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Order Asymptotically Isometric Copies of c_o in the Subspaces of Order Continuous Elements in Orlicz Spaces

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Necessary and sufficient conditions in order that the subspace of order continuous elements of an Orlicz sequence space contains an order asymptotically isometric copy of c_0 are given for both, the Luxemburg and the Amemiya-Orlicz norm. In case of a non-atomic, complete and $\sigma-$finite measure space (T, Σ, μ) and the Luxemburg norm (the Amemiya-Orlicz norm) such criteria are obtained under the additional assumption that the space $L^{\Phi}(T, \Sigma, \mu)$ is a dual space (resp. the space $L^{\Phi}_{A}(T, \Sigma, \mu)$ is a dual and non-square space). In both cases, the Luxemburg and the Amemiya-Orlicz norm the criteria are given under the necessary assumption that the spaces $E^{\Phi}(T, \Sigma, \mu)$ and $E^{\Phi}_{A}(T, \Sigma, \mu)$ are non-trivial. The asymptotically isometric copies of c_0 that are built in our theorems are order copies.

Keywords: Orlicz space, subspace of order continuous elements, Luxemburg norm, Amemiya-Orlicz norm, condition Δ_2, asymptotically isometric copy of c_o, the fixed point property

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1. Introduction

The concept of asymptotically isometric copy of c_o was introduced in [16], where it was also proved that Banach spaces containing asymptotically isometric copy of c_o or of l_1 fail the fixed point property. Let us recall that a Banach space

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$(X, \|\cdot\|)$ is said to contain an asymptotically isometric copy of c_o if there exist a null sequence $(\epsilon_n) \subset (0,1)$ and a sequence $(x_n)_{n=1}^{\infty} \subset X$ such that the series $\sum_{n=1}^{\infty} t_n x_n$ converges in X and

$$\sup_n (1-\epsilon_n)|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sup_n |t_n|$$

for any sequence $(t_n)_{n=1}^{\infty}$ in c_o.
A map $\Phi : \mathbb{R} \to \mathbb{R}_+^e = [0, +\infty]$ is said to be an Orlicz function if Φ is convex and even, $\Phi(0) = 0$, $\Phi(u) \to +\infty$ as $u \to +\infty$ and Φ is left-continuous at the point

$$b(\Phi) = \sup\{u \geq 0 : \Phi(u) < \infty\}, \tag{1}$$

that is, $\lim_{u \to b(\Phi)_-} \Phi(u) = \Phi(b(\Phi))$. Note that by convexity and by $\Phi(0) = 0$, Orlicz functions are continuous in the interval $[0, b(\Phi))$ and thanks to the assumption concerning the left limit of Φ at $b(\Phi)$, they are continuous in the extended sense on $\mathbb{R}$, which means that infinite limits are admitted. Let us define another important parameter $a(\Phi)$ of an Orlicz function Φ by

$$a(\Phi) = \sup\{u \geq 0 : \Phi(u) = 0\}. \tag{2}$$

Given any σ-finite and complete measure space (T, Σ, μ) and any Orlicz function Φ we can define on the space $L^o(T, \Sigma, \mu)$ of all (equivalence classes with respect to the equality μ-a.e. in T of) Σ-measurable functions $x : T \to \mathbb{R}$, the convex and even modular

$$I_\Phi(x) = \int_T \Phi(x(t)) d\mu(t).$$

In the case of counting measure, we can assume that $T = \mathbb{N}$ (the set of all natural numbers), $\Sigma = 2^{\mathbb{N}}$ and $\mu(A) = \text{card}(A)$ for any $A \subset \mathbb{N}$. In this case $L^o(T, \Sigma, \mu)$ is denoted by l^o and it is the space of all real-valued sequences. Then

$$I_\Phi(x) = \sum_{n=1}^{\infty} \Phi(x_n)$$

for any $x = (x_n)_{n=1}^{\infty} \in l^o$. The Orlicz space generated by an Orlicz function Φ and a σ-finite and complete measure space (T, Σ, μ) is denoted by $L^\Phi(T, \Sigma, \mu)$ and defined by the formula

$$\begin{aligned} &\mathrm{L}^\Phi(T, \Sigma, \mu) \\ &= \{x \in L^o(T, \Sigma, \mu) : I_\Phi(\lambda x) < +\infty \text{ for some } \lambda > 0 \text{ depending on } x\}. \end{aligned} \tag{3}$$

In the sequence case, that is, in the case of counting measure μ on $2^{\mathbb{N}}$ we write simply l^Φ instead of $L^\Phi(\mathbb{N}, 2^{\mathbb{N}}, \mu)$. All Orlicz spaces considered in this paper are equipped with the Luxemburg norm given by

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leq 1\} \tag{4}$$

for any $x \in L^\Phi(T,\Sigma,\mu)$ (or any $x \in l^\Phi$), if it is not assumed that another norm in $L^\Phi(T,\Sigma,\mu)$ or l^Φ is considered. All Orlicz spaces equipped with this norm are Banach lattices which are symmetric spaces (for the definition of Banach lattices we refer to [3], [12], [29], [32] and for the definition of symmetric spaces we refer to [4], [31]).

The regularity Δ_2-condition for a generating Orlicz function Φ should be defined accordingly to the measure space (T,Σ,μ) in such a way that the corresponding subspace $E^\Phi(T,\Sigma,\mu)$ of order continuous elements in the Orlicz space $L^\Phi(T,\Sigma,\mu)$ should be equal to the whole space $L^\Phi(T,\Sigma,\mu)$.

Let us recall that an element x of a Banach lattice $(X,\leq,\|\cdot\|)$ is said to be order continuous if for any sequence $(x_n)_{n=1}^\infty$ in X such that $0 \leq x_n \leq |x|$, and $\inf_n x_n = 0$, we have $\|x_n\| \to 0$, as $n \to \infty$. In Banach function lattices the partial order "$\leq$" is defined via inequality μ-almost everywhere, that is, for $x, y \in X$ the relation $x \leq y$ means that $x(t) \leq y(t)$ for μ-almost all $t \in T$ and the condition $\inf_n x_n = 0$ in the definition of order continuous elements in X should be replaced by the condition "$x_n \to 0$ μ-a.e. in T". For the theory of Banach lattices we refer to [3], [12], [29], [32].

Given any Banach lattice X the subspace of all order continuous elements from X is denoted by X_a and called the subspace of order continuous elements. A Banach lattice X is called *order continuous* if $X_a = X$. In the case of Orlicz spaces $L^\Phi = L^\Phi(T,\Sigma,\mu)$ a simple characterization of L_a^Φ is well-known. Namely, in the case when the measure space (T,Σ,μ) is non-atomic it is known that

$$(L^\Phi)_a = \{x \in L^o(T,\Sigma,\mu) : I_\Phi(\lambda x) < +\infty \text{ for any } \lambda > 0\}.$$

In the case of counting measure μ on $2^{\mathbb{N}}$ it is known that

$$(l^\Phi)_a = \left\{ x \in l^o : \text{ for any } \lambda > 0 \text{ there exists } n_\lambda \in \mathbb{N} \text{ such that } \sum_{n=n_\lambda}^{\infty} \Phi(\lambda x_n) < \infty \right\} \tag{5}$$

(see [27]).

Now, we return to the definition of the regularity Δ_2-condition. We say that an Orlicz function Φ satisfies suitable Δ_2-condition if:

(i) Φ satisfies the Δ_2-condition on the whole $\mathbb{R}_+$ ($\Phi \in \Delta_2(\mathbb{R}_+)$ for short) which means that there exists $K \geq 2$ such that $\Phi(2u) \leq K\Phi(u)$ for any $u \in \mathbb{R}$ whenever the measure μ is nonatomic and infinite;

(ii) Φ satisfies the Δ_2-condition at infinity ($\Phi \in \Delta_2(\infty)$ for short),that is, there exists $K \geq 2$ and $u_o > 0$ with $\Phi(u_o) < \infty$ such that $\Phi(2u) \leq K\Phi(u)$ for any $u \in \mathbb{R}$ satisfying $|u| \geq u_o$ if μ is nonatomic and finite;

(iii) Φ satisfies the Δ_2-condition at zero ($\Phi \in \Delta_2(0)$ for short),that is, there exists $K \geq 2$ and $u_1 > 0$ with $\Phi(u_1) > 0$ such that $\Phi(2u) \leq K\Phi(u)$ for any $u \in \mathbb{R}$ satisfying $|u| \leq u_1$ if μ is the counting measure on $2^{\mathbb{N}}$.

It is well known that $(L^\Phi)_a = L^\Phi$ if and only if Φ satisfies the suitable Δ_2-condition (see [30], [33], [34], [35], [37], [8]). The space L_a^Φ is denoted shortly by

E^Φ if the measure μ is non-atomic and by h^Φ if μ is the counting measure on $2^{\mathbb{N}}$. The sufficiency of the suitable Δ_2-condition for order continuity of the norm $\|\cdot\|_\Phi$ in the whole space L^Φ (or l^Φ) follows from the Lebesgue Dominated Convergence Theorem. The necessity of the suitable Δ_2-condition for order continuity of the norm $\|\cdot\|_\Phi$ in the whole space L^Φ (or l^Φ) follows from the fact that if Φ fails to satisfy the suitable Δ_2-condition, then the space l^∞ is embedded into L^Φ (or l^Φ) order linearly isometrically (see [23], [38], [8]).

The problem of the existence of an isomorphic, almost isometric, asymptotically isometric or isometric copy of c_o or l^1 or l^∞ in a Banach space X has a long history. A lot of papers have been devoted to these problems, because Banach spaces with such copies are "bad" geometrically and topologically (they fail reflexivity, uniform rotundity, uniform non-squareness and so on). In order to see this, we refer to the papers [1], [2], [21], [22], [13]–[20], [5]–[7], [23]–[25], [36], [9], [28] and the monographs [37], [8]. It is obvious that isometric copies are asymptotically isometric, and the last ones are almost isometric and that possessing the isomorphic copies is the weakest property. Recently, asymptotically isometric copies of l^1 and c_o began important when Dowling, Lennard and Turret proved in [13] that Banach spaces having such copies fail the fixed point property.
Therefore, it is natural and profitable to look for criteria of the existence in the subspaces E^Φ and h^Φ of Orlicz spaces L^Φ and l^Φ of asymptotically isometric copies of c_o. In the following we will consider the spaces E^Φ and h^Φ equipped with the Luxemburg norm or with the Amemiya-Orlicz norm $\|\cdot\|_\Phi^A$ defined as

$$\|x\|_\Phi^A = \inf\left\{\frac{I_\Phi(kx)+1}{k} : k > 0\right\}. \tag{6}$$

2. Results

In the next three theorems we will show under which conditions we have order asymptotically isometric copies of c_o in h^Φ spaces. For brevity we denote h^Φ as $(h^\Phi, \|\cdot\|_\Phi)$ and h_A^Φ as $(h^\Phi, \|\cdot\|_\Phi^A)$.

Theorem 2.1. *Let Φ be an Orlicz function with $a(\Phi) > 0$. Then both spaces h^Φ and h_A^Φ contain an order asymptotically isometric copy of c_o.*

Proof. Assume that $a(\Phi) > 0$ and choose a sequence $(k_n)_{n=1}^\infty$ of natural numbers such that

$$k_n\Phi((1+1/n)a(\Phi)) \geq 1 \tag{7}$$

for any $n \in \mathbb{N}$. Let us define for any $j \in \mathbb{N}$, $e_j = (0, ..., 0, 1_j, 0, ...)$ and $m_n = \sum_{p<n} k_p$ for $n > 1$. Put $x_1 = a(\Phi)(\sum_{j=1}^{k_1} e_j)$ and $x_n = a(\Phi)(\sum_{j=m_n+1}^{m_{n+1}} e_j)$ for $n \in \mathbb{N}\setminus\{1\}$. Now, define an operator $P : c_o \to l^o$ by the formula

$$Pt = \sum_{n=1}^\infty t_n x_n,$$

for any $t=(t_1,t_2,...)\in c_o$, It is obvious that P is linear. First we want to show that $Pt\in h^\Phi$ for any $t=(t_1,t_2,...)\in c_o$. Given any $\lambda>0$ we can find $n_\lambda\in\mathbb{N}$ such that $\lambda|t_p|<1$ for any $p\geq n_\lambda$. Hence

$$I_\Phi\left(\lambda P\left(\sum_{p=n_\lambda}^{\infty} t_pe_p\right)\right)=I_\Phi\left(\lambda\sum_{p=n_\lambda}^{\infty} t_px_p\right)=\sum_{p=n_\lambda}^{\infty} k_p\Phi(\lambda t_pa(\Phi))=0,$$

which by (5) means that $Pt\in h^\Phi$. Notice that for any $t=(t_1,t_2,...)\in c_o$

$$I_\Phi\left(\frac{Pt}{\|t\|_\infty}\right)=\sum_{n=1}^{\infty}k_n\Phi\left(\frac{t_n}{\|t\|_\infty}a(\Phi)\right)=0, \tag{8}$$

whence $\|\frac{Pt}{\|t\|_\infty}\|_\Phi\leq 1$, and so $\|Pt\|_\Phi\leq\|t\|_\infty$. From equality (8), we have

$$1+I_\Phi(P(t/\|t\|_\infty))=1$$

whence $\|\frac{Pt}{\|t\|_\infty}\|_\Phi^A\leq 1$ and $\|Pt\|_\Phi^A\leq\|t\|_\infty$. On the other hand, given any $\lambda\in(0,1)$, there exists $m=m_\lambda\in\mathbb{N}$ such that

$$\frac{(1-\epsilon_m)|t_m|}{\lambda\sup\{(1-\epsilon_n)|t_n|:n\in\mathbb{N}\}}>1,$$

where $\epsilon_n=\frac{1}{n+1}$ for any $n\in\mathbb{N}$. This gives that

$$\frac{|t_m|}{\lambda\sup\{(1-\epsilon_n)|t_n|:n\in\mathbb{N}\}}>\frac{1}{1-\epsilon_m}=1+\frac{1}{m}.$$

Notice that by (7)

$$I_\Phi\left(\frac{Pt}{\lambda\sup\{(1-\epsilon_n)|t_n|:n\in\mathbb{N}\}}\right)\geq k_m\Phi((1+1/m)a(\Phi))>1,$$

whence

$$\|Pt\|_\Phi^A\geq\|Pt\|_\Phi\geq\lambda\sup\{(1-\epsilon_n)|t_n|:n\in\mathbb{N}\}.$$

Since the above inequality holds for any $\lambda\in(0,1)$, the inequality

$$\|Pt\|_\Phi^A\geq\|Pt\|_\Phi\geq\sup\{(1-\epsilon_i)|t_n|:n\in\mathbb{N}\}$$

holds for any $t\in c_o$, which completes the proof. □

Theorem 2.2. *If Φ is an Orlicz function such that $a(\Phi)=0$ and $\Phi\notin\Delta_2(0)$, then h_Φ contains an order asymptotically isometric copy of c_o.*

Proof. Since $\Phi\notin\Delta_2(0)$ we can easily get that for any $a>1$ and any $v>0$, $\Phi\notin\Delta_a([0,v])$, which means that

$$\text{for any } k\geq 2 \text{ there exists } u_k\in[0,v] \text{ with } \Phi(au_k)>k\Phi(u_k). \tag{9}$$

Then it is obvious that any u_k satisfying (9) must be strictly positive. In consequence we can find inductively a sequence $(u_k)_{k=1}^{\infty}$ of positive numbers such that

$$\Phi(u_k) \le \frac{1}{2^{k+1}} \quad \text{for any } k \in \mathbb{N} \tag{10}$$

and

$$\Phi\left(\left(1+\frac{1}{k}\right)u_k\right) > 2^{k+1}\Phi(u_k) \quad \text{for any } k \in \mathbb{N}. \tag{11}$$

Let for any $k \in \mathbb{N}$, n_k be the biggest natural number such that $n_k\Phi(u_k) \le \frac{1}{2^k}$. Then

$$\frac{1}{2^{k+1}} < n_k\Phi(u_k) \le \frac{1}{2^k} \quad \text{for any } k \in \mathbb{N}. \tag{12}$$

Now, we define an operator $P : c_o \to l^o$ by

$$Pt = \sum_{n=1}^{\infty} t_n x_n,$$

where $t = (t_1, t_2, ...) \in c_o$, $x_1 = u_1(\sum_{j=1}^{n_1} e_j)$ and $x_n = u_n(\sum_{j=m_n+1}^{m_{n+1}} e_j)$ for $n > 1$. Here $e_j = (0, ..., 0, 1_j, 0, ...)$ for all $j \in \mathbb{N}$ and $m_n = \sum_{p<n} n_p$ for $n > 1$. It is obvious that P is a linear and non-negative, so order preserving ("order" for short) operator. Now we need to show that $Pt \in h^{\Phi}$ for any $t = (t_1, t_2, ...) \in c_o$. Given any $\lambda > 0$ we can find $j \in \mathbb{N}$ such that $\lambda|t_i| < 1$ for any $i \ge j$. Hence

$$I_{\Phi}\left(\lambda P\left(\sum_{p=j}^{\infty} t_p e_p\right)\right) = I_{\Phi}\left(\lambda \sum_{p=j}^{\infty} t_p x_p\right) = \sum_{p=j}^{\infty} n_p\Phi(\lambda t_p u_p) \le \sum_{p=j}^{\infty} \frac{1}{2^p} < 1,$$

which by (5) means that $Pt \in h^{\Phi}$. Notice that for any $t = (t_1, t_2, ...) \in c_o$,

$$I_{\Phi}\left(\frac{Pt}{\|t\|_{\infty}}\right) = \sum_{k=1}^{\infty} n_k\Phi\left(\lambda\frac{t_k}{\|t\|_{\infty}}u_k\right) \le \sum_{k=1}^{\infty}\frac{1}{2^k} = 1,$$

whence $\|Pt\|_{\Phi} \le \|t\|_{\infty}$ for any $t \in c_o$. On the other hand, given any $\lambda \in (0, 1)$, there exists $j \in \mathbb{N}$ such that

$$\frac{(1-\epsilon_j)|t_j|}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}} > 1,$$

where $\epsilon_i = 1/(i+1)$ for any $i \in \mathbb{N}$. This gives that

$$\frac{|t_j|}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}} > \frac{1}{1-\epsilon_j} = 1 + \frac{1}{j}.$$

Therefore

$$\begin{aligned} &I_{\Phi}\left(\frac{Pt}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}}\right) \\ &\ge n_j\Phi((1+1/j)u_j) > 2^{j+1}n_j\Phi(u_j) > 2^{j+1}/2^{j+1} = 1, \end{aligned}$$

whence

$$\|Pt\|_\Phi \geq \lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}.$$

Since the above inequality holds for any $\lambda \in (0,1)$, we obtain

$$\|Pt\|_\Phi \geq \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}$$

for any $t \in c_o$, which completes the proof. □

Theorem 2.3. *If $\Phi \in \Delta_2(0)$, then the space $h_A^\Phi = l_A^\Phi$ contains asymptotically isometric copy of c_o if and only if $a(\Phi) > 0$.*

Proof. Fix an Orlicz function Φ. By Theorem 2.1, if $a(\Phi) > 0$, then h_A^Φ contains asymptotically isometric copy of c_o.
To prove the converse, assume that $a(\Phi) = 0$. We will apply Theorem 2 from [18] which states that a Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy of c_o if and only if there exist a sequence $(x^n)_{n=1}^\infty$ in X and a constant $0 < m < 1$ such that for any finite collection of scalars (t_n)

$$m \sup_n |t_n| \leq \left\| \sum_n t_n x^n \right\| \leq \sup_n |t_n| \tag{13}$$

and

$$\lim_n \|x^n\| = 1. \tag{14}$$

Assume on the contrary that h_A^Φ contains asymptotically isometric copy of c_o and fix a sequence $(x^n)_{n=1}^\infty \subset h^\Phi$ satisfying the above two conditions. First notice that for any $j \in \mathbb{N}$,

$$\Phi\left(\sum_{n=1}^\infty |x_j^n|\right) \leq 1. \tag{15}$$

If not, then $\Phi(\sum_{n=1}^N |x_{j_o}^n|) > 1 + d_o$ for some $j_o \in \mathbb{N}$, $N \in \mathbb{N}$ and $d_o > 0$. Set for $j = 1, ..., N$, $t_j = \operatorname{sgn}(x_{j_o}^n)$ and $t = (t_1, ..., t_N)$. Note that for any $k \geq 1$,

$$\frac{I_\Phi(k\sum_{j=1}^N t_j x^j) + 1}{k} \geq \frac{I_\Phi(k\sum_{j=1}^N t_j x^j)}{k} \geq I_\Phi\left(\sum_{j=1}^N t_j x^j\right)$$
$$\geq \Phi\left(\sum_{j=1}^N (|(t_j x^j)_{j_o}|)\right) = \Phi\left(\sum_{j=1}^N (|x_{j_o}^j|)\right) > 1 + d_o.$$

Hence, by the left-continuity of the function $k \to \frac{I_\Phi(k\sum_{j=1}^N t_j x^j)+1}{k}$ at $k = 1$ and the above inequality, there exists $0 < k_o < 1$ and such that

$$\frac{I_\Phi(k\sum_{j=1}^N t_j x^j) + 1}{k} > 1 + d_o$$

for $k \geq k_o$. Notice that $\frac{I_\Phi(k\sum_{j=1}^N t_j x^j)+1}{k} \geq \frac{1}{k} \geq \frac{1}{k_o} > 1$ for $0 < k < k_o$. Consequently

$$\inf\left\{\frac{I_\Phi(k\sum_{j=1}^N t_j x^j)+1}{k} : k > 0\right\} \geq \min\{1+d_o, 1/k_o\} > 1.$$

By (6), $\|\sum_{j=1}^N t_j x^j\|_\Phi^A > 1$; a contradiction.
Now, we will show that there exists $d > 0$ such that for any $j_o, n \in \mathbb{N}$ there exists $m_n \geq n$ such that

$$\sum_{j=j_o}^{\infty} \Phi(|x_j^{m_n}|) > d. \tag{16}$$

If not, then for any $d > 0$ there exists $j_o, n_o \in \mathbb{N}$ such that for any $m \geq n_o$,

$$\sum_{j=j_o}^{\infty} \Phi(|x_j^m|) \leq d.$$

Let for $x = (x_1, x_2, ...) \in h_\Phi$, $P_{j_o}x = (x_1, ..., x_{j_o}, 0, ...)$. Notice that for any $m \in \mathbb{N}$,

$$\begin{aligned} I_\Phi(x^m) &= I_\Phi(P_{j_o}(x^m)) + I_\Phi(x^m - P_{j_o}(x^m)) \\ &= \sum_{j=1}^{j_o-1} \Phi(|x_j^m|) + \sum_{j=j_o}^{\infty} \Phi(|x_j^m|) \leq \sum_{j=1}^{j_o-1} \Phi(|x_j^m|) + d \end{aligned}$$

for $m \geq n_o$. Since by (15), for any $j \in \mathbb{N}$, $\lim_n x_j^n = 0$, $I_\Phi(x^m) \leq 2d$ for m sufficiently large. This means that $I_\Phi(x^m) \to_m 0$. Since $(x^m)_{m=1}^\infty \subset h^\Phi$, we conclude that $\|x^m\|_\Phi^A \to_m 0$; a contradiction with (14), which shows that the inequality (16) holds true with some $d > 0$.
Now fix $d > 0$ satisfying (16). Since $\|x^n\|_\Phi^A \to 1$, there exists $n_o \in \mathbb{N}$ such that $\|x^{n_o}\|_\Phi^A > 1 - d/4$. Since $x^{n_o} \in h^\Phi$, there exists $j_o \in \mathbb{N}$ such that $\|P_{j_o}(x^{n_o})\|_\Phi^A > 1 - d/2$. By (15), $\lim_m P_{j_o}(x^m) = 0$ coordinatewise. Hence

$$\|P_{j_o}(x^{n_o}) \pm x^m\|_\Phi^A \to_m \|P^{j_o}(x^{n_o})\|_\Phi^A$$

and consequently

$$\|P_{j_o}(x^{n_o} \pm x^m)\|_\Phi^A > 1 - d/2$$

for $m \geq m_o$. Fix $k \geq 1$. Then for any $m \in \mathbb{N}$,

$$\begin{aligned} &\frac{I_\Phi(k(x^{n_o} \pm x^m)) + 1}{k} \\ &= \frac{I_\Phi(k(P_{j_o}(x^{n_o} \pm x^m))) + 1}{k} + \frac{I_\Phi(k(Id - P_{j_o})(x^{n_o} \pm x^m))}{k} \\ &\geq 1 - d/2 + I_\Phi((Id - P_{j_o})(x^{n_o} \pm x^m)) = 1 - d/2 + \sum_{j=j_o+1}^{\infty} \Phi(|x_i^m \pm x_j^{n_o}|). \end{aligned} \tag{17}$$

for $m \geq m_o$. By (16) we can choose m_o in such a way that

$$\sum_{j=j_o+1}^{\infty} \Phi(|x_j^{m_o}|) > d. \tag{18}$$

Now assume that

$$M = \max\left\{ \sum_{j=j_o+1}^{\infty} \Phi(|x_i^{m_o} + x_j^{n_o}|), \sum_{j=j_o+1}^{\infty} \Phi(|x_i^{m_o} - x_j^{n_o}|) \geq d. \right. \tag{19}$$

By (19), without loss of generality we can assume that $\sum_{j=j_o+1}^{\infty} \Phi(|x_i^{m_o}+x_j^{n_o}|) \geq d$. Then by (17) for any $k \geq 1$,

$$\frac{I_\Phi(k(x^{n_o} + x^{m_o}) + 1}{k} > 1 + d/4. \tag{20}$$

Since the function $g(k) = \frac{I_\Phi(k(x^{n_o}+x^{m_o}))+1}{k}$ is left-continuous and by (20) $g(1) > 1 + d/4$, there exists $0 < k_o < 1$ such that for any $k \geq k_o$,

$$\frac{I_\Phi(k(x^{n_o} + x^{m_o}) + 1}{k} > 1 + d/4.$$

Notice that for $0 < k < k_o$, $\frac{I_\Phi(k\sum_{j=1}^N t_j x^j)+1}{k} \geq \frac{1}{k} \geq \frac{1}{k_o} > 1$. Consequently

$$\inf\left\{ \frac{I_\Phi(k\sum_{j=1}^N t_j x^j) + 1}{k} : k > 0 \right\} \geq \min\{1 + d/4, 1/k_o\} > 1.$$

By (6), $\|x^{n_o} + x^{m_o}\|_\Phi^A > 1$; a contradiction with (13). Hence to end our proof, we should demonstrate (19). Assume on the contrary that $M < d$. Hence

$$\begin{aligned} d &> \frac{\sum_{j=j_o+1}^{\infty} \Phi(|x_i^{m_o} + x_j^{n_o}|) + \sum_{j=j_o+1}^{\infty} \Phi(|x_i^{m_o} - x_j^{n_o}|)}{2} \\ &\geq \sum_{j=j_o+1}^{\infty} \Phi\left(\frac{|x_i^{m_o} + x_j^{n_o}| + |x_i^{m_o} - x_j^{n_o}|}{2} \right) \geq \sum_{j=j_o+1}^{\infty} \Phi(|x_i^{m_o}|); \end{aligned}$$

a contradiction with (18). The proof of our theorem is fully complete. □

It is obvious that in the case of non-atomic measure space (T, Σ, μ) if Φ is an Orlicz function with $b(\Phi) < \infty$, then $E^\Phi = \{0\}$. Therefore, considering the case of non-atomic measure, we will always assume that $b(\Phi) = \infty$.

Remark 2.4. If we assume additionally that the Orlicz function Φ is an N-function, that is, $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$ and $\lim_{u\to 0} \frac{\Phi(u)}{u} = 0$, then assuming also that $a(\Phi) = 0$, we can prove that h_A^Φ does not contain an order asymptotically isometric copy of c_0 without the assumption that $\Phi \in \Delta_2(0)$. Really, if h_A^Φ contains such a copy, then ℓ_A^Φ does. Since ℓ_A^Φ is the dual space of h^Ψ, where Ψ is the complementary function of Φ, by Theorem 1 in [14], ℓ_A^Φ contains an isometric copy of ℓ^∞. Therefore, ℓ_A^Φ is not locally uniformly non-square, which contradicts Theorem 3.26 in [8].

Theorem 2.5. *Assume that (T, Σ, μ) is a non-atomic, complete and infinite measure space and Φ is an Orlicz function such that $b(\Phi) = \infty$ and $\Phi \notin \Delta_2(\mathbb{R}_+)$. Then the space $E^\Phi = E_\Phi(T, \Sigma, \mu)$ contains an order asymptotically isometric copy of c_o.*

Proof. First consider the case $a(\Phi) > 0$. Choose a sequence $(A_n)_{n=1}^\infty$ of pairwise disjoint and measurable subsets of T such that

$$\Phi\left(\left(1 + \frac{1}{n}\right) a(\Phi)\right) \mu(A_n) = 1 \quad \text{for any } n \in \mathbb{N}, \tag{21}$$

and define an operator $P : c_o \to L^o(T, \Sigma, \mu)$ by

$$Pt = \sum_{n=1}^{\infty} t_n a(\Phi) \chi_{A_n},$$

for any $t = (t_n)_{n=1}^\infty \in c_o$. It is obvious that P is linear and non-negative, so order preserving ("order" for short). Now we need to show that $Pt \in E^\Phi$ for any $t = (t_1, t_2, ...) \in c_o$. Given any $\lambda > 0$ we can find $j \in \mathbb{N}$ such that $\lambda |t_i| < 1$ for any $i \geq j$. Hence

$$\begin{aligned} I_\Phi(\lambda P(t) &= \sum_{n=1}^{j-1} \Phi(\lambda t_n a(\Phi)) \mu(A_n) + \sum_{n=j}^{\infty} \Phi(\lambda t_n a(\Phi)) \mu(A_n) \\ &= \sum_{n=1}^{j-1} \Phi(\lambda t_n a(\Phi)) \mu(A_n) < \infty, \end{aligned}$$

which means that $Pt \in E^\Phi$. Notice that for any $t = (t_1, t_2, ...) \in c_o$,

$$I_\Phi\left(\frac{Pt}{\|t\|_\infty}\right) = \sum_{n=1}^{\infty} \Phi\left(\frac{t_n}{\|t\|_\infty} a(\Phi)\right) \mu(A_n) \leq \sum_{n=1}^{\infty} \Phi(a(\Phi)) \mu(A_n) = 0,$$

whence $\|Pt\|_\Phi \leq \|t\|_\infty$ for any $t \in c_o$. On the other hand, given any $\lambda \in (0, 1)$, there exists $j \in \mathbb{N}$ such that

$$\frac{(1 - \epsilon_j)|t_j|}{\lambda \sup\{(1 - \epsilon_i)|t_i| : i \in \mathbb{N}\}} > 1,$$

where $\epsilon_i = 1/(i + 1)$ for $i \in \mathbb{N}$. This gives

$$\frac{|t_j|}{\lambda \sup\{(1 - \epsilon_i)|t_i| : i \in \mathbb{N}\}} > \frac{1}{1 - \epsilon_j} = 1 + \frac{1}{j}.$$

Therefore

$$\begin{aligned} I_\Phi\left(\frac{Pt}{\lambda \sup\{(1 - \epsilon_i)|t_i| : i \in \mathbb{N}\}}\right) &\geq \Phi\left(\frac{t_j a(\Phi)}{\lambda \sup\{(1 - \epsilon_i)|t_i| : i \in \mathbb{N}\}}\right) \mu(A_j) \\ &> \Phi\left(\left(1 + \frac{1}{j}\right) a(\Phi)\right) \mu(A_j) = 1, \end{aligned}$$

whence

$$\|Pt\|_\Phi \geq \lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}.$$

Since the above inequality holds for any $\lambda \in (0,1)$, we get

$$\|Pt\|_\Phi \geq \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\},$$

for any $t \in c_o$, which completes the proof in the first case.

Now assume that $a(\Phi) = 0$ and $\Phi \notin \Delta_2(\mathbb{R}_+)$. Then there exists a sequence $(u_n)_{n=1}^\infty$ of positive numbers such that $\Phi((1+\frac{1}{n})u_n) > 2^n\Phi(u_n)$ for any $n \in \mathbb{N}$. Choose a sequence $(A_n)_{n=1}^\infty$ of pairwise disjoint and measurable subsets of T such that

$$\Phi(u_n)\mu(A_n) = 2^{-n} \text{ for any } n \in \mathbb{N}, \tag{22}$$

and define an operator $P : c_o \to L^o(T, \Sigma, \mu)$ by

$$Pt = \sum_{n=1}^\infty t_n u_n \chi_{A_n}.$$

Now we need to show that $Pt \in E^\Phi$ for any $t = (t_1, t_2, ...) \in c_o$. Given any $\lambda > 0$ we can find $j \in \mathbb{N}$ such that $\lambda|t_i| < 1$ for any $i \geq j$. Hence

$$\begin{aligned} I_\Phi(\lambda P(t) &= \sum_{n=1}^{j-1}\Phi(\lambda t_n u_n)\mu(A_n) + \sum_{n=j}^\infty \Phi(\lambda t_n u_n)\mu(A_n) \\ &\leq \sum_{n=1}^{j-1}\Phi(\lambda t_n u_n)\mu(A_n) + \sum_{n=j}^\infty \Phi(u_n)\mu(A_n) \\ &\leq \sum_{n=1}^{j-1}\Phi(\lambda t_n u_n)\mu(A_n) + \sum_{n=j}^\infty 2^{-n} < \infty, \end{aligned}$$

because of $b(\Phi) = \infty$, which means that $Pt \in E^\Phi$. Notice that for any $t = (t_1, t_2, ...) \in c_o$,

$$I_\Phi\left(\frac{Pt}{\|t\|_\infty}\right) = \sum_{n=1}^\infty \Phi\left(\frac{t_n}{\|t\|_\infty}u_n\right)\mu(A_n) \leq \sum_{n=1}^\infty \Phi(u_n)\mu(A_n) = \sum_{n=1}^\infty 2^{-n} = 1,$$

whence $\|Pt\|_\Phi \leq \|t\|_\infty$ for any $t \in c_o$. On the other hand, given any $\lambda \in (0,1)$ and any $t = (t_i) \in c_0$, there exists $j \in \mathbb{N}$ such that

$$\frac{(1-\epsilon_j)|t_j|}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}} > 1,$$

where $\epsilon_i = 1/(i+1)$ for $i \in \mathbb{N}$. This gives that

$$\frac{|t_j|}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}} > \frac{1}{1-\epsilon_j} = 1 + \frac{1}{j}.$$

Therefore

$$
\begin{aligned}
& I_\Phi\left(\frac{Pt}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}}\right) \\
& \geq \Phi\left(\frac{t_j u_j}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}}\right)\mu(A_j) \\
& > \Phi\left(\left(1+\frac{1}{j}\right)u_j\right)\mu(A_j) > 2^j \Phi(u_j)\mu(A_j) = 1,
\end{aligned}
$$

whence

$$\|Pt\|_\Phi \geq \lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}.$$

Since the above inequality holds for any $\lambda \in (0,1)$, we get

$$\|Pt\|_\Phi \geq \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}$$

for any $t \in c_o$, which completes the proof in this case, and the theorem is proved. $\square$

Theorem 2.6. *Assume that (T, Σ, μ) is a non-atomic, complete and finite measure space with $\mu(T) > 0$ and Φ is an Orlicz function such that $b(\Phi) = \infty$ and $\Phi \notin \Delta_2(\infty)$. Then the space $E^\Phi = E^\Phi(T, \Sigma, \mu)$ contains an order asymptotically isometric copy of c_o.*

Proof. Since $\Phi \notin \Delta_2(\infty)$, we know that $\Phi \notin \Delta_{1+1/n}([v, +\infty))$, for any $n \in \mathbb{N}$, where $\Phi(v)\mu(T) = 1$. In consequence, there exists a sequence $(u_n)_{n=1}^\infty$ such that

$$u_n \in [v, +\infty) \tag{23}$$

and

$$\Phi\left(\left(1+\frac{1}{n}\right)u_n\right) > 2^n \Phi(u_n) \tag{24}$$

for any $n \in \mathbb{N}$. Since μ is non-atomic, $\Phi(v)\mu(T) = 1$ and $\sum_{n=1}^\infty 2^{-n} = 1$, one can find a sequence $(A_n)_{n=1}^\infty$ of pairwise disjoint and measurable subsets of T such that

$$\Phi(u_n)\mu(A_n) = 2^{-n} \quad \text{for any } n \in \mathbb{N}. \tag{25}$$

Define an operator $P : c_o \to L^o(T, \Sigma, \mu)$ by

$$Pt = \sum_{n=1}^\infty t_n u_n \chi_{A_n}.$$

It is obvious that P is a linear and non-negative, so order preserving ("order" for short) operator. Now we need to show that $Pt \in E^\Phi$ for any $t = (t_1, t_2, ...) \in c_o$.

Given any $\lambda > 0$ we can find $j \in \mathbb{N}$ such that $\lambda|t_i| < 1$ for any $i \geq j$. Hence

$$\begin{aligned} I_\Phi(\lambda P(t) &\leq \sum_{n=1}^{j-1} \Phi(\lambda t_n u_n)\mu(A_n) + \sum_{n=j}^{\infty} \Phi(u_n)\mu(A_n) \\ &\leq \sum_{n=1}^{j-1} \Phi(\lambda t_n u_n)\mu(A_n) + \sum_{n=j}^{\infty} 2^{-n} < \infty, \end{aligned}$$

which means that $Pt \in E^\Phi$. Notice that for any $t = (t_1, t_2, ...) \in c_o$,

$$I_\Phi\left(\frac{Pt}{\|t\|_\infty}\right) \leq \sum_{n=1}^{\infty} \Phi(u_n)\mu(A_n) \leq \sum_{n=1}^{\infty} 2^{-n} = 1,$$

whence $\|Pt\|_\Phi \leq \|t\|_\infty$ for any $t \in c_o$. On the other hand, given any $\lambda \in (0,1)$, there exists $j \in \mathbb{N}$ such that

$$\frac{(1-\epsilon_j)|t_j|}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}} > 1,$$

where $\epsilon_i = 1/(i+1)$ for any $i \in \mathbb{N}$. This gives

$$\frac{|t_j|}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}} > \frac{1}{1-\epsilon_j} = 1 + \frac{1}{j}.$$

Therefore

$$\begin{aligned} &I_\Phi\left(\frac{Pt}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}}\right) \\ &\geq \Phi\left(\frac{t_j u_j}{\lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}}\right)\mu(A_j) \\ &> \Phi\left(\left(1 + \frac{1}{j}\right)u_j\right)\mu(A_j) > 2^j \Phi(u_j)\mu(A_j) = 2^j 2^{-j} = 1, \end{aligned}$$

whence

$$\|Pt\|_\Phi \geq \lambda \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}.$$

Since the above inequality holds for any $\lambda \in (0,1)$, we get

$$\|Pt\|_\Phi \geq \sup\{(1-\epsilon_i)|t_i| : i \in \mathbb{N}\}$$

for any $t \in c_o$, which completes the proof. □

Now, let (T, Σ, μ) be a non-atomic, σ-finite and complete measure space with $\mu(T) > 0$. Let Φ be an Orlicz function such that $b(\Phi) = +\infty$ and the function $R(u) = A|u| - \Phi(u)$, where $A = \lim_{u\to\infty} \Phi(u)/u$, be unbounded on $\mathbb{R}_+$. Assume furthermore that $a(\Phi)\mu(T) < 2$ and $\Psi(d_\Psi)\mu(T) < 2$ (with $0 \cdot (+\infty) = 0$), and

$b(\Psi) = +\infty$, where Ψ is the function complementary to Φ in the sense of Young and

$$d_\Psi = \sup\{u > 0 : \Psi \text{ is linear on } [0,u]\}$$

(with $\sup\{\emptyset\} = 0$). Let $E_A^\Phi(T,\Sigma,\mu)$ ($L_A^\Phi(T,\Sigma,\mu)$, resp.) denote the space $E^\Phi(T,\Sigma,\mu)$ ($L^\Phi(T,\Sigma,\mu)$, resp.) equipped with the Amemiya-Orlicz norm $\|.\|_\Phi^A$ (see formula (6)). Assuming the above conditions we can prove

Theorem 2.7. *The space $E_A^\Phi(T,\Sigma,\mu)$ contains an asymptotically isometric copy of c_o if and only if $a(\Phi) > 0$ and $\mu(T) = +\infty$.*

Proof. *Sufficiency.* Assume that $a(\Phi) > 0$ and $\mu(T) = +\infty$ and let $(A_n)_{n=1}^\infty$ be a sequence of pairwise disjoint measurable sets such that

$$\Phi((1+1/n)a(\Phi))\mu(A_n) = 1 \quad \text{for } n \in \mathbb{N}.$$

Define an operator $P : c_o \to L^o(T,\Sigma,\mu)$ by

$$P(t) = \sum_{n=1}^{\infty} t_n a(\Phi)\chi_{A_n}(t),$$

for any $t = (t_1, t_2, ...) \in c_o$. It is obvious that P is linear and non-negative, so order preserving. It can be shown in the same way as in the proof of Theorem 2.5 that $Pt \in E_A^\Phi(T,\Sigma,\mu)$ for any $t = (t_1,t_2,...) \in c_o$. Moreover, for any $t = (t_1,t_2,...) \in c_o$,

$$\left\|\frac{Pt}{\|t\|_\infty}\right\|_\Phi^A \le 1 + I_\Phi\left(\frac{Pt}{\|t\|_\infty}\right) = 1 + \sum_{n=1}^{\infty}\Phi\left(\frac{t_n}{\|t\|_\infty}a(\Phi)\right)\mu(A_n) = 1,$$

whence

$$\|Pt\|_\Phi^A \le \|t\|_\infty.$$

On the other hand, by the proof of Theorem 2.5,

$$\|Pt\|_\Phi^A \ge \|Pt\|_\Phi \ge \sup\{(1-\epsilon_n)|t_n| : n \in \mathbb{N}\},$$

which completes the proof of the sufficiency.

Necessity. If $a(\Phi) = 0$ or $\mu(T) < \infty$ then the conditions $a(\Phi)\mu(T) < \infty$ and $a(\Phi) < b(\Phi)$ are satisfied. Moreover, the unboundedness of the function $R(u)$ on $\mathbb{R}_+$ gives that Φ is k_1^*-finite, that is, for any $x \in L^\Phi(T,\Sigma,\mu)$ there is $k > 0$ such that $I_\Psi(p \circ (k|x|)) \ge 1$, where p is the right-hand side derivative of Φ. Consequently, by the assumptions fixed just before formulating this theorem, the space $L_A^\Phi(T,\Sigma,\mu)$ is non-square (see Theorem 3.4, p. 3979 in [11]). By $b(\Psi) = +\infty$, the space $L_A^\Phi(T,\Sigma,\mu)$ is a dual space. Namely, $L_A^\Phi(T,\Sigma,\mu)$ is the dual space to $E^\Psi(T,\Sigma,\mu)$, equipped with the Luxemburg norm (see [8], [30], [35] and [37]). Assuming that $E_A^\Phi(T,\Sigma,\mu)$ contains an asymptotically isometric copy of c_o, we also have that $L_A^\Phi(T,\Sigma,\mu)$ contains such a copy. Since $L_A^\Phi(T,\Sigma,\mu)$ is a dual space, we know (see [14]) that $L_A^\Phi(T,\Sigma,\mu)$ contains an asymptotically isometric copy of l^∞ and finally, $L_A^\Phi(T,\Sigma,\mu)$ contains an isometric copy of l^∞. Consequently, the space $L_A^\Phi(T,\Sigma,\mu)$ is not non-square, a contradiction which finishes the proof. □

Remark 2.8. If we restrict ourselves in Theorem 2.7 to the class of Orlicz functions Φ such that $\lim_{u\to\infty}\Phi(u)/u = +\infty$, then the function $R(u)$ is unbounded on $\mathbb{R}_+$ and $b(\Psi) = +\infty$. Therefore, under these assumptions about Φ, our assumptions concerning Theorem 2.7 reduces only to the following conditions:

(i) $a(\Phi)\mu(T) = +\infty$ (with $0 \cdot \infty = 0$)
(ii) $\Psi(d_\Psi)\mu(T) < 2$.

Remark 2.9. In Theorems 2.2, 2.5 and 2.6 the assumption that Φ does not satisfy the suitable Δ_2-condition is necessary if Φ is an N-function. Really, the assumption that Φ satisfies a suitable Δ_2-condition implies that ℓ^Φ and ℓ^Φ_A are dual spaces. In case of a non-atomic measure space the suitable Δ_2-condition and the assumptions that $b(\Psi) = \infty$ imply that L^Φ and L^Φ_A are also dual spaces. Therefore, they contain isometric copies of l^∞ whenever E^Φ (resp. h^Φ) contains an asymptotically isometric copy of c_0 (see the argumentation in Remark 2.4 and in the necessity part of the proof of Theorem 2.7). But this contradicts the local uniform non-squareness of L^Φ (resp. ℓ^Φ) if the suitable Δ_2-condition for Φ is satisfied.

3. Final remarks

If the space $E^\Phi(T,\Sigma,\mu)$ contains an order asymptotically isometric copy of c_o then it must be infinitely dimensional, so the measure space (T,Σ,μ) cannot reduce to finite numbers of atoms of positive measure. Therefore, considering the problem of the existence of asymptotically isometric copy of c_o in $E^\Phi(T,\Sigma,\mu)$, if the measure μ is purely atomic, we may assume that $T = \mathbb{N}$ and that the atoms are singletons $\{n\}$, $n \in \mathbb{N}$, with $w_n = \mu(\{n\}) \in (0,\infty)$, for any $n \in \mathbb{N}$. Then the Orlicz space $l^\Phi(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with $\mu(A) = \sum_{n\in A} w_n$ is nothing else but the weighted Orlicz sequence space $l^\Phi(\{w_n\}_{n=1}^\infty)$.
The following theorem can be proved analogously as the above theorems.

Theorem 3.1.

(i) *Assume that Φ is an Orlicz function and $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ is such a purely atomic measure space as we just have established and that $\sum_{n=1}^\infty w_n < \infty$. Then the space $h^\Phi(\{w_n\}_{n=1}^\infty)$ contains an order asymptotically isometric copy of c_o whenever $\Phi \notin \Delta_2(\infty)$.*
(ii) *Assume that Φ is an Orlicz function and the measure space is as in (i) but $\sum_{n=1}^\infty w_n = \infty$ and $\inf\{w_n : n \in \mathbb{N}\} = 0$. Then the space $h^\Phi(\{w_n\}_{n=1}^\infty)$ contains an order asymptotically isometric copy of c_o whenever $\Phi \notin \Delta_2(\mathbb{R}_+)$.*
(iii) *Assume that Φ is an Orlicz function and the measure space is such that $\inf\{w_n : n \in \mathbb{N}\} > 0$. Then the space $h^\Phi(\{w_n\}_{n=1}^\infty)$ contains an order asymptotically isometric copy of c_o whenever $\Phi \notin \Delta_2(0)$.*

Remark 3.2. It is natural to ask why we consider in this paper the problem of the existence in the spaces $(E^\Phi(T,\Sigma,\mu)$ an order asymptotically isometric copy of c_o but not an order isometric copy of c_o, which is a stronger property than asymptotically isometric copy of c_o. The answer follows from the result presented in [26] which says that if $a(\Phi) = 0$ and $\Phi(b(\Phi)) \geq 1$, then the space $(E^\Phi(T, \Sigma, \mu)$ is strictly monotone (even lower locally uniformly monotone), so it cannot contain an order isometric copy of c_o.

Remark 3.3. If (T, Σ, μ) is an arbitrary measure space with $\mu(T) > 0$, then denoting by T_a the collection of all atoms in Σ, and defining $T_{n-a} = T \setminus T_a$, we can decompose the measure space (T, Σ, μ) as a direct sum of a non-atomic measure space and of a purely atomic measure space:

$$(T, \Sigma, \mu) = (T_{n-a}, \Sigma \cap T_{n-a}, \mu|_{\Sigma \cap T_{n-a}}) \oplus (T_a, \Sigma \cap T_a, \mu|_{\Sigma \cap T_a}).$$

Then it is easy to see that for any Orlicz function Φ the space $(E^\Phi(T, \Sigma, \mu)$ contains an order asymptotically isometric copy of c_o if and only if at least one of the spaces $E^\Phi(T_{n-a}, \Sigma \cap T_{n-a}, \mu|_{\Sigma \cap T_{n-a}})$ or $E^\Phi(T_a, \Sigma \cap T_a, \mu|_{\Sigma \cap T_a})$ contains an order asymptotically isometric copy of c_o.

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