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Separation of B^{-1}-Convex Sets by B^{-1}-Measurable Maps

Gultekin Tinaztepe
Vocational School of Technical Sciences, Akdeniz University,
Dumlupinar Boulevard, 07058 Campus Antalya, Turkey
gtinaztepe@akdeniz.edu.tr

Ilknur Yesilce
Faculty of Science and Letters, Mersin University,
Ciftlikkoy Campus, 33343 Mersin, Turkey
ilknuryesilce@gmail.com

Gabil Adilov
Faculty of Education, Akdeniz University,
Dumlupinar Boulevard, 07058 Campus Antalya, Turkey
gabiladilov@gmail.com

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A subset A of $\mathbb{R}^n_{++}$ is B^{-1}-convex if for all $x_1, x_2 \in A$ and all $t \geq 1$ one has $tx_1 \wedge x_2 \in A$. These sets were first investigated in [2, 3, 7]. In this paper, we establish separation and Hahn-Banach like Theorem for B^{-1}-convex sets.

Keywords: B-convexity, B^{-1}-convexity, half spaces, gauges, co-gauges, separation, B^{-1}-measurable maps

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1. Introduction

B-convexity concept is determined in [4], properties B-convex sets and functions are given in [1, 4, 5, 6], separation properties are investigated in [5, 6].

B^{-1}-convexity concept is given in [2, 7], some important properties of B^{-1}-convex sets are investigated in [2, 3], the application to economy is given in [7]. In this paper, functional Hahn-Banach like separation properties in B^{-1}-convexity are established.

The outline of the paper as follows:

In second section, required attainments to article are introduced; B^{-1}-convex sets, B^{-1}-convex hull, gauge and co-gauge functions are defined and some algebraic separation properties are given.

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In third section, properties of B^{-1}-measurable maps and the relations of them with half-spaces are investigated and analytic separation theorems are proved for disjoint B^{-1}-convex sets.

In this paper, the following notations are used:

$I = \{1, 2, ..., n\}$;

$\mathbb{R}^n_+ = \left\{x = (x_1, x_2, ..., x_n) : x_i \geq 0, i = \overline{1, n}\right\}$;

$\mathbb{R}^n_{++} = \left\{x = (x_1, x_2, ..., x_n) : x_i > 0, i = \overline{1, n}\right\}$;

for $x \in \mathbb{R}^n_+, I(x) = \{i \in I : x_i \neq 0\}$;

$[\![A]\!]$ is the B^{-1}-convex hull of A.

$K = \mathbb{R} \setminus \{0\}$

$\overline{\mathbb{R}} =]-\infty, +\infty]$

2. Preliminaries

Let us give some definitions and theorems with respect to B^{-1}-convexity (given in [2]).

Definition 2.1. A set A in $\mathbb{R}^n_{++}$ is B^{-1}-convex if for all $x_1, ...x_m \in A$ and all $t_1, ..., t_m \in [1, \infty)$ such that $\min\{t_1, ..., t_m\} = 1$ one has $\wedge_{i=1}^m t_i x_i \in A$, where $\wedge$ denotes the minimum with respect to partial order relation on $\mathbb{R}^n_+$ associated to the positive cone, that is, the coordinatewise minimum.

The definition of B^{-1}-convexity in K^n as $K = \mathbb{R} \setminus \{0\}$ was given in [2].

Below theorem allows to define B^{-1}-convex sets more simply [2].

Theorem 2.2. *A subset A of $\mathbb{R}^n_{++}$ is* B^{-1}*-convex if and only if for all $x_1, x_2 \in A$ and all $t \in [1, \infty)$ one has $tx_1 \wedge x_2 \in A$.*

In [4], the following proposition is proved for B-convexity.

Proposition 2.3. *If A is a* B*-convex subset of $\mathbb{R}^n_+$ then*

$$U_\infty (A, \delta) = \left\{x \in \mathbb{R}^n_+ : \exists y \in A, \|x - y\|_\infty < \delta\right\}$$

is also B*-convex set.*

This proposition is not valid in B^{-1}-convexity, that is, for a B^{-1}-convex subset A of $\mathbb{R}^n_{++}$, $U_\infty (A, \delta)$ may not be always B^{-1}-convex. A counterexample is given in [3].

Now, for a set $A \subset \mathbb{R}^n_{++}$, we define a different neighborhood concept.

Definition 2.4. Let A be a subset of $\mathbb{R}^n_{++}$. A set

$$U^r_\infty (A, \delta) = \left\{x \in \mathbb{R}^n_{++} : \exists y \in A, \exists \lambda > 0, x = \lambda y, |1 - \lambda| \, \|y\|_\infty < \delta\right\}$$

is called radial δ-neighborhood of A.

Proposition 2.5. *If A is a* B^{-1}*-convex subset of* $\mathbb{R}^n_{++}$*, then* $U^r_\infty(A,\delta)$ *is also* B^{-1}*-convex set.*

Proof. Let $x', x'' \in U^r_\infty(A,\delta)$. We will show that for all $t \geq 1$, $x = tx' \wedge x'' \in U^r_\infty(A,\delta)$.

$$x' \in U^r_\infty(A,\delta) \Rightarrow \exists y' \in A, \exists \lambda' > 0, x' = \lambda' y', \qquad |1-\lambda'| \, \|y'\|_\infty < \delta \tag{1}$$

$$x'' \in U^r_\infty(A,\delta) \Rightarrow \exists y'' \in A, \exists \lambda'' > 0, x'' = \lambda'' y'', \qquad |1-\lambda''| \, \|y''\|_\infty < \delta. \tag{2}$$

Then, $x = tx' \wedge x'' = t\lambda' y' \wedge \lambda'' y''$.

There are two cases:

Case I. Let $t\lambda' \geq \lambda''$. Hence, we have $x = \lambda'' \left(\frac{t\lambda'}{\lambda''} y' \wedge y''\right) = \lambda y$. Here, $\lambda = \lambda'' > 0$ and $y = \frac{t\lambda'}{\lambda''} y' \wedge y'' \in A$ (since $\frac{t\lambda'}{\lambda''} \geq 1$ and A is B^{-1}-convex.). We have to show that $|1-\lambda| \, \|y\|_\infty < \delta$. Coordinates of y can equal to one of the following two value:

$$y_i = \begin{cases} \dfrac{t\lambda'}{\lambda''} y'_i, & \text{if } \dfrac{t\lambda'}{\lambda''} y'_i < y''_i, \\[2ex] y''_i, & \text{if } \dfrac{t\lambda'}{\lambda''} y'_i \geq y''_i. \end{cases} \tag{3}$$

If $\|y\|_\infty = \frac{t\lambda'}{\lambda''} y'_i$, from (3) and (2), we obtain

$$|1-\lambda| \, \|y\|_\infty = |1-\lambda''| \frac{t\lambda'}{\lambda''} y'_i < |1-\lambda''| \, y''_i < \delta.$$

If $\|y\|_\infty = y''_i$, from (2) we have

$$|1-\lambda| \, \|y\|_\infty = |1-\lambda''| \, y''_i < \delta.$$

Case II. Let $t\lambda' < \lambda''$. Thus, $x = t\lambda' \left(y' \wedge \frac{\lambda''}{t\lambda'} y''\right) = \lambda y$. Here, $\lambda = t\lambda' > 0$ and $y = y' \wedge \frac{\lambda''}{t\lambda'} y'' \in A$ (since $\frac{\lambda''}{t\lambda'} > 1$ and A is B^{-1}-convex.). We have to show that $|1-\lambda| \, \|y\|_\infty < \delta$. Coordinates of y can take one of the following two value:

$$y_i = \begin{cases} y'_i, & \text{if } y'_i < \dfrac{\lambda''}{t\lambda'} y''_i, \\[2ex] \dfrac{\lambda''}{t\lambda'} y''_i, & \text{if } y'_i \geq \dfrac{\lambda''}{t\lambda'} y''_i. \end{cases} \tag{4}$$

When $\|y\|_\infty = y'_i$,

$$|1-\lambda| \, \|y\|_\infty = |1-t\lambda'| \, y'_i.$$

If $1 - t\lambda' \geq 0$, then from (1), we obtain

$$|1-\lambda| \, \|y\|_\infty = (1-t\lambda') \, y'_i < (1-\lambda') \, y'_i < \delta.$$

And, if $1-t\lambda'<0$, then taking into account that $1<t\lambda'<\lambda''$ and (4), (2); we have

$$\begin{aligned} |1-\lambda|\,\|y\|_\infty &= (t\lambda'-1)\,y_i' < (t\lambda'-1)\,\frac{\lambda''}{t\lambda'}y_i'' \\ &= \left(\lambda''-\frac{\lambda''}{t\lambda'}\right)y_i'' < (\lambda''-1)\,y_i'' < \delta. \end{aligned}$$

In the other case, that is, when $\|y\|_\infty = \frac{\lambda''}{t\lambda'}y_i''$, it is $|1-\lambda|\,\|y\|_\infty = |1-t\lambda'|\,\frac{\lambda''}{t\lambda'}y_i''$. Again, if $1-t\lambda'\geq 0$, then from $t\geq 1$ and (4), (1), we obtain

$$|1-\lambda|\,\|y\|_\infty = (1-t\lambda')\,\frac{\lambda''}{t\lambda'}y_i'' < (1-\lambda')\,y_i' < \delta.$$

If $1-t\lambda'<0$, then from $1<t\lambda'<\lambda''$ and (2), we have

$$|1-\lambda|\,\|y\|_\infty = (t\lambda'-1)\,\frac{\lambda''}{t\lambda'}y_i'' = \left(\lambda''-\frac{\lambda''}{t\lambda'}\right)y_i'' < (\lambda''-1)\,y_i'' < \delta.$$

Consequently, we have shown that $x = tx'\wedge x'' \in U_\infty^r(A,\delta)$.

Thus it has been proved that the set $U_\infty^r(A,\delta)$ is B^{-1}-convex set. □

Closure and interior of B^{-1}-convex sets are also B^{-1}-convex [3].

Of particular interest are the B^{-1}-convex sets whose complements are also B^{-1}-convex, those sets are called half-spaces.

Definition 2.6. For a set $A\subset\mathbb{R}^n_{++}$, the intersection of all the B^{-1}-convex subset of $\mathbb{R}^n_{++}$ containing A is called the B^{-1}-convex hull of A and denoted by $[\![A]\!]$.

The following properties will be necessary in the future.

Lemma 2.7. *For all subsets A of $\mathbb{R}^n_{++}$ and for all point p, we have*

$$\left[\!\!\left[A\bigcup\{p\}\right]\!\!\right] = \bigcup_{x\in[\![A]\!]}[\![\{x,p\}]\!].$$

Proof. We assume that $A\neq\emptyset$ and $y\in[\![A\bigcup\{p\}]\!]$. Then, there exist $x_1,...,x_k\in A$ and $\eta_1,...,\eta_{k+1}\in[1,\infty)$ such that $\min\{\eta_1,...,\eta_k,\eta_{k+1}\}=1$ and $y=\eta_1x_1\wedge ...\wedge\eta_kx_k\wedge\eta_{k+1}p$. Let $\mu=\min\{\eta_1,...,\eta_k\}$ and $\mu_i=\frac{\eta_i}{\mu}$ for $i=\overline{1,k}$. Hence $x=\mu_1x_1\wedge...\wedge\mu_kx_k\in[\![A]\!]$ and $y=\mu x\wedge\eta_{k+1}p\in[\![\{x,p\}]\!]$ since $\min\{\mu,\eta_{k+1}\}=1$.

We prove the other side: If $y\in\bigcup_{x\in[\![A]\!]}[\![\{x,p\}]\!]$ then, there exist $x'\in[\![A]\!]$ such that $y\in[\![\{x',p\}]\!]$. Since $x'\in[\![A]\!]$, for $\min\{\mu_1,...,\mu_k\}=1$ and $x_1',...,x_k'\in A$ we can write $x'=\mu_1x_1'\wedge...\wedge\mu_kx_k'$. $y\in[\![\{x',p\}]\!]$ hence, for $\eta_1,\eta_2\in[1,\infty)$, $\min\{\eta_1,\eta_2\}=1$,

$$\begin{aligned} y = \eta_1x'\wedge\eta_2p &= \eta_1\left(\mu_1x_1'\wedge...\wedge\mu_kx_k'\right)\wedge\eta_2p \\ &= (\eta_1\mu_1)\,x_1'\wedge...\wedge(\eta_1\mu_k)\,x_k'\wedge\eta_2p. \end{aligned}$$

Consequently, for

$$x'_1, ..., x'_k, p \in A \bigcup \{p\}, \quad \eta_1\mu_1, \eta_1\mu_2, ..., \eta_1\mu_k, \eta_2 \in [1, \infty)$$

and

$$\min\{\eta_1\mu_1, \eta_1\mu_2, ..., \eta_1\mu_k, \eta_2\} = 1$$

thus

$$y = (\eta_1\mu_1)\, x'_1 \wedge ... \wedge (\eta_1\mu_k)\, x'_k \wedge \eta_2 p \in \left[\!\!\left[A \bigcup \{p\} \right]\!\!\right]. \qquad \square$$

Proposition 2.8 (Pash-Peano). *For all quintuple (a, b_1, b_2, c_1, c_2) of points of $\mathbb{R}^n_{++}$ such that $c_i \in [\![\{a, b_i\}]\!]$ we have $[\![\{b_1, c_2\}]\!] \cap [\![\{b_2, c_1\}]\!] \neq \emptyset$.*

Proof. We have to show that the following system of equations has a solution

$$\begin{cases} \eta_1 c_1 \wedge \mu_2 b_2 = \eta_2 c_2 \wedge \mu_1 b_1 \\ \min\{\eta_1, \mu_2\} = 1 \\ \min\{\eta_2, \mu_1\} = 1 \end{cases} \tag{5}$$

Since $c_i \in [\![a, b_i]\!]$ we can write $c_i = \rho_i a \wedge \alpha_i b_i$ with $\min\{\rho_i, \alpha_i\} = 1$; a substitution in the first line of (5) yields

$$(\eta_1\rho_1)\, a \wedge (\eta_1\alpha_1)\, b_1 \wedge \mu_2 b_2 = (\eta_2\rho_2)\, a \wedge (\eta_2\alpha_2)\, b_2 \wedge \mu_1 b_1 \tag{6}$$

Given ρ_i and $\alpha_i, i = 1, 2$, one can easily solve (6) for η_i and $\mu_i, i = 1, 2$ in the following way

$$\eta_1 = 1 \quad \text{and} \quad \eta_2 = \frac{\rho_1}{\rho_2} \quad \text{if } \rho_1 > \rho_2 \tag{7}$$

$$\eta_1 = \frac{\rho_2}{\rho_1} \quad \text{and} \quad \eta_2 = 1 \quad \text{if } \rho_1 < \rho_2 \tag{8}$$

$$\eta_1 = \eta_2 = 1 \quad \text{if } \rho_1 = \rho_2 \tag{9}$$

$$\mu_i = \eta_i \alpha_i \tag{10}$$

This solution of (6) is also a solution of (5) if $\min\{\eta_1, \mu_2\} = \min\{\eta_2, \mu_1\} = 1$; let us see that this is indeed the case. There are, formally, three cases to consider, namely (7), (8) and (9). In the first case, (7) we have $\min\{\eta_1, \mu_2\} = \eta_1 = 1$ and $\min\{\rho_1, \alpha_1\} = 1$ with $\rho_1 > \rho_2 \geq 1$ from which we obtain $\alpha_1 = 1$ and $\mu_1 = \eta_1 = 1$; we have shown that $\min\{\eta_2, \mu_1\} = 1$. The second case, that is (8), is treated similarly; as for (9) there is nothing to prove since $\eta_1 = \eta_2 = 1$. $\square$

Proposition 2.9 (The Stone-Kakutani Property). *If C_1 and C_2 are disjoint B^{-1}-convex sets, then there exists a B^{-1}-convex set $D \subset \mathbb{R}^n_{++}$ such that $\mathbb{R}^n_{++} \setminus D$ is also B^{-1}-convex, $C_1 \subset D$ and $C_2 \subset \mathbb{R}^n_{++} \setminus D$.*

Proof. Let Z be the family of pairs of disjoint B^{-1}-convex sets (D_1, D_2) such that $C_i \subset D_i$ partially ordered by $(D_1, D_2) \subset (D'_1, D'_2)$ if $D_i \subset D'_i$. The pair (C_1, C_2) belong to Z and if $\mathcal{C} = \{(D_{1,\lambda}, D_{2,\lambda}) : \lambda \in \Lambda\}$ is a chain in Z, then $\left(\bigcup_{\lambda\in\Lambda} D_{1,\lambda}, \bigcup_{\lambda\in\Lambda} D_{2,\lambda}\right) \in Z$ since an up-directed union of B^{-1}-convex sets is B^{-1}-convex and, as can easily be seen, $\left(\bigcup_{\lambda\in\Lambda} D_{1,\lambda}\right) \bigcap \left(\bigcup_{\lambda\in\Lambda} D_{2,\lambda}\right) = \emptyset$; by Zorn's lemma there is a maximal element (H_1, H_2) in Z. Assume that there is a point a in $\mathbb{R}^n_{++} \setminus (H_1 \bigcup H_2)$; from the maximality of the pair (H_1, H_2) we have $[\![H_1 \bigcup \{a\}]\!] \bigcap H_2 \neq \emptyset$ and $[\![H_2 \bigcup \{a\}]\!] \bigcap H_1 \neq \emptyset$, take a point $c_1 \in [\![H_1 \bigcup \{a\}]\!] \bigcap H_2$ and a point $c_2 \in [\![H_2 \bigcup \{a\}]\!] \bigcap H_1$. By Lemma 2.7 there exist $b_i \in H_i$ such that $c_i \in [\![\{a, b_i\}]\!]$. By the Pash-Peano Property there exists a point u in $[\![\{b_1, c_2\}]\!] \bigcap [\![\{b_2, c_1\}]\!]$. From $b_1, c_2 \in H_1$ and $b_2, c_1 \in H_2$ we obtain $u \in H_1 \bigcap H_2$ which is impossible since the pair (H_1, H_2) is in Z. $\square$

We need the following definitions (see [8] for details). A set $U \subset \mathbb{R}^n$ is called radiant if $x \in U, t \in]0, 1]$ implies $tx \in U$. A set $V \in \mathbb{R}^n$ is called co-radiant if $x \in V, t \geq 1$ implies $tx \in V$. The Minkowski gauge $\mu_U : \mathbb{R}^n \to [0, +\infty]$ of the radiant set U is defined by

$$\mu_U(x) = \inf \{\lambda \in]\, 0, +\infty\, [: x \in \lambda U\}, \quad x \in \mathbb{R}^n.$$

The Minkowski co-gauge $\nu_V : \mathbb{R}^n \to [0, +\infty]$ of the co-radiant set V is defined by

$$\nu_V(x) = \sup \{\lambda \in]\, 0, +\infty\, [: x \in \lambda V\}, \quad x \in \mathbb{R}^n.$$

As is known, $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$.

If U is radiant set then $V = \mathbb{R}^n \setminus U$ is co-radiant and $\nu_V = \mu_U$. If V is co-radiant then $U = \mathbb{R}^n \setminus V$ is radiant and $\mu_U = \nu_V$.

Note that both μ_U and ν_V are positively homogeneous functions (A function $f : \mathbb{R}^n \to [0, +\infty]$ is called positively homogeneous if $f(\lambda x) = \lambda f(x)$ for all $\lambda \in]0, +\infty[$).

A set $U \subset \mathbb{R}^n$ is called radially closed if

$$(x \in \mathbb{R}^n, \lambda_k > 0, \lambda_k x \in U, k = 1, 2, ..., \lambda_k \to \lambda) \Rightarrow \lambda x \in U.$$

A set U is called radially open if its complement $\mathbb{R}^n \setminus U$ is radially closed.

3. Separation in B^{-1}-convexity

3.1. B^{-1}-measurable maps and half-spaces

Definition 3.1. Let $K = \mathbb{R} \setminus \{0\}$ and $\overline{\mathbb{R}} =]-\infty, +\infty]$. A map $f : K^n \to \overline{\mathbb{R}}^m$ is B^{-1}-measurable if for all B^{-1}-convex set C of $\overline{\mathbb{R}}^m$ the inverse image $f^{-1}(C)$ is a B^{-1}-convex set in K^n.

Theorem 3.2. *For a map $f : K^n \to \overline{\mathbb{R}}$, f is* B^{-1}*-measurable if and only if for all $t \in \mathbb{R}$ the sets $\{x : f(x) \leq t\}$ and $\{x : f(x) \geq t\}$ are* B^{-1}*-convex.*

Proof. Let f be B^{-1}-measurable. Then, for all subset $C \subset \mathbb{R}$ which is B^{-1}-convex, $f^{-1}(C)$ is B^{-1}-convex set. The sets $C_1 = (-\infty, t]$ and $C_2 = [t, +\infty)$ are B^{-1}-convex for all $t \in \mathbb{R}$. Hence, the sets

$$f^{-1}(C_1) = \{x \mid f(x) \leq t\} \quad \text{and} \quad f^{-1}(C_2) = \{x \mid f(x) \geq t\}$$

are B^{-1}-convex.

Now, for all $t \in \mathbb{R}$, let $\{x \mid f(x) \leq t\}$ and $\{x \mid f(x) \geq t\}$ be B^{-1}-convex sets. In $\mathbb{R}$, each B^{-1}-convex sets are intervals (finite or infinite). For instance, let C=[a,b), in this case

$$\begin{aligned} f^{-1}(C) &= f^{-1}\left(\bigcup_{n=1}^{\infty}\left(-\infty, b-\frac{1}{n}\right]\bigcap[a,+\infty)\right) \\ &= \left(\bigcup_{n=1}^{\infty} f^{-1}\left(-\infty, b-\frac{1}{n}\right]\right)\bigcap f^{-1}\left([a,+\infty)\right) \\ &= \left(\bigcup_{n=1}^{\infty}\left\{x \mid f(x) \leq b-\frac{1}{n}\right\}\right)\bigcap\{x \mid f(x) \geq a\} .. \end{aligned}$$ □

Theorem 3.3. *A map $f : \mathbb{R}^n_{++} \to [0, \infty]$ is B^{-1}-measurable if and only if*

$$\min\{f(x_1), f(x_2)\} \leq f(\lambda x_1 \wedge x_2) \leq \max\{f(x_1), f(x_2)\}$$

for all $x_1, x_2 \in \mathbb{R}^n_{++}$ and for all $\lambda \in [1, \infty)$.

Proof. If f is B^{-1}-measurable, it follows from Theorem 3.2 that the sets $\{x : f(x) \leq t\}$ and $\{x : f(x) \geq t\}$ are B^{-1}-convex for all t. Hence, for $x_1, x_2 \in \mathbb{R}^n_{++}$, $\{x : f(x) \leq \max\{f(x_1), f(x_2)\}\}$ and $\{x : f(x) \geq \min\{f(x_1), f(x_2)\}\}$ are B^{-1}-convex. Since $x_1, x_2 \in \{x : f(x) \leq \max\{f(x_1), f(x_2)\}\}$ and the set is B^{-1}-convex, $f(\lambda x_1 \wedge x_2) \leq \max\{f(x_1), f(x_2)\}$ for all $\lambda \in [1, +\infty)$. In a similar way we can show the left side of the inequality.

Now, let us assume that the inequality is valid and we show that the sets of $\{x : f(x) \leq t\}$ and $\{x : f(x) \geq t\}$ are B^{-1}-convex for all t. If $x_1, x_2 \in \{x : f(x) \leq t\}$, then $f(x_1) \leq t, f(x_2) \leq t$. From the inequalities, for $\lambda \in [1, +\infty)$ we have $f(\lambda x_1 \wedge x_2) \leq \max\{f(x_1), f(x_2)\} \leq t$, consequently, $\lambda x_1 \wedge x_2 \in \{x : f(x) \leq t\}$. B^{-1}-convexity of the second set is shown by the same way. □

Half-spaces of B-convexity are either radiant or co-radiant. For B^{-1}-convexity, it can be shown that same property is obtained, that is, a half-space in B^{-1}-convexity is radiant or co-radiant.

Lemma 3.4.

1) *Let $U \in \mathbb{R}^n_{++}$ be a radiant set.*
 (*a*) *If U is B^{-1}-convex, then*

$$\mu_U(tx \wedge y) \leq \max\{\mu_U(x), \mu_U(y)\} \quad \text{for all } x, y \in \mathbb{R}^n_{++}, t \in [1, \infty) \tag{11}$$

(*b*) *If* U *is radially closed or radially open and* (11) *holds, then* U *is* B^{-1}-*convex.*

2) *Let* $V \subset \mathbb{R}^n_{++}$ *be a co-radiant set.*

(*a*) *If* V *is* B^{-1}-*convex, then*

$$\nu_V(tx \wedge y) \geq \min\{\nu_V(x), \nu_V(y)\} \quad \text{for all } x, y \in \mathbb{R}^n_{++}, t \in [1, \infty) \tag{12}$$

(*b*) *If* V *is radially closed or radially open and* (12) *holds, then* V *is* B^{-1}-*convex.*

Proof. *1*) Let U be a radiant B^{-1}-convex set. Consider points $x, y \in \mathbb{R}^n_{++}$ and $t \in [1, \infty)$. Assume for the sake of definiteness that $\mu_U(x) \leq \mu_U(y) < \infty$. If $r > \mu_U(y)$ then $x \in rU, y \in rU$. From $r^{-1}(tx \wedge y) = r^{-1}tx \wedge r^{-1}y$ we have $r^{-1}(tx \wedge y) \in U$; this shows that (11) holds. If either $\mu_U(tx) = \infty$ or $\mu_U(y) = \infty$ then (11) is obvious.

Let U be radiant. If U is radially closed, then $U = \{x \in \mathbb{R}^n_{++} : \mu_U(x) \leq 1\}$, if U is radially open then $U = \{x \in \mathbb{R}^n_{++} : \mu_U(x) < 1\}$. In both cases (11) implies that U is a B^{-1}-convex set.

2) Let V be a co-radiant and B^{-1}-convex set. Consider points $x, y \in \mathbb{R}^n_{++}$ and a number $t \in [1, \infty)$. Assume for the sake of definiteness that $0 < \nu_V(y) \leq \nu_V(x) \leq \infty$. Let $0 < r < \nu_V(y)$ then $x \in rV, y \in rV$. Since V is B^{-1}-convex set, $r^{-1}(tx \wedge y) = tr^{-1}x \wedge r^{-1}y \in V$, in other words, $tx \wedge y \in rV$. Hence (12) holds. If either $\nu_V(x) = 0$ or $\nu_V(y) = 0$ then (12) is obvious.

If V is either radially closed or open and (12) holds, then clearly V is a B^{-1}-convex set. □

Proposition 3.5. (*1*) *Let* H *be a radiant half-space. Then for all* $x, y \in \mathbb{R}^n_{++}$ *and* $t \in [1, +\infty)$

$$\min\{\mu_H(x), \mu_H(y)\} \leq \mu_H(tx \wedge y) \leq \max\{\mu_H(x), \mu_H(y)\} \tag{13}$$

If H *is* B^{-1}-*convex, radiant, either radially closed or open and the left-hand inequality in* (13) *holds, then* H *is a half-space.*

(*2*) *Let* H *be a co-radiant half-space. Then for all* $x, y \in \mathbb{R}^n_{++}$ *and* $t \in [1, +\infty)$

$$\min\{\nu_H(x), \nu_H(y)\} \leq \nu_H(tx \wedge y) \leq \max\{\nu_H(x), \nu_H(y)\} \tag{14}$$

If H *is* B^{-1}-*convex, co-radiant, either radially closed or open and the right-hand inequality in* (14) *holds, then* H *is a half-space.*

Proof. (*1*) Let H be a radiant half-space and $x, y \in \mathbb{R}^n_{++}, t \in [1, \infty)$. Since H is B^{-1}-convex and radiant it follows from (11) that the right-hand inequality in (13) holds. Since $C = \mathbb{R}^n_{++} \setminus H$ is B^{-1}-convex and co-radiant it follows from (12) that

$$\nu_C(tx \wedge y) \geq \min\{\nu_C(x), \nu_C(y)\}$$

From $\mu_H = \nu_C$, we obtain the left-hand inequality in (13).

Let H be B^{-1}-convex and radially closed, then C is radially open. The left-hand inequality in (13) can be presented as $\min\{\nu_C(x), \nu_C(y)\} \leq \nu_C(tx \wedge y)$. Applying Lemma 3.4, we conclude that C is B^{-1}-convex, hence H is half-space. A similar argument can be used if H is radially open.

(*2*) Let H be a co-radiant half-space and $x, y \in \mathbb{R}^n_{++}, t \in [1, \infty)$. Since H is B^{-1}-convex and co-radiant it follows from (12) that left-hand inequality in (14) holds. Since C is B^{-1}-convex and radiant it follows from (11) that

$$\mu_C(tx \wedge y) \leq \max\{\mu_C(x), \mu_C(y)\}$$

This implies the right-hand inequality in (14).

Let H be B^{-1}-convex and radially closed,then C is radially open. The right-hand inequality in (14) can be presented as $\mu_C(tx \wedge y) \leq \max\{\mu_C(x), \mu_C(y)\}$. Applying Lemma 3.4 we conclude that C is B^{-1}-convex, hence H is a half-space. The similar argument can be used if H is radially open. $\square$

As one can easily see that the gauge of a radiant half-space and the co-gauge of a co-radiant half-space are positively homogeneous B^{-1}-measurable maps.

3.2. Separation of B^{-1}-convex sets by a map

Definition 3.6. Let A and B be two sets.

- (i) A and B are weakly separated by a map f if $\sup_{x\in A} f(x) \leq \inf_{x\in B} f(x)$ or $\sup_{x\in B} f(x) \leq \inf_{x\in A} f(x)$.
- (ii) A and B are separated by a map f if there exists a real number r such that either $\forall(x, y) \in A \times B\ f(x) < r < f(y)$ or $\forall(x, y) \in A \times B\ f(y) < r < f(x)$.
- (iii) A and B are strictly separated by a map f if $\sup_{x\in A} f(x) < \inf_{x\in B} f(x)$ or $\sup_{x\in B} f(x) < \inf_{x\in A} f(x)$.

Theorem 3.7. *Let C_1 and C_2 be disjoint and* B^{-1}*-convex sets.*

- (*1*) *C_1 and C_2 can be weakly separated by a* B^{-1}*-measurable positively homogeneous map.*
- (*2*) *If, $\inf_{(x,y)\in C_1\times C_2} \|x - y\|_\infty > 0$ and there exists a vector $u \in \mathbb{R}^n_{++}$ such that $C_1 \cup C_2 \subset \mathbb{R}^n_{++} + u$, then C_1 and C_2 can be separated by a* B^{-1}*-measurable positively homogeneous map.*
- (*3*) *If, with the condition in (2), either C_1 or C_2 is bounded then they can be strictly separated by a* B^{-1}*-measurable positively homogeneous map.*

Proof. (*1*) Let C_1 and C_2 be disjoint and B^{-1}-convex subsets of $\mathbb{R}^n_{++}$; by Proposition 2.9 (Stone-Kakutani Property) there exists a half-space B such that $C_1 \subset B$ and $C_2 \subset \mathbb{R}^n_{++} \setminus B$ and we can assume that B is radiant. Hence, for all $x \in C_1, \mu_B(x) \leq 1$ and for all $x \in C_2, \mu_B(x) \geq 1$.

(*2*) Let $\eta = \inf_{(x,y)\in C_1\times C_2} \|x - y\|_\infty$ and, for all subset $A \subset \mathbb{R}^n_{++}$ let $U^r_\infty(A; \delta) = \{x \in \mathbb{R}^n_{++} : \exists y \in A,\ \exists\lambda > 0,\ x = \lambda y,\ |1 - \lambda|\, \|y\|_\infty < \delta\}$; if A is B^{-1}-convex then, for all $\delta \geq 0, U^r_\infty(A; \delta)$ is also B^{-1}-convex by Proposition 2.5. Since $\eta > 0$ we can choose $u \in \mathbb{R}^n_{++}, \epsilon > 0$ such $\epsilon < \min\{\eta, \|u\|_\infty\}$ and $U^r_\infty(C_1;\epsilon) \cap U^r_\infty(C_2;\epsilon) = \emptyset$. From part (*1*) we find a radiant half-space H such that $U^r_\infty(C_1;\epsilon) \subset H, U^r_\infty(C_2;\epsilon) \subset \mathbb{R}^n_{++} \setminus H$ and $\sup_{x\in U^r_\infty(C_1;\epsilon)} \mu_H(x) \leq 1 \leq \inf_{x\in U^r_\infty(C_2;\epsilon)} \mu_H(x)$. If $x \in C_1$ and $t = (1 + \epsilon/(2\|x\|_\infty))$ then $tx \in U^r_\infty(C_1;\epsilon) \subset H$, and therefore,

$$\mu_H(x) \leq \left(1 + \frac{\epsilon}{2\|x\|_\infty}\right)^{-1} < 1$$

If $y \in C_2$ then $y \in \mathbb{R}^n_{++} + u$ and $\|y\|_\infty > \epsilon$ and therefore $s = (1 - \epsilon/(2\|y\|_\infty))$ is strictly positive. Since $\|sy - y\|_\infty = \epsilon/2$ we have $sy \in U^r_\infty(C_2;\epsilon) \subset \mathbb{R}^n_{++} \setminus H$, and therefore $sy \notin H$ which implies that

$$1 < \left(1 - \frac{\epsilon}{2\|y\|_\infty}\right)^{-1} \leq \mu_H(y)$$

(*3*) Now, assume that C_1 is bounded. There is a $v \in \mathbb{R}^n_{++}$ such that, for all $x \in C_1, \|x\|_\infty \leq \|v\|_\infty$ and therefore

$$\sup_{x\in C_1} \mu_H(x) \leq \left(1 + \frac{\epsilon}{\|v\|_\infty}\right)^{-1} < 1 \leq \inf_{y\in C_2} \mu_H(y).$$

If C_2 is bounded there is a $v \in \mathbb{R}^n_{++}$ such that, for all $y \in C_2, \|y\|_\infty \leq \|v\|_\infty$, and $\epsilon < \|v\|_\infty$; we then have

$$\sup_{x\in C_1} \mu_H(x) \leq 1 < \left(1 - \frac{\epsilon}{\|v\|_\infty}\right)^{-1} \leq \inf_{y\in C_2} \mu_H(y). \qquad \square$$

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