \in B$, such that $x \leq y$, the inequality $f(x) \leq f(y)$ holds. We shall also call such an f as an order monotone function.

Schur-convex functions, which preserve an other partial order, the Hardy-Littlewood-Pólya (HLP) order of vectors in $\mathbb{R}^n$, will be also briefly discussed in this paper.

Definition 1.3. A vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is majorized by a vector $y = (y_1, ..., y_n) \in \mathbb{R}^n$ in the sense of HLP, if

$$\sum_{i=1}^{k} x_i^* \leq \sum_{i=1}^{k} y_i^* \quad (k = 1, ..., n), \tag{2}$$

with equality for $k = n$, where $(x_1^*...., , x_n^*)$ denotes the non-increasing rearrangement of x.

We need some more definitions. A set $B \subset \mathbb{R}^n$ is symmetric, when $x \in B$, if and only if, $Px \in B$ for every permutation matrix P. If $B \subset \mathbb{R}^n$ is symmetric, then $F : B \to \mathbb{R}$ is symmetric on B if $F(Px) = F(x)$ for all permutation matrices P and for all $x \in B$.

Definition 1.4. Let B be a symmetric subset of $\mathbb{R}^n$. The function $F : B \to \mathbb{R}$ is said to be Schur-convex on B, if F is symmetric on B, and $F(x_1, ..., x_n) \leq F(y_1, ..., y_n)$ for all $(x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{R}^n$ such that $(x_1, ..., x_n)$ is majorized by $(y_1, ..., y_n)$ in the sense of HLP.

The next definition is a slight extension of the one given in [7].

Definition 1.5. Let X be an arbitrary set and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$. Then f and g are said to be similarly ordered on X, if for any $x, y \in X$ we have that $(f(x) - f(y))(g(x) - g(y)) \geq 0$, and oppositely ordered if f and $-g$ are similarly ordered on X.

The definition of convex majorization of sets was introduced in [4].

Definition 1.6. Let A and B be bounded Lebesgue measurable subsets of $\mathbb{R}^n$ of positive Lebesgue measure. The set A is said to be convexly majorized by B, if the inequality $v_A \leq v_B$ holds for every real, continuous and convex function v defined on $\overline{\text{co}}(A \cup B)$. This relation will be denoted by $A \prec B$.

Let A be a bounded Lebesgue measurable subset of positive Lebesgue measure of $\mathbb{R}^n$. The barycenter of A, denoted by $\overline{x}_A$, is defined by the formula

$$\overline{x}_A := \frac{1}{\mathcal{L}^n(A)} \int_A x d\mathcal{L}^n(x).$$

The following necessary conditions for $A \prec B$ were established in [4].

Theorem 1.7. *Let A and B be bounded Lebesgue measurable sets of positive (Lebesgue) measure in $\mathbb{R}^n$. If the relation $A \prec B$ holds then*

(*i*) *A and B have the same barycenter, i.e. $\overline{x}_A = \overline{x}_B$.*
(*ii*) *If, in addition, A and B are compact, then $\mathcal{L}^n(A \setminus \text{co}(B)) = 0$.*
(*iii*) *If, in addition, A and B are convex and compact, then $A \subset B$.*

The next result, also from [4], presents sufficient conditions for the relation $A \prec B$.

Theorem 1.8. *Let A and B be bounded Lebesgue measurable sets in $\mathbb{R}^n$ with $0 < \mathcal{L}^n(A) < \mathcal{L}^n(B)$ and such that they have the same barycenter at $\overline{x}_A$. If A and B are similar with respect to their common barycenter, i.e. if there exists a $c > 1$ such that $c(A - \overline{x}_A) = B - \overline{x}_A$, then for every real function v, which is continuous and convex on $\overline{\text{co}}(B)$ the inequality $v_A \leq v_B$ holds.*

The next few results describe some additional properties of the relation $\prec$.

Lemma 1.9. *Let A and B be bounded subsets of $\mathbb{R}^n$ with $A \subset B$ and $\mathcal{L}^n(A) > 0, \mathcal{L}^n(B \setminus A) > 0$. Then*

a) *the relation $A \prec B$ holds if and only if $A \prec B \setminus A$;*
b) *the relation $A \prec B$ holds if and only if $B \prec B \setminus A$.*

If A and B are both convex then the conclusions of Theorem 1.8 are true in $\mathbb{R}$, but the next example, which is from [4], where such a v was explicitly given, shows they are not remain true already in $\mathbb{R}^2$.

Example 1.10. Let $A = [-1,1] \times [-1,1]$ and let B be the parallelogram bounded by the lines $x = -1$, $x = 1$, $y = 2 + x$ and $y = -2 + x$. Then there is a convex function $v(x, y)$ with $v_A > v_B$ even though $A \subset B$ and A and B are convex and compact sets which are symmetric with respect to their common barycenter $\overline{x}_A = \overline{x}_B = (0, 0)$.

The next result, also from [4], shows that the integral mean is invariant and the relation $\prec$ is preserved under a nonsingular affine map.

Theorem 1.11. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a nonsingular affine map. Let A and B be bounded Lebesgue measurable sets in $\mathbb{R}^n$ of positive Lebesgue measure. Then*

(*i*) *If v is a continuous function on $T(A)$ then $v_{T(A)} = (v \circ T)_A$,*
(*ii*) *If $A \prec B$ then $T(A) \prec T(B)$.*

The following, particularly striking result is from [4].

Theorem 1.12. *Let A and B be compact ellipsoids in $\mathbb{R}^n$ with common barycenter $\overline{x}_A = \overline{x}_B$. Then $A \prec B$, if and only if $A \subset B$.*

The importance of the previous theorem consists of the fact that it derives $A \prec B$ from the necessary condition having a common barycenter, together with the purely geometric properties of the sets of A and B. In other words A and B are both ellipsoids. In this paper, we obtain results similar to Theorem 1.12, under different geometric assumptions.

2. The Multidimensional Chebysev Inequality

The discrete, one-dimensional Chebysev inequality can be stated as follows.

Theorem 2.1. *If $a_1, ..., a_n$ and $b_1, ..., b_n$ are sequences of reals, which are both increasing or both decreasing, then the inequality*

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \tag{3}$$

holds between the arithmetic means, while the reverse inequality holds when one of the sequences is increasing and the other is decreasing.

The following functional extension of the inequality (3) follows easily from Theorem 2.1.

Theorem 2.2. *Let f and g be real functions, which are defined on the bounded interval $[a, b]$. If f and g are both non-decreasing or both non-increasing on $[a, b]$,* *then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx. \tag{4}$$

If one of the functions is non-increasing, while the other is non-decreasing, then the reverse inequality holds.

In [8], L. Vietoris gave the following generalization of (3), as a discrete inequality, for multidimensional arrays, i.e. for multidimensional matrices.

Theorem 2.3. *Let $A := (a_{i_1,\dots,i_r})$ and $B := (b_{i_1,\dots,i_r})$ be two real $r-$dimensional matrices of the same type, such that $i_k = 1, \dots, m_k$ for $1 \leq k \leq r$. Then*

$$\prod_k m_k \sum_i a_i b_i \geq \sum_i a_i \sum_i b_i \tag{5}$$

holds, where the summation is over all the positions $i = (i_1, \dots, i_r)$ of the $r-$dimensional matrix A, if for every two positions $i = (i_1, \dots i_r)$ and $j = (j_1, \dots, j_r)$ for which $i_k \leq j_k$ for $k = 1, \dots, r$, we have both $a_i \leq a_j$ and $b_i \leq b_j$,

Next, the measurability of order monotone functions will be discussed. It will be shown first by an example, that an order monotone function does not need to be Borel measurable,

Example 2.4. Let $X = [0,1] \times [0,1]$. Let $E \subset [0,1]$ be a non (Lebesgue) measurable set. Let

$$f(x,y) = \begin{cases} 0 & x+y<1; \\ 1 & x+y=1,\ x \in E; \\ 3/2 & x+y=1,\ x \notin E; \\ 2 & x+y>1; \end{cases} \tag{6}$$

We claim, that f is Lebesgue measurable, but it is not Borel measurable. Indeed, consider the level set $f^{-1}((1/2, 5/4)) = \{(x,y) : x+y=1,\ x \in E\}$. This is not a Borel measurable set, because its projection into the $x-$axis is E, which is not a Lebesgue measurable set. But f is a Lebesgue measurable function, because the previous level set is a two dimensional Lebesgue measure zero set. Furthermore, we can see easily, that all the other level sets are also Lebesgue measurable. Also, for every fixed $0 \leq x_0 \leq 1$, and for every fixed $0 \leq y_0 \leq 1$, the functions $g(y) = f(x_0, y)$ and $h(x) = f(x, y_0)$ are increasing step-functions.

Now, an easy modification of (6) yields a non convex and non Borel measurable Schur- convex function on X.

Example 2.5. Let $X = [0,1] \times [0,1]$. Let $E \subset [0,1]$ be a non (Lebesgue) measurable set. Let

$$F(x,y) = \begin{cases} 2 & (x,y) \in X,\, y-x \neq 0; \\ 1 & y-x=0,\, x \in E; \\ 3/2 & y-x=0,\, x \notin E; \end{cases}$$

The non Borel measurability of F is clear in view of (6), and F is a non convex function. The set X is symmetric and F is a symmetric function on X. Now, $(u,u) \in X, (v,v) \in X$ are comparable in the sense of HLP, if and only if $u = v$. If $(x,y) \in X$ and $(u,u) \in X$ are comparable in the sense of HLP, then $x + y = 2u$ and $\max(x,y) \geq u$. Thus (x,y) majorizes (u,u) in the sense of HLP, and since $F(x,y) \geq F(u,u)$, it follows that F is a Schur-convex function.

The following result can be found in [3].

Theorem 2.6. *Let $v : \mathbb{R}^n \to \mathbb{R}$ be an order monotone function. Then v is a Lebesgue-measurable function, which is continuous $\mathcal{L}^n$ a.e. on $\mathbb{R}^n$.*

The previous result has been extended to more general settings in [1] and [2]. Next, we will need the following result.

Lemma 2.7. *Let $v : X \to \mathbb{R}$ be a bounded Lebesgue measurable function on a compact $n-$dimensional box $X = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. If the set of points of discontinuity of v is an n-dimensional Lebesgue zero set, then v is Riemann integrable on X.*

Proof. The $n-$dimensional form of Lebesgue's criterion of Riemann integrability shows that v is Riemann integrable on X, (see, volume 2, p. 111 of [9],) because v is a bounded function, which is $\mathcal{L}^n$ a.e. continuous on X. □

Now, we can state and prove the integral inequality of Chebysev-Vietoris.

Theorem 2.8 (Chebysev-Vietoris). *If $f(x_1, ..., x_r)$ and $g(x_1, ..., x_r)$ are order monotone functions on $X = [a_1, b_1] \times \cdots \times [a_r, b_r]$, then the functions f, g and fg are Riemann integrable on X, and we have the inequality*

$$\prod_{k=1}^{r}(b_k - a_k) \int_X fg d\mathcal{L}^r \geq \int_X f d\mathcal{L}^r \int_X g d\mathcal{L}^r \tag{7}$$

Proof. Since f and g are order-monotone functions on X, it follows that

$$f(a_1, ..., a_r) \leq f(x) \leq f(b_1, ..., b_n), \qquad g(a_1, ..., a_r) \leq g(x) \leq g(b_1, ..., b_n)$$

for all $x \in X$. Thus f, g and fg are bounded functions on X. Theorem 2.6 indicates that f, g and fg are Riemann integrable on X. Next, we construct Riemann sums for f, g and fg so that X is partitioned uniformly into sub-boxes, where the sample points are chosen to be at the centroids of the vertices of the sub-boxes. Then the inequality (7) follows from the inequality (5). □

An alternative proof of the previous result is also possible. Integral version could be proven directly (along Vietoris' lines) and applying it to step functions on a suitable grid would give the discrete version.

An extension of a result of Hardy, Littlewood and Pólya (p. 168 in [7]) into a general measure space will be stated without proof, since the two results can be proven very similarly.

Theorem 2.9. *Let $(X, \mathcal{S}, \mu)$ be a complete measure space with $\mu(X) < \infty$, and let f and g be real-valued functions, which are defined and similarly ordered on X. If $f, g, fg \in L^1(X, \mu)$, then one has the inequality*

$$\frac{1}{\mu(X)} \int_X fg d\mu \geq \frac{1}{\mu(X)} \int_X f d\mu \frac{1}{\mu(X)} \int_X g d\mu. \tag{8}$$

3. Convex Majorization

In this section, we obtain results similar to Theorem 1.12, under different geometric assumptions.

Theorem 3.1. *Let $B \subset \mathbb{R}^n$ be a box of the form $B = X_{i=1}^n [-b_i, b_i]$, where $b_i > 0$ for $i = 1, ..., n$. Let $A \subset B$ be a convex and compact set of positive Lebesgue measure, which is preserved by reflections with respect to all coordinate hyperplanes $x_i = 0$ for $i = 1, ..., n$. Then A is convexly majorized by B, i.e. $A \prec B$.*

Proof. Denote by G, the group of linear transformations of $\mathbb{R}^n$, generated by the simple reflections $g_i : \mathbb{R}^n \to \mathbb{R}^n$, where

$$g_i(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n) = (x_1, ..., x_{i-1}, -x_i, x_{i+1}, ..., x_n) \text{ for } i = 1, ..., n\,.$$

Note that the group G has 2^n elements.

The statement of this theorem is clearly true when $A = B$. Thus, we need to consider only the case when A is a proper subset of B. In this case $\mathcal{L}^n(B \setminus A) > 0$, since A and B are convex and compact sets. By assumption, we have that $\mathcal{L}^n(A) > 0$.

Let $v : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary continuous and convex function. To prove this theorem, it has to be shown that $v_A \leq v_B$. Equivalently, in view of Lemma 1.9, it has to be shown that $v_B \leq v_{B \setminus A}$. To do that, symmetrize first the function v, by letting

$$f(x) := \frac{1}{2^n} \sum_{g \in G} v(g(x)). \tag{9}$$

Note that $f(g(x)) = f(x)$ for every $g \in G$ and $x \in \mathbb{R}^n$. Furthermore, $g(A) = A$ and $g(B) = B$ for every $g \in G$, because A and B are symmetric with respect to all coordinate hyperplanes. Let $A_+ = A \cap \mathbb{R}_+^n = A \cap B_+$, where $B_+ = B \cap \{x \in \mathbb{R}^n : x \geq 0\} = X_{i=1}^n [0, b_i]$ and $\mathbb{R}_+^n$ denotes the positive orthant of $\mathbb{R}^n$. Denote by

$C_+ = (B \setminus A) \cap \mathbb{R}^n_+ = B_+ \setminus A_+$. Then

$$\int_B v d\mathcal{L}^n = \int_B f d\mathcal{L}^n = 2^n \int_{B_+} f d\mathcal{L}^n.$$

Furthermore, since $\mathcal{L}^n(B) = 2^n \mathcal{L}^n(B_+)$, it follows that

$$v_B = \frac{1}{\mathcal{L}^n(B)} \int_B v(x) d\mathcal{L}^n(x) = f_{B_+} \quad \text{and} \quad \int_{B\setminus A} v d\mathcal{L}^n = 2^n \int_{C_+} f d\mathcal{L}^n. \tag{10}$$

Observe next, that for every fixed

$$x_1 = x_1^\star,\ x_2 = x_2^\star, ..., x_{i-1} = x_{i-1}^\star, x_{i+1} = x_{i+1}^\star, ..., x_n = x_n^\star,$$

such that

$$-b_1 \le x_1^\star \le b_1, ..., -b_{i-1} \le x_{i-1}^\star \le b_{i-1}, -b_{i+1} \le x_{i+1}^\star \le b_{i+1}, ..., -b_n \le x_n^\star \le b_n,$$

the single variable function

$$k_i := x_i :\longrightarrow f(x_1^\star, ..., x_{i-1}^\star, x_i, x_{i+1}^\star, ..., x_n^\star)$$

is convex and even on $-b_i \le x_i \le b_i$, since f was obtained by symmetrizing v. Consequently, the function $k_i(x_i)$ is nondecreasing on the interval $[0, b_i]$. Thus, whenever

$$(0, ..., 0) \le (x_1, ..., x_n) \le (y_1, ..., y_n) \in B_+,$$

it follows that

$$\begin{aligned} f(x_1, x_2, ..., x_n) &\le f(y_1, x_2, ..., x_n) \le f(y_1, y_2, x_3, ..., x_n) \\ &\le ... \le f(y_1, y_2, ..., y_{n-1}, x_n) \le f(y_1, y_2, ..., y_n), \end{aligned}$$

i.e. f is order nondecreasing on B_+.

Let the characteristic function of C_+ be denoted by h. Thus $h(x) = \chi_{C_+}(x) = \chi_{(B_+ \setminus A_+)}(x)$. We claim that h is order-nondecreasing on B_+.

Indeed, if this were not true, then there were x and y in B_+ such that $0 \le x \le y$ and $h(x) = 1$, $h(y) = 0$, i.e. $y \in A_+$ and $x \notin A$. Since A is symmetric, with respect to all coordinate hyperplanes, all the points $(\pm y_1, ..., \pm y_n)$ belong to A. The convex hull of these points, which is the box $[-y_1, y_1] \times [-y_2, y_2] \times \cdots \times [-y_n, y_n]$, is also contained in A, because A is convex. Since x belongs to the above box, and $x \ge 0$, it follows that $x \in A_+$, and so $h(x) = 0$, which is a contradiction with the above relation $x \notin A$.

Hence, Chebyshev-Vietoris inequality (7) applied to f and h yields that

$$\int_{B_+} f h d\mathcal{L}^n \ge \frac{1}{\mathcal{L}^n(B_+)} \int_{B_+} f d\mathcal{L}^n \int_{B_+} h d\mathcal{L}^n.$$

Observe that

$$\int_{B_+} fhd\mathcal{L}^n = \int_{C_+} fd\mathcal{L}^n \quad \text{and} \quad \int_{B_+} hd\mathcal{L}^n = \mathcal{L}^n(C_+).$$

Thus, we obtain

$$\int_{C_+} fd\mathcal{L}^n \geq (f_{B_+})\mathcal{L}^n(C_+). \tag{11}$$

Hence, using (10) and (11), it follows that

$$\begin{aligned} v_B = f_{B_+} &\leq \frac{1}{\mathcal{L}^n(C_+)} \int_{C^+} fd\mathcal{L}^n \\ &= \frac{1}{2^n \mathcal{L}^n(C_+)} \int_{B\setminus A} vd\mathcal{L}^n = \frac{1}{\mathcal{L}^n(B\setminus A)} \int_{B\setminus A} vd\mathcal{L}^n = v_{B\setminus A}, \end{aligned}$$

which proves that $v_B \leq v_{B\setminus A}$. □

The following example shows that in Theorem 3.1, the condition of a 2-fold symmetry of A in $\mathbb{R}^2$ can not be omitted, and can not be replaced by the weaker condition $A = -A$.

Example 3.2. Let $B = [-1,1] \times [-1,1]$. There is a convex and compact set $A \subset B$, satisfying $-A = A$, for which there is a convex function v defined on B such that $v_A > v_B$, that is $A \not\prec B$.

Proof. It follows from Example 1.10 that there is a parallelogram $A_1 \subset \mathbb{R}^2$ containing the square $B = [-1,1]\times[-1,1]$ such that A_1 and B are symmetric with respect to their common barycenter $\overline{x}_{A_1} = \overline{x}_B = (0,0)$, but $B \not\prec A_1$. Consider an invertible and affine transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, which maps A_1 onto the square B, i.e. $T(A_1) = B$. Clearly T can be chosen to be linear, i.e. $T((0,0)) = (0,0)$. Let $A := T(B)$. It follows that $A \subset B$ and that A is convex and compact because T is linear and B is convex and compact. Since Theorem 1.11 implies that a nonsingular affine map preserves the relation $\prec$, we can deduce that $A \not\prec B$.

Since $-B = B$, it follows that $T(-B) = T(B) = A$. The linearity of T implies that $T(-B) = -T(B) = -A$. Thus $-T(B) = -A = A$. □

Next, a generalization of Theorem 3.1 will be presented. First, another definition will be introduced. A set $A \subset \mathbb{R}^n$ will be said order saturated or order closed if for all $x, y, z \in \mathbb{R}^n$ such that $x, z \in A$ and $x \leq y \leq z$ implies that $y \in A$. The following result holds.

Theorem 3.3. *Let $B = X_{i=1}^n[-b_i, b_i]$ where $b_i > 0$ for $i = 1, ..., n$, and let $A \subset B$ be a compact set of positive Lebesgue measure, which is preserved by the group of linear transformations G. If the set $A_+ := A \cap \mathbb{R}^n_+ = A \cap B_+$contains the origin and order closed, then A is convexly majorized by B, i.e. $A \prec B$.*

The proof of this theorem is practically identical with the proof of Theorem 3.1, once the following claim is verified. For every point $x = (x_1, ..., x_n) \in A$, $x_i \neq 0$ for $i = 1, ..., n$, the box $E := X_{i=1}^n [-|x_i|, |x_i|]$ is contained in A.

To verify this claim, we let $|x_i| = b_i$, for $i = 1, ..., n$ and $b = (b_1, ..., b_n)$ for a given $x = (x_1, ..., x_n) \in A$, $x_i \neq 0$ for $i = 1, ..., n$. Then $x_i = \epsilon_i b_i$, where ϵ_i is one or minus one. Let the transformation $g_0 : \mathbb{R}^n \to \mathbb{R}^n$ be defined as $g_0 : (y_1, ..., y_n) \to (\epsilon_1 y_1, ..., \epsilon_n y_n)$, where ϵ_i is uniquely determined by the relationship $x_i = \epsilon_i b_i$ for $i = 1, \cdots, n$. It follows that $g_0 \in G$, and thus for $x \in A$ we have that $g_0(x) = b \in A_+$. Since A_+ is order closed, the box $E_0 := X_{i=1}^n [0, b_i]$ is contained in $A_+ \subset A$. Therefore, $E = \cup_{g \in G}\, g(E_0)$, and so $E \subset A$, as required.

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