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Calmness of Nonsmooth Constraint Systems: Dual Conditions via Scalarized Exhausters

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This paper is dedicated to Vladimir Fëdorovich Demyanov,
as a small token of my respect, esteem and affection.

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This paper considers the problem of establishing sufficient conditions for calmness, a property describing the Lipschitzian behaviour of set-valued mappings, whose introduction was strongly motivated by needs in optimization and mathematical programming. The study of such problem is undertaken in the specific context of nonsmooth constraint systems. As analysis tools, proper adaptations of known dual constructions in generalized differentiation theory and abstract convex analysis are employed. By means of them, some settings are singled out, where it is possible to formulate calmness conditions, in the case the set appearing in the constraint system is convex or prox-regular.

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1. Introduction

Among the notable achievements of modern variational analysis it is certainly to be mentioned the recognition of the crucial role played by the Lipschitzian properties within such and related areas. Indeed, Lipschitzian properties are the very subject of the ninth chapter of [30], where one reads:

> But the Lipschitzian properties haven't only been important as favorable *assumptions*; they have also been sought as favorable *conclusions* in quantifying the variations of optimal values, optimal solutions, and the behavior of solution elements more generally

(see the commentary concluding that chapter). The forth chapter of [24] and the third chapter of [16] are devoted to the same topic, whereas, after the appearance of the seminal papers [1, 29], a great amount of shorter works can be counted,

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which are focussed on such theme[1]. Of course, some strictly intertwined properties like metric regularity, metric subregularity, openness at a linear rate were formulated even earlier, but, as recognized in [21], it was the definition proposed in [1]

> that led to the development of a new chapter of non-smooth analysis devoted to the study of Lipschitz properties of set-valued maps.

According to the common understanding, under the label of Lipschitzian properties fall actually various quantitative formalizations of the continuity behaviour of general set-valued mappings (e.g. Aubin property, upper Lipschitz continuity in the sense of Robinson, Lipschitz lower semicontinuity, sub-Lipschitzian continuity, Lipschitz continuity with respect to Pompeiu-Hausdorff distance, and so on). All of them have found strong motivations in optimization and mathematical programming, of which modern variational analysis can be viewed as an outgrowth, with particular reference to the issue of stability/sensitivity analysis with respect to perturbations. This is true also for the concept of calmness, which has been developed by several researchers working in different contexts, where some kind of Lipschitzian behaviour was required (see, for instance, [5, 8, 16, 22, 30, 35]). A set-valued mapping $\Phi : P \longrightarrow 2^X$ between metric spaces (P, d) and (X, d) is said to be *calm* at $(\bar{p}, \bar{x}) \in \operatorname{gph} \Phi = \{(p, x) \in P \times X : \ x \in \Phi(p)\}$ if there exist positive reals δ, ζ and $\ell > 0$ such that

$$\Phi(p) \cap \mathrm{B}(\bar{x}, \delta) \subseteq \mathrm{B}(\Phi(\bar{p}), \ell d(p, \bar{p})), \quad \forall p \in \mathrm{B}(\bar{p}, \zeta), \tag{1}$$

where $\mathrm{B}(x, r)$ denotes the closed ball centered at $x \in X$ with radius $r \geq 0$, and, given $S \subseteq X$, $\mathrm{B}(S, r) = \{x \in X : \ \operatorname{dist}(x, S) = d_S(x) = \inf_{v \in S} d(v, x) \leq r\}$ denotes the r-enlargement of S. Any such constant as ℓ will be called *calmness constant* for Φ at $(\bar{p}, \bar{x})$. The nonnegative value

$$\operatorname{clm} \Phi(\bar{p}, \bar{x}) = \inf\{\ell > 0 : \ \exists \ \delta, \zeta > 0 \text{ for which (1) is valid}\}$$

is called *calmness modulus* of Φ at $(\bar{p}, \bar{x})$. From inclusion (1) one sees that calmness is a property relaxing at the same time the requirements of Aubin continuity and those of upper Lipschitz continuity. As such, it revealed to be a convenient conceptual tool in the analysis of optimization problems.

The aim of the present paper is to bring some contributions to the study of calmness conditions for constraint systems, i.e. for feasible regions of extremum problems, whose constraints depend on a parameter. A wide variety of functional and geometrical types of constraints appearing in optimization problems can be put in the abstract form

$$(\mathcal{CS}) \qquad f(p, x) \in C$$

by a proper choice of the mapping $f : P \times X \longrightarrow Y$ and of the closed subset $C \subset Y$. A special but meaningful case are, for example, equality/inequality

[1] For the sake of completeness, one should add that certain Lipschitzian properties, in connection with variational problems, were considered ante litteram by some authors much ahead of their times, such as L. A. Lyusternik and L. M. Graves.

constraints in a standard mathematical programming problem, namely systems of the form

$$\begin{cases} f_i(p,x) \leq 0, & i = 1,\dots,m_1, \\ f_i(p,x) = 0, & i = m_1+1,\dots,m_1+m_2, \end{cases}$$

where $f : P \times X \longrightarrow \mathbb{R}^{m_1+m_2}$, $C = \mathbb{R}^{m_1}_{-} \times \{\mathbf{0}\}$, and m_1, $m_2 \in \mathbb{N}$, with $\mathbb{R}$ and $\mathbb{N}$ denoting the real and the positive integer number sets, respectively, whereas $\mathbb{R}^{m_1}_{-}$ stands for the nonpositive orthant of $\mathbb{R}^{m_1}$ and $\mathbf{0}$ is the null vector of $\mathbb{R}^{m_2}$. As one experiences, the analytical expression of the set-valued solution mapping $S : P \longrightarrow 2^X$ associated with $(\mathcal{CS})$, i.e.

$$S(p) = \{x \in X :\ f(p,x) \in C\} = f(p,\cdot)^{-1}(C),$$

can be hardly derived explicitly in practice. Therefore a key issue in the constraint system analysis is to establish verifiable conditions on the problem data f and C, which are able to ensure the implicit mapping S to be calm at reference values. In turn, calmness of constraint systems is known to propitiate such desirable phenomena as local error bounds, constraint qualifications, the viability of penalty function methods. For a complete discussion of the role played by calmness in constraint system analysis the reader is referred to [19, 20, 22].

Not surprisingly, the study of calmness conditions for constraint systems has been undertaken by many authors in various settings. The present paper focusses on constraint systems of the form $(\mathcal{CS})$, which are defined by (possibly) nonsmooth mappings f. On motivations justifying the interest in considering such class of constraint systems much has been written and said. The author's opinion on the issue has been formed after reading works such as [4, 9, 14, 22, 24, 25, 28, 30, 32], that distil the attitude of several schools of thought during almost half century. To such references something better can be hardly added (at least, by the author's side). Instead, some word is to be spent in the attempt of explaining the approach proposed in the current paper. Roughly speaking, it results from the combination of variational techniques, employing recently refined tools of analysis in metric spaces, with certain achievements concerning the dual representations of positively homogeneous functions, which have been developed upon the effect of abstract convexity suggestions. More precisely, the analysis of sufficient conditions of calmness is carried out according to the following scheme:

Step 1: formulation of a sufficient condition for the calmness of the solution mapping S, associated with $(\mathcal{CS})$, in terms of a derivative-like object, by using a variational technique in metric spaces;

Step 2: estimation of the employed derivative-like object in terms of Hadamard directional derivatives, in the case of scalar functions defined in normed spaces;

Step 3: introduction of Hadamard directional derivatives of mappings via a scalarization approach and study of their dual representation in "inf max" form;

Step 4: estimation of Hadamard directional derivatives of certain composite functions, by exploiting geometrical/variational properties of C (i.e. convexity/prox-regularity).

The above scheme of analysis leads to single-out the class of locally uniformly convex Banach spaces as a proper setting, where verifiable sufficient calmness conditions in the case of nonsmooth constraint systems can be formulated.

The material contained in the rest of the paper is arranged as follows. In Section 2 the mentioned sufficient condition for calmness of parameterized constraint systems is established in a purely metric setting. Such condition is formulated in terms of nondegeneracy of a certain derivative-like object (the strict outer slope) of a function involving the problem data, which can work in the absence of any linear structure. The investigations stemming from that result can be regarded as an attempt to single out in more structured settings tools and notions able to translate the nondegeneracy into verifiable requirements. To this aim, in Section 3 known elements of nonsmooth analysis needed in the sequel are recalled; the class of scalarly exhaustible mappings is introduced and several related examples are furnished; some technical lemmata providing estimates of exhausters of composite functions are proved. Section 4 is reserved for presenting the main calmness conditions and for their discussion. In doing so, two cases will be considered, according to the assumption made on the subset C: the prox-regular case in Hilbert spaces and the convex case in general Banach spaces. Both of them appear to be relevant for applications.

2. A calmness condition in metric spaces

Throughout this section, (P, d), (X, d) and (Y, d) will denote metric spaces. In order to formulate a calmness condition for parameterized constraint systems in such setting, one needs the following elements. Given a (nonempty) closed set $C \subset Y$ and a positive real δ, a function $\psi_{C,\delta} : Y \longrightarrow [0, +\infty)$ having the properties:

(1) $\psi_{C,\delta}(y) \le \operatorname{dist}(y, C)$, for every $y \in \mathrm{B}(C, \delta)$;
(2) $\psi_{C,\delta}(y) = 0$ implies $y \in C$;
(3) $\psi_{C,\delta}$ is continuous on Y,

will be called *subdistance functional for C*. Unlike the obvious subdistance functional d_C, other subdistance functionals may not be necessarily Lipschitz continuous.

Another element to be considered concerns the specialization of the notion of calmness to single-valued mappings defined on the Cartesian product of metric spaces. More precisely, what is needed is a calmness property with respect to the parameter variable, which is uniform in the state variable. Consequently, a single-valued mapping $f : P \times X \longrightarrow Y$ between metric spaces is said to be *partially calm* at $(\bar{p}, \bar{x}) \in P \times X$ *with respect to p, uniformly in x*, if if there exist ζ, $\ell > 0$ such that

$$d(f(p, x), f(\bar{p}, x)) \le \ell d(p, \bar{p}), \quad \forall x \in \mathrm{B}(\bar{x}, \zeta),\ \forall p \in \mathrm{B}(\bar{p}, \zeta). \tag{2}$$

In such case, the nonnegative value

$$\operatorname{clm}_p f(\bar{p}, \bar{x}) = \inf\{\ell > 0 :\ \exists\, \zeta > 0 \text{ for which (2) is valid}\}$$

is called *modulus of partial calmness* of f at $(\bar{p}, \bar{x})$.

In order to express the nondegeneracy condition, on which the calmness result relies, a recently refined derivative-like object is required, which enables to conduct a local variational analysis in the abstract setting of metric spaces. Such tool can be presented as a regularization of the well-known notion of strong slope (see [10]). Given a function $\varphi : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ and $\bar{x} \in \operatorname{dom} \varphi = \{x \in X : \varphi(x) \in \mathbb{R}\}$, the real-extended value

$$|\nabla\varphi|(\bar{x}) = \begin{cases} 0, & \text{if } \bar{x} \text{ is a local minimizer for } \varphi, \\ \limsup\limits_{x \to \bar{x}} \dfrac{\varphi(\bar{x}) - \varphi(x)}{d(x, \bar{x})}, & \text{otherwise,} \end{cases}$$

is called (*strong*) *slope* of φ at $\bar{x}$. The real-extended value

$$\overline{|\nabla\varphi|}^{>}(\bar{x}) = \lim_{\epsilon \to 0^+} \inf\{|\nabla\varphi|(x) : x \in \mathrm{B}(\bar{x}, \epsilon),\ \varphi(\bar{x}) < \varphi(x) \leq \varphi(\bar{x}) + \epsilon\}$$

is called *strict outer slope* of φ at $\bar{x}$. To the best of the author's knowledge, the notion of strict outer slope was introduced in [17], among other concepts of slopes, and it was already employed in the analysis of calmness in [7]. After having recalled such notion, one is in a position to formulate the mentioned condition for the calmness of the solution mapping to parameterized constraint systems in metric spaces.

Theorem 2.1. *Let a mapping $f : P \times X \longrightarrow Y$ and a closed set $C \subset Y$ be given, and let $\bar{p} \in P$ and $\bar{x} \in X$ be such that $\bar{x} \in S(\bar{p})$. Suppose that:*

(h_0) *(X, d) is metrically complete;*
(h_1) *there exists $\delta_1 > 0$ such that ψ_{C,δ_1} is a subdistance functional for C;*
(h_2) *there exists $\delta_2 > 0$ such that $f(\bar{p}, \cdot)$ is continuous on $\mathrm{B}(\bar{x}, \delta_2)$;*
(h_3) *mapping f is partially calm at $(\bar{p}, \bar{x})$ with respect to p, uniformly in x, with modulus $\operatorname{clm}_p f(\bar{p}, \bar{x})$;*
(h_4) *it holds*

$$\overline{|\nabla(\psi_{C,\delta_1} \circ f(\bar{p}, \cdot))|}^{>}(\bar{x}) > 0.$$

Then, the solution mapping S to the related constraint system $(\mathcal{CS})$ is calm at $(\bar{p}, \bar{x})$ and the following estimate holds

$$\operatorname{clm} S(\bar{p}, \bar{x}) \leq \frac{\operatorname{clm}_p f(\bar{p}, \bar{x})}{\overline{|\nabla(\psi_{C,\delta_1} \circ f(\bar{p}, \cdot))|}^{>}(\bar{x})}.$$

Proof. According to the definition of calmness and related modulus, both the assertions in the thesis will be proved if it is shown that for every ρ, $\ell \in \mathbb{R}$, with

$$0 < \rho < \overline{|\nabla(\psi_{C,\delta_1} \circ f(\bar{p}, \cdot))|}^{>}(\bar{x}) \quad \text{and} \quad \operatorname{clm}_p f(\bar{p}, \bar{x}) < \ell,$$

there exists $\zeta > 0$ such that

$$S(p) \cap \mathrm{B}(\bar{x}, \zeta) \subseteq \mathrm{B}\left(S(\bar{p}), \frac{\ell}{\rho}\right), \quad \forall p \in \mathrm{B}(\bar{p}, \zeta). \tag{3}$$

To this aim, note that, by virtue of hypothesis (h_3), corresponding to ℓ there exists $\delta_3 > 0$ such that

$$d(f(p,x), f(\bar{p},x)) \le \ell d(p,\bar{p}), \quad \forall p \in \mathrm{B}(\bar{p},\delta_3),\ \forall x \in \mathrm{B}(\bar{x},\delta_3).$$

Define $\delta = \min\{\delta_1,\, \delta_2,\, \delta_3\}$, where δ_1 and δ_2 are as postulated in hypothesis (h_1) and (h_2), respectively. Again, by hypothesis (h_4), there exists $\delta_* \in (0,\delta)$ such that

$$|\nabla(\psi_{C,\delta_1} \circ f(\bar{p},\cdot))|(x) > \rho, \quad \forall x \in \mathrm{B}(\bar{x},\delta_*) :\ 0 < \psi_{C,\delta_1}(f(\bar{p},x)) \le \delta_*.$$

This means that for every $x \in \mathrm{B}(\bar{x},\delta_*)$, with $0 < \psi_{C,\delta_1}(f(\bar{p},x)) \le \delta_*$, and for every $\eta > 0$ there exists $x_\eta \in \mathrm{B}(x,\eta)\backslash\{x\}$ such that

$$\psi_{C,\delta_1}(f(\bar{p},x)) > \psi_{C,\delta_1}(f(\bar{p},x_\eta)) + \rho d(x_\eta, x). \tag{4}$$

Since, according to (h_2), mapping $f(\bar{p},\cdot)$ is continuous in particular at $\bar{x}$, corresponding to δ_* there exists $\tilde{\delta} \in (0,\delta_*)$ such that

$$d(f(\bar{p},z), f(\bar{p},\bar{x})) < \delta_*, \quad \forall z \in \mathrm{B}(\bar{x},\tilde{\delta}).$$

Now, choose ζ in such a way that

$$0 < \zeta < \min\left\{\tilde{\delta}, \frac{\delta_*}{\ell}, \frac{\rho\delta_*}{2\rho+\ell}\right\}. \tag{5}$$

Take arbitrary $p \in \mathrm{B}(\bar{p},\zeta)\backslash\{\bar{p}\}$ and $z \in S(p) \cap \mathrm{B}(\bar{x},\zeta)$. Then, one has

$$d(z,\bar{x}) \le \zeta$$

and, being $f(p,z) \in C$,

$$\psi_{C,\delta_1}(f(p,z)) = 0.$$

If it happens also that $z \in S(\bar{p})$, then a fortiori it is $z \in \mathrm{B}(S(\bar{p}), \frac{\ell}{\rho})$, so the proof is complete. Otherwise, one has $f(\bar{p},z) \notin C$ and hence

$$0 < \psi_{C,\delta_1}(f(\bar{p},z)).$$

Moreover, since it holds

$$\mathrm{dist}(f(\bar{p},z), C) \le d(f(\bar{p},z), f(p,z))$$

and $z \in \mathrm{B}(\bar{x},\zeta) \subset \mathrm{B}(\bar{x},\delta_3)$, while $p \in \mathrm{B}(\bar{p},\zeta) \subset \mathrm{B}(\bar{p},\delta_3)$, one obtains

$$\mathrm{dist}(f(\bar{p},z), C) \le \ell d(p,\bar{p}) \le \ell\zeta < \delta_* < \delta_1.$$

In other words, $f(\bar{p},z) \in \mathrm{B}(C,\delta_1)$. This fact, by recalling hypothesis (h_1), implies

$$\psi_{C,\delta_1}(f(\bar{p},z)) \le \mathrm{dist}(f(\bar{p},z), C) \le \ell d(p,\bar{p}). \tag{6}$$

Thus, being $\psi_{C,\delta_1}(y) \geq 0$ for every $y \in Y$, it follows

$$\psi_{C,\delta_1}(f(\bar{p}, z)) \leq \inf_{x \in \mathrm{B}(\bar{x},\delta_*)} \psi_{C,\delta_1}(f(\bar{p}, x)) + \ell d(p, \bar{p}).$$

Observe that, since $\mathrm{B}(\bar{x}, \delta_*) \subset \mathrm{B}(\bar{x}, \delta_2)$ and $f(\bar{p}, \cdot)$ is continuous on $\mathrm{B}(\bar{x}, \delta_2)$, the composition $\psi_{C,\delta_1} \circ f(\bar{p}, \cdot)$ still remains continuous on $\mathrm{B}(\bar{x}, \delta_*)$, which is a complete metric space, if endowed with the metric induced by that of X (recall hypothesis (h_0)). Consequently, it is possible to apply the Ekeland variational principle. According to it, corresponding to the value

$$\lambda = \frac{\ell}{\rho} d(p, \bar{p}),$$

there exists $x_\lambda \in \mathrm{B}(\bar{x}, \delta_*)$ such that

$$\psi_{C,\delta_1}(f(\bar{p}, x_\lambda)) \leq \psi_{C,\delta_1}(f(\bar{p}, z)), \tag{7}$$

$$d(x_\lambda, z) \leq \frac{\ell}{\rho} d(p, \bar{p}), \tag{8}$$

and

$$\psi_{C,\delta_1}(f(\bar{p}, x_\lambda)) < \psi_{C,\delta_1}(f(\bar{p}, x)) + \rho d(x, x_\lambda), \quad \forall x \in \mathrm{B}(\bar{x}, \delta_*) \backslash \{x_\lambda\}. \tag{9}$$

Notice that, by virtue of inequalities (6) and (7), it is

$$\psi_{C,\delta_1}(f(\bar{p}, x_\lambda)) \leq \ell d(p, \bar{p}) \leq \ell \zeta < \delta_*.$$

One can show that actually it holds

$$\psi_{C,\delta_1}(f(\bar{p}, x_\lambda)) = 0. \tag{10}$$

Indeed, assume ab absurdo that $\psi_{C,\delta_1}(f(\bar{p}, x_\lambda)) > 0$. Then, since $x_\lambda \in \mathrm{B}(\bar{x}, \delta_*)$, according to (4), for $\eta = \zeta$ there exists $x_\eta \in \mathrm{B}(x_\lambda, \zeta) \backslash \{x_\lambda\}$ such that

$$\psi_{C,\delta_1}(f(\bar{p}, x_\lambda)) > \psi_{C,\delta_1}(f(\bar{p}, x_\eta)) + \rho d(x_\eta, x_\lambda). \tag{11}$$

Notice that, being by virtue of choice made in (5)

$$d(x_\eta, \bar{x}) \leq d(x_\eta, x_\lambda) + d(x_\lambda, z) + d(z, \bar{x}) \leq \left(2 + \frac{\ell}{\rho}\right) \zeta < \delta_*,$$

x_η must belong to $\mathrm{B}(\bar{x}, \delta_*) \backslash \{x_\lambda\}$. Thus, one sees that inequality (11) contradicts (9). The validity of (10) leads to conclude that

$$f(\bar{p}, x_\lambda) \in C,$$

whence one obtains that $x_\lambda \in S(\bar{p})$. Such inclusion entails, in the light of inequality (8),

$$\mathrm{dist}(z, S(\bar{p})) \leq d(z, x_\lambda) \leq \frac{\ell}{\rho} d(p, \bar{p}).$$

By arbitrariness of $z \in S(p) \cap \mathrm{B}(\bar{x}, \zeta)$, the above argument proves that inclusion (3) is true, thereby completing the proof. $\square$

Remark 2.2. (r_1) As a matter of fact, Theorem 2.1 works as an implicit multifunction theorem for generalized equations (inclusions) in metric spaces. Indeed, from assuming that the mapping defining the inclusion enjoys a certain property (calmness), under a nondegeneracy condition it guarantees the same property to hold for the related solution (set-valued) mapping. Besides, it provides an estimate in terms of initial data quantifying such property (calmness modulus). Unfortunately, it suffers from the same drawback as classical implicit function theorems: it is far from being also a necessary condition. To see this, it suffices to consider the simple constraint system with $P = X = Y = \mathbb{R}$, defined by $f(p, x) = x^2 - p^4$ and $C = \{0\}$, at the reference values $\bar{p} = \bar{x} = 0$. In such case, one has $S(p) = \{-p^2\} \cup \{p^2\}$, so S is calm at $(0, 0)$. Nevertheless, since $d_{\{0\}}(f(0, x)) = x^2$ is strictly differentiable at 0, one finds

$$\overline{|\nabla(d_{\{0\}} \circ f(0, \cdot))|^{>}}(0) = \|\overline{\mathrm{D}}(d_{\{0\}} \circ f(0, \cdot))(0)\| = 0,$$

where $\overline{\mathrm{D}}$ denotes the strict derivative operator.

(r_2) It is worth noting that Theorem 2.1, included the estimate of the calmness modulus, remains true even if $\overline{|\nabla(\psi_{C,\delta_1} \circ f(\bar{p}, \cdot))|^{>}}(\bar{x}) = +\infty$, provided that the convention $\ell/+\infty = 0$, for every $\ell \geq 0$, is introduced. The example below shows that such situation may happen even in simple cases.

Let $P = X = Y = \mathbb{R}$ be endowed with the usual Euclidean metric structure. Consider the constraint system defined by $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, where

$$f(p, x) = |xp|,$$

and by the subset $C = (-\infty, 0]$, at the reference pair $\bar{p} = 0$ and $\bar{x} = 0$. In this case S can be derived explicitly at once, being

$$S(p) = \begin{cases} \{0\}, & \text{if } p \in \mathbb{R}\backslash\{0\}, \\ \mathbb{R}, & \text{if } p = 0. \end{cases}$$

It is not difficult to check by direct verification that $0 \in S(0)$ and S is calm at $(0, 0)$, with modulus $\operatorname{clm} S(0, 0) = 0$. Consistently, notice that S is a polyhedral set-valued mapping, so it is upper Lipschitz continuous (and hence calm) at each point of its graph. If taking $\psi_{C,\delta_1} = d_{(-\infty,0]}$, one sees that all hypotheses of Theorem 2.1 are fulfilled. In particular, since

$$f(0, x) = 0, \quad \forall x \in \mathbb{R} \quad \text{and hence} \quad d_{(-\infty,0]}(f(0, x)) = 0, \quad \forall x \in \mathbb{R},$$

one has

$$\{x \in [-\epsilon, \epsilon] : \ 0 < d_{(-\infty,0]}(f(0, x)) \leq \epsilon\} = \varnothing, \quad \forall \epsilon > 0.$$

It follows

$$\overline{|\nabla(d_{(-\infty,0]} \circ f(0, \cdot))|^{>}}(0) = \lim_{\epsilon \to 0^+} \inf \varnothing = +\infty.$$

The reader should also observe that S fails to be Aubin continuous at $(0, 0)$. Therefore, the above example allows also to see that the condition provided in

Theorem 2.1 is specifically tailored for detecting calmness, without implying in general Aubin continuity.

(r_3) The use of a subdistance functional in the place of d_C shows that the condition expressed by Theorem 2.1 does not rely on the special properties of the distance function. Nonetheless, for practical calculations, function d_C will be mainly employed.

(r_4) As an implicit multifunction theorem focussing on the calmness property, Theorem 2.1 can not prevent S to be empty-valued near a reference pair. For example, it can be applied (with $\psi_{C,\delta_1} = d_{(-\infty,0]}$) at the reference points $\bar{p} = \bar{x} = 0$ to the constraint system defined by

$$f(p, x) = |x| + p^2 \in (-\infty, 0] = C,$$

whose solution mapping

$$S(p) = \begin{cases} \{0\}, & \text{if } p = 0, \\ \varnothing, & \text{otherwise,} \end{cases}$$

is indeed calm at $(0, 0)$.

(r_5) Since parameterized constraint systems are a special case of variational system, one might believe that Theorem 2.1 can be derived by the sufficient condition for calmness of variational system, which has been established in [34] (see Theorem 5.1 therein), under a similar nondegeneracy condition. Even though the technique of the proof is essentially the same, Theorem 2.1 can not be formally obtained from that result: indeed, because of the use of a different displacement function, the latter requires a metric completeness assumption on both X and Y, while it involves a strict outer slope nondegeneracy, considering variations in both x and y.

3. Nonsmooth analysis tools

3.1. Generalized derivatives and exhausters

Throughout the present subsection $(\mathbb{X}, \|\cdot\|)$ will denote a (at least) normed space. The unit sphere centered at its null vector $\mathbf{0}$ is indicated by $\mathbb{S}$. The dual space of $\mathbb{X}$ is indicated by $\mathbb{X}^*$, with $\langle \cdot, \cdot \rangle : \mathbb{X}^* \times \mathbb{X} \longrightarrow \mathbb{R}$ denoting their duality pairing. The null vector of $\mathbb{X}^*$ is marked by $\mathbf{0}^*$. The class consisting of all nonempty convex and weak* compact subsets of $\mathbb{X}^*$ is denoted by $\mathcal{KC}_0(\mathbb{X}^*)$.

In order to study conditions for calmness in the more structured context of linear spaces, from Theorem 2.1 it is clear that one has to analyze the term involved in hypothesis (h_4), in the light of the features of the new setting. A tool that seems to be proper for such kind of analysis is the Hadamard directional derivative, in its lower and upper version. Given a function $\varphi : \mathbb{X} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$, a point $\bar{x} \in \operatorname{dom} \varphi$ and $v \in \mathbb{X}$, by *Hadamard lower directional derivative* of φ at $\bar{x}$, in the direction $v \in \mathbb{X}$, the value

$$\varphi^{\downarrow}_{\mathcal{H}}(\bar{x}; v) = \liminf_{\substack{v' \to v \\ t \downarrow 0}} \frac{\varphi(\bar{x} + tv') - \varphi(\bar{x})}{t}$$

is meant. The *Hadamard upper directional derivative* of φ at $\bar{x}$, here denoted by $\varphi^{\uparrow}_{\mathcal{H}}(\bar{x}; \cdot)$, is defined similarly, by replacing lim inf with lim sup. Notice that such derivative-like objects can be always considered, even though they may not be finite. Thus, the following estimation of the strong slope of a function turns out to be useful in a wide variety of situations. Even though such kind of result is rather typical in variational analysis (it can be derived, for instance, as a special case from Proposition 4.1 in [2]), an elementary proof is provided for the sake of completeness.

Lemma 3.1. *Given a function $\varphi : \mathbb{X} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ and an element $\bar{x} \in \operatorname{dom} \varphi$, it holds*

$$|\nabla \varphi|(\bar{x}) \geq \sup_{u \in \mathbb{S}} [-\varphi^{\downarrow}_{\mathcal{H}}(\bar{x}; u)]. \tag{12}$$

Proof. If $\bar{x}$ is a local minimizer of φ, evidently one has

$$\varphi^{\downarrow}_{\mathcal{H}}(\bar{x}; u) \geq 0, \quad \forall u \in \mathbb{S},$$

and therefore

$$-\varphi^{\downarrow}_{\mathcal{H}}(\bar{x}; u) \leq 0 = |\nabla \varphi|(\bar{x}), \quad \forall u \in \mathbb{S},$$

whence inequality (12). Otherwise, by making use of the inequality

$$\inf_{b \in B} \sup_{a \in A} \nu(a, b) \geq \sup_{a \in A} \inf_{b \in B} \nu(a, b),$$

which holds for any function $\nu : A \times B \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ and any pair of sets A and B, one obtains

$$\begin{aligned}
&|\nabla \varphi|(\bar{x}) \\
&= \limsup_{h \to \mathbf{0}} \frac{\varphi(\bar{x}) - \varphi(\bar{x} + h)}{\|h\|} = -\liminf_{h \to \mathbf{0}} \frac{\varphi(\bar{x} + h) - \varphi(\bar{x})}{\|h\|} \\
&= -\sup_{\delta > 0} \inf_{h \in \mathrm{B}(\mathbf{0}, \delta) \setminus \{\mathbf{0}\}} \frac{\varphi(\bar{x} + h) - \varphi(\bar{x})}{\|h\|} = -\sup_{\delta > 0} \inf_{u \in \mathbb{S}} \inf_{\substack{v \in \mathrm{B}(u, \delta) \cap \mathbb{S} \\ t \in (0, \delta]}} \frac{\varphi(\bar{x} + tv) - \varphi(\bar{x})}{t} \\
&\geq -\inf_{u \in \mathbb{S}} \sup_{\delta > 0} \inf_{\substack{v \in \mathrm{B}(u, \delta) \cap \mathbb{S} \\ t \in (0, \delta]}} \frac{\varphi(\bar{x} + tv) - \varphi(\bar{x})}{t} = -\inf_{u \in \mathbb{S}} \varphi^{\downarrow}_{\mathcal{H}}(\bar{x}; u),
\end{aligned}$$

as it was to be proved. □

Remark 3.2. Clearly, one can establish estimates from below of $|\nabla\varphi|(\bar{x})$ that are analogous to (12), by employing different generalized derivatives of φ. For instance, since for every $u \in \mathbb{S}$ it trivially holds $\varphi^{\uparrow}_{\mathcal{H}}(\bar{x}; u) \geq \varphi^{\downarrow}_{\mathcal{H}}(\bar{x}; u)$, as an immediate consequence of (12) one obtains

$$|\nabla\varphi|(\bar{x}) \geq \sup_{u\in\mathbb{S}} [-\varphi^{\uparrow}_{\mathcal{H}}(\bar{x}; u)]. \tag{13}$$

Nonetheless, it is to be noted that, in view of the application of Theorem 2.1, an estimation of the strong slope turns out to be as more appreciable[2] as more accurate (with equality being the limit case) it happens to be.

The estimate expressed by inequality (12) may be made more effective, provided that the Hadamard lower directional derivative, which is in general a positively homogeneous function, admits a simple description in dual terms. In what follows, ∂ will denote the subdifferential operator in the sense of convex analysis, whereas $\varsigma(\cdot, E) : \mathbb{X} \longrightarrow \mathbb{R}$ the support function of a given element $E \in \mathcal{KC}_0(\mathbb{X}^*)$, i.e.

$$\varsigma(x, E) = \max_{x^*\in E}\langle x^*, x\rangle.$$

Given a subset $S \subseteq \mathbb{X}$ and an element $\bar{x} \in S$, $\mathrm{N}(\bar{x}, S)$ will denote the normal cone to S at $\bar{x}$, in the sense of convex analysis.

Let $\phi : \mathbb{X} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ be a positively homogeneous of degree one (henceforth, for short, p.h.) function defined on a real Banach space. Since the Minkowski-Hörmander duality was discovered, it has been well understood that, under adequate assumptions, such a functional may be characterized in various fashions by making use of subsets of the dual space $\mathbb{X}^*$, in particular handling elements of $\mathcal{KC}_0(\mathbb{X}^*)$. A line of research within abstract convex analysis deals essentially with such problem (see [31]). In the light of recent developments in such area, a rather general dual characterization of ϕ is that provided by a family of exhausters. A class $\mathcal{E}_*(\phi) \subseteq \mathcal{KC}_0(\mathbb{X}^*)$ is said to be a *lower exhauster* of ϕ if

$$\phi(v) = \inf_{E\in\mathcal{E}_*(\phi)} \max_{x^*\in E}\langle x^*, v\rangle = \inf_{E\in\mathcal{E}_*(\phi)} \varsigma(v, E), \quad \forall v \in \mathbb{X}.$$

An *upper exhauster* $\mathcal{E}^*(\phi)$ of ϕ is defined similarly, referring instead to the following dual representation

$$\phi(v) = \sup_{E\in\mathcal{E}^*(\phi)} \min_{x^*\in E}\langle x^*, v\rangle, \quad \forall v \in \mathbb{X}.$$

Lower and upper exhausters were introduced in [11]. A related calculus was developed in [12, 13]. A crucial issue in concern with such notion is to single out algebraic/topological properties of ϕ able to guarantee the existence of lower or/and upper exhausters. For instance, if $\phi : \mathbb{X} \longrightarrow \mathbb{R}$ can be expressed as a difference of two (Lipschitz) continuous sublinear functionals, say $\underline{\phi}$ and $\overline{\phi}$, then

[2]in the sense that it allows for applications to broader classes of problems.

one has as an immediate consequence of known properties of the Minkowski-Hörmander duality

$$\phi(v) = \max_{x^* \in \partial\underline{\phi}(\mathbf{0})} \langle x^*, v\rangle - \max_{z^* \in \partial\overline{\phi}(\mathbf{0})} \langle z^*, v\rangle = \min_{z^* \in \partial\overline{\phi}(\mathbf{0})} \max_{w^* \in \partial\underline{\phi}(\mathbf{0})+\{-z^*\}} \langle w^*, v\rangle,$$

and

$$\phi(v) = \max_{x^* \in \partial\underline{\phi}(\mathbf{0})} \langle x^*, v\rangle + \min_{z^* \in -\partial\overline{\phi}(\mathbf{0})} \langle z^*, v\rangle = \max_{x^* \in \partial\underline{\phi}(\mathbf{0})} \min_{w^* \in \{x^*\}-\partial\overline{\phi}(\mathbf{0})} \langle w^*, v\rangle.$$

Thus ϕ admits in this case both lower ad upper exhausters, being

$$\mathcal{E}_*(\phi) = \{\partial\underline{\phi}(\mathbf{0}) + \{-z^*\} : \ z^* \in \partial\overline{\phi}(\mathbf{0})\}$$
$$\text{and} \quad \mathcal{E}^*(\phi) = \{\{x^*\} - \partial\overline{\phi}(\mathbf{0}) : \ x^* \in \partial\underline{\phi}(\mathbf{0})\}.$$

In order to quantify a nondegeneracy condition involving exhausters in constraint system analysis, the following "norm" of a lower exhauster $\mathcal{E}_*(\phi)$ will be utilized in the subsequent section:

$$|||\mathcal{E}_*(\phi)||| = \sup\{\mathrm{dist}(\mathbf{0}^*, E) : \ E \in \mathcal{E}_*(\phi)\}.$$

Of course, since $\mathcal{KC}_0(\mathbb{X}^*)$ fails to possess a vector space structure, this is not a true norm.

The generalized differentiation tools so far presented must be employed in dealing with data of problem $(\mathcal{CS})$, which are vector-valued functions. Therefore, the next definition is intended to extend to such kind of functions dual representations of Hadamard directional derivatives via a scalarization approach.

Definition 3.3. A mapping $f : \mathbb{X} \longrightarrow \mathbb{Y}$ between Banach spaces is said to be *lower scalarly exhaustible from below* [3] at $\bar{x} \in \mathbb{X}$ if for every $y^* \in \mathbb{Y}^*$ the p.h. functional $(y^* \circ f)^{\downarrow}_{\mathcal{H}}(\bar{x}; \cdot)$ admits a lower exhauster, i.e., in the above notations, if

$$\forall y^* \in \mathbb{Y}^* \ \exists \mathcal{E}_*((y^* \circ f)^{\downarrow}_{\mathcal{H}}(\bar{x}; \cdot)) \subseteq \mathcal{KC}_0(\mathbb{X}^*) :$$
$$(y^* \circ f)^{\downarrow}_{\mathcal{H}}(\bar{x}; v) = \inf_{E \in \mathcal{E}_*((y^* \circ f)^{\downarrow}_{\mathcal{H}}(\bar{x}; \cdot))} \max_{x^* \in E} \langle x^*, v\rangle.$$

The set-valued mapping $\mathcal{E}^{\downarrow}_*(f, \bar{x}) : \mathbb{Y}^* \longrightarrow 2^{\mathcal{KC}_0(\mathbb{X}^*)}$, defined by

$$\mathcal{E}^{\downarrow}_*(f, \bar{x})(y^*) = \mathcal{E}_*((y^* \circ f)^{\downarrow}_{\mathcal{H}}(\bar{x}; \cdot)),$$

is said to be a *lower scalarized exhauster (from below)* of f at $\bar{x}$.

Analogously, mutatis mutandis, one defines the notion of mapping *upper scalarly exhaustible from below* and that of *upper scalarized exhauster*, which will be denoted by $\mathcal{E}^{\uparrow}_*$. The definition of mappings which are lower and upper scalarly exhaustible from above is readily obtained by replacing the inf max with a sup min dual representation.

[3]The adverb "lower" indicates the use of the Hadamard lower directional derivative, whereas "from below" indicates that the existence of a dual representation via lower exhauster is postulated.

Remark 3.4. Whenever it is $\mathcal{E} \subseteq \mathcal{KC}_0(\mathbb{X}^*)$ and $\alpha > 0$, by $\alpha\mathcal{E}$ the following family of $\mathcal{KC}_0(\mathbb{X}^*)$

$$\alpha\mathcal{E} = \{F \in \mathcal{KC}_0(\mathbb{X}^*) :\ F = \alpha E,\ E \in \mathcal{E}\}$$

is meant. Since with such meaning it holds

$$\mathcal{E}^{\downarrow}_{*}(f, \bar{x})(\alpha y^*) = \alpha\mathcal{E}^{\downarrow}_{*}(f, \bar{x})(y^*), \quad \forall y^* \in \mathbb{Y}^*,\ \forall \alpha > 0,$$

the set-valued mapping $\mathcal{E}^{\downarrow}_{*}(f, \bar{x})$ appears to inherit a form of positive homogeneity (in a broad sense, because $\mathcal{KC}_0(\mathbb{X}^*)$ is not a vector space). The same, of course, can be repeated for $\mathcal{E}^{\uparrow}_{*}(f, \bar{x})$.

The examples provided below are aimed at exhibiting entire classes of scalarly exhaustible mappings. The first two show that all smooth and all locally Lipschitz mappings are both lower and upper scalarly exhaustible. Since well-known calculi devoted to smooth and locally Lipschitz mappings are already at disposal, such examples serve merely to connect the introduced notion with important “categories” in nonsmooth analysis. The third one singles out a specific environment, which is adequate for the analysis tools under consideration. The last one illustrates a situation in which a lower exhauster can be computed, whereas another important tool of nonsmooth analysis such as the Fréchet subdifferential lacks.

Example 3.5 (Smooth mappings). Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be Fréchet differentiable at $\bar{x} \in \mathbb{X}$, with Fréchet derivative $\widehat{\mathrm{D}}f(\bar{x})$. Since for every $y^* \in \mathbb{Y}^*$ one has

$$(y^* \circ f)^{\downarrow}_{\mathcal{H}}(\bar{x}; v) = (y^* \circ f)^{\uparrow}_{\mathcal{H}}(\bar{x}; v) = \widehat{\mathrm{D}}(y^* \circ f)(\bar{x})[v] = \langle \widehat{\mathrm{D}}f(\bar{x})^*[y^*], v\rangle, \quad \forall v \in \mathbb{X},$$

where $\widehat{\mathrm{D}}f(\bar{x})^* : \mathbb{Y}^* \longrightarrow \mathbb{X}^*$ denotes the adjoint operator to $\widehat{\mathrm{D}}f(\bar{x})$, then f is both lower and upper scalarly exhaustible from below at $\bar{x}$ and a scalarized exhauster of f at $\bar{x}$ coincides with the singleton

$$\mathcal{E}^{\downarrow}_{*}(f, \bar{x})(y^*) = \mathcal{E}^{\uparrow}_{*}(f, \bar{x})(y^*) = \{\, \{\widehat{\mathrm{D}}f(\bar{x})^*[y^*]\} \,\}.$$

Example 3.6 (Locally Lipschitz mappings in general Banach spaces). A relevant achievement in abstract convex analysis states that a p.h. function $\phi : \mathbb{X} \longrightarrow \mathbb{R}$ defined on a normed space is Lipschitz continuous iff it admits bounded both lower and upper exhausters (see [6]). Recall that a family $\mathcal{E} \subseteq \mathcal{KC}_0(\mathbb{X}^*)$ is said to be bounded provided that there exists a constant $\mu > 0$ such that

$$E \subseteq \mathrm{B}(\mathbf{0}^*, \mu), \quad \forall E \in \mathcal{E}.$$

Whenever a mapping $f : \mathbb{X} \longrightarrow \mathbb{Y}$ is locally Lipschitz at $\bar{x} \in \mathbb{X}$, function $(y^* \circ f)$ clearly maintains such property for every $y^* \in \mathbb{Y}^*$, and hence $(y^* \circ f)^{\downarrow}_{\mathcal{H}}(\bar{x}; \cdot) : \mathbb{X} \longrightarrow \mathbb{R}$ and $(y^* \circ f)^{\uparrow}_{\mathcal{H}}(\bar{x}; \cdot) : \mathbb{X} \longrightarrow \mathbb{R}$ turn out to be Lipschitz continuous. Thus, by virtue of the dual characterization above recalled, it is possible to see that a locally Lipschitz mapping is both lower and upper scalarly exhaustible from below.

Example 3.7 (Nonsmooth mappings in locally uniformly convex Banach spaces). Let us recall that a Banach space $(\mathbb{X}, \|\cdot\|)$ is said to be *locally uniformly convex* if

$$\forall u \in \mathbb{S},\ \forall \epsilon > 0 \ \text{ it is } \sup_{v \in \mathbb{S} \setminus \operatorname{int} \mathrm{B}(u,\epsilon)} \left\| \frac{u+v}{2} \right\| < 1,$$

where $\operatorname{int} S$ denotes the topological interior of a set S. As a result of other investigations in abstract convex analysis, it has been established (see, for instance, [31, 33]) that a real-valued p.h. function defined on a locally uniformly convex Banach space admits lower (resp. upper) exhausters iff it is u.s.c. (resp. l.s.c.). Thus, any mapping $f : \mathbb{X} \longrightarrow \mathbb{Y}$, which is defined on a locally uniformly convex Banach space and has the property that $(y^* \circ f)^{\downarrow}_{\mathcal{H}}(\bar{x}; \cdot) : \mathbb{X} \longrightarrow \mathbb{R}$ is continuous (recall that such functional is automatically l.s.c.), turns out to be lower scalarly exhaustible from below at $\bar{x}$. Furthermore, any mapping $f : \mathbb{X} \longrightarrow \mathbb{Y}$ admitting real-valued scalarized Hadamard directional derivative $(y^* \circ f)^{\uparrow}_{\mathcal{H}}(\bar{x}; \cdot)$ (which is u.s.c.) is upper scalarly exhaustible from below at $\bar{x}$.

It is clear that every uniformly convex Banach space is, in particular, locally uniformly convex. Moreover, every reflexive Banach space can be equivalently renormed in such a way to be locally uniformly convex (see [15]). These facts seem to indicate that the class of locally uniformly convex Banach spaces is sufficiently wide for many purposes.

Example 3.8 (Non Fréchet subdifferentiable function). Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = -|x|.$$

As it is readily seen, f fails to be Fréchet subdifferentiable at $\bar{x} = 0$, in the sense that, if denoting by $\widehat{\partial} f(0)$ its Fréchet subdifferential at 0, one has $\widehat{\partial} f(0) = \varnothing$. Nonetheless, being

$$(y^* \circ f)^{\downarrow}_{\mathcal{H}}(0; v) = (y^* \circ f)^{\uparrow}_{\mathcal{H}}(0; v) = -y^* \cdot |v| = \begin{cases} \inf\limits_{E \in \{\{-y^*\}, \{y^*\}\}} \varsigma(v, E), & \text{if } y^* \geq 0, \\ \inf\limits_{E \in \{[y^*, -y^*]\}} \varsigma(v, E), & \text{if } y^* < 0, \end{cases}$$

f is both lower and upper exhaustible from below, with

$$\mathcal{E}^{\downarrow}_{*}(f, 0)(y^*) = \mathcal{E}^{\uparrow}_{*}(f, 0)(y^*) = \begin{cases} \{\{-y^*\}, \{y^*\}\}, & \text{if } y^* \geq 0, \\ \{\, [y^*, -y^*] \,\}, & \text{if } y^* < 0, \end{cases} \quad y^* \in \mathbb{R}.$$

Remark 3.9. (r_1) Note that Definition 3.3 applies, in particular, when $\mathbb{Y} = \mathbb{R}$, that is when f is a functional. In any such case it seems to be reasonable to omit the term "scalarized".

(r_2) Observe that, according to Definition 3.3, if $f : \mathbb{X} \longrightarrow \mathbb{Y}$ is upper scalarly exhaustible from above at $\bar{x} \in \mathbb{X}$, with upper exhauster $\mathcal{E}^{*\uparrow}(f, \bar{x})(y^*)$, then it is also lower exhaustible from below at the same point, with lower exhauster

$$\mathcal{E}^{\downarrow}_{*}(f, \bar{x})(y^*) = -\mathcal{E}^{*\uparrow}(f, \bar{x})(-y^*) = \{-E :\ E \in \mathcal{E}^{*^{\uparrow}}(f, \bar{x})(-y^*)\}, \quad y^* \in \mathbb{Y}^*.$$

When handling exhausters, the following technical lemma, establishing a dual characterization of a nondegeneracy condition, will be exploited. Even if its proof relies on well-known facts from convex analysis in normed spaces, it is provided below for the reader's convenience.

Lemma 3.10. *Let $E \in \mathcal{KC}_0(\mathbb{X}^*)$ and $\gamma > 0$. The following assertions are equivalent:*

(a_1) *there exists $u_0 \in \mathbb{S}$ such that $\varsigma(u_0, E) < -\gamma$;*
(a_2) $\mathrm{dist}(\mathbf{0}^*, E) > \gamma$.

Proof. $(a_1) \Rightarrow (a_2)$ Suppose, ab absurdo, that $\mathrm{dist}(\mathbf{0}^*, E) \leq \gamma$ and choose $\epsilon > 0$ in such a way that

$$\epsilon + \gamma + \varsigma(u_0, E) < 0. \tag{14}$$

By hypothesis (a_1) such an ϵ does exist. Corresponding to ϵ there is $x^*_\epsilon \in E$ such that $\|x^*_\epsilon\| < \mathrm{dist}(\mathbf{0}^*, E) + \epsilon \leq \gamma + \epsilon$. Notice that, being $-x^*_\epsilon \in \mathrm{B}(\mathbf{0}^*, \gamma + \epsilon)$, $\mathbf{0}^*$ must belong to $E + \mathrm{B}(\mathbf{0}^*, \gamma + \epsilon)$. This fact implies

$$\varsigma(u, E + \mathrm{B}(\mathbf{0}^*, \gamma + \epsilon)) \geq 0, \quad \forall u \in \mathbb{S}.$$

By a known property of the support function of elements of $\mathcal{KC}_0(\mathbb{X}^*)$, one obtains

$$0 \leq \varsigma(u, E + \mathrm{B}(\mathbf{0}^*, \gamma + \epsilon)) = \varsigma(u, E) + \varsigma(u, \mathrm{B}(\mathbf{0}^*, \gamma + \epsilon)) \leq \varsigma(u, E) + \gamma + \epsilon, \quad \forall u \in \mathbb{S},$$

which contradicts inequality (14).

$(a_2) \Rightarrow (a_1)$ Suppose (a_1) to be false, i.e.

$$\varsigma(u, E) \geq -\gamma, \quad \forall u \in \mathbb{S},$$

and hence

$$\inf_{u \in \mathbb{S}} \varsigma(u, E) \geq -\gamma.$$

Since, for every $u \in \mathbb{S}$, it is $\varsigma(u, \mathrm{B}(\mathbf{0}^*, \gamma)) = \gamma$, it follows

$$\inf_{u \in \mathbb{S}} \varsigma(u, E + \mathrm{B}(\mathbf{0}^*, \gamma)) \geq \inf_{u \in \mathbb{S}} \varsigma(u, E) + \inf_{u \in \mathbb{S}} \varsigma(u, \mathrm{B}(\mathbf{0}^*, \gamma)) \geq 0.$$

The above inequality entails that $\mathbf{0}^* \in E + \mathrm{B}(\mathbf{0}^*, \gamma)$. Thus, there must exist $x^* \in E$ such that $\|x^*\| \leq \gamma$, what contradicts the assertion (a_2). $\square$

In the proof of the main results of the paper a key step consists in estimating Hadamard directional derivatives of composite functions. The next lemma accomplishes such task by a scalarization method in the upper case.

Lemma 3.11. *Let $g : \mathbb{X} \longrightarrow \mathbb{Y}$ be a mapping between Banach spaces, let $C \subset \mathbb{Y}$, and let $x \in \mathbb{X}$ be such that $g(x) \notin C$. Suppose that*

(h_1) *C is closed and convex;*
(h_2) *g is continuous at x.*

Then, defined for any $\eta \in (0, d_C(g(x)))$ *the set* $\mathcal{U}(g(x),\eta) = \bigcup_{y \in \mathrm{B}(g(x),\eta)} \partial d_C(y)$, *the following estimation holds*

$$(d_C \circ g)^{\uparrow}_{\mathcal{H}}(x;u) \le \lim_{\eta \to 0^+} \sup_{y^* \in \mathcal{U}(g(x),\eta)} (y^* \circ g)^{\uparrow}_{\mathcal{H}}(x;u), \quad \forall u \in \mathbb{S}. \tag{15}$$

Proof. Since C is closed and $g(x) \notin C$, then it is $d_C(g(x)) > 0$. Fix arbitrary $u \in \mathbb{S}$ and $\eta \in (0, d_C(g(x))$. Clearly, one has $\mathrm{B}(g(x),\eta) \cap C = \varnothing$. By virtue of hypothesis (h_2), corresponding to η there exists $\delta_\eta > 0$ such that

$$g(x+tv) \in \mathrm{B}(g(x),\eta), \quad \forall v \in \mathrm{B}(u,\delta_\eta),\ \forall t \in [0,\delta_\eta]. \tag{16}$$

Observe that, being $\mathrm{B}(g(x),\eta)$ a convex set containing $g(x)$ and $g(x+tv)$, provided that $v \in \mathrm{B}(u,\delta_\eta)$ and $t \in [0,\delta_\eta]$, it contains the whole line segment of extremes $g(x)$ and $g(x+tv)$, here denoted by $[g(x), g(x+tv)]$. Consequently, one has

$$[g(x), g(x+tv)] \subseteq \mathbb{Y} \backslash C, \quad \forall v \in \mathrm{B}(u,\delta_\eta),\ \forall t \in [0,\delta_\eta].$$

Now, since function $d_C : \mathbb{Y} \longrightarrow \mathbb{R}$ is convex and (Lipschitz) continuous, for any value of $v \in \mathrm{B}(u,\delta_\eta)$ and $t \in [0,\delta_\eta]$ it is possible to apply the mean value theorem (see, for instance, [18]), according to which there exist proper $y_{v,t} \in [g(x), g(x+tv)] \backslash \{g(x), g(x+tv)\}$ and $y^*_{v,t} \in \partial d_C(y_{v,t})$ such that

$$d_C(g(x+tv)) - d_C(g(x)) = \langle y^*_{v,t}, g(x+tv) - g(x) \rangle.$$

Notice that inclusion (16) continues being true if replacing δ_η with every $\delta \in (0,\delta_\eta)$, as well as the possibility of applying the mean value theorem. Therefore, one obtains

$$\begin{aligned}
(d_C \circ g)^{\uparrow}_{\mathcal{H}}(x;u) &= \inf_{\delta > 0} \sup_{\substack{v \in \mathrm{B}(u,\delta)\backslash\{u\} \\ t \in (0,\delta]}} \frac{d_C(g(x+tv)) - d_C(g(x))}{t} \\
&= \inf_{\delta \in (0,\delta_\eta]} \sup_{\substack{v \in \mathrm{B}(u,\delta)\backslash\{u\} \\ t \in (0,\delta]}} \frac{d_C(g(x+tv)) - d_C(g(x))}{t} \\
&= \inf_{\delta \in (0,\delta_\eta]} \sup_{\substack{v \in \mathrm{B}(u,\delta)\backslash\{u\} \\ t \in (0,\delta]}} \left\langle y^*_{v,t}, \frac{g(x+tv) - g(x)}{t} \right\rangle.
\end{aligned}$$

Since by inclusion (16) it is $y_{v,t} \in \mathrm{B}(g(x),\eta)$ and hence $y^*_{v,t} \in \mathcal{U}(g(x),\eta)$, provided that $v \in \mathrm{B}(u,\delta)$ and $t \in [0,\delta]$, with $\delta \in (0,\delta_\eta]$, then from the above inequalities one gets

$$\begin{aligned}
(d_C \circ g)^{\uparrow}_{\mathcal{H}}(x;u) &\le \sup_{y^* \in \mathcal{U}(g(x),\eta)} \inf_{\delta \in (0,\delta_\eta]} \sup_{\substack{v \in \mathrm{B}(u,\delta)\backslash\{u\} \\ t \in (0,\delta]}} \left\langle y^*, \frac{g(x+tv) - g(x)}{t} \right\rangle \\
&= \sup_{y^* \in \mathcal{U}(g(x),\eta)} (y^* \circ g)^{\uparrow}_{\mathcal{H}}(x;u).
\end{aligned}$$

Owing to the arbitrariness of $\eta \in (0, d_C(g(x))$, all of the above reasoning as well as the consequent estimate of $(d_C \circ g)^{\uparrow}_{\mathcal{H}}(x;u)$ remains valid for every $\eta \in (0, d_C(g(x))$. Thus, passing to the limit as $\eta \to 0^+$, one can achieve inequality (15). This completes the proof. $\square$

Remark 3.12. Recalling that the set-valued mapping $\partial d_C : \mathbb{Y} \longrightarrow \mathcal{KC}_0(\mathbb{Y}^*)$ is norm-to-weak* upper semicontinuous on $\mathbb{Y}$ allows one, whenever the Banach space $\mathbb{Y}$ has finite dimension, to replace the inequality in (15) with a simpler estimate of $(d_C \circ g)^{\uparrow}_{\mathcal{H}}(x; u)$. Indeed, for any fixed $\epsilon > 0$ there exists $\eta_\epsilon > 0$ such that

$$\partial d_C(y) \subseteq \mathrm{B}(\partial d_C(g(x)), \epsilon), \quad \forall y \in \mathrm{B}(g(x), \eta_\epsilon).$$

Therefore, in this case one has

$$\mathcal{U}(g(x), \eta_\epsilon) \subseteq \mathrm{B}(\partial d_C(g(x)), \epsilon), \quad \forall \epsilon > 0.$$

As, without loss of generality, it is possible to assume $\eta_\epsilon \in (0, \epsilon)$, so that $\epsilon \to 0^+$ implies $\eta_\epsilon \to 0^+$, then one deduces

$$(d_C \circ g)^{\uparrow}_{\mathcal{H}}(x; u) \le \lim_{\epsilon \to 0^+} \sup_{y^* \in \mathrm{B}(\partial d_C(g(x)), \epsilon)} (y^* \circ g)^{\uparrow}_{\mathcal{H}}(x; u), \quad \forall u \in \mathbb{S}.$$

The reader should notice that, in contrast to (15), such an estimate requires the evaluation of ∂d_C only at $g(x)$.

3.2. A generalization of convex sets

Let $(\mathbb{H}, (\cdot\,, \cdot))$ be a Hilbert space. Given a nonempty closed set $C \subseteq \mathbb{H}$ and $y \in C$, let us denote by $\mathrm{N}(y, C)$ the cone of all normals to C at y in a *generalized sense* (or *à la Kruger-Mordukhovich*), i.e.

$$\begin{aligned} &\mathrm{N}(y, C) \\ &= \{y^* \in \mathbb{H}^* : \ \exists (y_n)_{n \in \mathbb{N}},\, y_n \in C,\, y_n \to y,\, \exists (y_n^*)_{n \in \mathbb{N}},\, y_n^* \in \widehat{\mathrm{N}}(y_n, C),\, y_n^* \rightharpoonup y^*\}, \end{aligned}$$

where

$$\widehat{\mathrm{N}}(y_n, C) = \left\{ y^* \in \mathbb{H}^* : \ \limsup_{z \xrightarrow{C} y_n} \frac{\langle y^*, z - y_n \rangle}{\|z - y_n\|} \le 0 \right\}$$

stands for the *Fréchet normal cone* to C at y_n and $\rightharpoonup$ indicates weak* convergence (for a complete account on such constructions the reader is referred to [23, 24]). According to [27], a closed subset $C \subset \mathbb{H}$ is said to be *prox-regular at* $\bar{y}$, with $\bar{y} \in C$, if there exist positive ϵ and ρ such that for every $y \in C \cap \mathrm{int}\, \mathrm{B}(\bar{y}, \epsilon)$ and for every $y^* \in \mathrm{N}(y, C) \cap \mathrm{int}\, \mathrm{B}(\mathbf{0}^*, \epsilon)$ it holds

$$\langle y^*, z - y \rangle \le \frac{\rho}{2} \|z - y\|^2, \quad \forall z \in C \cap \mathrm{int}\, \mathrm{B}(\bar{y}, \epsilon).$$

A set is called *prox-regular* if it is prox-regular at each of its points[4]. Of course, every closed convex set is prox-regular. Moreover, a good deal of examples of sets exhibiting prox-regularity in finite dimensions has been provided in [26], in particular sets admitting a smooth constraint representation. Thus the notion of

[4]Actually, this is not the original definition, but an alternative one, as resulting from Proposition 1.2 in [27]. The notion of prox-regularity was introduced in [26] in a finite-dimensional setting.

prox-regularity allows to extend the present analysis beyond the realm of convexity. The key property making prox-regularity of C so appealing within the proposed approach is the deep connection with the local differentiability behaviour of the distance function d_C, which motivated its introduction. In this vein, for the purposes of the present analysis, some facts concerning prox-regular sets collected in the next lemma will be relevant (for its proof and further equivalences see Theorem 1.3 in [27]).

Lemma 3.13. *For a closed set $C \subset \mathbb{H}$ and any point $\bar{y} \in C$, the following assertions are equivalents:*

(a_1) *C is prox-regular at $\bar{y}$;*

(a_2) *function d_C is continuously differentiable on $[\mathrm{int}\,\mathrm{B}(\bar{y}, r)]\backslash C$, for some $r > 0$;*

(a_3) *function d_C^2 is $\mathrm{C}^{1,1}$ on $[\mathrm{int}\,\mathrm{B}(\bar{y}, r)]\backslash C$, for some $r > 0$.*

Prox-regularity of a set C allows one to obtain the following simple representation of lower scalarized exhausters for composite functions, in the case the outer factor in the composition is d_C.

Lemma 3.14. *Let $g : \mathbb{X} \longrightarrow \mathbb{Y}$ be a mapping defined on a normed vector space and taking values in a Hilbert space, let $\bar{x} \in \mathbb{X}$ and $C \subset \mathbb{Y}$, with $g(\bar{x}) \in C$. Suppose that:*

(h_1) *g is calm at each point of a ball $\mathrm{B}(\bar{x}, \delta_1)$;*

(h_2) *g is lower scalarly exhaustible from below at each point x of a ball $\mathrm{B}(\bar{x}, \delta_2)$, with scalarized lower exhauster $\mathcal{E}^{\downarrow}_*(g, x) : \mathbb{Y}^* \longrightarrow 2^{\mathcal{KC}_0(\mathbb{X}^*)}$;*

(h_3) *C is prox-regular at $g(\bar{x})$.*

Then, there exists $\zeta > 0$ such that the functional $(d_C \circ g)^{\downarrow}_{\mathcal{H}}(x; \cdot)$ admits lower exhausters and it results in

$$\mathcal{E}^{\downarrow}_*(d_C \circ g, x) = \mathcal{E}^{\downarrow}_*(g, x)(\widehat{\mathrm{D}} d_C(g(x))), \quad \forall x \in \mathrm{B}(\bar{x}, \zeta)\backslash g^{-1}(C).$$

Proof. By virtue of Lemma 3.13, hypothesis (h_3) ensures the existence of $r > 0$ such that function d_C is Fréchet differentiable on $[\mathrm{int}\,\mathrm{B}(g(\bar{x}), r)]\backslash C$. Since g is in particular continuous at $\bar{x}$, there is $\zeta \in (0, \min\{\delta_1,\, \delta_2\})$ such that

$$g(x) \in \mathrm{int}\,\mathrm{B}(g(\bar{x}), r), \quad \forall x \in \mathrm{B}(\bar{x}, \zeta).$$

Thus, fix $x \in \mathrm{B}(\bar{x}, \zeta)\backslash g^{-1}(C)$ and $v \in \mathbb{X}$. Since d_C is Fréchet differentiable at $g(x)$, one obtains

$$(d_C \circ g)^{\downarrow}_{\mathcal{H}}(x; v) = \liminf_{\substack{v' \to v \\ t \downarrow 0}} \frac{d_C(g(x + tv')) - d_C(g(x))}{t}$$

$$= \liminf_{\substack{v' \to v \\ t \downarrow 0}} \left[\frac{\widehat{\mathrm{D}} d_C(g(x))[g(x + tv') - g(x)]}{t} + \frac{\mathrm{o}(\|g(x + tv') - g(x)\|)}{t} \right],$$

where

$$\lim_{h \to \mathbf{0}} \frac{\mathrm{o}(\|h\|)}{\|h\|} = 0. \tag{17}$$

Notice that it holds

$$\lim_{\substack{v'\to v\\ t\downarrow 0}} \frac{\mathrm{o}(\|g(x+tv')-g(x)\|)}{t}=0. \tag{18}$$

To see this, fix an arbitrary $\epsilon>0$. Because of the calmness of g at $x\in\mathrm{B}(\bar{x},\zeta)\subseteq\mathrm{B}(\bar{x},\delta_1)$, there exist positive δ_x and ℓ_x such that

$$\|g(x+tv')-g(x)\|\le \ell_x t\|v'\|\le \ell_x t(\|v\|+\delta_x),\quad \forall t\in[0,\delta_x],\ \forall v'\in\mathrm{B}(v,\delta_x). \tag{19}$$

Owing to equality (17), corresponding to $\epsilon>0$ there exists $\delta_{\mathrm{o}}>0$ such that

$$|\mathrm{o}(\|h\|)|\le \frac{\epsilon\|h\|}{\ell_x(\|v\|+\delta_x)},\quad \forall h\in\mathbb{X}:\ \|h\|\le\delta_{\mathrm{o}}. \tag{20}$$

Thus, take δ_ϵ so that

$$0<\delta_\epsilon<\min\left\{\frac{\delta_{\mathrm{o}}}{\ell_x(\|v\|+\delta_x)},\ \delta_x\right\}.$$

With such position, by recalling the inequalities in (19), one finds

$$\|g(x+tv')-g(x)\|<\delta_{\mathrm{o}},\quad \forall t\in[0,\delta_\epsilon],\ \forall v'\in\mathrm{B}(v,\delta_\epsilon),$$

whence, in the light of inequality (20), it follows

$$|\mathrm{o}(\|g(x+tv')-g(x)\|)|\le\epsilon\frac{\|g(x+tv')-g(x)\|}{\ell_x(\|v\|+\delta_x)}\le\epsilon t,\quad \forall t\in[0,\delta_\epsilon],\ \forall v'\in\mathrm{B}(v,\delta_\epsilon).$$

The existence of δ_ϵ making the last inequality satisfied shows the existence of the limit and the validity of the equality in (18). Then, being $\widehat{\mathrm{D}}d_C(g(x))\in\mathbb{Y}^*$, it results in

$$(d_C\circ g)^{\downarrow}_{\mathcal{H}}(x;v)=(\widehat{\mathrm{D}}d_C(g(x))\circ g)^{\downarrow}_{\mathcal{H}}(x;v),\quad \forall v\in\mathbb{X}.$$

According to Definition 3.3, it remains to exploit hypotesis (h_2) to achieve the thesis. □

Remark 3.15. For a better understanding of nondegeneracy conditions appearing in the subsequent section, but related to the above result, it is worth noting that, d_C being Lipschitz continuous with constant 1, whenever $\widehat{\mathrm{D}}d_C(y)$ exists, one has $\|\widehat{\mathrm{D}}d_C(y)\|\le 1$. To be more precise, by virtue of a known characterization of the Fréchet subdifferential of d_C at out-of-set points (see Theorem 1.99 in [24]), whenever $\widehat{\mathrm{D}}d_C(y)$ exists, it holds

$$\{\widehat{\mathrm{D}}d_C(y)\}=\widehat{\partial}d_C(y)\subseteq\mathbb{S}^*\cap\widehat{\mathrm{N}}(y,\mathrm{B}(C,d_C(y))),\quad y\notin C,$$

where $\widehat{\partial}d_C(y)$ stands for the Fréchet subdifferential of d_C at y.

4. Calmness conditions for scalarly exhaustible constraints

4.1. The prox-regular case in Hilbert spaces

In view of the employment of a prox-regularity assumption on the set C, throughout this subsection $(\mathbb{Y}, (\cdot,\cdot))$ will be a Hilbert space, whereas $(\mathbb{X}, \|\cdot\|)$ continues being a Banach space. In such setting, given a parameterized constraint system $(\mathcal{CS})$ and a pair $(\bar p, \bar x) \in \operatorname{gph} S$, it is possible to define the following constant

$$\begin{aligned}&\mathrm{c}^{\downarrow}(f(\bar p,\cdot),\bar x)\\ =\ &\lim_{\epsilon\to 0^+} \inf\{|||\mathcal{E}^{\downarrow}_*(f(\bar p,\cdot),x)(\widehat{\mathrm{D}}d_C(f(\bar p,x)))|||:\ x\in \mathrm{B}(\bar x,\epsilon),\ 0<d_C(f(\bar p,x))\le\epsilon\},\end{aligned}$$

provided that adequate assumptions of lower scalar exhaustibility on mapping $f(\bar p,\cdot)$ near $\bar x$ are introduced. If so, such constant enables one to formulate a nondegeneracy condition, on which the first main calmness result is based. In fact, it allows one to provide a key estimate for the strict outer slope of function $d_C \circ f(\bar p,\cdot)$.

Theorem 4.1. *With reference to a constraint system $(\mathcal{CS})$, having a solution mapping $S : P \longrightarrow 2^{\mathbb{X}}$, let $\bar p \in P$ and $\bar x \in S(\bar p)$. Suppose that:*

(h_1) *C is prox-regular at $f(\bar p,\bar x)$;*

(h_2) *mapping $f(\bar p,\cdot)$ is calm and lower scalarly exhaustible from below at each point of a neighbourhood of $\bar x$;*

(h_3) *mapping f is partially calm at $(\bar p,\bar x)$ with respect to p, uniformly in x, with modulus $\mathrm{clm}_p f(\bar p,\bar x)$;*

(h_4) *it holds*

$$\mathrm{c}^{\downarrow}(f(\bar p,\cdot),\bar x) > 0.$$

Then, it results in

$$\overline{|\nabla(d_C\circ f(\bar p,\cdot))|}^{>}(\bar x) \ge \mathrm{c}^{\downarrow}(f(\bar p,\cdot),\bar x). \tag{21}$$

Consequently, mapping S is calm at $(\bar p,\bar x)$ and the following estimate holds

$$\mathrm{clm}\ S(\bar p,\bar x) \le \frac{\mathrm{clm}_p f(\bar p,\bar x)}{\mathrm{c}^{\downarrow}(f(\bar p,\cdot),\bar x)}.$$

Proof. The argument proposed in order to prove the thesis relies on the application of the sufficient condition for calmness of constraint system provided by Theorem 2.1, with $\psi_{C,\delta_1} = d_C$. Notice that, by virtue of hypotheses (h_1) and (h_2) it is possible to invoke Lemma 3.14, where $f(\bar p,\cdot)$ plays now the role of mapping g. In force of such lemma, there exists $\zeta > 0$ such that

$$\begin{aligned}&\mathcal{E}^{\downarrow}_*(d_C\circ f(\bar p,\cdot),x)\\ &= \mathcal{E}^{\downarrow}_*(f(\bar p,\cdot),x)(\widehat{\mathrm{D}}d_C(f(\bar p,x))),\quad \forall x\in \mathrm{B}(\bar x,\zeta),\ \text{with } d_C(f(\bar p,x))>0.\end{aligned} \tag{22}$$

Now, let us verify that all hypotheses of Theorem 2.1 are satisfied under the current assumptions. Hypothesis (h_2) is clearly fulfilled, inasmuch as mapping

$f(\bar{p},\cdot)$ is calm at each point of a neighbourhood of $\bar{x}$. Thus, it remains to show that

$$\overline{|\nabla(d_C \circ f(\bar{p},\cdot))|}^{>}(\bar{x}) > 0. \tag{23}$$

According to hypothesis (h_4), there exist $\gamma > 0$ and $\epsilon_\gamma > 0$ such that

$$\begin{aligned} &|||\mathcal{E}^{\downarrow}_{*}(f(\bar{p},\cdot),x)(\widehat{\mathrm{D}}d_C(f(\bar{p},x)))||| > \gamma, \\ &\qquad \forall x \in \mathrm{B}(\bar{x},\epsilon_\gamma),\ \text{with } 0 < d_C(f(\bar{p},x)) \le \epsilon_\gamma. \end{aligned} \tag{24}$$

Without any loss of generality it is possible to assume that $\epsilon_\gamma < \zeta$. Fix an arbitrary $x \in \mathrm{B}(\bar{x},\epsilon_\gamma)$, with $0 < d_C(f(\bar{p},x)) \le \epsilon_\gamma$. According to inequality (24) there must exist $E_0 \in \mathcal{E}^{\downarrow}_{*}(f(\bar{p},\cdot),x)(\widehat{\mathrm{D}}d_C(f(\bar{p},x)))$ such that

$$\mathrm{dist}(\mathbf{0}^*, E_0) > \gamma.$$

By virtue of Lemma 3.10 the last inequality implies the existence of $u_0 \in \mathbb{S}$ with the property

$$\varsigma(u_0, E_0) < -\gamma.$$

Thus, it follows

$$\inf_{E\in\mathcal{E}^{\downarrow}_{*}(f(\bar{p},\cdot),x)(\widehat{\mathrm{D}}d_C(f(\bar{p},x)))} \varsigma(u_0,E) \le \varsigma(u_0,E_0) < -\gamma,$$

whence one gets

$$\sup_{u\in\mathbb{S}}[-\inf_{E\in\mathcal{E}^{\downarrow}_{*}(f(\bar{p},\cdot),x)(\widehat{\mathrm{D}}d_C(f(\bar{p},x)))} \varsigma(u,E)] > \gamma. \tag{25}$$

Since $x \in \mathrm{B}(\bar{x},\zeta)$ and $d_C(f(\bar{p},x)) > 0$, equality (22) can be applied, so from (25) along with Lemma 3.1 one obtains

$$|\nabla(d_C \circ f(\bar{p},\cdot))|(x) \ge \sup_{u\in\mathbb{S}}[-(d_C\circ f(\bar{p},\cdot))^{\downarrow}_{\mathcal{H}}(\bar{x};u)] > \gamma.$$

Thus, it has been shown that there exist γ and $\epsilon_\gamma > 0$ such that

$$\inf\{|\nabla(d_C \circ f(\bar{p},\cdot))|(x) :\ x \in \mathrm{B}(\bar{x},\epsilon_\gamma),\ 0 < d_C(f(\bar{p},x)) \le \epsilon_\gamma\} \ge \gamma.$$

By recalling the definition of strict outer slope of a function, the last inequality enables to deduce the estimate (21) to be proved in the first assertion of the thesis. As a by-product of the above reasoning, one finds that also condition (23) is fulfilled. Thus, the second assertion in the thesis as well as the related estimate become a trivial consequence of Theorem 2.1. This completes the proof. □

Within the proposed approach, there is nothing to prevent one from considering how the above result works in the case of constraint systems defined by a smooth mapping $f(\bar{p},\cdot)$. As one expects, this circumstance simplifies the formulation of Theorem 4.1. Indeed, in such event it is possible to define the constant

$$\begin{aligned} &\hat{\mathrm{c}}^{\downarrow}(f(\bar{p},\cdot),\bar{x}) \\ &= \lim_{\epsilon\to 0^+} \inf\{\|\widehat{\mathrm{D}}f(\bar{p},\cdot)(x)^*[\widehat{\mathrm{D}}d_C(f(\bar{p},x))]\| :\ x \in \mathrm{B}(\bar{x},\epsilon),\ 0 < d_C(f(\bar{p},x)) \le \epsilon\}. \end{aligned}$$

By employing such new constant, one achieves the next condition for calmness, which is specific for smooth constraint systems.

Corollary 4.2. *Given a constraint system* $(\mathcal{CS})$ *with solution mapping* $S : P \longrightarrow 2^{\mathbb{X}}$, *let* $\bar{p} \in P$ *and* $\bar{x} \in S(\bar{p})$. *Suppose that:*

(h_1) *C is prox-regular at $f(\bar{p}, \bar{x})$;*
(h_2) *mapping $f(\bar{p}, \cdot)$ is Fréchet differentiable at each point of an open neighbourhood of $\bar{x}$;*
(h_3) *mapping f is partially calm at $(\bar{p}, \bar{x})$ with respect to p, uniformly in x, with modulus* $\mathrm{clm}_p\, f(\bar{p}, \bar{x})$;
(h_4) *it holds*

$$\hat{\mathrm{c}}^{\downarrow}(f(\bar{p}, \cdot), \bar{x}) > 0.$$

Then, it results in

$$\overline{|\nabla(d_C \circ f(\bar{p}, \cdot))|}^{>}(\bar{x}) \geq \hat{\mathrm{c}}^{\downarrow}(f(\bar{p}, \cdot), \bar{x}).$$

Consequently, mapping S is calm at $(\bar{p}, \bar{x})$ and the following estimate holds

$$\mathrm{clm}\, S(\bar{p}, \bar{x}) \leq \frac{\mathrm{clm}_p\, f(\bar{p}, \bar{x})}{\hat{\mathrm{c}}^{\downarrow}(f(\bar{p}, \cdot), \bar{x})}.$$

Proof. It suffices to observe that $f(\bar{p}, \cdot)$ is calm at x, as a consequence of hypothesis (h_2). Indeed, by the Fréchet differentiability assumption on $f(\bar{p}, \cdot)$ at x one has the first-order expansion

$$f(\bar{p}, x + h) - f(\bar{p}, x) = \widehat{\mathrm{D}} f(\bar{p}, x)(h) + \mathrm{o}(\|h\|)$$

whence, taking into account that for some $\delta > 0$ it is

$$\|\mathrm{o}(\|h\|)\| \leq \|h\|, \quad \forall h \in \mathrm{B}(\mathbf{0}, \delta),$$

one obtains

$$\|f(\bar{p}, x + h) - f(\bar{p}, x)\| = (\|\widehat{\mathrm{D}} f(\bar{p}, x)\| + 1)\|h\|, \quad \forall h \in \mathrm{B}(\mathbf{0}, \delta).$$

Thus, the thesis follows from being

$$|||\mathcal{E}_*(f(\bar{p}, \cdot), x)(\widehat{\mathrm{D}} d_C(f(\bar{p}, x)))||| = \|\widehat{\mathrm{D}} f(\bar{p}, \cdot)(x)^*[\widehat{\mathrm{D}} d_C(f(\bar{p}, x))]\|,$$

according to what observed in Example 3.5, and hence

$$\mathrm{c}^{\downarrow}(f(\bar{p}, \cdot), \bar{x}) = \hat{\mathrm{c}}^{\downarrow}(f(\bar{p}, \cdot), \bar{x}). \qquad \square$$

Remark 4.3. (r_1) In the light of Example 3.7, Theorem 4.1 indicates the class of locally uniformly convex Banach spaces as a proper setting, where verifiable sufficient conditions allow one to perform the analysis of calmness for nonsmooth constraint systems. Indeed, the continuity of the scalarized Hadamard lower directional derivative of the mapping $f(\bar{p}, \cdot)$ paves the way to the dual construction of the constant $\mathrm{c}^{\downarrow}(f(\bar{p}, \cdot), \bar{x})$, whose nondegeneracy gives answer to the question.

(r_2) After the submission of the present paper the author's attention has been drawn to the fact that, quite recently, the notion of prox-regular set has been extended beyond the Hilbert space setting, to uniformly convex Banach space (see [3]). Since to deal with such an extension requires a good amount of technicalities and the case of Hilbert space already covers significant applications, consequent extensions of the result presented above are left for a future research work.

(r_3) In the light of what noticed in Remark 3.15, it holds

$$\begin{aligned}&\mathrm{c}^{\downarrow}(f(\bar{p},\cdot),\bar{x})\\ \geq\ &\lim_{\epsilon\to 0^+}\inf\{\inf_{y^*\in\mathbb{S}^*}|||\mathcal{E}^{\downarrow}_*(f(\bar{p},\cdot),x)(y^*)|||:\ x\in\mathrm{B}(\bar{x},\epsilon),\ 0<d_C(f(\bar{p},x))\leq\epsilon\}.\end{aligned}$$

In linear functional analysis, given a linear bounded operator $\Lambda:\mathbb{X}\longrightarrow\mathbb{Y}$ between Banach spaces, the value

$$\inf_{y^*\in\mathbb{S}^*}\|\Lambda^* y^*\|$$

is sometimes referred to as the *dual Banach constant* of Λ. According to a modern reformulation of the Banach-Schauder theorem, Λ is open at a linear rate (equivalently, the solution mapping $\Lambda^{-1}:\mathbb{Y}\longrightarrow 2^{\mathbb{X}}$ for the equation $\Lambda x=y$ is Lipschitz-like and, hence, calm) iff its dual Banach constant does not vanish. Thus, the nondegeneracy condition expressed by hypothesis (h_4) can be regarded as a nonsmooth/nonlinear development of a condition for the Lipschitzian behaviour of solution mappings historically tracing back to the celebrated open mapping theorem.

4.2. The convex case in general Banach spaces

In the previous subsection, the requirement of prox-regularity imposed on the set C has forced to limit the analysis to constraint systems defined by mappings with values in Hilbert spaces. Here such limitation is removed, in order to consider what can be done, if coming back to mappings between general Banach spaces. Therefore, throughout this subsection $(\mathbb{X},\|\cdot\|)$ and $(\mathbb{Y},\|\cdot\|)$ will be both Banach spaces. The price to be paid for such an extension in the approach under study is the reintroduction of a convexity assumption on C. In this setting, define the constant

$$\begin{aligned}\mathrm{c}^{\uparrow}(f(\bar{p},\cdot),\bar{x})=\lim_{\epsilon\to 0^+}\inf\{\lim_{\eta\to 0^+}\inf_{y^*\in\mathcal{U}(f(\bar{p},x),\eta)}|||\mathcal{E}^{\uparrow}_*(f(\bar{p},\cdot),x)(y^*)|||:\\ x\in\mathrm{B}(\bar{x},\epsilon),\ 0<d_C(f(\bar{p},x))\leq\epsilon\},\end{aligned}$$

where $\mathcal{U}(f(\bar{p},x),\eta)$ has the same meaning as in Lemma 3.11. By means of it, it is possible to formulate the next condition for calmness of the solution mapping to parameterized constraint systems.

Theorem 4.4. *With reference to a constraint system $(\mathcal{CS})$, having a solution mapping $S:P\longrightarrow 2^{\mathbb{X}}$, let $\bar{p}\in P$ and $\bar{x}\in S(\bar{p})$. Suppose that:*

(h_1) *C is closed and convex;*

(h_2) *mapping* $f(\bar{p},\cdot)$ *is continuous and upper scalarly exhaustible from below at each point of a neighbourhood of* $\bar{x}$*;*

(h_3) *mapping* f *is partially calm at* $(\bar{p},\bar{x})$ *with respect to* p*, uniformly in* x*, with modulus* $\mathrm{clm}_p\, f(\bar{p},\bar{x})$*;*

(h_4) *it holds*

$$\mathrm{c}^{\uparrow}(f(\bar{p},\cdot),\bar{x}) > 0.$$

Then, it results in

$$\overline{|\nabla(d_C \circ f(\bar{p},\cdot))|}^{>}(\bar{x}) \geq \mathrm{c}^{\uparrow}(f(\bar{p},\cdot),\bar{x}).$$

Consequently, mapping S *is calm at* $(\bar{p},\bar{x})$ *and the following estimate holds*

$$\mathrm{clm}\ S(\bar{p},\bar{x}) \leq \frac{\mathrm{clm}_p\, f(\bar{p},\bar{x})}{\mathrm{c}^{\uparrow}(f(\bar{p},\cdot),\bar{x})}.$$

Proof. It suffices to prove the first assertion in the thesis, inasmuch as in the presence of the current assumptions the second one becomes a trivial consequence of the sufficient condition for constraint system calmness established in Theorem 2.1, with $\psi_{C,\delta_1} = d_C$, being

$$\overline{|\nabla(d_C \circ f(\bar{p},\cdot))|}^{>}(\bar{x}) > 0.$$

This will be done by exploiting hypothesis (h_4). Indeed, according to it there exist $\gamma > 0$ and $\epsilon_\gamma > 0$ such that

$$\lim_{\eta\to 0^+} \inf_{y^*\in\mathcal{U}(f(\bar{p},x),\eta)} |||\mathcal{E}^{\uparrow}_*(f(\bar{p},\cdot),x)(y^*)||| > \gamma,$$
$$\forall x \in \mathrm{B}(\bar{x},\epsilon_\gamma),\ \text{with } 0 < d_C(f(\bar{p},x)) \leq \epsilon_\gamma. \tag{26}$$

Fix an arbitrary $x \in \mathrm{B}(\bar{x},\epsilon_\gamma)$, with $0 < d_C(f(\bar{p},x)) \leq \epsilon_\gamma$. From inequality (26) one deduces the existence of $\eta_0 \in (0, d_C(f(\bar{p},x))$ such that

$$\forall y^* \in \mathcal{U}(f(\bar{p},x),\eta_0)\ \ |||\mathcal{E}^{\uparrow}_*(f(\bar{p},\cdot),x)(y^*)||| > \gamma.$$

For every $y^* \in \mathcal{U}(f(\bar{p},x),\eta_0)$ the last inequality entails that for some $E_0 \in \mathcal{E}^{\uparrow}_*(f(\bar{p},\cdot),x)(y^*)$ it must be

$$\mathrm{dist}(\mathbf{0}^*, E_0) > \gamma.$$

According to Lemma 3.10, this means that there is $u_0 \in \mathbb{S}$ with the property

$$\varsigma(u_0, E_0) < -\gamma.$$

It follows that

$$\inf_{E\in\mathcal{E}^{\uparrow}_*(f(\bar{p},\cdot),x)(y^*)} \varsigma(u_0, E) \leq \varsigma(u_0, E_0) < -\gamma. \tag{27}$$

As, on account of hypothesis (h_2), mapping $f(\bar{p},\cdot)$ is upper scalarly exhaustible from below at x, from inequality (27) one obtains

$$(d_C \circ f(\bar{p},\cdot))^{\uparrow}_{\mathcal{H}}(x;u_0) = \inf_{E\in\mathcal{E}^{\uparrow}_{*}(f(\bar{p},\cdot),x)(y^*)} \varsigma(u_0,E) < -\gamma.$$

Such estimation being valid for every $y^* \in \mathcal{U}(f(\bar{p},x),\eta_0)$, from the above inequality one deduces

$$\begin{aligned} &\lim_{\eta\to 0^+} \sup_{y^*\in\mathcal{U}(f(\bar{p},x),\eta)} (d_C \circ f(\bar{p},\cdot))^{\uparrow}_{\mathcal{H}}(x;u_0) \\ \le &\sup_{y^*\in\mathcal{U}(f(\bar{p},x),\eta_0)} (d_C \circ f(\bar{p},\cdot))^{\uparrow}_{\mathcal{H}}(x;u_0) \le -\gamma. \end{aligned}$$

In the light of the estimate established in Lemma 3.11, what has been so far obtained leads to the inequality

$$-(d_C \circ f(\bar{p},\cdot))^{\uparrow}_{\mathcal{H}}(x;u_0) \ge \gamma,$$

and hence, by recalling inequality (13) in Remark 3.2,

$$|\nabla(d_C \circ f(\bar{p},\cdot))|(x) \ge \sup_{u\in\mathbb{S}}[-(d_C \circ f(\bar{p},\cdot))^{\uparrow}_{\mathcal{H}}(x;u)] \ge -(d_C \circ f(\bar{p},\cdot))^{\uparrow}_{\mathcal{H}}(x;u_0) \ge \gamma.$$

By arbitrariness of $x \in \mathrm{B}(\bar{x},\epsilon_\gamma)$, with $0 < d_C(f(\bar{p},x)) \le \epsilon_\gamma$, the above argument shows that

$$\overline{|\nabla(d_C \circ f(\bar{p},\cdot))|}^{>}(\bar{x}) \ge \gamma.$$

Notice that such inequality is true for every $\gamma \in (0, \mathrm{c}^{\uparrow}(f(\bar{p},\cdot),\bar{x}))$, so that it is possible to assert that

$$\overline{|\nabla(d_C \circ f(\bar{p},\cdot))|}^{>}(\bar{x}) \ge \mathrm{c}^{\uparrow}(f(\bar{p},\cdot),\bar{x}).$$

This allows one to obtain the estimate of the calmness modulus of S in the thesis, thereby completing the proof. □

Again, in the presence of a smoothness property of the mapping $f(\bar{p},\cdot)$, from the above general calmness condition it is possible to derive a simplified version. To do so, define the constant

$$\begin{aligned} \hat{\mathrm{c}}^{\uparrow}(f(\bar{p},\cdot),\bar{x}) = &\lim_{\epsilon\to 0^+} \inf\{ \lim_{\eta\to 0^+} \inf_{y^*\in\mathcal{U}(f(\bar{p},x),\eta)} \|\widehat{\mathrm{D}}f(\bar{p},\cdot)(x)^*[y^*]\| : \\ &x \in \mathrm{B}(\bar{x},\epsilon),\ 0 < d_C(f(\bar{p},x)) \le \epsilon\}. \end{aligned}$$

The resulting calmness condition runs as follows.

Corollary 4.5. *Let $(\mathcal{CS})$ be a constraint system with solution mapping $S: P \longrightarrow 2^{\mathbb{X}}$, let $\bar{p} \in P$ and $\bar{x} \in S(\bar{p})$. Suppose that:*

(h_1) *C is closed and convex;*

(h_2) *mapping $f(\bar{p},\cdot)$ is Fréchet differentiable at each point of an open neighbourhood of $\bar{x}$;*

(h_3) *mapping f is partially calm at $(\bar{p}, \bar{x})$ with respect to p, uniformly in x, with modulus* $\mathrm{clm}_p f(\bar{p}, \bar{x})$;

(h_4) *it holds*

$$\hat{\mathrm{c}}^{\uparrow}(f(\bar{p}, \cdot), \bar{x}) > 0.$$

Then, it results in

$$\hat{\mathrm{c}}^{\uparrow}(f(\bar{p}, \cdot), \bar{x}) = \mathrm{c}^{\uparrow}(f(\bar{p}, \cdot), \bar{x}).$$

Consequently, mapping S is calm at $(\bar{p}, \bar{x})$ and the following estimate holds

$$\mathrm{clm}\, S(\bar{p}, \bar{x}) \leq \frac{\mathrm{clm}_p f(\bar{p}, \bar{x})}{\hat{\mathrm{c}}^{\uparrow}(f(\bar{p}, \cdot), \bar{x})}.$$

Proof. It suffices to observe that, by hypothesis (h_2), mapping $f(\bar{p}, \cdot)$ is upper scalarly exhaustible from below near $\bar{x}$, with upper scalar exhausters

$$\mathcal{E}^{\uparrow}_{*}(f(\bar{p}, \cdot), x)(y^*) = \{\, \{\widehat{\mathrm{D}} f(\bar{p}, \cdot)(x)^*[y^*]\}\, \}, \quad y^* \in \mathbb{Y}^*$$

(remember Example 3.5). This leads to the equalitity in the thesis. Thus, being

$$\overline{|\nabla(d_C \circ f(\bar{p}, \cdot))|}^{>}(\bar{x}) \geq \hat{\mathrm{c}}^{\uparrow}(f(\bar{p}, \cdot), \bar{x}) > 0$$

according to (h_4), it is possible to apply Theorem 4.4 along with the related estimate. □

Remark 4.6. (r_1) What has been noted in Remark 4.3 can be repeated as a comment to Theorem 4.4, with the additional remark that, in the setting of locally uniformly convex Banach spaces, for assumption (h_2) to be satisfied one has only to suppose that each function $(y^* \circ f(\bar{p}, \cdot))^{\uparrow}_{\mathcal{H}}(\bar{x}; \cdot)$ is real-valued (remember Example 3.7).

(r_2) It is worth noting that, whenever $\mathbb{Y}$ is a Hilbert space, then, being C prox-regular as a closed convex subset of $\mathbb{Y}$, for every $\eta > 0$ sufficiently small it results in

$$\mathcal{U}(f(\bar{p}, x), \eta) = \bigcup_{y \in \mathrm{B}(f(\bar{p}, x), \eta)} \{\widehat{\mathrm{D}} d_C(y)\}, \quad \forall x \in \mathrm{B}(\bar{x}, \epsilon), \text{ with } 0 < d_C(f(\bar{p}, x)) \leq \epsilon$$

(recall Lemma 3.13). Taking into account that d_C is continuously differentiable on $\mathrm{B}(f(\bar{p}, x), \eta)$ and $f(\bar{p}, \cdot)$ is smooth near $\bar{x}$, one obtains

$$\begin{aligned}
&\lim_{\eta \to 0^+} \inf_{y^* \in \mathcal{U}(f(\bar{p}, x), \eta)} \|\widehat{\mathrm{D}} f(\bar{p}, \cdot)(x)^*[y^*]\| \\
&= \lim_{\eta \to 0^+} \inf_{y \in \mathrm{B}(f(\bar{p}, x), \eta)} \|\widehat{\mathrm{D}} f(\bar{p}, \cdot)(x)^*[\widehat{\mathrm{D}} d_C(y)]\| \\
&= \lim_{y \to f(\bar{p}, x)} \|\widehat{\mathrm{D}} f(\bar{p}, \cdot)(x)^*[\widehat{\mathrm{D}} d_C(y)]\| = \|\widehat{\mathrm{D}} f(\bar{p}, \cdot)(x)^*[\widehat{\mathrm{D}} d_C(f(\bar{p}, x))]\|.
\end{aligned}$$

Thus, in such event, one finds

$$\hat{\mathrm{c}}^{\downarrow}(f(\bar{p}, \cdot), (\bar{x})) = \hat{\mathrm{c}}^{\uparrow}(f(\bar{p}, \cdot), (\bar{x})),$$

so Corollary 4.5 becomes a consequence of Corollary 4.2.

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