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Approximation by DC Functions and Application to Representation of a Normed Semigroup

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Let Ω be a nonempty compact set of a locally convex space L, and let $C(\Omega)$ be the Banach space of all real-valued continuous functions on Ω endowed with the sup-norm. In this paper, we show first that for every $f \in C(\Omega)$, and for every $\varepsilon > 0$, there are continuous affine functions $(g_i)_{i=1}^m, (h_j)_{j=1}^n$ on L for some $m, n \in \mathbb{N}$ such that

$$|f(\omega) - [(g_1 \vee g_2 \vee \cdots \vee g_m) - (h_1 \vee h_2 \vee \cdots \vee h_n)](\omega)| < \varepsilon$$

uniformly for $\omega \in \Omega$. We prove then that if $\Omega = B_{X^*}$, the closed unit ball of X^* of a Banach space X endowed with the w^*-topology, then $C(\Omega)^*$ is just the dual of the normed semigroup $\mathrm{b}(X)$ generated closed balls in X.

1. Introduction

There is a large literature on difference of convex functions, also called DC, or, Δ-convex functions – functions which are the difference of two continuous convex functions. The class of DC functions is a remarkable subclass of locally Lipschitz functions that is of interest both in analysis and optimization. See, Bac̄ák-Borwein's survey [1]. It appears not only very naturally as the smallest vector space containing all continuous convex functions on a given set, but also nice properties. For example, all DC functions defined on a convex set of a normed space form both a lattice and an algebra [7] (see, also, [1]). There are still many interesting and useful properties about DC functions, which can be found in [1].

Cepedello Boiso [2] has used some interesting properties of DC functions to characterize superreflexivity of Banach spaces ([2], [3]). He showed that a Banach space X is superreflexive if and only if for every (real-valued) Lipschitz function

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f defined on X there exists a sequence (f_n) of DC functions converging to f uniformly on each bounded subset of X [3].

Let L be a real locally convex space, and L^* its dual, and let X be a Banach space. We denote by B_X, resp. S_X, the closed unit ball, resp. the unit sphere of X. The pointwise maximum, resp. minimum of $f, g : X \to \mathbb{R}$ is denoted $f \vee g$, resp. $f \wedge g$. We denote by $A(L)$ the space of all continuous affine functions on L.

It is shown in [6] that for every $\varepsilon > 0$, for every positively homogenous w^*-continuous (resp. weakly continuous) function defined on X^* (resp. X), and for every w^*-compact (resp. weakly compact) set $K \subset X^*$ (resp. $K \subset X$), there exist $(x_i)_{i=1}^m, (y_j)_{j=1}^n \subset X$, resp. $(x_i^*)_{i=1}^m, (y_j^*)_{j=1}^n \subset X^*$ for some $m, n \in \mathbb{N}$ such that

$$|f(x^*) - [(x_1 \vee \cdots \vee x_m) - (y_1 \vee \cdots \vee y_n)](x^*)| < \varepsilon, \quad \text{for all } x^* \in K,$$

(resp.

$$|f(x) - [(x_1^* \vee \cdots \vee x_m^*) - (y_1^* \vee \cdots \vee y_n^*)](x)| < \varepsilon, \text{ for all } x \in K.)$$

In this paper, making use of the Kakutani-Krein theorem, we generalize first the result mentioned above to general continuous functions defined on a compact subset of a locally convex space.

Theorem 1.1. *Let Ω be a nonempty compact set of a locally convex space L, and let $C(\Omega)$ be the Banach space of all real-valued continuous functions on Ω endowed with the* sup*-norm. Then for every $f \in C(\Omega)$, and for every $\varepsilon > 0$, there are finitely many continuous affine functions $(g_i)_{i=1}^m, (h_j)_{j=1}^n \subset \mathrm{A}^+(L)$ for some $m, n \in \mathbb{N}$ such that*

$$|f(\omega) - [(g_1 \vee g_2 \vee \cdots \vee g_m) - (h_1 \vee h_2 \vee \cdots \vee h_n)](\omega)| < \varepsilon$$

uniformly for $\omega \in \Omega$.

We refer to Section 2 for the definition of $A^+(L)$.

Comparing this result with Cepedello Boiso's theorem, we can see that a w^*-continuous function defined on a dual space, or, a weakly continuous function defined on a reflexive space acts much like a function defined on a superreflexive space.

Let G be an Abelian semigroup and let $\mathbb{R} \times G \to G, (\lambda, g) \mapsto \lambda g$ be a given operation. If it satisfies the following conditions:

$$(\lambda\mu)g = \lambda(\mu g) \text{ for all } \lambda, \mu \in \mathbb{R}, \text{ and } g \in G;$$

$$\lambda(f + g) = \lambda f + \lambda g, \text{ for all } \lambda \in \mathbb{R}, \text{ and } f, g \in G,$$

and

$$1g = g \text{ and } 0g = 0, \text{ for all } g \in G,$$

then G is called a module over $\mathbb{R}$. A norm of G is a function $\|\cdot\| : G \to \mathbb{R}^+$ satisfying

$$\|g\| \geq 0, \quad \text{and} \quad \|g\| = 0 \Longleftrightarrow g = 0$$
$$\|\lambda g\| = |\lambda|\|g\|, \quad \text{for all } \lambda \in \mathbb{R}, \text{ and } g \in G,$$

and

$$\|f + g\| \leq \|f\| + \|g\|, \quad \text{for all } f, g \in G.$$

A (real) normed semigroup is a module G endowed with a norm. A function ϕ on a normed semigroup G is called a linear functional if it satisfies that

$$\phi(\alpha g_1 + \beta g_2) = \alpha\phi(g_1) + \beta\phi(g_2), \quad \forall \alpha, \beta \in \mathbb{R}^+ \text{ and } g_1, g_2 \in G.$$

It is said to be bounded provided $\|\phi\| = \sup\{\phi(g) : g \in G, \|g\| \leq 1\} < \infty$. We denote by G^* the Banach space of all bounded functionals on G, and call it the dual of G.

A normed semigroup has both rich topological structure and algebraic structure (see, for instance, [10]). There are various kinds of normed semigroups. For example, given a (real) Banach space X, let

$$\mathrm{bc}(X) = \{C \subset X : \; C \text{ is nonempty closed bounded and convex}\}.$$

Clearly, $\mathrm{bc}(X)$ with the usual operations $C_1 \oplus C_2 = \overline{\{c_1 + c_2 : c_1 \in C_1 \text{ and } c_2 \in C_2\}}$ and $\lambda C = \{\lambda c : c \in C\}$ is a module. We endow the Hausdorff metric d_H on $\mathrm{bc}(X)$, i.e.,

$$d_H(C_1, C_2) = \max\{\sup_{x \in C_1} d(x, C_2), \sup_{y \in C_2} d(C_1, y)\}, \quad \text{for } C_1,\, C_2 \in \mathrm{bc}(X).$$

This metric induces further a norm $\|\cdot\|_H$ for $C \in \mathrm{bc}(X)$

$$\|C\|_H = d_H(0, C) = \sup\{\|c\| : c \in C\}.$$

Thus, endowed with this norm, $\mathrm{bc}(X)$ is a normed semigroup. By a normed semigroup generated by subsets of a Banach space, we always mean the semigroup endowed with the norm $\|\cdot\|_H$ in the sequel.

Representation properties of the following normed subsemigroups of $\mathrm{bc}(X)$ and their duals are studied in [5] and [6].

$$\mathrm{cc}(X) = \{K \subset X : \; K \text{ is nonempty convex and compact}\},$$
$$\mathrm{wcc}(X) = \{K \subset X : \; K \text{ is nonempty convex and weakly compact}\},$$
$$\mathrm{swcc}(X) = \{K \subset X : \; K \text{ is convex super weakly compact}\}.$$

The notion of super weakly compact set is defined in [4].

Let $B(x, r)$ be the closed ball centered at $x \in X$ with radius $r \geq 0$, and let $\mathrm{b}(X)$ be the normed semigroup generated by all closed balls, that is,

$$\mathrm{b}(X) = \left\{ C = \overline{\mathrm{co}} \sum_{j=1}^{n} C_j : C_j = \overline{\mathrm{co}} \cup_{i=1}^{m} B(x_{ij}, r_{ij}), m, n \in \mathbb{N} \right\}.$$

Clearly, it is the smallest normed semigroup containing all closed balls of X, and closed under operations of closure of vector addition of sets, and the closed convex hull of the union of finitely many closed balls. Through discussion of "w^*-continuity restriction and extension" properties of support functions, we show then that the dual $\mathrm{b}^*(X)$ of $\mathrm{b}(X)$ is just $C(B_{X^*}, w^*)$.

2. Approximation by DC functions on locally convex spaces

In this section, the letter L is always a real locally convex space, and L^* its dual. For a set $A \subset L$, $\overline{\mathrm{co}}A$ stands for the closed convex hull of A, and the support function σ_A of A is defined for $x^* \in L^*$ by $\sigma_A(x^*) = \sup_{x \in A}\langle x^*, x\rangle$. Clearly, $\sigma_A = \sigma_{\overline{\mathrm{co}}A}$. If A is in a function space L and has only finitely many affine functions, then $\mathrm{co}A = \overline{\mathrm{co}}A$ is a convex polyhedron with vertices of affine functions, and which is called an affine (convex) polyhedron. Therefore, a function of the form $(a_1 \vee \cdots \vee a_m) - (b_1 \vee \cdots \vee b_n)$, with $a_i, b_j \in A(L)$, is said to be a difference affine (convex) polyhedral support function, or DAPS function. We also denote $\{f = x^* + c : x^* \in L^*, c \in \mathbb{R}, c \geq 0\}$ by $\mathrm{A}^+(L)$.

Recall that a subspace of a function space is said to be a lattice provided it is closed under the operations of $\vee$ and $\wedge$.

Theorem 2.1. *Let L be a locally convex space. Then the subspace*

$$\mathrm{DAPS}(L) \equiv \{(a_1 \vee \cdots \vee a_m) - (b_1 \vee \cdots \vee b_n) : a_i, b_j \in A^+(L), m, n \in \mathbb{N}\}$$

is a lattice in $C(\Omega)$.

Proof. Note $\sigma_A + \sigma_B = \sigma_{A+B}$ and $\lambda\sigma_A = \sigma_{\lambda A}$ for all $A, B \subset L$ and $\lambda \geq 0$. We see that $\{a_1 \vee \cdots \vee a_m : a_i \in A^+(L), m \in \mathbb{N}, i = 1, 2, \cdots, m\}$ is a convex cone. Therefore, $\mathrm{DAPS}(L)$ is a linear subspace of $C(\Omega)$. To show it is a lattice, it suffices to prove it is closed under the operation of $\vee$.

Let

$$f = (a_1 \vee \cdots \vee a_m) - (b_1 \vee \cdots \vee b_n),$$

and

$$g = (c_1 \vee \cdots \vee c_p) - (d_1 \vee \cdots \vee d_q),$$

for some $(a_i)_{i=1}^m \equiv A, (b_j)_{j=1}^n \equiv B, (c_k)_{k=1}^p \equiv C, (d_l)_{l=1}^q \equiv D \subset A^+(L)$ and some nonnegative integers $m, n, p, q \geq 0$. We rewrite f and g as

$$f = \sigma_A - \sigma_B, g = \sigma_C - \sigma_D. \tag{1}$$

Then for all $x \in L$,

$$\begin{aligned}(f \vee g)(x) &= (\sigma_A - \sigma_B)(x) \vee (\sigma_C - \sigma_D)(x)\\ &= [(\sigma_A - \sigma_B) - (\sigma_C - \sigma_D)](x) \vee 0 + (\sigma_C - \sigma_D)(x)\\ &= (\sigma_{A+D} - \sigma_{B+C})(x) \vee 0 + (\sigma_C - \sigma_D)(x)\\ &= (\sigma_{(A+D)\cup(B+C)} - \sigma_{B+C})(x) + (\sigma_C - \sigma_D)(x)\\ &= (\sigma_{(A+D)\cup(B+C)+C} - \sigma_{B+C+D})(x).\end{aligned}$$

Therefore, $f \vee g \in \mathrm{DAPS}(L)$. □

The following result is the Kakutani-Krein theorem.

Theorem 2.2 (Kakutani-Krein). *Let Ω be a compact Hausdorff space and $C(\Omega)$ the space of real-valued continuous functions on Ω endowed with the* sup-*norm. If $\mathcal{A} \subset C(\Omega)$ satisfies the following three conditions: (i) $\mathcal{A}$ is a vector lattice of $C(\Omega)$; (ii) the constant function $1 \in \mathcal{A}$, and (iii) $\mathcal{A}$ separates points of Ω. Then $\mathcal{A}$ is dense in $C(\Omega)$.*

Theorem 2.3. *Let Ω be a nonempty compact set of a locally convex space L, and let $C(\Omega)$ be the Banach space of all real-valued continuous functions on Ω endowed with the* sup-*norm. Then for every $f \in C(\Omega)$, and for every $\varepsilon > 0$, there are finitely many continuous affine functions $(g_i)_{i=1}^m, (h_j)_{j=1}^n \subset A^+(L)$ for some $m, n \in \mathbb{N}$ such that*

$$|f(\omega) - [(g_1 \vee g_2 \vee \cdots \vee g_m) - (h_1 \vee h_2 \vee \cdots \vee h_n)](\omega)| < \varepsilon$$

uniformly for $\omega \in \Omega$.

Proof. For proof of the theorem, what we need is to note that it is a direct consequence of Theorem 2.1 via the Kakutani-Krein theorem, since DAPS functions contain constants and separate points of the underlying space Ω. □

Corollary 2.4. *Let Ω be a nonempty weakly (resp. w^*) compact set of a Banach space X (resp. the dual X^* of X), and let $C(\Omega)$ be the Banach space of all real-valued continuous functions on (Ω, w) (resp. (Ω, w^*)) endowed with the* sup-*norm. Then for every $f \in C(\Omega)$, and for every $\varepsilon > 0$, there are finitely many continuous affine functions $(g_i)_{i=1}^m, (h_j)_{j=1}^n \subset A^+(X)$ (resp. $A^+(X^*)$) for some $m, n \in \mathbb{N}$ such that*

$$|f(\omega) - [(g_1 \vee g_2 \vee \cdots \vee g_m) - (h_1 \vee h_2 \vee \cdots \vee h_n)](\omega)| < \varepsilon$$

uniformly for $\omega \in \Omega$.

Comparing this result with Cepedello Boiso's theorem, we can see that a w^*-continuous function defined on a dual space, or, a weakly continuous function defined on a reflexive space acts much like a function defined on a superreflexive space.

3. Support functions of convex sets having a w^*-continuous restriction-extension

Let X be a Banach space and $C \subset X$ be a nonempty closed bounded convex set. Then its support function σ_C, defined for $x^* \in X^*$ by $\sigma_C(x^*) = \sup_{x \in C}\langle x^*, x\rangle$, is continuous and w^*-lower semi continuous on X^*. For any w^*-compact set $K \subset X^*$, we denote by σ_C^K, the restriction of σ_C to K. In this section, we shall discuss continuous restriction and extension property of such functions σ_C^K. When no confusion arises, it is simply denoted by σ_C instead of σ_C^K.

The following property is classical (see, for instance, [8]).

Proposition 3.1. *Let Ω be a Hausdorff space, and $f : \Omega \to \mathbb{R}$ be a lower semi continuous function. Then the set consisting of all points, at which g is not continuous, is a first category subset of Ω. In particular, if Ω is compact, then the set $\mathrm{CP}(f)$ consisting of all continuity points of f is of Baire category. Hence, CP(f) contains a dense G_δ-subset of Ω.*

Definition 3.2. Let f be a real-valued function defined on a topological space Ω. We say a continuous function $g : \Omega \to \mathbb{R}$ is a continuous restriction-extension (CRE) of f, if $g = f$ on $\mathrm{CP}(f)$. We also denote such a CRE by f_{cre}.

Recall that a function defined on a subset A of a linear topological space E is said to be uniformly continuous, if for all $\varepsilon > 0$, there is an open neighborhood U of the origin, such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ with $x - y \in U$.

Proposition 3.3. *Suppose that f is a real-valued lower semi continuous function defined on a compact subset Ω of a Hausdorff topological linear space. Then it has a unique CRE if and only if it is uniformly continuous on a dense (G_δ-) subset of Ω.*

Proof. Suppose that D is a dense subset of Ω and $f|_D$ is uniformly continuous on D. It is clear that $f|_D$ has a natural (uniformly) continuous extension $\overline{f}$ from D to Ω, and f is necessarily uniformly continuous on $\mathrm{CP}(f) \supset D$. These facts further imply that $f_{\text{cre}} = \overline{f}|_D$. Conversely, let g be the unique CRE of f. Then g is uniformly continuous since Ω is compact. By Proposition 3.1, $f = g$ on a dense (G_δ-) subset D of Ω. Therefore, f is uniformly continuous on D. $\square$

Example 3.4. Let X be a Banach space, $K \subset X$ be a nonempty compact set, and let $\Omega = (B_{X^*}, w^*)$. Let $C(\Omega)$ denote the Banach space of all w^*-continuous functions endowed with the sup-norm, $\|\cdot\|_\infty$, i.e., $\|f\|_\infty = \sup_{x^* \in \Omega} |f(x^*)|$. Clearly, $\sigma_K \in C(\Omega)$. Therefore, $(\sigma_K)_{\text{cre}} = \sigma_K$.

Next example shows that if K is not compact then $(\sigma_K)_{\text{cre}}$ can be quite different from σ_K.

Example 3.5. With the same spaces X, Ω and $C(\Omega)$ as in Example 3.4, let $C = rB_X$ for some $r > 0$. Then $\sigma_C(x^*) = r\|x^*\|$, for all $x^* \in \Omega$. If $\dim X < \infty$, then $\sigma_C \in C(\Omega)$, and this entails $(\sigma_C)_{\text{cre}} = \sigma_C$. If $\dim X = \infty$, then $\mathrm{CP}(f) = S_{X^*}$. Indeed, for any $x^* \in S_{X^*}$, and every net $(x^*_\alpha) \subset B_{X^*}$ w^*-converging to x^*, we obtain

$$r = r\|x^*\| = \sigma_C(x^*) \le \liminf_\alpha \sigma(x^*_\alpha) = \liminf_\alpha r\|x^*_\alpha\| \le r = \sigma_C(x^*).$$

Thus, σ_C is w^*-continuous at x^*. Conversely, suppose that $x^* \in B_{X^*}$ is w^*-continuity point of σ_C. Since $\dim X = \infty$, S_{X^*} is w^*-dense in B_{X^*}. Let $(x^*_\alpha) \subset S_{X^*}$ be a net w^*-converging to x^*. Then

$$r\|x^*\| = \sigma_C(x^*) = \lim_\alpha \sigma_C(x^*_\alpha) = \lim_\alpha r\|x^*_\alpha\| = r.$$

Hence, $x^* \in S_{X^*}$, and which entails that $(\sigma_C)_{\text{cre}} = r$.

The following example shows that not every σ_K admits a CRE, if K is not compact.

Example 3.6. Let $C = \{x \in c_0 : 1 \geq x(j) \geq 0, \forall j \in \mathbb{N}\}$, and let $\Omega = (B_{\ell_1}, w^*)$. Let $(e_n)_{n\in\mathbb{N}}$ be the standard unit vector basis of ℓ_1. Then σ_C is w^*-continuous at $(\pm e_j)_{j=1}^{\infty}$. In fact, since c_0 is separable, (B_{ℓ_1}, w^*) is metrizable. w^*-continuity of functions on B_{ℓ_1} coincides with w^*-sequential continuity. Let $(x_n^*) \subset B_{\ell_1}$ be a sequence w^*-converging to $x^* \in (\pm e_j)_{j\in\mathbb{N}}$. If $x^* = e_m$, for some $m \in \mathbb{N}$, then $\sigma_C(x^*) = 1$, and w^*-lower semi continuity of σ_C entails that $\sigma_C(x_n^*) \to 1$. If $x^* = -e_m$, then $\sigma_C(x^*) = 0$ and $x_n^*(j) \to x^*(j) = -1$, if $j = m$; $= 0$, if $j \neq m$. Therefore, $\sum_{j\neq m} |x_n^*(j)| \to 0$, and $\sigma_C(x_n^*) \to 0 = \sigma_C(x^*)$, as $n \to \infty$. Note that both (e_n) and $(-e_n)$ w^*-sequentially converges to 0. Assume that $(\sigma_C)_{\mathrm{cre}} = g$ exists. Then

$$1 = \sigma_C(e_n) = g(e_n) \to g(0) \leftarrow g(-e_n) = \sigma_C(-e_n) = 0,$$

and this is a contradiction.

Let $\Omega \subset X^*$ be a nonempty w^*-compact set. We denote by $\mathcal{E}$, the set of all closed bounded convex sets C such that $(\sigma_C)_{\mathrm{cre}}$ exists on Ω. The following theorem shows that $\mathcal{E}$ is closed under operations of vector addition of sets, scalar multiplication, and of the closed convex hull of the union of finitely many sets.

Theorem 3.7. *Let $A, B \in \mathcal{E}$. Then*

$$A + B \in \mathcal{E}; \quad \lambda A, \quad \text{for all } \lambda \geq 0; \quad \text{and} \quad \overline{\mathrm{co}}(A \cup B) \in \mathcal{E}.$$

If, in addition, Ω is symmetric, then

$$\lambda A \in \mathcal{E}, \quad \text{for all } \lambda \in \mathbb{R}.$$

Proof. Let $A, B \in \mathcal{E}$. Note $\sigma_{A+B} = \sigma_A + \sigma_B$. According to Proposition 3.3, σ_A (σ_B, resp.) is uniformly continuous on $\mathrm{CP}\sigma_A$ (σ_B, resp.), which contains a dense G_δ-subset of Ω. Therefore, $\mathrm{CP}(\sigma_A) \cup \mathrm{CP}(\sigma_B)$ contains again a dense G_δ-subset of Ω and σ_{A+B} is uniformly continuous on it. By Proposition 3.3 again, $A + B \in \mathcal{E}$. If $\lambda \geq 0$, then we have $\sigma_{\lambda A} = \lambda\sigma_A$, which entails that $(\sigma_{\lambda A})_{\mathrm{cre}}$ exists. To show $\overline{\mathrm{co}}(A \cup B) \in \mathcal{E}$, we note first that $\sigma_{\overline{\mathrm{co}}(A\cup B)} = \sigma_A \vee \sigma_B$, and that both σ_A and σ_B are uniformly continuous on $D \equiv \mathrm{CP}(\sigma_A) \cap \mathrm{CP}(\sigma_B)$. $\overline{\mathrm{co}}(A \cup B) \in \mathcal{E}$ follows from that D contains a dense G_δ subset of Ω.

It remains to prove that if Ω is symmetric, and if $A \in \mathcal{E}$, then $-A \in \mathcal{E}$. Note $\sigma_{-A}(x^*) = \sigma_A(-x^*)$ for all $x^* \in \Omega$. Symmetry of Ω and Proposition 3.3 together say that $\mathrm{CP}\sigma_{-A} = \mathrm{CP}(\sigma_A)$ and σ_{-A} is uniformly continuous on it. Thus, $-A \in \mathcal{E}$. □

Theorem 3.8. *Let $A, B \in \mathcal{E}$ and $\lambda \in \mathbb{R}^+$. Then*

i) $(\sigma_{\lambda A})_{\mathrm{cre}} = \lambda(\sigma_A)_{\mathrm{cre}}$*;*

ii) $(\sigma_{A+B})_{\mathrm{cre}} = (\sigma_A)_{\mathrm{cre}} + (\sigma_B)_{\mathrm{cre}}$*, and* *iii)* $(\sigma_{\overline{\text{co}}(A\cup B)})_{\text{cre}} = (\sigma_A)_{\text{cre}} \vee (\sigma_B)_{\text{cre}}$.

Proof. We need only show *iii*). It follows from Theorem 3.7 that $(\sigma_{\overline{\text{co}}(A\cup B)})_{\text{cre}}$ exists and equals $(\sigma_A \vee \sigma_B)_{\text{cre}}$. Note that (a) both σ_A and σ_B are uniformly continuous on $D \equiv \text{CP}(\sigma_A) \cup \text{CP}(\sigma_B)$; and that (b) D contains a dense G_δ-subset of Ω. We see that $(\sigma_{\overline{\text{co}}(A\cup B)})_{\text{cre}} = \sigma_A \vee \sigma_B = (\sigma_A)_{\text{cre}} \vee (\sigma_B)_{\text{cre}}$ on D. w^*-density of D in Ω and w^*-uniformly continuity of $(\sigma_{\overline{\text{co}}(A\cup B)})_{\text{cre}}$, $(\sigma_A)_{\text{cre}}$ and $(\sigma_B)_{\text{cre}}$ together imply *iii*). □

We denote by $\mathcal{C}$, the set of all elements $C \in \mathcal{E}$ such that $(\sigma_C)_{\text{cre}}$ is convex. The following results are direct consequences of Theorem 3.8.

Corollary 3.9. *Let $A, B \in \mathcal{C}$. Then $\lambda A \in \mathcal{C}$ for all $\lambda \geq 0$, $A + B \in \mathcal{C}$, and $\overline{\text{co}}(A \cup B) \in \mathcal{C}$. Thence, $\mathcal{C}$ is a convex cone.*

The corollary above says that $\mathcal{C}$ is closed under the operations of closed convex hull of finite union, finite addition, and nonnegative scalar multiplication.

Corollary 3.10. *Let $\Omega \subset X^*$ be a nonempty w^*-compact set endowed with the w^*-topology, and let*

$$\text{scre}(\mathcal{C}) = \{(\sigma_C)_{\text{cre}} : C \in \mathcal{C}\}.$$

Then scre$(\mathcal{C})$ is a convex cone in $C(\Omega)$.

Though every $C \in \mathcal{E}$ guarantees its support function σ_C has a continuous restriction-extension $(\sigma_C)_{\text{cre}} \in C(\Omega)$, we do not know whether all such $(\sigma_C)_{\text{cre}}$ are convex, or, equivalently, $\mathcal{C} = \mathcal{E}$.

4. Representation of $\text{b}(X)^*$

We know that support functions are very useful to embed isometrically some classes of closed bounded convex sets into Banach function spaces (see, for instance, [5], [6], [9] and [11]). It is shown that $M_{\text{cc}(X)} \equiv \{\sigma_C - \sigma_D : C, D \subset X$ are convex and compact$\}$, $M_{\text{wcc}} \equiv \{\sigma_C - \sigma_D : C, D \subset X$ are convex and weakly compact$\}$ ([6]), and $M_{\text{swcc}} \equiv \{\sigma_C - \sigma_D : C, D \subset X$ are convex and super weakly compact$\}$ [5] are lattices of $C(B_{X^*})$. In this section, we shall show that $\text{b}(X)^*$ is isometric to $C(B_{X^*}, w^*)$. First, we use restriction-extension properties of support functions to prove $\Delta(\mathcal{C}) \equiv \text{scre}(\mathcal{C}) - \text{scre}(\mathcal{C})$ is a lattice.

Theorem 4.1. *Let $\Omega \subset X^*$ be a nonempty w^*-compact set endowed with the w^*-topology. Then*

$$i) \qquad \Delta(\mathcal{C}) \equiv \text{scre}(\mathcal{C}) - \text{scre}(\mathcal{C})$$

is a vector lattice of $C(\Omega)$; ii) $\Delta(\mathcal{C})$ separates points of Ω, hence, $\Delta(\mathcal{C})$ is dense in $C(B_{X^*}, w^*)$.

Proof. *i)* Since $\text{scre}(\mathcal{C})$ is a convex cone of $C(\Omega)$, $\Delta(\mathcal{C})$ is clearly a linear subspace. Note that for all $f, g \in \text{scre}(\mathcal{C})$, $f \wedge g = -(-f \vee -g)$. To show $\Delta(\mathcal{C})$ is a vector lattice, it suffices to prove that it is closed under operation of $\vee$. For all $f, g \in \Delta(\mathcal{C})$, by definition, there exist A, B, C and $D \in \mathcal{C}$ such that

$$f = (\sigma_A)_{\rm cre} - (\sigma_B)_{\rm cre} \quad \text{and} \quad g = (\sigma_C)_{\rm cre} - (\sigma_D)_{\rm cre}.$$

Let

$$\begin{aligned}\mathcal{U} = \{&A, B, C, D, A+D, B+C, B+C+D,\\ &\overline{\rm co}((A+D)\cup(B+C)), \overline{\rm co}[(A+D)\cup(B+C)]+C\}.\end{aligned}$$

According to Theorem 3.7, $\mathcal{U} \subset \mathcal{E}$. Let $G = \cap_{U\in\mathcal{U}}{\rm CP}(\sigma_U)$. Then by Proposition 3.1, G contains a dense G_δ-subset of Ω. If we restrict all the eleven functions f, g, σ_U $(U \in \mathcal{U})$ to G, then for all $x^* \in G$, we obtain

$$\begin{aligned}(f \vee g)(x^*) &= (\sigma_A - \sigma_B)(x^*) \vee (\sigma_C - \sigma_D)(x^*)\\ &= [(\sigma_A - \sigma_B) - (\sigma_C - \sigma_D)](x^*) \vee 0 + (\sigma_C - \sigma_D)(x^*)\\ &= (\sigma_{A+D} - \sigma_{B+C})(x^*) \vee 0 + (\sigma_C - \sigma_D)(x^*)\\ &= (\sigma_{\overline{\rm co}((A+D)\cup(B+C))} - \sigma_{B+C})(x^*) + (\sigma_C - \sigma_D)(x^*)\\ &= (\sigma_{\overline{\rm co}((A+D)\cup(B+C))+C} - \sigma_{B+C+D})(x^*)\\ &= (\sigma_{\overline{\rm co}((A+D)\cup(B+C))})_{\rm cre}(x^*) - (\sigma_{B+D})_{\rm cre}(x^*).\end{aligned}$$

Uniform continuity of $f \vee g, (\sigma_{\overline{\rm co}((A+D)\cup(B+C))})_{\rm cre}$, and $(\sigma_{B+D})_{\rm cre}$ on the whole Ω entails that

$$f \vee g = (\sigma_{\overline{\rm co}((A+D)\cup(B+C))})_{\rm cre} - (\sigma_{B+D})_{\rm cre}.$$

By Theorem 3.8,

$$(\sigma_{B+D})_{\rm cre} = (\sigma_B)_{\rm cre} + (\sigma_D)_{\rm cre}$$

and

$$\begin{aligned}(\sigma_{\overline{\rm co}((A+D)\cup(B+C))})_{\rm cre} &= (\sigma_{A+D})_{\rm cre} \vee (\sigma_{B+C})_{\rm cre}\\ &= ((\sigma_A)_{\rm cre} + (\sigma_D)_{\rm cre}) \vee ((\sigma_B)_{\rm cre} + (\sigma_C)_{\rm cre}).\end{aligned}$$

Since $(\sigma_U)_{\rm cre}(U = A, B, C$ and $D)$ are convex,

$$f \vee g = (\sigma_{\overline{\rm co}((A+D)\cup(B+C))})_{\rm cre} - (\sigma_{B+D})_{\rm cre} \in \Delta(\mathcal{C}).$$

ii) To show that $\Delta(\mathcal{C})$ separates points of Ω, it suffices to note that $\mathcal{C}$ contains all singletons of X, and which separate points of X^*. □

Theorem 4.2. *Let X be an infinite dimensional Banach space, and let $b(X)$ be the normed semigroup generated by closed balls endowed with the Hausdorff norm defined in Section* 1*, i.e.*

$$\mathrm{b}(X) = \left\{C = \overline{\rm co}\sum_{j=1}^{n} C_j : C_j = \overline{\rm co}\cup_{i=1}^{m} B(x_{ij}, r_{ij}), m, n \in \mathbb{N}\right\}.$$

Then $b(X)$ is isometric to $mA^+(X^*) \equiv \{a_1 \vee a_2 \vee \cdots \vee a_n : n \in \mathbb{N}, a_j \in A^+(X^*)$ for $j = 1, 2, \cdots, n\} \subset C(B_{X^*}, w^*)$.

Proof. Given $C = \overline{\mathrm{co}} \sum_{j=1}^{n} C_j$, where $C_j = \overline{\mathrm{co}} \cup_{i=1}^{m} B(x_{ij}, r_{ij}), m, n \in \mathbb{N}$ with $x_{ij} \in X$ and $r_{ij} \geq 0$, note $B(x_{ij}, r_{ij}) = x_{ij} + r_{ij} B_X$. We see

$$\sigma_C = \sum_{j=1}^{n} \sigma_{C_j} = \sum_{j=1}^{n} (x_{1j} + r_{1j} \|\cdot\|) \vee (x_{2j} + r_{2j}\|\cdot\|) \vee \cdots \vee (x_{mj} + r_{mj}\|\cdot\|).$$

According to Theorem 3.8 and Example 3.5,

$$\begin{aligned}
(\sigma_C)_{\mathrm{cre}} &= \sum_{j=1}^{n} (\sigma_{C_j})_{\mathrm{cre}} \\
&= \sum_{j=1}^{n} (x_{1j} + r_{1j}(\|\cdot\|)_{\mathrm{cre}}) \vee (x_{2j} + r_{2j}(\|\cdot\|)_{\mathrm{cre}}) \vee \cdots \vee (x_{mj} + r_{mj}(\|\cdot\|)_{\mathrm{cre}} \\
&= \sum_{j=1}^{n} (x_{1j} + r_{1j}) \vee (x_{2j} + r_{2j}) \vee \cdots \vee (x_{mj} + r_{mj}) \in mA^+(X^*).
\end{aligned}$$

Conversely, given $a_1 \vee a_2 \vee \cdots \vee a_n \in mA^+(X^*)$, note $a_i = x_i + r_i \in A^+(X^*)$ for $i = 1, 2, \cdots, n$. Let $C = \overline{\mathrm{co}} \cup_{i=1}^{n} B(x_i, r_i)$. Then it is easy to see $(\sigma_C)_{\mathrm{cre}} = a_1 \vee a_2 \vee \cdots \vee a_n$. Thus, $M : b(X) \to mA^+(X^*)$ defined for $C \in b(X)$ by $M(C) = (\sigma_C)_{\mathrm{cre}}$ is a bijection satisfying

$$M(\lambda C) = \lambda M(C), M(C + D) + M(C) + M(D)$$

for all $C, D \in b(X)$ and $\lambda \geq 0$.

It remains to show that M is norm-preserving. By Example 3.5, the norm $\|\cdot\| = \sigma_{B_{X^*}} \in C(B_{X^*})$ is w^*-continuous on S_{X^*}. This and

$$\sigma_C = \sum_{j=1}^{n} (x_{1j} + r_{1j}\|\cdot\|) \vee (x_{2j} + r_{2j}\|\cdot\|)) \vee \cdots \vee (x_{mj} + r_{mj}\|\cdot\|)$$

(where $C = \overline{\mathrm{co}} \sum_{j=1}^{n} C_j$, $C_j = \overline{\mathrm{co}} \cup_{i=1}^{m} B(x_{ij}, r_{ij}), m, n \in \mathbb{N}$ with $x_{ij} \in X$ and $r_{ij} \geq 0$) entail that σ_C is w^*-continuous on S_{X^*} for all $C \in b(X)$. Let $C, D \in b(X)$. Then

$$\begin{aligned}
d_H(C, D) = \|\sigma_C - \sigma_D\| &= \max_{\omega \in B_{X^*}} |\sigma_C(\omega) - \sigma_D(\omega)| \\
&= \max_{\omega \in S_{X^*}} |\sigma_C(\omega) - \sigma_D(\omega)| = \max_{\omega \in S_{X^*}} |(\sigma_C)_{\mathrm{cre}}(\omega) - (\sigma_D)_{\mathrm{cre}}(\omega)| \\
&= \|(\sigma_C)_{\mathrm{cre}} - (\sigma_D)_{\mathrm{cre}}\|.
\end{aligned}$$

Therefore, M is a surjective isometry. □

Theorem 4.3. *Suppose that X is a Banach space. Then*

$$b(X)^* = C(B_{X^*}, w^*)^*.$$

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