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Geometric Properties of Maximal Monotone Operators and Convex Functions which May Represent Them

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We study the relations between some geometric properties of maximal monotone operators and generic geometric and analytical properties of the functions on the associate Fitzpatrick family of convex representations. We present some new technical properties of the Fitzpatrick families associated to bounded-range and bounded-domain maximal monotone operator which infer, among other properties, that the convex hull of the range of bounded domain maximal monotone operators are weak-$*$ dense; we also present sufficient conditions for a convex function to represent a bounded-range maximal monotone operator; and finally give an example which makes clear that the result of Zagrodny that maximal monotone operators with a relatively compact range are of type (D) can not be extended to the weak* topology.

Keywords: Fitzpatrick function, maximal monotone operator, bounded range, non-reflexive Banach spaces

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1. Introduction

Fitzpatrick functions [5] are convex functions which represent maximal monotone operators. A natural question is whether geometric features of maximal monotone operators are related with generic analytical or geometric features of their convex representations. We present in this work some answers to this question. Using these results, we prove that the convex hull of the range of bounded domain maximal monotone operators are weak-$*$ dense.

Another natural question is whether a convex function represents a maximal monotone operator. Although this question is reasonably settled in reflexive Banach spaces [3], in non-reflexive Banach spaces we have, up to now, what seems to be partial results [7, 9, 11, 10] (see also [8, 12]). In this work we answer this question for the case of bounded-range maximal monotone operators and

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show that there exist non-type (D) operators in this class.

This work is heavily based on results and technique previously developed in collaboration with J. E. Martínez-Legaz [14, 15], M. Marques Alves [7, 9, 11, 10, 8, 12] and O. Bueno [1].

This work is organized as follows. In Section 2 we establish the notation and recall some basic results. In Section 3 we analyze some properties of the domains of extended real valued functions defined in the Cartesian product of a Banach space with its dual. In Section 4 we characterize the properties of the Fitzpatrick families associated to bounded-range and bounded-domain maximal monotone operators. In Section 5 we characterize the convex functions which represent bounded range maximal monotone operators and provide an example of a non-type (D) operator in this class.

Throughout this work, $\overline{\mathbb{R}}$ stands for the extended real set $\mathbb{R} \cup \{\infty, -\infty\}$ and, in a Cartesian product $A \times B$, we will use the notation P_1 and P_2 for the canonical projections onto the first and onto the second term of the product respectively, that is

$$P_1 : A \times B \to A,\ P_1(a,b) = a, \qquad P_2 : A \times B \to B,\ P_2(a,b) = b.$$

2. Notation and Basic Results

Let X be a real Banach space, possible non-reflexive, with topological dual denoted by X^* and endowed with the canonical dual norm. We will use the notation $\langle \cdot, \cdot \rangle$ for the duality product in $X \times X^*$,

$$\langle x, x^* \rangle = \langle x^*, x \rangle = x^*(x), \quad x \in X,\ x^* \in X^*.$$

In this work, X^{**}, X^{***}, etc., stands for $(X^*)^*$, $((X^*)^*)^*$, etc. Recall that X^*, $X^{**}, \ldots,$ are Banach spaces. We will identify X with its image by the canonical injection of X into X^{**}, that is, each $x \in X$ is identified with the (continuous, linear) functional

$$X^* \to \mathbb{R}, \quad x^* \mapsto \langle x, x^* \rangle.$$

The same identification will be used for X^*, X^{**}, etc. with respect to X^{***}, X^{****}, etc. The Banach space X is *reflexive* if this injection, of X into X^{**}, is onto, and non-reflexive otherwise. The weak-$*$ topology of X^* is the smallest topology (in X^*) in which the functional $x^* \mapsto \langle x, x^* \rangle$ are continuous for all $x \in X$.

A closed ball in X, with radius r and center 0, will be denoted by $B_X[r]$,

$$B_X[r] = \{x \in X \mid \|x\| = r\}.$$

The closure and convex hull of $C \subseteq X$ will be denoted by $\operatorname{cl} C$ and $\operatorname{conv} C$, respectively. The closure of $C \subset X^*$ in the weak-$*$ topology will be denoted by $\operatorname{cl}_{w*}(C)$. The set of *directions of recession* of $C \subseteq X$ or *recession cone* of C is defined as

$$0^+C = \{u \in X \mid x + \lambda u \in C,\ \forall \lambda \geq 0,\ \forall x \in C\}.$$

The *indicator function* of $C \subseteq X$ is

$$\delta_C : X \to \overline{\mathbb{R}}, \quad \delta_C(x) = \begin{cases} 0, & x \in C \\ \infty, & \text{otherwise.} \end{cases} \tag{1}$$

A point-to-set operator $T : X \rightrightarrows X^*$ is a relation $T \subset X \times X^*$ and

$$T(x) = \{x^* \in X^* \mid (x, x^*) \in T\}, \quad x \in X.$$

The domain and range of $T : X \rightrightarrows X^*$ are (respectively)

$$D(T) = \{x \in X \mid T(x) \neq \emptyset\} = P_1 T,$$
$$R(T) = \{x^* \in X \mid \exists x \in X,\, x^* \in T(x)\} = P_2 T.$$

An operator $T : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in T$$

and it is *maximal monotone* if it is monotone and maximal in the family of monotone operators with respect to the inclusion. A maximal monotone $T : X \rightrightarrows X^*$ is of *Gossez type (D)* [6] if any point in the set

$$\{(x^{**}, x^*) \in X^{**} \times X^* \mid \langle x^{**} - y, x^* - y^* \rangle \geq 0,\ \forall (y, y^*) \in T\}$$

is the weak-$*\times$strong limit of a *bounded* net of points in T.

Let $f : X \to \overline{\mathbb{R}}$. The function f is *proper* if $f \not\equiv \infty$ and $f > -\infty$. The function f is *closed* if it is lower semi-continuous and $\operatorname{cl} f$, the closure of f, is the largest closed function majorized by f. The *domain* of f is

$$D(f) = \{x \in X \mid f(x) < \infty\}.$$

We use the notation $\operatorname{conv} f$ for the lower convex envelop of f, which is the largest convex function majorized by f. The *conjugate* of $f : X \to \overline{\mathbb{R}}$ is

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(x^*) = \sup_{x \in X} \langle x, x^* \rangle - f(x).$$

Latter on we will need the following elementary result

Proposition 2.1. *Let $f : X \to \overline{\mathbb{R}}$ be a proper closed convex function. For any $\lambda \geq 0$, $z^*, w^* \in X^*$*

$$f^*(z^*) + \lambda \delta^*_{D(f)}(w^*) \geq f^*(z^* + \lambda w^*)$$

where $\delta^*_{D(f)}$ stands for $(\delta_{D(f)})^*$, the support function of $D(f)$.

Proof. See Appendix. □

In the study of convex functions in $X \times X^*$ it is convenient to define the operator $\mathcal{J}$ [2],

$$\mathcal{J} : \overline{\mathbb{R}}^{X \times X^*} \to \overline{\mathbb{R}}^{X \times X^*}, \quad (\mathcal{J}\, h)(x, x^*) = h^*(x^*, x). \tag{2}$$

As remarked in [2], the operator $\mathcal{J}$ is a generalized Moreau conjugation by means of the (symmetric) coupling function

$$\Phi : (X \times X^*) \times (X \times X^*) \to \mathbb{R}, \quad \Phi((x, x^*), (y, y^*)) = \langle x, y^* \rangle + \langle y, x^* \rangle$$

so that

$$\mathcal{J}h(z) = h^{\Phi}(z) = \sup_{z' \in X \times X^*} \Phi(z, z') - h(z').$$

Hence, we may call the operator $\mathcal{J}$ also the Φ-conjugation. It will also be useful to have a notation for the (ordered) duality product, say

$$\pi : X \times X^* \to \mathbb{R}, \quad \pi(x, x^*) = \langle x, x^* \rangle. \tag{3}$$

Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. Fitzpatrick family $\mathcal{F}_T$ associated with T is defined as

$$\mathcal{F}_T = \left\{ h : X \times X^* \to \overline{\mathbb{R}} \;\middle|\; \begin{array}{l} h \text{ is convex and closed} \\ h(x, x^*) \geq \langle x, x^* \rangle, \ \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^* \rangle \end{array} \right\}. \tag{4}$$

The smallest function in $\mathcal{F}_T$ (which is always non-empty) is Fitzpatrick minimal function φ_T

$$\varphi_T : X \times X^* \to \overline{\mathbb{R}}, \quad \varphi_T(x, x^*) = \sup_{(y, y^*) \in T} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle. \tag{5}$$

Any function in the family $\mathcal{F}_T$ characterizes T in the following sense.

Theorem 2.2 ([5, Theorem 3.10]). *Let $T : X \rightrightarrows X^*$ be maximal monotone. For any $h \in \mathcal{F}_T$,*

$$T = \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}.$$

For any maximal monotone $T : X \rightrightarrows X^*$, the largest element of $\mathcal{F}_T$ is σ_T [2]

$$\sigma_T : X \times X \to \overline{\mathbb{R}}, \quad \sigma_T = \operatorname{cl} \operatorname{conv}(\pi + \delta_T) \tag{6}$$

and

$$\varphi_T(x, x^*) = \sigma_T{}^*(x^*, x) = (\pi + \delta_T)^*(x^*, x), \quad \forall (x, x^*) \in X \times X^*. \tag{7}$$

The elementary properties of σ_T, together with its inclusion in $\mathcal{F}_T$ will be instrumental for proving Lemma 4.1. Using (2) we obtain a very simple reformulation of the above relation,

$$\varphi_T = \mathcal{J}\sigma_T = \mathcal{J}(\pi + \delta_T). \tag{8}$$

The family $\mathcal{F}_T$ is invariant under $\mathcal{J}$, that is $\mathcal{J}(\mathcal{F}_T) \subseteq \mathcal{F}_T$. In particular, for any $h \in \mathcal{F}_T$,

$$h \geq \pi, \qquad \mathcal{J}h \geq \pi. \tag{9}$$

In *reflexive* Banach spaces the two above inequalities are a sufficient condition for a proper closed convex function h to represent a maximal monotone operator [2]. In non-reflexive Banach spaces, up to now, additional conditions are required on h [7, 9, 11, 10]. The next result was essentially proved in [10, Theorem 3.1, Corollary 3.2]. We will present it as stated in [13, Theorem 2.1]

Theorem 2.3. *If $h : X \times X^* \to \overline{\mathbb{R}}$ is a closed convex function, $h \geq \pi$, $\mathcal{J}h \geq \pi$ and, for some $x_0 \in X$,*

$$\bigcup_{\lambda>0} \lambda(P_1 D(h) - x_0)$$

is a closed subspace, then

$$T = \{\zeta \in X \times X^* \mid \mathcal{J}h(\zeta) = \pi(\zeta)\}$$

is maximal monotone, $\mathcal{J}h \in \mathcal{F}_T$. If, additionally h is lower semi-continuous in the strong$\times$weak-$*$ topology, then $h \in \mathcal{F}_T$.

3. Extended real valued functions in $X \times X^*$ and their conjugates

In this section we study extended real valued functions in $X \times X^*$. We start with a general and simple result.

Proposition 3.1. *If $h : X \times X^* \to \overline{\mathbb{R}}$ is a proper closed convex function then $D(\mathcal{J}\delta_{D(h)}) \subseteq 0^+ D(\mathcal{J}h)$.*

Proof. Use definition (2) and Proposition 2.1 with $f = h$ □

Now we study those extended real valued functions in $X \times X^*$ which satisfy one of the inequalities in (9). The two next propositions are the main technical tools in this work.

Lemma 3.2. *If $h : X \times X^* \to \overline{\mathbb{R}}$ and $\mathcal{J}h \geq \pi$ (where $\mathcal{J}$, π are as in (2), (3)), then*

$$P_2(D(\mathcal{J}h)) = P_1(D(h^*) \cap (X^* \times X)) \subseteq \mathrm{cl}_{w*}\,\mathrm{conv}\, P_2(D(h)),$$
$$P_1(D(\mathcal{J}h)) = P_2(D(h^*) \cap (X^* \times X)) \subseteq \mathrm{cl}\,\mathrm{conv}\, P_1(D(h)).$$

Proof. If $h(x, x^*) = -\infty$ for some (x, x^*) the proposition holds trivialy. Assume that h is proper. The two equalities follows triviality from definition (2). To prove the first inclusion, suppose that

$$u^* \notin \mathrm{cl}_{w*}\,\mathrm{conv}\, P_2(D(h)), \quad u^* \in P_1(D(h^*) \cap (X^* \times X)). \tag{10}$$

From the first above relation and the geometric form of Hahn-Banach Lemma in X^* endowed with the weak-$*$ topology it follows that there exist $\hat{x} \in X$ and $\varepsilon > 0$ such that

$$\langle \hat{x}, u^* \rangle \geq \langle \hat{x}, y^* \rangle + \varepsilon, \quad \forall y^* \in P_2(D(h^*)),$$

while the second relation in (10) means that $\infty > h^*(u^*, u) \geq \langle u^*, u \rangle$ for some $u \in X$. By the definition of the conjugation

$$h^*(u^*, u) \geq \langle y, u^* \rangle + \langle u, y^* \rangle - h(y, y^*), \quad \forall (y, y^*) \in X \times X^*.$$

Combining the two above inequalities we conclude that, for any $\lambda \geq 0$,

$$h^*(u^*, u) + \lambda[\langle \hat{x}, u^* \rangle - \varepsilon] \geq \langle y, u^* \rangle + \langle u + \lambda \hat{x}, y^* \rangle - h(y, y^*).$$

Therefore, taking the sup for $(y, y^*) \in X \times X^*$ in the right hand side of the above inequality we have

$$h^*(u^*, u) + \lambda[\langle \hat{x}, u^* \rangle - \varepsilon] \geq h^*(u^*, u + \lambda \hat{x}) \geq \langle u + \lambda \hat{x}, u^* \rangle.$$

Dividing these inequalities by λ, noting that $h^*(u^*, u) \in \mathbb{R}$, and taking the limit $\lambda \to \infty$ we obtain $\langle \hat{x}, u^* \rangle - \varepsilon \geq \langle \hat{x}, u^* \rangle$ which is absurd. Therefore, there is no u^* as in (10) and the first inclusion on the lemma holds.

To prove the second inclusion, suppose that

$$u \notin \operatorname{cl} \operatorname{conv} P_1(D(h)), \quad u \in P_2(D(h^*) \cap (X^* \times X)). \tag{11}$$

From the first above relation and the geometric form of Hahn-Banach Lemma in X endowed with the strong topology it follows that there exist $\hat{x}^* \in X^*$ and $\varepsilon > 0$ such that

$$\langle u, \hat{x}^* \rangle \geq \langle y, \hat{x}^* \rangle + \varepsilon, \quad \forall y \in P_1(D(h)),$$

while the second relation in (11) means that there exists $u^* \in X^*$ such that $\infty > h^*(u^*, u) \geq \langle u, u^* \rangle$. Combining this inequality with the definition of the conjugation we conclude that for any $\lambda \geq 0$

$$h^*(u^*, u) + \lambda[\langle u, \hat{x}^* \rangle - \varepsilon] \geq \langle y, u^* + \lambda \hat{x}^* \rangle + \langle u, y^* \rangle - h(y, y^*).$$

Therefore, taking the sup in $y \in X, y^* \in X^*$ at the right hand side of the above inequality we have

$$h^*(u^*, u) + \lambda[\langle u, \hat{x}^* \rangle - \varepsilon] \geq h^*(u^* + \lambda \hat{x}^*, u) \geq \langle u, u^* + \lambda \hat{x}^* \rangle.$$

Dividing this inequality by λ, noting that $h^*(u^*, u) \in \mathbb{R}$, and taking the limit $\lambda \to \infty$ we obtain $\langle u, \hat{x}^* \rangle - \varepsilon \geq \langle u, \hat{x}^* \rangle$ which is absurd. Therefore, there is no u^* as in (11) and the second inclusion on the lemma holds. □

The next result is proved using the same reasoning as in Lemma 3.2.

Lemma 3.3. *If $h : X \times X^* \to \overline{\mathbb{R}}$ is convex and $h \geq \pi$ then*

$$P_2(D(h)) \subseteq \operatorname{cl}_{w*} P_1(D(h^*)), \qquad P_1(D(h)) \subseteq \operatorname{cl}_{w*} P_2(D(h^*)).$$

Proof. If $h \equiv \infty$ the lemma holds trivially. Since $\operatorname{cl} h$ is convex, majorizes the duality product, $h^* = (\operatorname{cl} h)^*$ and $D(h) \subset D(\operatorname{cl} h)$, we may assume that h is a proper closed convex function and, under this assumption, using Moreau Theorem we have

$$h(x, x^*) = h^{**}(x, x^*).$$

Suppose that

$$u^* \notin \operatorname{cl}_{w*} \operatorname{conv} P_1(D(h^*)), \quad u^* \in P_2(D(h)).$$

This means that there exists $\hat{x} \in X$ and $\varepsilon > 0$ such that

$$\langle \hat{x}, u^* \rangle \geq \langle \hat{x}, y^* \rangle + \varepsilon, \quad \forall y^* \in P_1(D(h^*)),$$

and that $\infty > h(u, u^*) \geq \langle u, u^* \rangle$ for some $u \in X$. Hence, by the definition of the conjugation

$$h(u, u^*) = h^{**}(u, u^*) \geq \langle y^{**}, u^* \rangle + \langle u, y^* \rangle - h^*(y^*, y^{**}) \quad \forall y^* \in X^*, y^{**} \in X^{**}.$$

Combining the two above equations we conclude that, for any $\lambda \geq 0$,

$$h(u, u^*) + \lambda[\langle \hat{x}, u^* \rangle - \varepsilon] \geq \langle y^{**}, u^* \rangle + \langle u + \lambda \hat{x}, y^* \rangle - h^*(y^*, y^{**}).$$

Therefore, taking the sup in $y^* \in X^*, y^{**} \in X^{**}$ at the right hand side of the above inequality we have

$$h(u, u^*) + \lambda[\langle \hat{x}, u^* \rangle - \varepsilon] \geq h^{**}(u + \lambda \hat{x}, u^*) = h(u + \lambda \hat{x}, u^*) \geq \langle u + \lambda \hat{x}, u^* \rangle.$$

Dividing this inequality by λ and taking the limit $\lambda \to \infty$ we obtain $\langle \hat{x}, u^* \rangle - \varepsilon \geq \langle \hat{x}, u^* \rangle$ which is absurd.

Suppose that

$$u \notin \operatorname{cl}_{w*} \operatorname{conv} P_2(D(h^*)), \quad u \in P_1(D(h)).$$

This means that there exists $\hat{x}^* \in X^*$ and $\varepsilon > 0$ such that

$$\langle u, \hat{x}^* \rangle \geq \langle y^{**}, \hat{x}^* \rangle + \varepsilon, \quad \forall y^{**} \in P_2(D(h^*)),$$

and that $\infty > h(u, u^*) \geq \langle u, u^* \rangle$ for some $u^* \in X^*$. Hence, by the definition of the conjugation

$$h(u, u^*) = h^{**}(u, u^*) \geq \langle y^{**}, u^* \rangle + \langle u, y^* \rangle - h^*(y^*, y^{**}) \quad \forall y^* \in X^*, y^{**} \in X^{**}.$$

Combining the two above equations we conclude that, for any $\lambda \geq 0$,

$$h(u, u^*) + \lambda[\langle u, \hat{x}^* \rangle - \varepsilon] \geq \langle y^{**}, u^* + \lambda \hat{x}^* \rangle + \langle u, y^* \rangle - h^*(y^*, y^{**}).$$

Therefore, taking the sup in $y^* \in X^*, y^{**} \in X^{**}$ at the right hand side of the above inequality we have

$$h(u, u^*) + \lambda[\langle u, \hat{x}^* \rangle - \varepsilon] \geq h^{**}(u, u^* + \lambda \hat{x}^*) = h(u, u^* + \lambda \hat{x}^*) \geq \langle u, u^* + \lambda \hat{x}^* \rangle.$$

Dividing this inequality by λ and taking the limit $\lambda \to \infty$ we obtain $\langle u, \hat{x}^* \rangle - \varepsilon \geq \langle u, \hat{x}^* \rangle$, which is absurd. □

Now we analyze those functions h for which $P_1D(h)$ or $P_2D(h)$ (or $P_1D(h^*)$, $P_2D(h^*)$) is bounded.

Proposition 3.4. *Let $h : X \times X^* \to \overline{\mathbb{R}}$ and $0 \leq L < \infty$.*

1. If $P_2D(h) \subseteq B_{X^*}[L]$ then

$$h^*(x^*, x^{**}) \leq h^*(x^*, z^{**}) + L\|x^{**} - z^{**}\|, \quad \forall x^* \in X^*, x^{**}, z^{**} \in X^{**}$$

and $D(h^*) = (P_1D(h^*)) \times X^{**}$.

2. If $P_1D(h) \subseteq B_X[L]$ then

$$h^*(x^*, x^{**}) \leq h^*(z^*, x^{**}) + L\|x^* - z^*\|, \quad \forall x^*, z^* \in X^*, x^{**} \in X^{**}$$

and $D(h^*) = X^* \times (P_2D(h^*))$.

Proof. Using the definition of the conjugate and the assumption on the domain of h we conclude that for any $x^* \in X^{**}$ and $x^{**}, z^{**} \in X^{**}$

$$\begin{aligned}
h^*(x^*, x^{**}) &= \sup_{(y,y^*)\in X\times X^*} \langle y, x^*\rangle + \langle x^{**}, y^*\rangle - h(y, y^*) \\
&= \sup_{(y,y^*)\in X\times X^*,\, \|y^*\|\leq L} \langle y, x^*\rangle + \langle x^{**}, y^*\rangle - h(y, y^*) \\
&= \sup_{(y,y^*)\in X\times X^*, \|y^*\|\leq L} \langle y, x^*\rangle + \langle z^{**}, y^*\rangle - h(y, y^*) + \langle x^{**} - z^{**}, y^*\rangle \\
&\leq \sup_{(y,y^*)\in X\times X^*, \|y^*\|\leq L} \langle y, x^*\rangle + \langle z^{**}, y^*\rangle - h(y, y^*) + L\|x^{**} - z^{**}\| \\
&= h^*(x^*, z^{**}) + L\|x^{**} - z^{**}\|
\end{aligned}$$

which proves the first result in item *1.* The second result in item *1.* follows triviality from the first result.

Item *2.* is proved by the same reasoning. □

Proposition 3.5. *Suppose that $h : X \times X^* \to \overline{\mathbb{R}}$ is a proper closed convex function and $0 \leq L < \infty$.*

1. If $P_1D(h^*) \subseteq B_{X^*}[L]$ then for any $x^* \in X^*$ and $x, z \in X$

$$h(x, x^*) \leq h(z, x^*) + L\|x - z\|, \quad \forall x^* \in X^*, x, z \in X$$

and $D(h) = X \times (P_2D(h))$;

2. if $P_2D(h^*) \subseteq B_{X^{**}}[L]$ then for any $x^* \in X^*$ and $x, z \in X$

$$h(x, x^*) \leq h(x, z^*) + L\|x^* - z^*\|, \quad \forall x^*, z^* \in X^*, x \in X$$

and $D(h) = (P_1D(h)) \times X^*$.

Proof. Since h is a proper closed convex function, according to Moreau Theorem we have

$$h(x, x^*) = h^{**}(x, x^*), \quad \forall (x, x^*) \in X \times X^*.$$

Items *1.* and *2.* follows directly from Proposition 3.4 applied to h^* and the above equation. □

We end this section combining the previous result.

Corollary 3.6. *If $h : X \times X^* \to \overline{\mathbb{R}}$, $\mathcal{J}h \geq \pi$ and $P_2(D(h)) \subseteq B_{X^*}[L]$ then*

1) $h^*(x^*, \cdot)$ *is L-Lipschitz continuous for any* $x^* \in P_1(D(h^*))$*;*
2) $D(h^*) = P_1(D(h^*)) \times X^{**} \subseteq B_{X^*}[L] \times X^{**}$*;*
3) $P_1 D(h^*) \subset B_{X^*}[L]$, $P_2 D(h^*) = X^{**}$.

Proof. Item *1)*, and the equality in item *2)* follows from Proposition 3.4 (item *1.*). In particular

$$D(h^*) \cap (X^* \times X) = (P_1 D(h^*)) \times X.$$

To end the prove of item *2)*, use the above equality and the first result in Lemma 3.2. Item *3)* follows trivially from item *2)*. □

Corollary 3.7. *If $h : X \times X^* \to \overline{\mathbb{R}}$ is a proper closed convex function, $h \geq \pi$ and $P_1(D(h^*)) \subseteq B_{X^*}[L]$ then*

1) $h(\cdot, x^*)$ *is L-Lipschitz continuous for any* $x^* \in P_2(D(h))$*;*
2) $D(h) = X \times P_2(D(h)) \subseteq X \times B_{X^*}[L]$*;*
3) $P_2 D(h) \subset B_{X^*}[L]$, $P_1(Dh) = X$.

Proof. Item *1)*, and the equality in item *2)* follows from Proposition 3.5 (item *1.*), while the inclusion in item *2)* follows from Lemma 3.3. Item *3)* follows trivially from item *2)*. □

4. Some geometric properties of maximal monotone operators and their relations with their Fitzpatrick families

In this section we discuss the relation of some elementary geometric properties of maximal monotone operators with elementary geometric and analytical (continuity) properties of the functions on Fitzpatrick families of these operators.

First we prove invariance of the closure of the projections of the domains along each Fitzpatrick family.

Lemma 4.1 (Invariant features of the domains). *Let $T : X \rightrightarrows X^*$ be maximal monotone. For any $h \in \mathcal{F}_T$,*

$$\operatorname{cl} P_1 D(h) = \operatorname{cl} \operatorname{conv} D(T), \qquad \operatorname{cl}_{w*} P_2 D(h) = \operatorname{cl}_{w*} \operatorname{conv} R(T).$$

Proof. Applying Lemma 3.2 to σ_T, $\varphi_T = \mathcal{J}\sigma_T$ and using definition (6) we conclude that

$$\operatorname{cl} P_1 D(\varphi_T) \subseteq \operatorname{cl} P_1(D\sigma_T) = \operatorname{cl} \operatorname{conv} D(T),$$

$$\operatorname{cl}_{w*} P_2(\varphi_T) \subseteq \operatorname{cl}_{w*} P_2(\sigma_T) = \operatorname{cl}_{w*} \operatorname{conv} R(T).$$

Take $h \in \mathcal{F}_T$. Since $\varphi_T \leq h$,

$$T \subseteq D(h) \subseteq D(\varphi_T)$$

where the first inclusion follows from Theorem 2.2. To end the proof, combine the two above equations. □

Lemma 4.1 is tight in the following sense: for a maximal monotone operator T and $h \in \mathcal{F}_T$,

$$T \subseteq D(h) \subseteq \operatorname{cl}\operatorname{conv} P_1 D(T) \times \operatorname{cl}_{w^*}\operatorname{conv} P_2 D(T) \tag{12}$$

and these inclusion may hold as equalities. For example, let

$$T : \mathbb{R} \to \mathbb{R}, \quad T(x) = x.$$

Then $T \subsetneq D(T) \times R(T) = \mathbb{R}^2$, and the first and the second inclusion in (12) holds as equalities for $h = \sigma_T$ and $h = \varphi_T$ respectively. Indeed, for this operator,

$$\sigma_T(x, x^*) = x^2 + \delta_0(x - x^*), \qquad \varphi_T(x, x^*) = (x + x^*)^2/4.$$

It had long been known that there exists a duality relation between the support function of the domain and the recession directions of the conjugate [16]. Lemma 4.1 will be used to prove that there exists a kind of "duality relation" between the domain and the range of a maximal monotone operator.

Lemma 4.2 (Domain/Range duality relation). *If $T : X \rightrightarrows X^*$ is maximal monotone then*

$$D\left(\delta^*_{D(T)}\right) \subseteq 0^+(\operatorname{cl}_{w*}\operatorname{conv} R(T)), \qquad D\left(\delta^*_{R(T)}\right) \cap X \subseteq 0^+(\operatorname{cl}\operatorname{conv} D(T)).$$

Proof. To prove the first inclusion, suppose that $u^* \in D\big(\delta^*_{D(T)}\big)$. Then

$$\infty > \delta^*_{D(T)}(u^*) = \delta^*_T(u^*, 0) = (\mathcal{J}\delta_T)(0, u^*) = (\mathcal{J}\delta_{D(\sigma_T)})(0, u^*),$$

and it follows from Proposition 3.1 that $(0, u^*) \in 0^+ D(\mathcal{J}\sigma_T) = 0^+ D(\varphi_T)$. Therefore, $u^* \in 0^+ P_2 D(\varphi_T)$ and the conclusion follows from this inclusion and Lemma 4.1.

To prove the second inclusion, suppose that $u \in D(\delta^*_{R(T)}) \cap X$. Then

$$\infty > \delta^*_{R(T)}(u) = \delta^*_T(0, u) = (\mathcal{J}\delta_T)(u, 0) = \left(\mathcal{J}\delta_{D(\sigma_T)}\right)(u, 0),$$

and it follows from Proposition 3.1 that $(u, 0) \in 0^+ D(\mathcal{J}\sigma_T) = 0^+ D(\varphi_T)$. Therefore, $u \in 0^+ P_1 D(\varphi_T)$ and the conclusion follows from this inclusion and Lemma 4.1. □

In view of the domain/range duality expressed in Lemma 4.2, maximal monotone operators with bounded domains/ranges shall have unbounded range/domains. One of these results is already well know. The other (which we do not know if it is new) is proved next.

Corollary 4.3. *If $T : X \rightrightarrows X^*$ is maximal monotone and $D(T)$ is bounded, then* $\operatorname{conv} R(T)$ *is weak-$*$ dense in X^*.*

Proof. Note that if $D(T)$ is bounded then $D((\delta_{D(T)})^*) = X^*$ and use Lemma 4.2. □

Now we will characterize the Fitzpatrick families of bounded range/domain maximal monotone operators.

Lemma 4.4. *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. The following conditions are equivalent:*

1. *T has a bounded range;*
2. *for any $h \in \mathcal{F}_T$, $P_2(D(h))$ is bounded;*
3. *there exists $h \in \mathcal{F}_T$ such that $P_2(D(h))$ is bounded;*
4. *the family*

$$\{h(\cdot, x^*) : X \to \overline{\mathbb{R}},\ x \mapsto h(x, x^*) \mid h \in \mathcal{F}_T, x^* \in P_2 D(h)\}$$

is an equi-Lipschitz family of real valued functions, that is, these functions are real-valued and there exists $0 \leq L < \infty$ such that,

$$|h(x, x^*) - h(z, x^*)| \leq L\|x - z\|, \quad \forall x, z \in X, h \in \mathcal{F}_T, x^* \in P_2 D(h);$$

5. *there exists $h \in \mathcal{F}_T$ such that the family*

$$\{h(\cdot, x^*) : X \to \overline{\mathbb{R}},\ x \mapsto h(x, x^*) \mid x^* \in P_2 D(h)\}$$

is an equi-Lipschitz family of real valued functions, that is, these functions are real-valued and there exists $0 \leq L < \infty$ such that,

$$|h(x, x^*) - h(z, x^*)| \leq L\|x - z\|, \quad \forall x, z \in X, x^* \in P_2 D(h).$$

Proof. The equivalence between items *1.*, *2.*, and *3.* follows trivially from Lemma 4.1.

Suppose that item *1.* holds, which means that there exists $0 \leq L < \infty$ such that

$$R(T) \subseteq B_{X^*}[L].$$

Take $h \in \mathcal{F}_T$. From Lemma 4.1 and these inclusion it follows that $P_2 D(h) \subseteq B_{X^*}[L]$. Using this inclusion and Corollary 3.6 we conclude that $P_1 D(h^*) \subseteq B_{X^*}[L]$. Hence, using also Proposition 3.5 (item *1.*) we conclude that item *4.* holds for such a L.

Item *4.* trivially implies item *5.*

Suppose that item *5.* holds for some $h \in \mathcal{F}_T$. Take

$$x^* \in X^*, \quad \|x^*\| > L.$$

There exists $(y_0, y_0^*) \in T$ and so, $y_0^* \in P_2 D(h)$

$$\begin{aligned} h^*(x^*, x) &= \sup_{(y,y^*)\in X\times X^*} \langle y, x^*\rangle + \langle x, y^*\rangle - h(y, y^*) \\ &\geq \sup_{y\in X} \langle y, x^*\rangle + \langle x, y_0^*\rangle - h(y, y_0^*) \\ &\geq \sup_{y\in X} \langle y, x^*\rangle + \langle x, y_0^*\rangle - h(0, y_0^*) - L\|y\| \\ &= \langle x, y_0^*\rangle - h(0, y_0^*) + \sup_{y\in X} \langle y, x^*\rangle - L\|y\| = \infty. \end{aligned}$$

Therefore, $P_2 D(\mathcal{J}h) \subseteq B_{X^*}[L]$ and using also Theorem 2.2 we conclude that item *1.* holds. □

Lemma 4.5. *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. The following conditions are equivalent:*

1. *T has a bounded domain;*
2. *for any $h \in \mathcal{F}_T$, $P_1(D(h))$ is bounded;*
3. *there exists $h \in \mathcal{F}_T$ such that $P_1(D(h))$ is bounded;*
4. *the family*

$$\{h(x,\cdot) : X^* \to \overline{\mathbb{R}},\ x^* \mapsto h(x, x^*) \mid h \in \mathcal{F}_T, h \text{ } s\times w* \text{ closed}, x \in P_1 D(h)\}$$

   is an equi-Lipschitz family of real valued functions;
5. *there exists $h \in \mathcal{F}_T$, h strong$\times$weak$*$ closed, such that the family*

$$\{h(x,\cdot) : X^* \to \overline{\mathbb{R}},\ x^* \mapsto h(x, x^*) \mid x^* \in P_2 D(h)\}$$

   is an equi-Lipschitz family of real valued functions.

Proof. The equivalence between items *1.*, *2.*, and *3.* follows trivially from Lemma 4.1.

Suppose that item *1.* holds. There exists $0 \leq L < \infty$ such that

$$D(T) \subseteq B_X[L].$$

Take $h \in \mathcal{F}_T$, h strong$\times$weak-$*$ closed. Using Lemma 4.1 we conclude that

$$P_1 D(h) \subseteq B_X[L].$$

Using this inclusion and the second part of Lemma 3.2 we conclude that

$$P_1 D(\mathcal{J}h) \subseteq B_X[L].$$

Applying Proposition 3.4 (item *2.*) to $\mathcal{J}h$ we conclude that

$$D((\mathcal{J}h)^*) = X^* \times P_2 D((\mathcal{J}h)^*)$$

and that for any $x^{**} \in P_2 D((\mathcal{J}h)^*)$,

$$(\mathcal{J}h)^*(\cdot, x^{**}) : X^* \to \mathbb{R},\ x^* \mapsto (\mathcal{J}h)^*(x^*, x^{**})$$

is L-Lipschitz continuous. Since h is strong$\times$weak$*$ closed

$$h(x, x^*) = \mathcal{J}^2 h(x, x^*) = (\mathcal{J}h)^*(x^*, x).$$

Combining the above results, we conclude that item *4.* holds.

If item *4.* holds, then item *5.* holds for $h = \varphi_T$, which is strong$\times$weak$*$ closed.

The implication *5.* $\Rightarrow$ *1.* is proved as in Lemma 4.4. □

Theorem 4.6. *If $T : X \rightrightarrows X^*$ is maximal monotone, then*

$$D(\mathcal{J}\delta_T) \subset \{(x, x^*) \in X \times X^* \mid \langle x, x^* \rangle \leq 0\}.$$

Proof. For any $(w, w^*), (x, x^*) \in X \times X^*$ and $\lambda > 0$

$$\begin{aligned} \varphi_T(w, w^*) + \lambda \mathcal{J}\delta_T(x, x^*) &= \sigma_T^*(w^*, w) + \lambda \delta^*_{D(\sigma_T)}(x^*, x) \\ &\geq \sigma_T^*(w^* + \lambda x^*, x + \lambda w) \end{aligned}$$

where the inequality follows from Proposition 2.1. Observe that

$$\sigma_T^*(w^* + \lambda x^*, x + \lambda w) = \varphi_T(w + \lambda x, w^* + \lambda x^*) \geq \langle w + \lambda x, w^* + \lambda x^* \rangle.$$

Therefore

$$\varphi_T(w, w^*) + \lambda \mathcal{J}\delta_T(x, x^*) \geq \langle w + \lambda x, w^* + \lambda x^* \rangle.$$

To end the proof, use $(w, w^*) \in T$ in the above inequality, divide it by λ and take the limit $\lambda \to \infty$. □

5. Convex functions which represent bounded-range maximal monotone operators and a non type (D) operator with bounded range

The next theorem is one of the main results of this work, and provides sufficient conditions for a convex function to represent a bounded-range maximal monotone operator.

Theorem 5.1. *If $h : X \times X^* \to \overline{\mathbb{R}}$ is convex $h \geq \pi$, $\mathcal{J}h \geq \pi$ and $P_2(D(h))$ is bounded then*

1. *$P_2(\mathcal{J}h) = P_1 D(h^*)$ is bounded;*
2. *$P_1 D(h) = X$;*
3. *$\mathcal{J}h$ and $\mathrm{cl}_{s\times w*}\, h$ (the lower semi-continuous closure of h in the strong$\times$weak$*$ topology) are Fitzpatrick functions of the maximal monotone operator*

$$T = \{(x, x^*) \mid \mathcal{J}h(x, x^*) = \langle x, x^* \rangle\};$$

4. *$R(T)$ is bounded and $D(T) = X$.*

Proof. Since the assumptions of the theorem remain valid if h is replaced by $\operatorname{cl} h$ and $(\operatorname{cl} h)^* = h^*$, we may assume that h is a proper closed convex function.

Item *1.* follows from the assumption $\mathcal{J}h \geq \pi$ and Corollary 3.6. Item *2.* follows from item *1.*, the assumption $h \geq \pi$ and Corollary 3.7.

Item *3.* follows item *2.*, the assumption $h \geq \pi$, $\mathcal{J}h \geq \pi$ and Theorem 2.3.

The first statement on item *4.* follows from items *3.* and *1.* The second part follows from the first, the maximal monotonicity of T, and Debrunner-Flor Theorem [4]. □

Zagrodny [17] proved that maximal monotone operators with a relatively compact range (in the strong topology) are of type (D). This result cannot be extended to the weak-$*$ topology, that is, to maximal monotone operators with a relatively weak-$*$ compact range. In this section we provide an example of a bounded range, non type (D) maximal monotone operator.

Theorem 5.2. *Let X be a non-reflexive real Banach space. Endow $Z = X \times X^*$ with the norm*

$$\|(x, x^*)\| = \sqrt{\|x\|^2 + \|x^*\|^2}$$

and define

$$\begin{aligned} h : Z \times Z^* \to \overline{\mathbb{R}}, \qquad & h((x, x^*), (y^*, y^{**})) \\ & = \|(x - y^{**}, x^* + y^*)\| + \delta_{B_{Z^*}[1]}(y^*, y^{**}) + \delta_X(y^{**}). \end{aligned} \tag{13}$$

Then:

1. *h is a closed convex function, $P_2D(h)$ is bounded, $h \geq \pi$, $\mathcal{J}h \geq \pi$;*
2. *$T = \{(z, z^*) \in Z \times Z^* \mid \mathcal{J}h(z, z^*) = \langle z, z^* \rangle\}$ is a bounded-range maximal monotone operator, $\mathcal{J}h \in \mathcal{F}_T$;*
3. *T is not of type (D);*

Proof. Note that $Z^* = X^* \times X^{**}$, $Z^{**} = X^{**} \times X^{***}$ and their corresponding norms are, respectively

$$\|(x^*, x^{**})\| = \sqrt{\|x^*\|^2 + \|x^{**}\|^2}, \qquad \|(x^{**}, x^{***})\| = \sqrt{\|x^{**}\|^2 + \|x^{***}\|^2}.$$

The function h is trivially proper, convex, lower semi-continuous and

$$\begin{aligned} D(h) &= \{((x, x^*), (y^*, y^{**})) \mid (x, x^*) \in X \times X^*,\ (y^*, y^{**}) \in X^* \times X,\ \|(y^*, y^{**})\| \leq 1\} \\ &= (X \times X^*) \times ((X^* \times X) \cap B_{X^* \times X^{**}}[1]). \end{aligned} \tag{14}$$

Hence $P_2D(h)$ is bounded. The conjugate of h is the function $h^* : Z^* \times Z^{**} \to \overline{\mathbb{R}}$,

$$\begin{aligned} & h^*((p^*, p^{**}), (q^{**}, q^{***})) \\ = & \sup_{\substack{x,y \in X, x^*, y^* \in X^*, \\ \|(y^*, y)\| \leq 1}} \langle x, p^* \rangle + \langle p^{**}, x^* \rangle + \langle q^{**}, y^* \rangle + \langle y, q^{***} \rangle - \|(x - y, x^* + y^*)\| \\ = & \sup_{\substack{x,y \in X, x^*, y^* \in X^*, \\ \|(y^*, y)\| \leq 1}} \langle x - y, p^* \rangle + \langle p^{**}, x^* + y^* \rangle - \|(x - y, x^* + y^*)\| \\ & \quad + \langle y, q^{***} + p^* \rangle + \langle q^{**} - p^{**}, y^* \rangle. \end{aligned}$$

Whence

$$\begin{aligned} & h^*((p^*,p^{**}),(q^{**},q^{***})) \\ = & \sup_{\substack{y\in X, y^*\in X^*, \\ \|(y^*,y)\| \leq 1}} \delta_{B_{Z^*}}(p^*,p^{**}) + \langle y, q^{***}+p^*\rangle + \langle q^{**}-p^{**}, y^*\rangle \\ = & \, \delta_{B_Z^*}(p^*,p^{**}) + \|(q^{**}-p^{**}, q^{***}|_X + p^*)\| \end{aligned} \tag{15}$$

where $q^{***}|_X$ stands for the functional q^{***} restricted to X.

Now we will prove that $h, \mathcal{J}h \geq \pi$. With this aim, take

$$(z,z^*) = ((x,x^*),(y^*,y^{**})) \in Z \times Z^*.$$

First note that

$$\begin{aligned} \langle (x,x^*),(y^*,y^{**})\rangle &= \langle x, y^*\rangle + \langle y^{**}, x^*\rangle \\ &= \langle x-y^{**}, y^*\rangle + \langle y^{**}, x^*+y^*\rangle \\ &\leq \|x-y^{**}\|\|y^*\| + \|x^*+y^*\|\|y^{**}\| \\ &\leq \sqrt{\|x-y^{**}\|^2+\|x^*+y^*\|^2}\,\sqrt{\|y^*\|^2+\|y^{**}\|^2} \\ &= \|(x-y^{**},x^*+y^*)\|\,\|(y^*,y^{**})\| \end{aligned}$$

where the second inequality follows from Cauchy-Schwarz inequality in $\mathbb{R}^2$. Using (15) we have

$$\mathcal{J}h(z^*,z) = h^*((y^*,y^{**}),(x,x^*)) = \delta_{B_{Z^*}[1]}(y^*,y) + \|(x-y^{**},x^*+y^*)\|.$$

Therefore $\mathcal{J}h \geq \pi$. Direct comparison of the above equation with (13) shows that $h \geq \mathcal{J}h$. Therefore, we also have $h \geq \pi$, which completes the proof of item *1.*

Item *2.* follows from item *1.* and Theorem 5.1.

Now we will prove that h^* does not majorizes the duality product in $Z^* \times Z^{**}$. Since X is non-reflexive, there exists

$$q_0^{**} \in X^{**} \setminus X, \quad \|q_0^{**}\| = 1.$$

Direct application of Hahn-Banach Lemma shows that there exists $q_0^{***} \in X^{***}$ such that

$$q_0^{***}|X \equiv 0, \quad \langle q_0^{**}, q_0^{***}\rangle = 1.$$

Observe that

$$1 = \langle (0,q_0^{**}),(q_0^{**},q_0^{***})\rangle > h^*((0,q_0^{**}),(q_0^{**},q_0^{***})) = 0.$$

Combining these results with the inclusion $h \in \mathcal{F}_T$ and [11, Theorem 4.4] we conclude that T is not of type (D). □

Observe that h defined in Theorem 5.2 is given by

$$h(x, x^*) = \inf_{u \in X} h_1(x - u, x^*) + h_2(u, x^*)$$

with h_1 the Fencehel-Young function for $\partial\|\cdot\|$,

$$h_1(z, z^*) = \|z\| + \delta_{B_Z[1]}(x^*),$$

and $h_2 = \delta_{T_2}$, where T_2 is the non-type (D) linear isometry studied in [1],

$$T_2 : X \times X^* \to X \times X^*, \quad T(x, x^*) = (-x^*, x).$$

A. Appendix

Proof of Proposition 2.1. For any $y \in X$

$$\delta_{D(f)}(w^*) \geq \langle y, w^* \rangle - \delta_{D(f)}(y), \qquad f^*(z^*) \geq \langle y, z^* \rangle - f(y).$$

Multiplying the first inequality by $\lambda \geq 0$ and adding it to the second inequality we obtain

$$f(z^*) + \lambda \delta^*_{D(f)}(w^*) \geq \langle y, z^* + \lambda w^* \rangle - f(y),$$

and the conclusion follows taking the sup in y at the right hand side of the above inequality □

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