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Differentiability of the "Positive Part" Map

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This is a 1964 work which was quoted in some papers (in 1966, 1976) but remained unpublished. We hope the results and the hypotheses of the statements will still find some echo. We give here the English translation of the article "Un problème de dérivabilité dans L^p", Séminaire d'Automatique Théorique, Université de Caen (France), November 27 (1964), only avoiding an archaical presentation of Uniform Integrability and suppressing two examples.

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Let $(X, \mathcal{X}, \mu)$ denote a finite measure space and $f_0 \in L^p(X)$. The differentiability of $f \mapsto f^+$ at f_0 cannot hold if $\mu(\{f_0 = 0\})$ is > 0 (here we have shortened $\{x \in X\,;\, f_0(x) = 0\}$ in $\{f_0 = 0\}$). So the hypothesis "$\{f_0 = 0\}$ is negligible" is natural. Keeping the same L^p space it seems difficult to obtain better than Gâteaux differentiability: see the Theorem below (Part 2) and the counter-example.

The author gave precise results at the end of 1964 in an unpublished paper [8] which was extended to the framework of Orlicz spaces in 1966 [9, Théorème 4, page 27] in a *troisième cycle* thesis. See also the references [7, 1966] (specially the statement in Section 5), and [3, 1976] and [4, Chapter 1, page 7]. In maybe different questions the same hypotheses appeared recently: [5, Proposition 4.1], [1, Th. 6], [2, Th. 3.2].

Here is the 1964 statement of [8]:

Theorem. *1) Let p and q such that $1 \le q < p \le +\infty$. Assume $f_0 \in L^p$ and $\{f_0 = 0\}$ is negligible, then the map (note that L^p is a subspace of L^q)*

$$\begin{cases} f \longmapsto f^+ \\ L^p \longrightarrow L^q \end{cases}$$

is Fréchet differentiable at f_0 with derivative

$$\begin{cases} h \longmapsto \mathbf{1}_{\{f_0 > 0\}}\, h \\ L^p \longrightarrow L^q \end{cases}$$

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2) If $p \in [1,+\infty[$, $f_0 \in L^p$ and $\{f_0 = 0\}$ is negligible the directional derivatives at f_0 do exist: for any $h \in L^p$ the following[1] *holds in the normed space L^p*

$$\lim_{\xi \searrow 0^+} \frac{(f_0 + \xi\, h)^+ - f_0^+}{\xi} = \mathbf{1}_{\{f_0 > 0\}}\, h\,.$$

Proof. For $h_n \in L^p$ and a fixed real $a \in]0,+\infty[$ (to be chosen later) let

$$A_n := \{-a \le f_0 \le a\} \cup \{|h_n| > a\}\,.$$

If $h_n \neq 0$, the following defines $\varepsilon_n \in L^p$

$$(f_0 + h_n)^+ = f_0^+ + \mathbf{1}_{\{f_0 > 0\}}\, h_n + \varepsilon_n\, \|h_n\|_p$$

or equivalently

$$\varepsilon_n = \frac{(f_0 + h_n)^+ - f_0^+ - \mathbf{1}_{\{f_0 > 0\}}\, h_n}{\|h_n\|_p}\,. \tag{1}$$

We have to prove for the first part that $\|h_n\|_p \to 0$ implies $\|\varepsilon_n\|_q \to 0$; and for the second part that if $h_n := \xi_n\, h$ with $\xi_n \searrow 0^+$, then[2] $\|\varepsilon_n\|_p \to 0$.

As a general fact one has

$$\text{on the whole space } X, \quad |(f_0 + h_n)^+ - f_0^+ - \mathbf{1}_{\{f_0 > 0\}}\, h_n| \le \mathbf{1}_{A_n}\, |h_n| \tag{2}$$

because, if $x \in X$: either $x \notin A_n$ which implies $|f_0(x)| > a$ and $|h_n(x)| \le a$ and therefore (2) holds because the first member is 0; or $x \in A_n$ and the inequality at x is obvious if $f_0(x) \le 0$ and easy to see when x satisfies $f_0(x) > 0$. Hence from (1) ($q = +\infty$ needs an easier discussion)

$$\|\varepsilon_n\|_q \le \left(\int_{A_n} \left(\frac{|h_n|}{\|h_n\|_p} \right)^q \right)^{1/q}\,.$$

1) For the first part: Let $\delta > 0$ be fixed. The sequence $\frac{|h_n|}{\|h_n\|_p}$ is bounded in L^p hence the set of powers $\left(\frac{|h_n|}{\|h_n\|_p}\right)^q$ are uniformly integrable (this comes from the de La Vallée Poussin criterion[3]: for a bounded subset B of L^p and for $\varphi(y) = |y|^{p/q}$ for all $y \in \mathbb{R}$, then $\sup_{g \in B} \int_X \varphi \circ |g|^q\, d\mu = \sup_{g \in B} \|g\|^p < +\infty$). So there exists $\eta > 0$ such that $\mu(A) < \eta$ implies $\forall n$, $\int_A \left(\frac{|h_n|}{\|h_n\|_p}\right)^q < \delta^q$. If a is sufficiently small,

[1] We could write $\lim_{\xi \neq 0,\, \xi \to 0}$ and speak of Gâteaux differentiability.

[2] Here (1) becomes $\varepsilon_n = \frac{1}{\|h\|_p} \Big(\frac{(f_0 + \xi_n\, h)^+ - f_0^+}{\xi_n} - \mathbf{1}_{\{f_0 > 0\}}\, h \Big)$.

[3] In 1964 I learned properties of uniform integrability in J. Neveu's book [6], specially the Exercise II-5-2 page 52. This notion is very useful in Martingale Theory, and for the characterization of weakly compact subsets of L^1. It says mainly that when a subset G of L^1 is uniformly integrable, $\sup_{g \in G} \int_A |g|\, d\mu$ tends to 0 with $\mu(A)$.

$\mu(\{-a \leq f_0 \leq a\})$ will be $< \eta/2$. And since $h_n \to 0$ in measure, for n large enough $\mu(\{|h_n| > a\})$ will be also $< \eta/2$, hence $\mu(A_n) < \eta$ and so $\|\varepsilon_n\|_q < \delta$.

2) As for the second part: Here $h_n = \xi_n\, h$ and from footnote 2

$$\left\| \frac{(f_0 + \xi_n\, h)^+ - f_0^+}{\xi_n} - \mathbf{1}_{\{f_0 > 0\}}\, h \right\|_p = \|h\|_p\, \|\varepsilon_n\|_p\,.$$

With the same notations for A_n, (1) and (2) give

$$|\varepsilon_n| \leq \frac{\mathbf{1}_{A_n}\, |h_n|}{\|h_n\|_p} = \mathbf{1}_{A_n}\, \frac{|h|}{\|h\|_p}$$

and

$$\|\varepsilon_n\|_p \leq \frac{1}{\|h_n\|_p} \left(\int_{A_n} |h|^p \right)^{1/p}$$

tends to 0 because $|h|^p \in L^1$ and $\mu(A_n) \to 0$. □

Counter-example (second Example in [8]). Our goal is to show that $p > q$ is generally a necessary condition. Let $p \in [1, +\infty[$. Suppose (this holds in non trivial measure spaces[4], i.e. when the L^p are infinite dimensional) there exists a sequence $(E_n)_n$ such that $\mu(E_n) > 0$ and $\mu(E_n) \to 0$. Take $f_0 = (1/2)\, \mathbf{1}_X$. If $f \mapsto f^+$ from L^p into itself had a Fréchet derivative, it would be $h \mapsto h$ (because $\{f_0 > 0\} = X$ and from Part 2 of the Theorem). Let $h_n = -\mathbf{1}_{E_n}$. Then $(f_0 - \mathbf{1}_{E_n})^+ - f_0^+ + \mathbf{1}_{E_n} = (1/2)\mathbf{1}_{E_n}$ Thus ε_n given by (1) is

$$\varepsilon_n = \frac{(1/2)\mathbf{1}_{E_n}}{\|\mathbf{1}_{E_n}\|_p}$$

whose L^p-norm does not tend to 0. With $q < p$ its L^q-norm, $(1/2)\, \mu(E_n)^{p/q}$ would tend to 0.

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