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Midpoint Locally Uniform Rotundity of Musielak-Orlicz-Bochner Function Spaces Endowed with the Orlicz Norm*

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The criteria for midpoint locally uniform rotundity in the classical Orlicz function spaces have been given already. However, because of the complicated structure of Musielak-Orlicz-Bochner function spaces, up to now the criteria for midpoint locally uniform rotundity have not been discussed yet. In this paper, criteria for midpoint locally uniform rotundity of Musielak-Orlicz-Bochner function spaces equipped with Orlicz norm are given.

Keywords: Strongly extreme point, midpoint locally uniformly rotund, Musielak-Orlicz-Bochner function spaces, Orlicz norm

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1. Introduction

A lot of rotundity concepts in Banach spaces are known. Among them rotundity (R for short) and midpoint locally uniform rotundity are important notions. One of reasons is that (see [11]) a Banach X space is midpoint locally uniformly rotund if and only if every closed ball in X is approximately compact Chebyshev set. In 2007, S. T. Chen, H. Hudzik, W. Kowalewski and Y. W. Wang (see [2]) proved

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that a nonempty closed convex C of a midpoint locally uniformly rotund space is approximately compact if and only if C is Chebyshev set and the metric projection operator P_C is continuous. The criteria for midpoint locally uniform rotundity in the classical Orlicz function spaces have been given in [3, 6] already. However, because of the complicated structure of Musielak-Orlicz-Bochner function spaces, up to now the criteria for midpoint locally uniform rotundity have not been discussed yet. In this paper, we will discuss criteria for midpoint locally uniform rotundity in Musielak-Orlicz-Bochner function spaces endowed with Orlicz norm. The topic of this paper is related to the topic of [1], [3]–[10] and [12]–[18].

Let $(X, \|\cdot\|)$ be a real Banach space. $S(X)$ and $B(X)$ denote the unit sphere and the unit ball, respectively. By X^* we denote the dual space of X. Let N, R and R^+ denote the set of natural numbers, reals and of nonnegative reals, respectively. Let us recall some geometrical notions concerning rotundity. A Banach space X is said to be rotund if for any $x, y \in S(X)$ with $\|x+y\| = 2$ we have $x = y$. A point $x \in S(X)$ is said to be strongly extreme point if for any $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \in X$ with $\|x_n\| \to 1, \|y_n\| \to 1$ and $x = \frac{1}{2}(x_n + y_n)$, there holds $\|x_n - y_n\| \to 0$ as $n \to \infty$. If the set of all strongly extreme points of $B(X)$ is equal to $S(X)$, then X is said to be midpoint locally uniformly rotund.

Let (T, Σ, μ) be a nonatomic finite measure space. Suppose that that a function $M : T \times [0, \infty) \to [0, \infty]$ statisfies the following conditions:

(1) $\lim_{u\to\infty} M(t, u) = \infty$ and $M(t, u') < \infty$ for some $u' > 0$, for μ-a.e., $t \in T$, $M(t, 0) = 0$.
(2) $M(t, u)$ is convex on $[0, \infty)$ with respect to u for μ-a.e., $t \in T$.
(3) $M(t, u)$ is a Σ-measurable function of t on T for each $u \in [0, \infty)$.

Let $p(t, u)$ denote the right derivative of $M(t, \cdot)$ at $u \in R^+$ (where if $M(t, u) = \infty$, let $p(t, u) = \infty$) and $q(t, \cdot)$ be the generalized inverse function of $p(t, \cdot)$ defined on R^+ by

$$q(t, v) = \sup_{u\geq 0}\{u \geq 0 : p(t, u) \leq v\}.$$

Then $N(t, v) = \int_0^v q(t, s)ds$ for any $v \in R$ and μ-a.e. $t \in T$. It is well known that the Young inequality $uv \leq M(t, u) + N(t, v)$ holds for μ-a.e. $t \in T$. Moreover, $uv = M(t, u) + N(t, v) \Leftrightarrow u = q(t, v)$ or $v = q(t, u)$.

Definition 1.1 (see [1]). We say that $M(t, u)$ satisfies condition Δ_2 ($M \in \Delta_2$) if there exist $K \geq 1$ and a measureable nonnegative function $\delta(t)$ on T such that $\int_T M(t, \delta(t))dt < \infty$ and $M(t, 2u) \leq KM(t, u)$ for almost all $t \in T$ and all $u \geq \delta(t)$.

Definition 1.2 (see [1]). A Musielak-Orlicz function $M(t, u)$ is said to be strictly convex with respect u for $t \in T$, a.e., if for almost every $t \in T$, a.e., and any $u, v \in R$, $u \neq v$, we have

$$M\left(t, \frac{u+v}{2}\right) < \frac{M(t, u) + M(t, v)}{2}.$$

Moreover, for a given Banach space $(X, \|\cdot\|)$, we denote by X_T the set of all strongly Σ-measurable functions from T to X, and for each $u \in X_T$, we define the modular of u by

$$\rho_M(u) = \int_T M(t, \|u(t)\|)dt.$$

Let us define the Musielak-Orlicz-Bochner function space

$$L_M(X) = \{u \in X_T : \rho_M(\lambda u) < \infty \text{ for some } \lambda > 0\}.$$

It is known that the space $L_M(X)$ is Banach spaces equipped with the Luxemburg norm

$$\|u\| = \inf\left\{\lambda > 0 : \rho_M\left(\frac{u}{\lambda}\right) \le 1\right\}$$

or the Orlicz norm

$$\|u\|^0 = \inf_{k>0} \frac{1}{k}[1 + \rho_M(ku)].$$

If $X = R$, then $L_M^0(R)$ is said to be the Musielak-Orlicz function space. Let us set

$$K(u) = \left\{k > 0 : \frac{1}{k}(1 + \rho_M(ku)) = \|u\|^0\right\}.$$

In particular, the set $K(u)$ can be nonempty. However, we have the following.

Proposition 1.3 (see [13]). *If $\lim_{u\to\infty} \frac{M(t,u)}{u} = \infty$ $\mu-a.e.$ $t \in T$, then $K(u) \neq \emptyset$ for any $u \in L_M^0(X)$.*

2. Main results

Theorem 2.1. *The space $L_M^0(X)$ is midpoint locally uniformly rotund if and only if:*

(a) *For any $u \in L_M^0(X)\backslash\{0\}$, the set $K(u)$ consists of one element from $(0, +\infty)$;*
(b) *$M \in \Delta_2$;*
(c) *$M(t, u)$ is strictly convex with respect u for almost all $t \in T$;*
(d) *X is midpoint locally uniformly rotund.*

In order to prove the theorem, we give some lemmas.

Lemma 2.2. *Let $z/\|z\|$ be a strongly extreme point. Then for any $\varepsilon > 0$ and $\alpha \in (0, 1)$ we have*

(a) $$\delta(\alpha, z) := \inf_{x\in X}\{\max(\|x\| - \|z\|, \|y\| - \|z\|) : \|x - y\| \ge \varepsilon, z = \alpha x + (1 - \alpha)y\} > 0;$$

(b) *if $z_n \to z$, $\alpha_n \to \alpha$, then $\delta(\alpha_n, z_n) \to \delta(\alpha, z)$.*

Proof. (a) Suppose that $\delta(\alpha, z) = 0$. Then there exists $\{x_n\}_{n=1}^{\infty} \subset X$, $\{y_n\}_{n=1}^{\infty} \subset X$ such that $\|x_n - y_n\| \geq \varepsilon$, $z = \alpha x_n + (1-\alpha) y_n$, but $\|x_n\| - \|z\| < \frac{1}{n}$ and $\|y_n\| - \|z\| < \frac{1}{n}$, hence

$$\left\| \frac{x_n}{\|z\|} \right\| \leq 1 + \frac{1}{n\|z\|}, \qquad \left\| \frac{y_n}{\|z\|} \right\| \leq 1 + \frac{1}{n\|z\|}, \qquad \alpha \frac{x_n}{\|z\|} + (1-\alpha) \frac{y_n}{\|z\|} = \frac{z}{\|z\|}.$$

We may assume without loss of generality that $\alpha \geq \frac{1}{2}$. Then there exists $z_n \in [x_n, y_n]$ such that $z = \frac{1}{2} x_n + \frac{1}{2} z_n$. It is easy to see that

$$\left\| \frac{x_n}{\|z\|} \right\| \leq 1 + \frac{1}{n\|z\|}, \qquad \left\| \frac{z_n}{\|z\|} \right\| \leq 1 + \frac{1}{n\|z\|}, \qquad \frac{1}{2} \cdot \frac{x_n}{\|z\|} + \frac{1}{2} \cdot \frac{z_n}{\|z\|} = \frac{z}{\|z\|}.$$

Since $z/\|z\|$ is a strongly extreme point, we have

$$\left\| \frac{x_n}{\|z\|} - \frac{z_n}{\|z\|} \right\| \to 0 \Rightarrow \left\| \frac{x_n}{\|z\|} - \frac{z}{\|z\|} \right\| \to 0 \Rightarrow \|x_n - z\| \to 0 (n \to \infty),$$

a contradiction!

(b) Suppose that $\limsup_{n\to\infty} \delta(\alpha_n, z_n) > \delta(\alpha, z)$, where $z_n \to z$, $\alpha_n \to \alpha$. Then there exist $r > 0$ and a subsequence of $\{n\}$ denoted again by $\{n\}$, such that $\delta(\alpha_n, z_n) - \delta(\alpha, z) \geq r$. By the definition of $\delta(\alpha, z)$, there exist $x_0, y_0 \in X$ such that

$$\delta(\alpha, z) + \frac{r}{8} > \max(\|x_0\| - \|z\|, \|y_0\| - \|z\|),$$

$$\|x_0 - y_0\| \geq \varepsilon, \qquad z = \alpha x_0 + (1-\alpha) y_0. \tag{1}$$

Since $z_n \to z$, $\alpha_n \to \alpha$, there exists $n_1 \in N$ such that $\|z_{n_1} - z\| < \frac{1}{8} r$ and $\|(\alpha_{n_1} - \alpha)(x_0 - y_0)\| < \frac{1}{8} r$. Let $h_n = (\alpha_n - \alpha)(x_0 - y_0)$. Hence we have

$$\begin{aligned}
z_{n_1} &= z_{n_1} \\
&= z_{n_1} + \alpha_{n_1} x_0 + (1-\alpha_{n_1}) y_0 - (\alpha_{n_1} - \alpha)(x_0 - y_0) - [\alpha x_0 + (1-\alpha) y_0] \\
&= z_{n_1} + \alpha_{n_1} x_0 + (1-\alpha_{n_1}) y_0 - h_{n_1} - z \\
&= z_{n_1} + \alpha_{n_1} x_0 + (1-\alpha_{n_1}) y_0 - [\alpha_{n_1} h_{n_1} + (1-\alpha_{n_1}) h_{n_1}] - [\alpha_{n_1} z + (1-\alpha_{n_1}) z] \\
&= z_{n_1} + \alpha_{n_1}(x_0 - h_{n_1} - z) + (1-\alpha_{n_1})(y_0 - h_{n_1} - z) \\
&= \alpha_{n_1} z_{n_1} + (1-\alpha_{n_1}) z_{n_1} + \alpha_{n_1}(x_0 - h_{n_1} - z) + (1-\alpha_{n_1})(y_0 - h_{n_1} - z) \\
&= \alpha_{n_1}(x_0 - h_{n_1} - z + z_{n_1}) + (1-\alpha_{n_1})(y_0 - h_{n_1} - z + z_{n_1})
\end{aligned}$$

and

$$\|(x_0 - h_{n_1} - z + z_{n_1}) - (y_0 - h_{n_1} - z + z_{n_1})\| = \|x_0 - y_0\| \geq \varepsilon.$$

By (1), we get the following inequality

$$\begin{aligned}\|x_0 - h_{n_1} - z + z_{n_1}\| - \|z_{n_1}\| &\le \|x_0\| + \|h_{n_1}\| + \|z_{n_1} - z\| - \|z_{n_1}\| \\ &\le \|x_0\| + \frac{1}{8}r + \frac{1}{8}r - (\|z\| - \|z_{n_1} - z\|) \\ &\le \|x_0\| + \frac{1}{8}r + \frac{1}{8}r - \|z\| + \frac{1}{8}r \\ &= \|x_0\| - \|z\| + \frac{3}{8}r \\ &< \delta(\alpha, z) + \frac{1}{8}r + \frac{3}{8}r \\ &= \delta(\alpha, z) + \frac{1}{2}r.\end{aligned}$$

Similarly, we have $\|y_0 - h_{n_1} - z + z_{n_1}\| - \|z_{n_1}\| \le \delta(\alpha, z) + \frac{1}{2}r$. Hence, $\delta(\alpha_{n_1}, z_{n_1}) \le \delta(\alpha, z) + \frac{1}{2}r$, a contradiction! Similarly, $\liminf_{n\to\infty} \delta(\alpha_n, z_n) < \delta(\alpha, z)$ is impossible. Thus, $\lim_{n\to\infty} \delta(\alpha_n, z_n) = \delta(\alpha, z)$. This completes the proof. □

Lemma 2.3 (see [13]). *Let $u \in L_M^0(X) \setminus \{0\}$. If $K(u) = \emptyset$, then $\|u\|^0 = \int_T A(t) \cdot \|u(t)\|\, dt$, where $A(t) = \lim_{u\to\infty} \frac{M(t,u)}{u}$.*

Lemma 2.4. *For any $u \in L_M^0(X)$, we have the inequalities $\|u\| \le \|u\|^0 \le 2\|u\|$.*

Proof. For any $k > 0$, we have

$$\rho_M\left(\frac{u}{\frac{1}{k}[\rho_M(ku) + 1]}\right) = \rho_M\left(\frac{ku}{\rho_M(ku) + 1}\right) \le \frac{\rho_M(ku)}{\rho_M(ku) + 1} \le 1.$$

Therefore

$$\|u\| \le \frac{1}{k}[\rho_M(ku) + 1] \text{ whence } \|u\| \le \inf_{k>0} \frac{1}{k}[\rho_M(ku) + 1] = \|u\|^0.$$

On the other hand,

$$\left\|\frac{u}{\|u\|}\right\|^0 \le 1 + \rho_M\left(\frac{u}{\|u\|}\right) \le 2 \text{ whence } \|u\|^0 \le 2\|u\|.$$ □

Lemma 2.5. *If $M \in \Delta_2$, then for any $u \in L_M^0(X)$ is norm absolotely continuous.*

Proof. Given any $\varepsilon > 0$, there exists $k > 0$ such that $\frac{1}{k} < \frac{1}{2}\varepsilon$. Since $M \in \Delta_2$, we have $\rho_M(ku) < \infty$. Since integral absolutely continuity, we can find $\delta > 0$ such that $\mu E < \delta$ implies that $\rho_M(ku \cdot \chi_E) < 1$. Hence, if $\mu E < \delta$, then

$$\|u \cdot \chi_E\|^0 \le \frac{1}{k}[1 + \rho_M(ku \cdot \chi_E)] < \frac{1}{k}[1 + 1] = \frac{2}{k} < \varepsilon.$$ □

Lemma 2.6 (see [1]). *Suppose $M \in \Delta_2$ and $e(t) = 0$ μ-a.e. on T. Then*

$$\rho_M(u_n) \to 0 \Leftrightarrow \|u_n\| \to 0 \quad \text{and} \quad \rho_M(u_n) \to 1 \Leftrightarrow \|u_n\| \to 1 (n \to \infty),$$

where $e(t) = \sup\{u \geq 0 : M(t, u) = 0\}$.

Lemma 2.7 (see [1]). *The following are equivalent:*

(*a*) $M \notin \Delta_2$

(*b*) *For each $\varepsilon \in (0, 1)$, there exists $u \in L_M(X)$ such that $\rho_M(u) = \varepsilon$, $\|u\| = 1$ and $\|u(t)\| < E(t)$ $\mu-$a.e. on T, where $E(t) = \sup\{u > 0 : M(t, u) < \infty\}$.*

Lemma 2.8 (see [13]). *The space $L^0_M(X)$ is rotund if and only if:*

(*a*) *For any $u \in S(L^0_M(X))$, the set $K(u)$ consists of one element from $(0, +\infty)$;*

(*b*) *X is rotund;*

(*c*) *$M(t, u)$ is strictly convex with respect to u for almost all $t \in T$.*

Proof. Necessity. By Lemma 2.8, (c) is obvious. For any $u \in L^0_M(X)\backslash\{0\}$, we have $\left\|\frac{u}{\|u\|^0}\right\|^0 = \frac{1}{k}[1+\rho_M(k\frac{u}{\|u\|^0})]$ by Lemma 2.8, where $k \in K(\frac{u}{\|u\|^0})$. Hence $\|u\|^0 = \frac{\|u\|^0}{k}[1+\rho_M(\frac{k}{\|u\|^0}\cdot u)]$. If $k_1, k_2 \in K(u)$, then $\left\|\frac{u}{\|u\|^0}\right\|^0 = \frac{1}{k_1\|u\|^0}[1+\rho_M(k_1\|u\|^0 \cdot \frac{u}{\|u\|^0})]$. This implies that $k_1\|u\|^0 \in K(\frac{u}{\|u\|^0})$. Similarly, we have that $k_2\|u\|^0 \in K(\frac{u}{\|u\|^0})$. By Lemma 2.8, we get $k_1 = k_2$.

(b) Suppose that $M \notin \Delta_2$. By Lemma 2.7, there exists $u \in L^0_M(X)$ such that $\rho_M(u) < \frac{1}{2}$, $\|u\| = 1$ and $\|u(t)\| < E(t)$ $\mu-$a.e. on T. Then for any $L > 1$, we have $\rho_M(Lu) = \infty$. Indeed, suppose that there exists $L_1 > 1$ such that $\rho_M(L_1u) < \infty$. We know that the function $F(k) = \int_T M(t, k\|u(t)\|)dt$ is continuous on $[1, L_1]$. Then there exists $L_2 > 1$ such that $\rho_M(L_2u) = 1$. This implies that $\|u\| \leq \frac{1}{L_2}$, contradicting $\|u\| = 1$.

Decompose T into E_1 and G_1 in such a way that $\mu E_1 = \mu G_1$. Then for any $L > 1$, we have $\int_{E_1} M(t, L\|u(t)\|)dt = \infty$ or $\int_{G_1} M(t, L\|u(t)\|)dt = \infty$. Without loss of generality, we may assume that $\int_{E_1} M(t, L\|u(t)\|)dt = \infty$. Decompose E_1 into E_2 and G_2 in such a way that $\mu E_2 = \mu G_2$. Then for any $L > 1$, we have $\int_{E_2} M(t, L\|u(t)\|)dt = \infty$ or $\int_{G_2} M(t, L\|u(t)\|)dt = \infty$. Without loss of generality, we may assume that $\int_{E_2} M(t, L\|u(t)\|)dt = \infty$. In generical, decompose E_n into E_{n+1} and G_{n+1} in such a way that $\mu E_{n+1} = \mu G_{n+1}$. Then for any $L > 1$, we have $\int_{E_{n+1}} M(t, L\|u(t)\|)dt = \infty$ or $\int_{G_{n+1}} M(t, L\|u(t)\|)dt = \infty$. Without loss of generality, we may assume that $\int_{E_{n+1}} M(t, L\|u(t)\|)dt = \infty$. Hence we have

$$E_1 \supset E_2 \supset E_3 \supset \cdots, \quad \mu E_i = \frac{1}{2}\mu E_{i+1} \quad \text{and} \quad \|u(t)\cdot\chi_{E_i}(t)\| = 1, \quad i = 1, 2, ...$$

Let $k \in K(u)$. It is easy to see that $k \leq 1$. We will derive a contradiction for each the following two cases.

Case 1. Let $k<1$. Define

$$u_n(t)=\sum_{i=1}^{n-1}u(t)\cdot\chi_{F_i}(t),\qquad v_n(t)=\sum_{i=1}^{n-1}u(t)\cdot\chi_{F_i}(t)+\frac{1}{k}\sum_{i=n}^{\infty}u(t)\cdot\chi_{F_i}(t),$$

where $F_i=E_i\backslash E_{i+1}$. Then $(1-k)u_n+kv_n=u$, $\|u_n\|^0\le\|u\|^0$ and $\|u_n-v_n\|^0\ge\|u_n-v_n\|=\left\|\frac{1}{k}u\cdot\chi_{E_n}\right\|=\frac{1}{k}$. For any $\varepsilon>0$, we have $\frac{1}{k}\sum_{i=n}^{\infty}\int_T M(t,\|k\cdot\frac{1}{k}u(t)\chi_{F_i}(t)\|)dt<\varepsilon$, when n is large enough. Therefore,

$$\begin{aligned}\|u\|^0\le\ &\|v_n\|^0\\ \le\ &\frac{1}{k}+\frac{1}{k}\sum_{i=1}^{n-1}\int_T M(t,\|ku(t)\cdot\chi_{F_i}(t)\|)dt\\ &\qquad+\frac{1}{k}\sum_{i=n}^{\infty}\int_T M\left(t,\left\|k\cdot\frac{1}{k}u(t)\chi_{F_i}(t)\right\|\right)dt\\ \le\ &\|u\|^0+\varepsilon\end{aligned}$$

This implies that $\|v_n\|^0\to\|u\|^0$ as $n\to\infty$, a contradiction!

Case 2. Let $k=1$. By the proof of Case 1, without loss of generality, we may assume that $\mu\{t\in T:\|u(t)\|=0\}>0$. Since $M(t,u)$ is strictly convex with respect u for almost all $t\in T$, then $p(t,u)<\infty$ for almost all $t\in T$. Hence, there exists $v\in L_M^0(X)\setminus\{0\}$ such that $\{t\in T:\|v(t)\|\ne0\}\subset\{t\in T:\|u(t)\|=0\}$, $\rho_N(p(\frac{1}{2}v))>1$ and $\rho_M(2v)<\infty$. Put $u_0=u+v$. Pick $0<k_1<1$, $0<k_2<1$, we have

$$\rho_M(k_1u_0\chi_{T_n})\ge\int_T k_1\|u_0(t)\chi_{T_n}(t)\|\cdot p(t,\|k_2u_0(t)\chi_{T_n}(t)\|)dt-\rho_N(p(k_2u_0\chi_{T_n}))$$

$$\rho_M(k_2u_0\chi_{T_n})=\int_T k_2\|u_0(t)\chi_{T_n}(t)\|\cdot p(t,\|k_2u_0(t)\chi_{T_n}(t)\|)dt-\rho_N(p(k_2u_0\chi_{T_n})),$$

where

$$T_n=\{t\in T:k_2\|u_0(t)\|\le n,\ p(t,\|k_2u_0(t)\|)\le n\}.$$

It is follows that

$$\begin{aligned}&\frac{1}{k_1}[1+\rho_M(k_1u_0\chi_{T_n})]-\frac{1}{k_2}[1+\rho_M(k_2u_0\chi_{T_n})]\\ =\ &\frac{k_1-k_2}{k_1k_2}\left(-1+\frac{k_2}{k_1-k_2}[\rho_M(k_1u_0\chi_{T_n})-\rho_M(k_2u_0\chi_{T_n})]-\rho_M(k_2u_0\chi_{T_n})\right)\\ \ge\ &\frac{k_1-k_2}{k_1k_2}\Bigg(-1+\frac{k_2}{k_1-k_2}\int_T(k_1-k_2)\|u_0(t)\chi_{T_n}(t)\|\cdot p(t,\|k_2u_0(t)\chi_{T_n}(t)\|)dt\\ &\qquad-\rho_M(k_2u_0\chi_{T_n})\Bigg)\\ =\ &\frac{k_1-k_2}{k_1k_2}(\rho_N(p(k_2u_0\chi_{T_n}))-1).\end{aligned}$$

Taking $n \to \infty$, we obtain

$$\frac{1}{k_1}[1+\rho_M(k_1u_0)] \geq \frac{k_1-k_2}{k_1k_2}(\rho_N(p(k_2u_0))-1)+\frac{1}{k_2}[1+\rho_M(k_2u_0)]. \tag{2}$$

By (2), it is easy to see that $h<1$, where $h \in K(u_0)$. Define

$$u_n^{'}(t)=\sum_{i=1}^{n-1}u_0(t)\cdot\chi_{F_i}(t), \qquad v_n^{'}(t)=\sum_{i=1}^{n-1}u_0(t)\cdot\chi_{F_i}(t)+\frac{1}{h}\sum_{i=n}^{\infty}u_0(t)\cdot\chi_{F_i}(t),$$

where $F_i=E_i\backslash E_{i+1}$. Then $(1-h)u_n^{'}+hv_n^{'}=u_0$, $\left\|u_n^{'}\right\|^0 \leq \|u_0\|^0$ and $\left\|u_n^{'}-v_n^{'}\right\|^0 \geq \left\|u_n^{'}-v_n^{'}\right\|=\left\|\frac{1}{k}u_0\cdot\chi_{E_n}\right\| \geq \frac{1}{h}$. Using the method of Case 1, we can easily deduce that $\left\|u_n^{'}-u_0\right\|^0 \to 0$, a contradiction!

Suppose that (d) is not true. Then there exist $x \in S(X)$, $\{x_n\}_{n=1}^{\infty} \subset X$, $\{y_n\}_{n=1}^{\infty} \subset X$ with $x=\frac{1}{2}(x_n+y_n)$, we have $\|x_n\| \to 1$, $\|y_n\| \to 1$ and $x_n-y_n \nrightarrow 0$. Pick $h(t) \in S(L_M^0(X))$, then there exists $d>0$ such that $\mu H>0$, where $H=\{t \in T:\|h(t)\| \geq 2d\}$. Since $h(t) \in S(L_M^0(X))$, then there exists $k^{'}>0$ such that

$$\int_H M(t,k^{'}\|h(t)\|)dt \leq \int_T M(t,k^{'}\|h(t)\|)dt<\infty.$$

Setting

$$u(t)=d\cdot x\cdot\chi_H(t), \qquad u_n(t)=d\cdot x_n\cdot\chi_H(t), \qquad v_n(t)=d\cdot y_n\cdot\chi_H(t).$$

we have

$$\int_T M(t,k^{'}\|u(t)\|)dt=\int_H M(t,k^{'}d)dt \leq \int_H M(t,k^{'}\|h(t)\|)dt<\infty,$$

$$\int_T M(t,k^{'}\|u_n(t)\|)dt \leq \int_H M(t,2k^{'}d)dt \leq \int_H M(t,k^{'}\|h(t)\|)dt<\infty$$

and

$$\int_T M(t,k^{'}\|v_n(t)\|)dt \leq \int_H M(t,2k^{'}d)dt \leq \int_H M(t,k^{'}\|h(t)\|)dt<\infty.$$

This implies that $u \in L_M^0(X)$, $\{u_n\}_{n=1}^{\infty} \subset L_M^0(X)$, and $\{v_n\}_{n=1}^{\infty} \subset L_M^0(X)$. Since $M \in \Delta_2$, we have $\int_T M(t,k\|2d\cdot x\cdot\chi_H(t)\|)dt<\infty$, where $k \in K(u)$. Notice that $M(t,k\|d\cdot x_n\cdot\chi_H(t)\|) \leq M(t,k\|2d\cdot x\cdot\chi_H(t)\|)$. By the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty}\|u_n\|^0 &= \lim_{n\to\infty}\frac{1}{k}\left[1+\int_T M(t,k\|d\cdot x_n\cdot\chi_H(t)\|)dt\right] \\
&= \frac{1}{k}\left[1+\int_T \lim_{n\to\infty} M(t,k\|d\cdot x_n\cdot\chi_H(t)\|)dt\right] \\
&= \frac{1}{k}\left[1+\int_T M(t,\lim_{n\to\infty} k\|d\cdot x_n\cdot\chi_H(t)\|)dt\right] \\
&= \frac{1}{k}\left[1+\int_T M(t,k\|d\cdot x\cdot\chi_H(t)\|)dt\right]=\|u\|^0.
\end{aligned}$$

Similarly, we have that $\lim_{n\to\infty} \|v_n\|^0 = \|u\|^0$. Since $x_n - y_n \nrightarrow 0$, there exist $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k} - y_{n_k}\}_{k=1}^{\infty}$ of $\{x_n - y_n\}_{n=1}^{\infty}$ such that $\|x_{n_k} - y_{n_k}\| \geq \varepsilon_0$. Then

$$\|u_n - v_n\|^0 \geq \inf_{k>0} \frac{1}{k}\left[1 + \int_T M(t, k\,\|d \cdot \varepsilon_0 \cdot \chi_H(t)\|)dt\right] > 0,$$

which implies that $u_n - v_n \nrightarrow 0$. This shows that $L_M^0(X)$ is not midpoint locally uniformly rotund, a contradiction!

Sufficiency. Put $\|u_n\|^0 \to 1$, $\|v_n\|^0 \to 1$, $u_n + v_n = 2u$, we want to prove that $\|u_n - v_n\|^0 \to 0$ as $n \to \infty$. The proof requires of considering of few cases separately.

Case 1. Let $\max\{\sup\{k_n\}, \sup\{h_n\}\} < \infty$, where $k_n = K(u_n)$, $h_n = K(v_n)$. Without loss of generality, we may assume that $k_n \to k$ and $h_n \to h$. Since $\|u_n\|^0 = \frac{1}{k_n}[1 + \rho_M(k_n u_n)] \to 1, \|v_n\|^0 = \frac{1}{h_n}[1 + \rho_M(h_n v_n)] \to 1$ as $n \to \infty$, without loss of generality, we may assume that $k_n \geq \frac{3}{4}$ and $h_n \geq \frac{3}{4}$. Let $l = \frac{2kh}{k+h}$, in view of the Fatou Lemma, we get

$$\begin{aligned}
1 = &\ \|u\|^0 \\
&\leq \frac{1}{l}[1 + \rho_M(lu)] \\
&\leq \liminf_{n\to\infty} \frac{k_n + h_n}{2k_n h_n}\left[1 + \rho_M\left(\frac{2k_n h_n}{k_n + h_n} u\right)\right] \\
&\leq \liminf_{n\to\infty} \frac{k_n + h_n}{2k_n h_n}\left[1 + \rho_M(\frac{h_n}{k_n + h_n}\left(k_n u_n) + \frac{k_n}{k_n + h_n}(h_n v_n)\right)\right] \\
&= \liminf_{n\to\infty} \frac{1}{2}\left[\frac{1}{k_n}(1 + \rho_M(k_n u_n)) + \frac{1}{h_n}(1 + \rho_M(h_n v_n))\right] \\
&= \liminf_{n\to\infty} \frac{1}{2}\left[\|u_n\|^0 + \|v_n\|^0\right] \\
&= 1.
\end{aligned}$$

Consequently, $l = K(u)$. Next we will prove that for any $\sigma > 0$, $\varepsilon > 0$ there exists N such that

$$\mu\{t \in T : \|k_n u_n(t) - h_n v_n(t)\| \geq \sigma\} < \varepsilon$$

whenever $n > N$. Otherwise, without loss of generality, we may assume that for each $n \in N$ there exists $E_n^{'} \subseteq T$, $\varepsilon_0 > 0$ and $\sigma_0 > 0$ such that $\mu E_n^{'} \geq \varepsilon_0$, where

$$E_n^{'} = \{t \in T : \|k_n u_n(t) - h_n v_n(t)\| \geq \sigma_0\}.$$

We define the function

$$\begin{aligned}
\eta(t) = \inf_{x\in X} \Big\{ &\max(\|x\| - \|lu(t)\|, \|y\| - \|lu(t)\|) : \|x - y\| \geq \sigma_0, \\
&lu(t) = \frac{h}{k+h}x + \frac{k}{k+h}y \Big\},
\end{aligned}$$

where $t \in \{t \in T : u(t) \neq 0\}$. By Lemma 2.2, we have $\eta(t) > 0$. Let $h_n(t) \to u(t)$ μ-a.e. on T, where h_n are simple functions. Then the functions

$$\eta_n(t) = \inf_{x\in X}\Big\{ \max(\|x\| - \|lh_n(t)\|, \|y\| - \|lh_n(t)\|) : \|x-y\| \geq \sigma_0,$$
$$lh_n(t) = \frac{h}{k+h}x + \frac{k}{k+h}y\Big\}$$

are Σ-measurable. By Lemma 2.2, we have $\eta_n(t) \to \eta(t)$ μ-a.e. on T. Then $\eta(t)$ is Σ-measurable. Using the inclusion

$$T \supset \bigcup_{n=1}^{\infty}\left\{t \in T : \frac{1}{n+1} < \eta(t) \leq \frac{1}{n},\ u(t) \neq 0\right\},$$

we get that there exists $0 < \eta_0 < \sigma_0$ such that $\mu H < \frac{1}{16}\varepsilon_0$. where

$$H = \{t \in T : \eta(t) \leq 2\eta_0,\ u(t) \neq 0\}.$$

Decompose E_n' into E_{n1}', E_{n2}' and E_{n3}', where

$$E_{n1}' = \left\{t \in E_n' : \big|\|k_n u_n(t)\| - \|h_n v_n(t)\|\big| \geq \frac{\eta_0}{4}\right\},$$
$$E_{n2}' = \left\{t \in E_n' : \big|\|k_n u_n(t)\| - \|h_n v_n(t)\|\big| < \frac{\eta_0}{4}, u(t) = 0\right\},$$
$$E_{n3}' = \left\{t \in E_n' : \big|\|k_n u_n(t)\| - \|h_n v_n(t)\|\big| < \frac{\eta_0}{4}, u(t) \neq 0\right\}.$$

If $t \in E_{n3}'$, then $\|k_n u_n(t) - h_n v(t)\| \geq \sigma_0$, $\big|\|k_n u_n(t)\| - \|h_n v_n(t)\|\big| < \frac{\eta_0}{4}$ and $u(t) \neq 0$. Since X is midpoint locally uniformly rotund, by Lemma 2.2, without loss of generality, we may assume that $\|k_n u_n(t)\| - \left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\| \geq \lambda_n(t)$, where

$$\lambda_n(t) = \inf_{x\in X}\Big\{ \max\left(\|x\| - \left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\|, \|y\| - \left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\|\right) :$$
$$\|x-y\| \geq \sigma_0,\ \frac{2k_nh_n}{k_n+h_n}u(t) = \frac{h_n}{k_n+h_n}x + \frac{k_n}{k_n+h_n}y\Big\}.$$

By Lemma 2.2, we have $\lambda_n(t) \to \eta(t)$ $\mu-$ a.e. on T. By the Egorov theorem, there exists N such that if $n > N$ then $|\lambda_n(t) - \eta(t)| < \frac{\eta_0}{2}$ on F, where $F \subset T$ and $\mu(T\backslash F) < \frac{1}{16}\varepsilon_0$. Let $E_{n3}'' = E_{n3}' \backslash (H \cup (T\backslash F))$, then $\mu(E_{n3}' - E_{n3}'') < \frac{1}{8}\varepsilon_0$. Hence, if $t \in E_{n3}''$, then

$$\|k_n u_n(t)\| - \left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\| \geq \lambda_n(t) > \eta(t) - \frac{\eta_0}{2} \geq 2\eta_0 - \frac{\eta_0}{2} > \eta_0, \tag{3}$$

when n is large enough. By $\left|\|k_n u_n(t)\| - \|h_n v_n(t)\|\right| < \frac{1}{4}\eta_0$ and (3), we get the following inequality

$$\begin{aligned}
&\frac{h_n}{k_n+h_n}\|k_n u_n(t)\| + \frac{k_n}{k_n+h_n}\|h_n v_n(t)\| - \left\|\frac{2k_n h_n}{k_n+h_n}u(t)\right\| \\
&= \frac{h_n}{k_n+h_n}\|k_n u_n(t)\| - \frac{h_n}{k_n+h_n}\left\|\frac{2k_n h_n}{k_n+h_n}u(t)\right\| \\
&\qquad + \frac{k_n}{k_n+h_n}\|h_n v_n(t)\| - \frac{k_n}{k_n+h_n}\left\|\frac{2k_n h_n}{k_n+h_n}u(t)\right\| \\
&\geq \frac{h_n}{k_n+h_n}\eta_0 + \frac{k_n}{k_n+h_n}\|h_n v_n(t)\| - \frac{k_n}{k_n+h_n}\left\|\frac{2k_n h_n}{k_n+h_n}u(t)\right\| \\
&\geq \frac{h_n}{k_n+h_n}\eta_0 + \frac{k_n}{k_n+h_n}\|k_n u_n(t)\| - \frac{k_n}{k_n+h_n}\cdot\frac{\eta_0}{4} - \frac{k_n}{k_n+h_n}\left\|\frac{2k_n h_n}{k_n+h_n}u(t)\right\| \\
&= \frac{h_n}{k_n+h_n}\eta_0 + \frac{k_n}{k_n+h_n}\|k_n u_n(t)\| - \frac{k_n}{k_n+h_n}\left\|\frac{2k_n h_n}{k_n+h_n}u(t)\right\| - \frac{k_n}{k_n+h_n}\cdot\frac{\eta_0}{4} \\
&\geq \frac{h_n}{k_n+h_n}\eta_0 + \frac{k_n}{k_n+h_n}\eta_0 - \frac{k_n}{k_n+h_n}\cdot\frac{\eta_0}{4} \\
&\geq \frac{h_n}{k_n+h_n}\eta_0 + \frac{k_n}{k_n+h_n}\cdot\frac{3}{4}\eta_0 \\
&\geq \frac{3}{4}\eta_0,
\end{aligned}$$

when n is large enough. Let $E_n = E'_{n1} \cup E'_{n2} \cup E''_{n3}$, then $\mu E_n \geq \frac{7}{8}\varepsilon_0$, when n is large enough. Let us define the sets

$$A_n = \left\{t \in T : M(t, \|k_n u_n(t)\|) > \frac{8}{\varepsilon_0}\right\},$$

$$B_n = \left\{t \in T : M(t, \|h_n v_n(t)\|) > \frac{8}{\varepsilon_0}\right\}.$$

Then

$$1 = \int_T M(t, \|k_n u_n(t)\|)dt \geq \int_{A_n} M(t, \|k_n u_n(t)\|)dt \geq \frac{8}{\varepsilon_0}\mu A_n.$$

This implies that $\mu A_n \leq \frac{1}{8}\varepsilon_0$. Similarly, we have that $\mu B_n \leq \frac{1}{8}\varepsilon_0$. For μ-a.e. $t \in T$, we define the bounded closed set

$$C_t = \left\{(u,v) \in R^2 : M(t,u) \leq \frac{8}{\varepsilon_0},\ M(t,v) \leq \frac{8}{\varepsilon_0},\ |u-v| \geq \frac{1}{4}\eta_0\right\}$$

in the two dimensional space R^2. Since C_t is compact, then for μ-a.e. $t \in T$ there exists $(u_t, v_t) \in C_t$ such that

$$1 > \frac{M(t, (\frac{h}{k+h}u_t + \frac{k}{k+h}v_t))}{\frac{h}{k+h}M(t,u_t) + \frac{k}{k+h}M(t,v_t)} \geq \frac{M(t, (\frac{h}{k+h}u + \frac{k}{k+h}v))}{\frac{h}{k+h}M(t,u) + \frac{k}{k+h}M(t,v)} \tag{4}$$

for any $(u,v) \in C_t$. If we define the function

$$1-\delta(t) = \frac{M(t,(\frac{h}{k+h}u_t + \frac{k}{k+h}v_t))}{\frac{h}{k+h}M(t,u_t) + \frac{k}{k+h}M(t,v_t)} \tag{5}$$

then $\delta(t)$ is Σ-measurable. Indeed, if we pick a dense set $\{r_i\}_{i=1}^{\infty}$ in $[0,\infty)$ and define the function

$$1-\delta_{r_i,r_j}(t) = \begin{cases} \dfrac{M(t,(\frac{h}{k+h}r_i + \frac{k}{k+h}r_j))}{\frac{h}{k+h}M(t,r_i) + \frac{k}{k+h}M(t,r_j)}, & M(t,r_i) \le \dfrac{8}{\varepsilon_0} \text{ and } M(t,r_j) \le \dfrac{8}{\varepsilon_0}, \\ 0, & M(t,r_i) > \dfrac{8}{\varepsilon_0} \text{ or } M(t,r_j) > \dfrac{8}{\varepsilon_0}, \end{cases}$$

then by the definition of $M(t,u)$, it is easy to see that $1-\delta_{r_i,r_j}(t)$ is Σ-measurable and

$$1-\delta(t) \ge \sup\left\{1-\delta_{r_i,r_j}(t) : |r_i - r_j| \ge \frac{1}{4}\eta_0\right\}.$$

On the other hand, since $\{r_i\}_{i=1}^{\infty}$ is dense in $[0,\infty)$ then $\{(r_i,r_j)\}_{i=1,j=1}^{\infty}$ is dense in $[0,\infty)\times[0,\infty)$. By definition of function $1-\delta(t)$, for μ-a.e. $t \in T, \varepsilon > 0$, there exists $(r_i,r_j) \in \{(u,v) \in R^2 : M(t,u) \le \frac{8}{\varepsilon_0}, M(t,v) \le \frac{8}{\varepsilon_0}, |u-v| \ge \frac{1}{4}\eta_0\}$ such that

$$1-\delta(t) - \varepsilon < 1-\delta_{r_i,r_j}(t) \le \sup\left\{1-\delta_{r_i,r_j}(t) : |r_i - r_j| \ge \frac{1}{4}\eta_0\right\}$$

μ-a.e. on T. Since $\varepsilon > 0$ was arbitrary, we have

$$1-\delta(t) \le \sup\left\{1-\delta_{r_i,r_j}(t) : |r_i - r_j| \ge \frac{1}{4}\eta_0\right\}$$

μ-a.e. on T. Therefore, $1-\delta(t) = \sup\{1-\delta_{r_i,r_j}(t) : |r_i - r_j| \ge \frac{1}{4}\eta_0\}$ μ-a.e. in T. This implies that $\delta(t)$ is Σ-measurable. By (4) and (5), we have

$$\begin{aligned} &M\left(t, \frac{h}{k+h}u + \frac{k}{k+h}v\right) \\ &\le (1-\delta(t))\left[\frac{h}{k+h}M(t,u) + \frac{k}{k+h}M(t,v)\right] \quad u,v \in C_t \end{aligned}$$

for μ-a.e. $t \in T$. Moreover, we have

$$T \supset \bigcup_{n=1}^{\infty}\left\{t \in T : \frac{1}{n+1} < \delta(t) \le \frac{1}{n}\right\}.$$

Since $M(t,u)$ is strictly convex with respect u for almost all $t \in T$, then there exists $2\delta_0 \in (0,1)$ such that $\mu G < \frac{1}{16}\varepsilon_0$, where

$$G = \{t \in T : \delta(t) \le 2\delta_0\}.$$

We have $W_n(t)-Q_n(t)\to 0(n\to\infty)$ $\mu-$a.e. on T, where

$$W_n(t)=\frac{M(t,\frac{h_n}{k_n+h_n}\|k_nu_n(t)\|+\frac{k_n}{k_n+h_n}\|h_nv_n(t)\|)}{\frac{h_n}{k_n+h_n}M(t,\|k_nu_n(t)\|)+\frac{k_n}{k_n+h_n}M(t,\|h_nv_n(t)\|)}\cdot\chi_{E'_{n1}\setminus(A_n\cup B_n)}(t)$$

$$Q_n(t)=\frac{M(t,\frac{h}{k+h}\|k_nu_n(t)\|+\frac{k}{k+h}\|h_nv_n(t)\|)}{\frac{h}{k+h}M(t,\|k_nu_n(t)\|)+\frac{k}{k+h}M(t,\|h_nv_n(t)\|)}\cdot\chi_{E'_{n1}\setminus(A_n\cup B_n)}(t).$$

By the Egorov theorem, there exists N such that if $n>N$, then $|W_n(t)-Q_n(t)|$ $<\frac{1}{4}\delta_0$ on $T\setminus E$, where $E\subset T$ and $\mu E<\frac{1}{16}\varepsilon_0$. Hence, if $t\in E'_{n1}\setminus(E\cup G\cup A_n\cup B_n)$, then

$$\begin{aligned}\frac{3}{2}\delta_0&=2\delta_0-\frac{1}{2}\delta_0\\&\le 1-\frac{M(t,\frac{h}{k+h}\|k_nu_n(t)\|+\frac{k}{k+h}\|h_nv_n(t)\|)}{\frac{h}{k+h}M(t,\|k_nu_n(t)\|)+\frac{k}{k+h}M(t,\|h_nv_n(t)\|)}-\frac{1}{2}\delta_0\\&\le 1-\frac{M(t,\frac{h_n}{k_n+h_n}\|k_nu_n(t)\|+\frac{k_n}{k_n+h_n}\|h_nv_n(t)\|)}{\frac{h_n}{k_n+h_n}M(t,\|k_nu_n(t)\|)+\frac{k_n}{k_n+h_n}M(t,\|h_nv_n(t)\|)},\end{aligned}$$

when n is large enough. This implies that

$$\begin{aligned}&M\left(t,\frac{h_n}{k_n+h_n}\|k_nu_n(t)\|+\frac{k_n}{k_n+h_n}\|h_nv_n(t)\|\right)\\&\le(1-\delta_0)\left[\frac{h_n}{k_n+h_n}M(t,\|k_nu_n(t)\|)+\frac{k_n}{k_n+h_n}M(t,\|h_nv_n(t)\|)\right]\end{aligned}$$

on $E'_{n1}\setminus(E\cup G\cup A_n\cup B_n)$. Let $K=\min\{\frac{1}{2}\sigma_0,\frac{3}{4}\eta_0\}$. Then we have $M(t,K)>0$ μ-a.e. on T. By $T\supset\cup_{i=1}^{\infty}\{t\in T:\frac{1}{i+1}<M(t,K)\le\frac{1}{i}\}$, there exists $a>0$ such that $\mu C<\frac{1}{8}\varepsilon_0$, where

$$C=\{t\in T:M(t,K)\le a\}.$$

Let $H_n=E_n\setminus(E\cup G\cup C\cup A_n\cup B_n)$, $H_n^1=E'_{n1}\setminus(E\cup G\cup C\cup A_n\cup B_n)$, $H_n^2=E'_{n2}\setminus(E\cup G\cup C\cup A_n\cup B_n)$, $H_n^3=E''_{n3}\setminus(E\cup G\cup C\cup A_n\cup B_n)$. Then $\mu H_n\ge\frac{1}{4}\varepsilon_0$. Let $\overline{k}=\sup\{\{k_n\},\{h_n\}\}$. Then

$$\begin{aligned}&\|u_n\|^0+\|v_n\|^0-2\|u\|^0\\&\ge\frac{1}{k_n}\left(1+\rho_M(k_nu_n)+\frac{1}{h_n}(1+\rho_M(h_nv_n))\right)-\frac{k_n+h_n}{k_nh_n}\left(1+\rho_M\left(\frac{2k_nh_n}{k_n+h_n}u\right)\right)\\&=\frac{k_n+h_n}{k_nh_n}\int_T\left[\frac{h_n}{k_n+h_n}M(t,\|k_nu_n(t)\|)+\frac{k_n}{k_n+h_n}M(t,\|h_nv_n(t)\|)\right.\\&\qquad\left.-M\left(t,\left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\|\right)\right]dt\\&\ge\frac{k_n+h_n}{k_nh_n}\int_{H_n^1}\left[\frac{h_n}{k_n+h_n}M(t,\|k_nu_n(t)\|)+\frac{k_n}{k_n+h_n}M(t,\|h_nv_n(t)\|)\right.\\&\qquad\left.-M\left(t,\frac{h_n}{k_n+h_n}\|k_nu_n(t)\|+\frac{k_n}{k_n+h_n}\|h_nv_n(t)\|\right)\right]dt\end{aligned}$$

$$
\begin{aligned}
&+\frac{k_n+h_n}{k_nh_n}\int_{H_n^2}\left[\frac{h_n}{k_n+h_n}M(t,\|k_nu_n(t)\|)+\frac{k_n}{k_n+h_n}M(t,\|h_nv_n(t)\|)\right.\\
&\qquad\left.-M\left(t,\left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\|\right)\right]dt\\
&+\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}\left[\frac{h_n}{k_n+h_n}M(t,\|k_nu_n(t)\|)+\frac{k_n}{k_n+h_n}M(t,\|h_nv_n(t)\|)\right.\\
&\qquad\left.-M\left(t,\left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\|\right)\right]dt\\
\geq\ &\frac{k_n+h_n}{k_nh_n}\int_{H_n^1}\delta_0\left[\frac{h_n}{k_n+h_n}M(t,\|k_nu_n(t)\|)+\frac{k_n}{k_n+h_n}M(t,\|h_nv_n(t)\|)\right]dt\\
&+\frac{k_n+h_n}{k_nh_n}\int_{H_n^2}\left[\frac{h_n}{k_n+h_n}M(t,\|k_nu_n(t)\|)+\frac{k_n}{k_n+h_n}M(t,\|h_nv_n(t)\|)\right]dt\\
&+\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}\left[M\left(t,\frac{h_n}{k_n+h_n}\|k_nu_n(t)\|+\frac{k_n}{k_n+h_n}\|h_nv_n(t)\|\right)\right.\\
&\qquad\left.-M\left(t,\left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\|\right)\right]dt\\
\geq\ &\frac{k_n+h_n}{k_nh_n}\int_{H_n^1\cup H_n^2}\delta_0\left[\frac{h_n}{k_n+h_n}M(t,\|k_nu_n(t)\|)+\frac{k_n}{k_n+h_n}M(t,\|h_nv_n(t)\|)\right]dt\\
&+\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}M\left(t,\frac{h_n}{k_n+h_n}\|k_nu_n(t)\|+\frac{k_n}{k_n+h_n}\|h_nv_n(t)\|\right.\\
&\qquad\left.-\left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\|\right)dt\\
=\ &\frac{1}{k_nh_n}\int_{H_n^1\cup H_n^2}\delta_0[h_nM(t,\|k_nu_n(t)\|)+k_nM(t,\|h_nv_n(t)\|)]dt\\
&+\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}M\left(t,\frac{h_n}{k_n+h_n}\|k_nu_n(t)\|+\frac{k_n}{k_n+h_n}\|h_nv_n(t)\|\right.\\
&\qquad\left.-\left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\|\right)dt\\
\geq\ &\frac{\delta_0}{k_nh_n}\int_{H_n^1\cup H_n^2}\frac{3}{4}M(t,\|k_nu_n(t)\|)+\frac{3}{4}M(t,\|h_nv_n(t)\|)dt\\
&+\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}M\left(t,\frac{h_n}{k_n+h_n}\|k_nu_n(t)\|+\frac{k_n}{k_n+h_n}\|h_nv_n(t)\|\right.\\
&\qquad\left.-\left\|\frac{2k_nh_n}{k_n+h_n}u(t)\right\|\right)dt\\
\geq\ &\frac{\delta_0}{k_nh_n}\int_{H_n^1\cup H_n^2}\frac{3}{4}M(t,\|k_nu_n(t)\|)+\frac{3}{4}M(t,\|h_nv_n(t)\|)dt\\
&+\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}M\left(t,\frac{3}{4}\eta_0\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3\delta_0}{2k_nh_n}\int_{H_n^1\cup H_n^2}\frac{1}{2}M\left(t,\|k_nu_n(t)\|\right)+\frac{1}{2}M(t,\|h_nv_n(t)\|)dt \\
&\qquad +\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}M\left(t,\frac{3}{4}\eta_0\right)dt \\
&\geq \frac{3\delta_0}{2k_nh_n}\int_{H_n^1\cup H_n^2}M\left(t,\frac{1}{2}\|k_nu_n(t)\|+\frac{1}{2}\|h_nv_n(t)\|\right)dt \\
&\qquad +\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}M\left(t,\frac{3}{4}\eta_0\right)dt \\
&\geq \frac{3\delta_0}{2k_nh_n}\int_{H_n^1\cup H_n^2}M\left(t,\frac{\|k_nu_n(t)-h_nv_n(t)\|}{2}\right)dt+\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}M\left(t,\frac{3}{4}\eta_0\right)dt \\
&\geq \frac{3\delta_0}{2\overline{k}^2}\int_{H_n^1\cup H_n^2}M\left(t,\frac{\|k_nu_n(t)-h_nv_n(t)\|}{2}\right)dt+\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}M\left(t,\frac{3}{4}\eta_0\right)dt \\
&\geq \frac{3\delta_0}{2\overline{k}^2}\int_{H_n^1\cup H_n^2}M\left(t,\frac{\sigma_0}{2}\right)dt+\frac{k_n+h_n}{k_nh_n}\int_{H_n^3}M\left(t,\frac{3}{4}\eta_0\right)dt \\
&\geq \frac{3\delta_0}{2\overline{k}^2}\int_{H_n^1\cup H_n^2}M\left(t,\frac{\sigma_0}{2}\right)dt+\frac{3}{2\overline{k}^2}\int_{H_n^3}M\left(t,\frac{3}{4}\eta_0\right)dt \\
&\geq \frac{3\delta_0}{2\overline{k}^2}\int_{H_n^1\cup H_n^2}M(t,K)dt+\frac{3\delta_0}{2\overline{k}^2}\int_{H_n^3}M(t,K)dt \\
&= \frac{3\delta_0}{2\overline{k}^2}\int_{H_n}M(t,K)dt \\
&\geq \frac{3\delta_0}{2\overline{k}^2}\cdot a\cdot\frac{1}{4}\varepsilon_0,
\end{aligned}
$$

when n is large enough. However, $\|u_n\|^0+\|v_n\|^0-2\|u\|^0\to 0$ as $n\to\infty$, a contradiction. Hence, for any $\sigma>0$ and $\varepsilon>0$ there exists N such that

$$\mu\{t\in T:\|k_nu_n(t)-h_nv_n(t)\|\geq\sigma\}<\varepsilon,$$

whenever $n>N$. By the Riesz theorem, there exists a subsequence of $\{n\}$ denoted again by $\{n\}$, such that $k_nu_n(t)-h_nv_n(t)\to 0$ μ-a.e. on T. Moreover,

$$\frac{h_n}{k_n+h_n}k_nu_n(t)+\frac{h_n}{k_n+h_n}k_nv_n(t)=\frac{2k_nh_n}{k_n+h_n}u(t)\to lu(t)\quad\text{as } n\to\infty.$$

Hence we have $k_nu_n(t)\to lu(t)$ μ-a.e. on T as $n\to\infty$. Then $k\geq l$. In fact, by the Fatou Lemma, it follows that

$$\frac{1}{l}[1+\rho_M(lu)]=\|u\|^0=\lim_{n\to\infty}\|u_n\|^0=\lim_{n\to\infty}\frac{1}{k_n}[1+\rho_M(k_nu_n)]\geq\frac{1}{k}[1+\rho_M(lu)],$$

which implies the inequality $k\geq l$. Similarly, we have $h\geq l$. Then $k=l$. In fact, suppose that $k-l=r_1>0,\ h-l=r_2\geq 0$. Then

$$\frac{2kh}{k+h}=\frac{2(l+r_1)(l+r_2)}{2l+r_1+r_2}=\frac{2l^2+2(r_1+r_2)l+2r_1r_2}{2l+r_1+r_2}\geq\frac{2l+2(r_1+r_2)}{2l+r_1+r_2}l>l,$$

contradicting the equality $\frac{2kh}{k+h} = l$. By $\frac{1}{l}[1+\rho_M(lu)] = 1$ and $\frac{1}{k_n}[1+\rho_M(k_n u_n)] \to 1$ as $n \to \infty$, we have $\rho_M(lu) - l + 1 = 0$ and $\rho_M(k_n u_n) - k_n + 1 \to 0$ as $n \to \infty$. Hence, $\rho_M(k_n u_n) \to \rho_M(lu)$ as $n \to \infty$. By the convexity of M, we have

$$\frac{M(t, \|k_n u_n(t)\|) + M(t, \|lu(t)\|)}{2} - M\left(t, \frac{\|k_n u_n(t) - lu(t)\|}{2}\right) \geq 0$$

for μ-a.e. $t \in T$. Therefore, by the Fatou Lemma, we obtain that

$$\begin{aligned}
&\rho_M(ku) \\
&= \int_T \lim_{n\to\infty} \left[\frac{M(t, \|k_n u_n(t)\|) + M(t, \|lu(t)\|)}{2} - M\left(t, \frac{\|k_n u_n(t) - lu(t)\|}{2}\right)\right] dt \\
&= \liminf_{n\to\infty} \int_T \left[\frac{M(t, \|k_n u_n(t)\|) + M(t, \|lu(t)\|)}{2} - M\left(t, \frac{\|k_n u_n(t) - lu(t)\|}{2}\right)\right] dt \\
&= \rho_M(ku) - \limsup_{n\to\infty} \rho_M\left(\frac{1}{2}(k_n u_n - lu)\right),
\end{aligned}$$

which implies that $\rho_M(\frac{1}{2}(k_n u_n - lu)) \to 0$ as $n \to \infty$. By Lemma 2.6, we have that $\|k_n u_n - lu\| \to 0$ as $n \to \infty$. By Lemma 2.4, we have that $\|k_n u_n - lu\|^0 \to 0$ as $n \to \infty$. By $\lim_{n\to\infty} k_n = k = l$, we have the convergence $\|u_n - u\|^0 \to 0$ as $n \to \infty$. Thus, $\|u_n - v_n\|^0 \to 0$ as $n \to \infty$.

Case 2. Let $\max\{\sup\{k_n\}, \sup\{h_n\}\} = \infty$, where $k_n = K(u_n), h_n = K(v_n)$. We consider the sequence $u_n' = \frac{1}{2}(u_n + u)$ and $v_n' = \frac{1}{2}(v_n + u)$ in place of $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$, because $\|u_n - u\|^0 \to 0$ and $\|v_n - u\|^0 \to 0$ as $n \to \infty$ if and only if $\|u_n' - u\|^0 \to 0$ and $\|v_n' - u\|^0 \to 0$ as $n \to \infty$, respectively. Moreover,

$$\left\|\frac{1}{2}(u_n + u)\right\|^0 \leq \frac{1}{2}(\|u_n\|^0 + \|u\|^0)$$

for every $n \in N$. Hence,

$$\limsup_{n\to\infty} \left\|\frac{1}{2}(u_n + u)\right\|^0 \leq 1.$$

In the same way we can prove that $\limsup_{n\to\infty} \left\|\frac{1}{2}(v_n + u)\right\|^0 \leq 1$. Since $\frac{1}{2}(u_n + u) + \frac{1}{2}(v_n + u) = 2u$, for every $n \in N$, we conclude that $\liminf_{n\to\infty} \left\|\frac{1}{2}(u_n + u)\right\|^0 \geq 1$ and $\liminf_{n\to\infty} \left\|\frac{1}{2}(v_n + u)\right\|^0 \geq 1$. Consequently, $\lim_{n\to\infty} \left\|\frac{1}{2}(u_n + u)\right\|^0 \to 1$ and $\lim_{n\to\infty} \left\|\frac{1}{2}(v_n + u)\right\|^0 \to 1$ as $n \to \infty$. Define

$$w_n = \frac{2k_n l}{k_n + l} \quad \text{and} \quad r_n = \frac{2h_n l}{h_n + l}$$

for any $n \in N$, where $l \in K(u)$. Then the sequences $\{w_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are bounded. Moreover,

$$\begin{aligned}
\left\|\frac{1}{2}(u_n+u)\right\|^0 &\leq \frac{1}{w_n}\left(1+\rho_M\left(w_n\cdot\frac{u_n+u}{2}\right)\right) \\
&= \frac{k_n+l}{2k_nl}\left(1+\rho_M\left(\frac{k_nl}{k_n+l}(u_n+u)\right)\right) \\
&\leq \frac{k_n+l}{2k_nl}\left(1+\rho_M\left(\frac{l}{k_n+l}(k_nu_n)\right)+\rho_M\left(\frac{k_n}{k_n+l}(lu)\right)\right) \\
&\leq \frac{1}{2}\left(\frac{1}{k_n}(1+\rho_M(k_nu_n))+\frac{1}{l}(1+\rho_M(lu))\right) \\
&= \frac{1}{2}(\|u_n\|^0+\|u\|^0)\to 1 \quad \text{as } n\to\infty
\end{aligned}$$

whence it follow that

$$\frac{k_n+l}{2k_nl}\left(1+\rho_M\left(\frac{k_nl}{k_n+l}(u_n+u)\right)\right)\to 1 \quad \text{as } n\to\infty.$$

Analogously,

$$\frac{r_n+l}{2r_nl}\left(1+\rho_M\left(\frac{r_nl}{r_n+l}(u_n+u)\right)\right)\to 1 \quad \text{as } n\to\infty.$$

Therefore, we can prove in the same way as in Case 1 that $\left\|u_n'-u\right\|^0\to 0$ and $\left\|v_n'-u\right\|^0\to 0$ as $n\to\infty$. This completes the proof. □

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