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On the Monotone Polar and Representable Closures of Monotone Operators

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Fitzpatrick proved that maximal monotone operators in topological vector spaces are representable by lower semi-continuous convex functions. A monotone operator is representable if it can be represented by a lower semi-continuous convex function. The smallest representable extension of a monotone operator is its *representable closure*. The intersection of all maximal monotone extensions of a monotone operator, its *monotone polar closure*, is also representable. A natural question is whether these two closures coincide. In finite dimensional spaces they do coincide. The aim of this paper is to analyze such a question in the context of topological vector spaces. In particular, we prove in this context that if the convex hull of a monotone operator is *not* monotone, then the representable closure and the monotone polar closure of such operator do coincide.

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1. Introduction

Fitzpatrick, in his seminal work [4], proved that maximal monotone operators in topological vector spaces are representable by lower semi-continuous convex functions. This result has been used in the study of maximal monotone operators in Banach spaces, a topic of intense research nowadays. In [6] the notion of repre-

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sentable (monotone) operators was introduced and analyzed in finite dimensional spaces and Banach spaces. Such a notion was subsequently also studied in [9, 7].

A natural question is whether a particular monotone operator is representable, and if not, which is the best outer approximation of it by representable operators. The smallest representable extension of a monotone operator is its *representable closure*. Since the intersections of representable operators are representable, the intersection of all maximal monotone extensions of a monotone operator, its *monotone polar closure*, is also representable. A natural question is whether these two closures coincide. In finite dimensional spaces they do coincide [6] and in infinite dimensional Banach spaces, an example where these closures do not coincide was presented in [7]. The aim of this paper is to analyze such a question in the context of topological vector spaces. In particular, we prove in this context that if the convex hull of a monotone operator is *not* monotone, then the representable closure and the monotone polar closure of such operator do coincide.

2. The representable closure and the monotone polar closure

From now on, X is a Hausdorff locally convex real topological vector space with topology τ, X^* is its topological dual, and w^* is the weak-$*$ topology of X^*, that is the $\sigma(X^*, X)$-topology.

A *point-to-set* (or multivalued) operator $A : X \rightrightarrows X^*$ is a relation $A \subset X \times X^*$. Given $A : X \rightrightarrows X^*$ and $x \in X$, we denote

$$A(x) := \{x^* \in x^* \mid (x, x^*) \in A\},$$

and the *domain* and the *range* of A are, respectively,

$$\operatorname{dom}(A) := \{x \in X \mid A(x) \neq \emptyset\}, \qquad \operatorname{ran}(A) := \{x^* \in X^* \mid \exists x \in X,\ x^* \in A(x)\}.$$

An operator $A : X \rightrightarrows X^*$ is *affine* (*affine linear*) if A is a subspace (an affine subspace) of $X \times X^*$.

The *duality product* $\pi : X \times X^* \to \mathbb{R}$ is defined as

$$\pi(x, x^*) := \langle x, x^* \rangle = x^*(x), \quad \forall x \in X,\ x^* \in X^*.$$

A point-to-set operator $A : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in A,$$

and it is *maximal* monotone if it is not properly contained in any other monotone operator in $X \times X^*$. The ε*-enlargement* of a maximal monotone operator $A : X \rightrightarrows X^*$ is defined as [2]

$$A^\varepsilon(x) := \{x^* \in X^* \mid \langle x - y, x^* - y^* \rangle \geq -\varepsilon,\ \forall (y, y^*) \in A\}, \quad \varepsilon \geq 0.$$

Although the original definition was given in the setting of finite dimensional spaces, its extensions to Banach spaces [3] and to topological vector spaces are trivial. Trivially, $A \subset A^{\varepsilon}$ for any $\varepsilon \geq 0$. A maximal monotone operator A is *non-enlargeable* if this inclusion holds as an equality for all $\varepsilon \geq 0$.

Given $h : X \times X^* \to \overline{\mathbb{R}}$ convex and proper, such that

$$h(x, x^*) \geq \pi(x, x^*), \quad \forall (x, x^*) \in X \times X^*,$$

the set $\{(x, x^*) \in X \times X^* | h(x, x^*) = \pi(x, x^*)\}$ is a monotone operator [6, Theorem 5]. In this case, such operator is *represented* by h. We are interested in monotone operators which are represented by lower semi-continuous functions.

Definition 2.1. Given $A \subset X \times X^*$, and a topology $\mathcal{O}$ in $X \times X^*$, we say that A is *representable in the $\mathcal{O}$ topology* if there exists $h : X \times X^* \to \overline{\mathbb{R}}$ convex, proper and $\mathcal{O}$-lower semi-continuous such that

$$h \geq \pi, \quad A = \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \pi(x, x^*)\}.$$

In this case, we say that h *represents* A.

For instance, maximal monotone operators in a Hausdorff locally convex real topological vector space are $\tau \times w^*$ representable, as proved by Fitzpatrick [4]. Moreover, if $A : X \rightrightarrows X^*$ is a maximal monotone operator, then it is represented – in the $\tau \times w^*$ topology – by its *Fitzpatrick function*, $\varphi_A : X \times X^* \to \overline{\mathbb{R}}$, defined as

$$\begin{aligned}\varphi_A(x, x^*) &:= \sup_{(y,y^*) \in A} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \\ &= \sup_{(y,y^*) \in A} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle. \end{aligned} \tag{1}$$

Let $\mathcal{R}$ be the family of the $\tau \times w^*$ representable operators in $X \times X^*$,

$$\mathcal{R} := \{R : X \rightrightarrows X^* \mid R \text{ is } \tau \times w^* \text{ representable}\}.$$

The *representable closure* of a monotone $A : X \rightrightarrows X^*$ is defined as

$$\mathrm{cl}_R(A) := \bigcap_{\substack{A \subset R \\ R \in \mathcal{R}}} R \tag{2}$$

and it is the smallest (in the sense of graph inclusion) representable extension of A, because $A : X \rightrightarrows X^*$ is monotone if, and only if, it has a representable extension and the intersection of representable operators is still representable [6, Corollary 10]. These facts were proved in Banach spaces, but their extensions to topological linear spaces are trivial.

Monotonicity can also be analyzed using the classical notion of polarity [1], as follows. Let μ be the symmetric and reflexive relation

$$(x, x^*) \mu (y, y^*) \iff \langle x - y, x^* - x^* \rangle \geq 0, \quad (x, x^*), (y, y^*) \in X \times X^*. \tag{3}$$

The *monotone polar* or *μ-polar* of $A : X \rightrightarrows X^*$ is defined as

$$A^\mu = \{(x, x^*) \in X \times X^* \mid \langle x - y, x^* - y^* \rangle \geq 0, \forall (y, y^*) \in A\}$$

and the *monotone polar closure* or *μ-polar closure* of A is

$$A^{\mu\mu} = (A^\mu)^\mu .$$

In [6] it was proved that if A is monotone, then

$$A^\mu = \bigcup_{\substack{A \subset M \\ M \text{ max mon.}}} M$$

and

$$A^{\mu\mu} = \bigcap_{\substack{A \subset M \\ M \text{ max mon.}}} M.$$

In view of the above definitions and the fact that maximal monotone operators are representable, if A is monotone, then $A^{\mu\mu}$ is representable and

$$\mathrm{cl}_R(A) \subset A^{\mu\mu}.$$

A monotone operator A is *pre-maximal* if there is only one maximal monotone extension of it [6]. An example where this inclusion is strict was provided in [7]. Our aim is to prove the following theorem. In this work, ed stands for the effective domain of a convex function.

Theorem 2.2. *Assume that $X \times X^*$ is endowed with the $\tau \times w^*$ topology. Let $A : X \rightrightarrows X^*$ be a monotone operator. If $\mathrm{cl}_R(A) \subsetneq A^{\mu\mu}$ then A^μ is affine linear, maximal monotone, non-enlargeable and there exists $(x_0, x_0^*) \in \mathrm{ed}(\varphi_A)$ and $\varepsilon > 0$ such that $\langle x - x_0, x^* - x_0^* \rangle \geq \varepsilon$, for all $(x, x^*) \in A$. In particular, A is pre-maximal monotone.*

Now we discuss some consequences of this theorem.

For those monotone operators that do not admit an *affine linear* maximal monotone extension, the representable closure and the monotone polar closure are identical:

Corollary 2.3. *If $A : X \rightrightarrows X^*$ is monotone, $X \times X^*$ is endowed with the $\tau \times w^*$ topology and the convex hull of A is* not *monotone then*

$$\mathrm{cl}_R(A) = A^{\mu\mu}.$$

Proof. If $\mathrm{cl}_R(A) \neq A^{\mu\mu}$ then A^μ is affine linear and maximal monotone. Since $A \subset A^\mu$, the convex hull of A is a subset of A^μ. Hence this convex hull must be monotone, otherwise $\mathrm{cl}_R(A) = A^{\mu\mu}$. $\square$

In the particular case where X is a Banach space and τ is the norm-topology, we have the following corollary.

Corollary 2.4. *Let X be a real Banach space and let $A \subset X \times X^*$ be a monotone operator. If $X \times X^*$ is endowed with the strong$\times$weak-$*$ topology, then one of the following cases holds:*

- $\mathrm{cl}_R(A) = A^{\mu\mu}$.
- *A^μ is affine linear, maximal monotone, non-enlargeable and there exists $(x_0, x_0^*) \in \mathrm{ed}(\varphi_A)$ and $\varepsilon > 0$ such that $\langle x - x_0, x^* - x_0^* \rangle \geq \varepsilon$, for all $(x, x^*) \in A$. In particular, A is pre-maximal monotone.*

Remark 2.5. If X is a reflexive Banach space, then $X \times X^*$ is also reflexive and, for proper convex functions, lower semi-continuity in the strong, weak, weak-$*$ and hence strong$\times$weak-$*$ topologies are equivalent. Therefore, in this case, Corollary 2.4 still holds if we replace the strong$\times$weak-$*$ topology by the strong topology of $X \times X^*$.

3. Additional properties of the closures

In the sequel we will need some results which were previously proved in [6] in the context of Banach spaces. Since the extension of these results to topological vector spaces is straightforward, we will omit the proofs.

Proposition 3.1.

1. *A is representable if, and only if, $\mathcal{S}_A = \mathrm{cl}\,\mathrm{conv}_{\tau \times w^*}(\pi + \delta_A)$ represents A.*
2. *If A is monotone then $\mathrm{cl}_R(A) = \{\mathcal{S}_A \leq \pi\}$.*
3. *Let A be monotone, $(x, x^*) \in X \times X^*$ and define*

$$\tau_{(x_0, x_0^*)}(A) := A - \{(x_0, x_0^*)\} = \{(x - x_0, x^* - x_0^*) \mid (x, x^*) \in A\}. \quad (4)$$

Then $\mathrm{cl}_R\left(\tau_{(x_0,x_0^*)}(A)\right) = \tau_{(x_0,x_0^*)}\left(\mathrm{cl}_R(A)\right)$.

The following proposition subsumes some properties of the monotone polar.

Proposition 3.2.

1. *If $A \subset B \subset X \times X^*$ then $B^\mu \subset A^\mu$.*
2. *A is monotone if, and only if, $A \subset A^\mu$.*
3. *A is maximal monotone if, and only if, $A = A^\mu$.*
4. *$A^\mu = \{\varphi_A \leq \pi\}$.*
5. *Denote $A^{\mu\mu} = (A^\mu)^\mu$. If A is monotone then*

$$A^\mu = \bigcup_{\substack{A \subset M \\ M \text{ max mon.}}} M \quad (5)$$

and

$$A^{\mu\mu} = \bigcap_{\substack{A \subset M \\ M \text{ max mon.}}} M \,. \quad (6)$$

6. *If A is monotone then $A \subset A^{\mu\mu} \subset A^\mu \subset \mathrm{ed}(\varphi_A)$.*

7. *Let A be monotone, $(x, x^*) \in X \times X^*$ and define $\tau_{(x_0,x_0^*)}(A)$ as in* (4). *Then*

$$\big(\tau_{(x_0,x_0^*)}(A)\big)^{\mu} = \tau_{(x_0,x_0^*)}\big(A^{\mu}\big).$$

Thus, if $A : X \rightrightarrows X^*$ is monotone, we define the *monotone closure* of A as $A^{\mu\mu}$. We refer the reader to [6, §3 and §4] for further properties of the representable and monotone closure of a monotone operator.

We are interested in the relationship between $\mathrm{cl}_R(A)$ and $A^{\mu\mu}$. From equations (2) and (6) we have

$$\mathrm{cl}_R(A) \subset A^{\mu\mu}. \tag{7}$$

In this work, we will simplify the proof of Theorem 31 in [6] as well as study the case when the monotone and representable closures of an operator don't coincide.

4. Auxiliary results and proof of Theorem 2.2

Define

$$N := \{(x, x^*) \in X \times X^* \mid \langle x, x^* \rangle < 0\}. \tag{8}$$

Observe that for any $A \subset X \times X^*$, $(0, 0) \notin A^{\mu\mu}$ if, and only if, $N \cap A^{\mu} \neq \emptyset$. The following proposition gives sufficient conditions for $(0, 0) \notin A^{\mu\mu}$, and it was proved in the context of Banach spaces in [6]. For the sake of completeness we provide its proof in the context of topological vector spaces.

Proposition 4.1 ([6, Propositions 26, 28 and 30]). *Let A be monotone. Then any of the following conditions are sufficient for $(0, 0) \notin A^{\mu\mu}$.*

1. $\varphi_A(0, 0) < 0$ *and* $N \cap \mathrm{ed}(\varphi_A) \neq \emptyset$.
2. $\varphi_A(0, 0) = 0$ *and there exists* $(x, x^*) \in N$ *such that* $\varphi_A(x, x^*) < 0$.
3. $\varphi_A(0, 0) > 0$.

Proof. Suppose that item *1.* holds. Take $(y, y^*) \in N \cap \mathrm{ed}(\varphi_A)$ and define

$$(y_t, y_t^*) := (1 - t)(0, 0) + t(y, y^*) = t(y, y^*), \quad t \in [0, 1].$$

Using the convexity of φ_A and the assumptions $\varphi_A(0, 0) < 0$, $\varphi_A(y, y^*) < \infty$ we conclude that there exists $t' \in (0, 1)$, small enough, such that

$$\varphi_A(y_{t'}, y_{t'}^*) \leq (1 - t')\varphi_A(0, 0) + t'\varphi_A(y, y^*) < (t')^2\langle y, y^* \rangle < 0$$

where the first inequality follows from the convexity of φ_A and the last inequality from the inclusion $(y, y^*) \in N$. Since $(t')^2\langle y, y^* \rangle = \langle y_{t'}, y_{t'}^* \rangle$, we conclude that

$$\varphi_A(y_{t'}, y_{t'}^*) < \langle y_{t'}, y_{t'}^* \rangle < 0.$$

Using item *4.* of Proposition 3.2 we conclude that $(y_{t'}, y_{t'}^*) \in A^{\mu}$, and the last inequality in the above equation implies that $(0, 0) \notin A^{\mu\mu}$.

Item *2.* is proved by the same reasoning. Indeed, if this item holds, take $(y, y^*) \in N$ such that $\varphi_A(y, y^*) < 0$ and define again $(y_t, y_t^*) := t(y, y^*)$ for $t \in [0, 1]$. Since $\varphi_A(0, 0) = 0$,

$$\varphi_A(y_t, y_t^*) \leq t\varphi_A(y, y^*), \quad t \in [0, 1].$$

Since $\varphi_A(y, y^*) < 0$, there exists $t' \in (0, 1)$ small enough, such that

$$t'\varphi_A(y, y^*) < (t')^2\langle y, y^*\rangle < 0$$

and the conclusion follows combining the above inequalities with the previous one at $t = t'$.

If item *3.* holds, direct use of Definition (1) shows that $0 \notin A^\mu$. Since A is monotone, $A \subset A^\mu$, $A^{\mu\mu} \subset A^\mu$ and the conclusion follows. □

The next lemma generalizes to topological linear spaces a result proved in reflexive Banach spaces in [6, Proposition 29].

Lemma 4.2. *Let A be monotone. If $\varphi_A(0, 0) = 0$ and $\mathcal{S}_A(0, 0) > 0$ then $(0, 0) \notin A^{\mu\mu}$.*

Proof. The assumption $\varphi_A(0, 0) = 0$ guarantees that A is non-empty and φ_A is a proper closed convex function. This, together with the monotonicity of A also implies that $\mathcal{S}_A$ is a proper closed convex function.

Since $\varphi_A(x, x^*) = (\pi + \delta_A)^*(x^*, x)$ for all $(x, x^*) \in X \times X^*$, by Fenchel-Moreau Theorem for locally convex spaces,

$$\mathcal{S}_A(x, x^*) = (\pi+\delta_A)^{**}(x, x^*) = (\varphi_A)^*(x^*, x) = \sup_{(y,y^*)} \langle y, x^*\rangle + \langle x, y^*\rangle - \varphi(y, y^*). \quad (9)$$

In particular, $\sup_{(y,y^*)} -\varphi(y^*, y) = \mathcal{S}_A(0, 0) > 0$ and there exists $(y, y^*) \in X \times X^*$ such that

$$\varphi_A(y, y^*) < 0. \qquad (10)$$

The above inequality trivially implies that there exists $t' \in (0, 1)$, small enough, such that

$$t'\big(t'\langle y, y^*\rangle + (1 - t')\varphi_A(y, y^*)\big) = t'^2\langle y, y^*\rangle + t'(1 - t')\varphi_A(y, y^*) < 0.$$

Since $\sup_{(x,x^*)\in A} -\langle x, x^*\rangle = \varphi_A(0, 0) = 0$ and $t', 1 - t' \in (0, 1)$, there exists $(z, z^*) \in A$ such that

$$\begin{aligned} t'^2\langle y, y^*\rangle + t'(1 - t')\varphi_A(y, y^*) + (1 - t')\langle z, z^*\rangle &< 0, \\ t'\varphi_A(y, y^*) + (1 - t')\langle z, z^*\rangle &< 0 \end{aligned}$$

where we used again (10) for the second inequality. Define

$$(z_0, z_0^*) = t'(y, y^*) + (1 - t')(z, z^*).$$

Then

$$\begin{aligned}\langle z_0, z_0^*\rangle &= t'^2\langle y, y^*\rangle + t'(1-t')(\langle z, y^*\rangle + \langle y, z^*\rangle) + (1-t')^2\langle z, z^*\rangle\\ &= t'^2\langle y, y^*\rangle + t'(1-t')\big(\langle z, y^*\rangle + \langle y, z^*\rangle - \langle z, z^*\rangle\big) + (1-t')\langle z, z^*\rangle\\ &\le t'^2\langle y, y^*\rangle + (1-t')t'\varphi_A(y, y^*) + (1-t')\langle z, z^*\rangle,\end{aligned}$$

where the last inequality follows from the inclusion $(z, z^*) \in A$ and (1). Using the convexity of φ_A and again the inclusion $(z, z^*) \in A$ we have

$$\begin{aligned}\varphi_A(z_0, z_0^*) &\le t'\varphi_A(y, y^*) + (1-t')\varphi_A(z, z^*)\\ &= t'\varphi_A(y, y^*) + (1-t')\langle z, z^*\rangle.\end{aligned}$$

Combining the three above equations we conclude that $(z_0, z_0^*) \in N$ and $\varphi_A(z_0, z_0^*) < 0$, which, combined with Proposition 4.1, item *2.*, ends the proof. □

Remark 4.3. Notice that the last equality in (9) uses the fact that X^* is endowed with w^* topology. This is the reason why in Theorem 2.2 and Corollary 2.4 the space $X \times X^*$ is endowed with the $\tau \times w^*$ and the strong×weak-$*$ topologies, respectively.

Lemma 4.4. *Let A be monotone. If $\mathrm{ed}(\varphi_A) \cap N = \emptyset$ then $A^\mu = \mathrm{ed}(\varphi_A)$ and $\mathrm{ed}(\varphi_A)$ is maximal monotone.*

Proof. Since $\mathrm{ed}(\varphi_A)$ is convex,

$$\left(\frac{x+y}{2}, \frac{x^*+y^*}{2}\right) \in \mathrm{ed}(\varphi_A), \quad \forall (x, x^*), (y, y^*) \in \mathrm{ed}(\varphi_A).$$

Hence

$$0 \le 4\left\langle \frac{x+y}{2}, \frac{x^*+y^*}{2}\right\rangle = \langle x+y, x^*+y^*\rangle, \quad \forall (x, x^*), (y, y^*) \in \mathrm{ed}(\varphi_A). \tag{11}$$

Fix $(x, x^*) \in \mathrm{ed}(\varphi_A)$, then, by (11),

$$\langle y - (-x), y^* - (-x^*)\rangle = \langle x+y, x^*+y^*\rangle \ge 0, \quad \forall (y, y^*) \in A \subset \mathrm{ed}(\varphi_A).$$

Thus, $(-x, -x^*) \in A^\mu \subset \mathrm{ed}(\varphi_A)$ and, again by (11),

$$\langle x - y, x^* - y^*\rangle = \langle (-x) + y, (-x^*) + y^*\rangle \ge 0, \quad \forall (y, y^*) \in \mathrm{ed}(\varphi_A),$$

which implies that $(x, x^*) \in \mathrm{ed}(\varphi_A)^\mu$. Therefore,

$$\mathrm{ed}(\varphi_A) \subset \mathrm{ed}(\varphi_A)^\mu,$$

proving that $\mathrm{ed}(\varphi_A)$ is monotone. Since A is monotone, $A \subset \mathrm{ed}(\varphi_A)$. Therefore, $\mathrm{ed}(\varphi_A)^\mu \subset A^\mu \subset \mathrm{ed}(\varphi_A)$, which, combined with the above inclusion, shows that $A^\mu = \mathrm{ed}(\varphi_A)$ and $\mathrm{ed}(\varphi_A)$ is maximal monotone. □

Lemma 4.5. *Let A be monotone. If $\varphi_A(0,0) < 0$ and $(0,0) \in A^{\mu\mu}$ then $A^\mu = \mathrm{ed}(\varphi_A)$ and $\mathrm{ed}(\varphi_A)$ is maximal monotone.*

Proof. First use Proposition 4.1, item *1.*, to conclude that under these assumptions, $N \cap \mathrm{ed}(\varphi_A) = \emptyset$. To end the proof, combine this equality with Lemma 4.4. □

Proof of Theorem 2.2. Suppose $\mathrm{cl}_R(A) \subsetneq A^{\mu\mu}$, that is, $A^{\mu\mu} \setminus \mathrm{cl}_R(A) \neq \emptyset$. Since $\mathrm{cl}_R(A)$ and $A^{\mu\mu}$ are preserved by translations, without loss of generality we can assume that $(0,0) \in A^{\mu\mu} \setminus \mathrm{cl}_R(A)$, that is,

$$(0,0) \in A^{\mu\mu}, \qquad (0,0) \notin \mathrm{cl}_R(A). \tag{12}$$

By Proposition 3.1, item *2.*, the second relation in the above equation is equivalent to $\mathcal{S}_A(0,0) > 0$. Combining this inequality with the first relation in (12) and Lemma 4.2 we conclude that $\varphi_A(0,0) \neq 0$. Combining the first relation in (12) with Proposition 4.1, item *3.*, we conclude that

$$\varphi_A(0,0) < 0.$$

This inequality, the first relation in (12) and Lemma 4.5 now imply that $A^\mu = \mathrm{ed}(\varphi_A)$ is maximal monotone. Moreover, A is pre-maximal monotone, A^μ is the unique maximal monotone extension of A and $A^{\mu\mu} = A^\mu = \mathrm{ed}(\varphi_A)$. Moreover, A^μ is affine linear by [5, Lemma 1.2].

Now we prove that A^μ is non-enlargeable. Since $A \subset A^\mu$, we have $\varphi_A \leq \varphi_{A^\mu}$. This means that $\mathrm{ed}(\varphi_{A^\mu}) \subset \mathrm{ed}(\varphi_A) = A^\mu$. Hence $A^\mu = \mathrm{ed}(\varphi_{A^\mu})$ which, in view of definition (1) is equivalent to A^μ being non-enlargeable.

Finally, if $(x_0, x_0^*) \in A^{\mu\mu} \setminus \mathrm{cl}_R(A)$, then $(0,0) \in (\tau_{(x_0,x_0^*)}(A))^{\mu\mu} \setminus \mathrm{cl}_R(\tau_{(x_0,x_0^*)}(A))$. Therefore $\varphi_{\tau_{(x_0,x_0^*)}(A)}(0,0) < 0$, which implies

$$\langle x - x_0, x^* - x_0^* \rangle \geq -\varphi_{\tau_{(x_0,x_0^*)}(A)}(0,0) > 0. \qquad \square$$

The general setting in which we wrote Theorem 2.2 allows us to have many important cases as a particular case. We begin giving a new proof of Theorem 31 in [6].

Theorem 4.6. *Let X be finite dimensional and $A \subset X \times X^*$ be monotone. Then $\mathrm{cl}_R(A) = A^{\mu\mu}$.*

Proof. If $A = \emptyset$ the theorem holds trivially. Assume that $A \neq \emptyset$. Since $\mathrm{cl}_R(A)$ and $A^{\mu\mu}$ are preserved by translations, without loss of generality we can assume that $(0,0) \in A$.

By Theorem 2.2, if $\mathrm{cl}_R(A) \neq A^{\mu\mu}$ then A^μ is affine linear, maximal monotone, non-enlargeable and there exist $(x_0, x_0^*) \in \mathrm{ed}(\varphi_A)$ and $\varepsilon > 0$ such that $\langle a - x_0, a^* - x_0^* \rangle \geq \varepsilon$, for all $(a, a^*) \in A$. The latter means that there exists (x_0, x_0^*) such that $\varphi_A(x_0, x_0^*) < \langle x_0, x_0^* \rangle$.

Let $T = A^{\mu}$ and define

$$T^{\vdash} = \{(x, x^*) \in X \times X^* \mid \langle x, y^* \rangle + \langle y, x^* \rangle = 0, \forall (y, y^*) \in T\}.$$

By Lemma 2.1 in [8], $T^{\vdash} \subset \{(x, x^*) \mid \varphi_T(x, x^*) = 0\} \subset \operatorname{ed}(\varphi_T) = T$. We now prove the opposite inclusions. Since X is finite dimensional, $\dim T^{\vdash} = 2n - \dim T$. Furthermore, since T is maximal monotone, $\dim T = n$ by Minty's Theorem. Hence $\dim T^{\vdash} = 2n - n = n = \dim T$ and

$$T^{\vdash} = T.$$

But this implies $T \subset \{(x, x^*) \in X \times X^* \mid \langle x, x^* \rangle = 0\}$; thus, since $(x_0, x_0^*) \in \operatorname{ed}(\varphi_A) = T$ and $(0, 0) \in A$,

$$0 = \langle x_0, x_0^* \rangle \geq \varepsilon,$$

clearly a contradiction. □

5. On non-Hausdorff spaces

If Z is an arbitrary topological vector space (may be non-Hausdorff) then,

1. $\operatorname{cl}(\{0\})$ is a closed subspace;
2. for any $A \subset Z$ open/closed, $A = A + \operatorname{cl}(\{0\})$;
3. $Z/\operatorname{cl}(\{0\})$, endowed with the canonical quotient topology, is a Hausdorff topological vector space;
4. if Z is locally convex, so is $Z/\operatorname{cl}(\{0\})$;
5. $A \subset Z$ is open (closed) if and only if
$$A = P^{-1}(P(A))$$
and $P(A)$ is open (closed) in $Z/\operatorname{cl}(\{0\})$, where P is the canonical projection of Z onto $Z/\operatorname{cl}(\{0\})$;
6. P^* is an isomorphism of $(Z/\operatorname{cl}(\{0\}))^*$ onto Z^*.

In view of these well-known and canonical results, Theorem 2.2 and Corollary 2.3 hold even in non-Hausdorff locally convex topological vector spaces. The proofs of these facts are left to the reader.

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