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Turnpike Properties of Approximate Solutions of Nonconcave Discrete-Time Optimal Control Problems

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In this work we study the structure of approximate solutions of an autonomous nonconcave discrete-time control system with a compact metric space of states. This control system is described by a bounded upper semicontinuous objective function which determines an optimality criterion. We are interested in turnpike properties of the approximate solutions which are independent of the length of the interval, for all sufficiently large intervals. We show that the turnpike property of approximate solutions on finite intervals follows from the asymptotic turnpike property of approximate solutions of the corresponding infinite horizon optimal control problem. We also show that this asymptotic turnpike property for the corresponding infinite horizon optimal control problem holds for most objective functions in the sense of Baire category.

Keywords: Compact metric space, generic property, good program, turnpike property

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1. Introduction

The study of (approximate) solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [2, 5–8, 11, 14, 16, 19–21, 23, 24, 27, 28] and the references mentioned therein. These problems arise in engineering [1, 12, 32], in models of economic growth [9, 10, 15, 18, 25, 28–30], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [4, 26] and in the theory of thermodynamical equilibrium for materials [13, 17]. In this paper we study the structure of solutions of a discrete-time optimal control system describing a general model of economic dynamics [10, 15, 18, 28–30].

Let (X, ρ) be a compact metric space and Ω be a nonempty closed subset of $X \times X$.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \geq 0$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$, where integers T_1, T_2 satisfy $0 \leq T_1 < T_2$, is called a

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program if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \in [T_1, T_2 - 1]$.

We consider the problem

$$\text{(P1)} \qquad \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \to \max, \quad \{(x_t, x_{t+1})\}_{t=0}^{T-1} \subset \Omega, \quad x_0 = z_1,\ x_T = z_2$$

studied in [30], the problem

$$\text{(P2)} \qquad \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \to \max, \quad \{(x_t, x_{t+1})\}_{t=0}^{T-1} \subset \Omega, \quad x_0 = z_1$$

studied in [29] and the problem

$$\text{(P3)} \qquad \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \to \max, \quad \{(x_t, x_{t+1})\}_{t=0}^{T-1} \subset \Omega,$$

where T is a natural number, $z_1, z_2 \in X$ and $v : \Omega \to R^1$ is a bounded upper semicontinuous objective function.

In models of economic growth the set X is the space of states, v is a utility function and $v(x_t, x_{t+1})$ evaluates consumption at moment t. The interest in discrete-time optimal problems of types (P1)–(P3) also stems from the study of various optimization problems which can be reduced to it, e.g., continuous-time control systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor [11], tracking problems in engineering [12, 32], the study of Frenkel-Kontorova model related to dislocations in one-dimensional crystals [4, 26] and the analysis of a long slender bar of a polymeric material under tension in [13, 17]. Optimization problems of the type (P) with $\Omega = X \times X$ were considered in [27, 28].

We are interested in a turnpike property of the approximate solutions of problems (P1)–(P3) which is independent of the length of the interval T, for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the optimal control problems are determined mainly by the cost function v, and are essentially independent of T and z_1, z_2. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [25]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path and a turnpike). This property was further investigated for optimal trajectories of models of economic dynamics (see, for example, [10, 15, 18, 28–30] and the references mentioned there).

In the classical turnpike theory [10, 15, 18] the space X is a compact convex subset of a finite-dimensional Euclidean space, the set Ω is convex and the objective function v is strictly concave. Under these assumptions the turnpike property can be established and the turnpike $\bar{x}$ is a unique solution of the maximization problem $v(x,x) \to \max$, $(x,x) \in \Omega$. In this situation it is shown that for each program $\{x_t\}_{t=0}^{\infty}$ either the sequence $\{\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x})\}_{T=1}^{\infty}$ is bounded (in this case the program $\{x_t\}_{t=0}^{\infty}$ is called (v)-good) or it diverges to $-\infty$. Moreover, it is also established that any (v)-good program converges to the turnpike $\bar{x}$. In the sequel this property is called as the asymptotic turnpike property.

In [29, 30] we studied the problems (P1) and (P2) and showed that the turnpike property follows from the asymptotic turnpike property. More precisely, we assumed that any (v)-good program converges to a unique solution $\bar{x}$ of the problem $v(x,x) \to \max$, $(x,x) \in \Omega$ and showed that the turnpike property holds and $\bar{x}$ is the turnpike. In the present paper we show that for the problem (P3) the turnpike property also follows from the asymptotic turnpike property (see Theorem 2.2 below). Note that we do not use convexity (concavity) assumptions. It should be mentioned that analogous results which show that turnpike properties follow from asymptotic turnpike properties for unconstrained variational problems were obtained in [17, 28].

The main goal of this paper is to show that the asymptotic turnpike property holds for most objective functions in the sense of Baire category (see Theorem 3.1 below). In other words, the asymptotic turnpike property holds for a generic (typical) objective function.

More precisely, we establish the existence of a set $\mathcal{F}$ in the complete metric space of objective functions which is a countable intersection of open everywhere dense sets such that each $v \in \mathcal{F}$ has the asymptotic turnpike property. Results of this kind for classes of unconstrained variational problems were obtained in [17, 28]. Thus, instead of considering the turnpike property for a single objective function, we investigate it for a space of all such functions equipped with some natural metric, and show that this property holds for most of these functions. Note that this generic approach is not limited to the turnpike property, but is also applicable to other problems in Mathematical Analysis [3, 22, 31]. The proof of Theorem 3.1 is based on Theorem 2.2.

The paper is organized as follows. The turnpike result for the problem (P3) is stated in Section 2 and proved in Section 5. Our generic turnpike result is stated in Section 3 and proved in Section 6. Section 4 contains auxiliary results for Theorem 2.2.

2. A turnpike result

Let (X, ρ) be a compact metric space and Ω be a nonempty closed subset of $X \times X$. Denote by $\mathcal{M}$ the set of all bounded functions $u : \Omega \to R^1$. For each $w \in \mathcal{M}$ set

$$||w|| = \sup\{|w(x,y)| : (x,y) \in \Omega\}. \tag{1}$$

For each $x, y \in X$, each integer $T \geq 1$ and each $u \in \mathcal{M}$ set

$$\sigma(u, T, x) = \sup\left\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \{x_i\}_{i=0}^{T} \text{ is a program and } x_0 = x\right\}, \tag{2}$$

$$\sigma(u,T,x,y)$$
$$= \sup\left\{\sum_{i=0}^{T-1} u(x_i,x_{i+1}) : \{x_i\}_{i=0}^{T} \text{ is a program and } x_0 = x,\ x_T = y\right\}, \tag{3}$$

$$\sigma(u,T) = \sup\left\{\sum_{i=0}^{T-1} u(x_i,x_{i+1}) : \{x_i\}_{i=0}^{T} \text{ is a program}\right\}. \tag{4}$$

(Here we use the convention that the supremum of an empty set is $-\infty$).

For each $x, y \in X$, each pair of integers T_1, T_2 satisfying $0 \le T_1 < T_2$ and each sequence $\{u_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}$ set

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x)$$
$$= \sup\left\{\sum_{t=T_1}^{T_2-1} u_t(x_t,x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = x\right\}, \tag{5}$$

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x, y)$$
$$= \sup\left\{\sum_{t=T_1}^{T_2-1} u_t(x_t,x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = x,\ x_{T_2} = y\right\}, \tag{6}$$

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) = \sup\left\{\sum_{t=T_1}^{T_2-1} u_t(x_t,x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program}\right\}. \tag{7}$$

Assume that $v \in \mathcal{M}$ is an upper semicontinuous function. Since in [29, 30] we assume that objective functions are defined on the set $X \times X$ in order to apply their results we set $v(x,y) = -||v|| - 1$ for all $(x,y) \in (X \times X) \setminus \Omega$.

We suppose that there exist $\bar{x} \in X$ and constants $\bar{c} > 0$ and $\bar{r} > 0$ such that the following assumptions hold.

(A1) $\{(x,y) \in X \times X : \ \rho(x,\bar{x}),\ \rho(y,\bar{x}) \le \bar{r}\} \subset \Omega$ and v is continuous at $(\bar{x},\bar{x})$.
(A2) $\sigma(v,T) \le Tv(\bar{x},\bar{x}) + \bar{c}$ for all integers $T \ge 1$.

It is easy to see that for each natural number T and each program $\{x_t\}_{t=0}^{T}$

$$\sum_{t=0}^{T-1} v(x_t,x_{t+1}) \le \sigma(v,T) \le Tv(\bar{x},\bar{x}) + \bar{c}. \tag{8}$$

Inequality (8) implies the following result.

Proposition 2.1. *For each program $\{x_t\}_{t=0}^{\infty}$ either the sequence*

$$\left\{\sum_{t=0}^{T-1} v(x_t,x_{t+1}) - Tv(\bar{x},\bar{x})\right\}_{T=1}^{\infty}$$

is bounded or $\lim_{T\to\infty}[\sum_{t=0}^{T-1} v(x_t,x_{t+1}) - Tv(\bar{x},\bar{x})] = -\infty$.

A program $\{x_t\}_{t=0}^{\infty}$ is called (v)-good if the sequence

$$\left\{\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x})\right\}_{T=1}^{\infty}$$

is bounded.

In this paper we use the following assumption.

(A3) (the asymptotic turnpike property or, briefly, (ATP)) For any (v)-good program $\{x_t\}_{t=0}^{\infty}$, $\lim_{t\to\infty} \rho(x_t, \bar{x}) = 0$.

Clearly, if (A3) holds and $(\bar{x}, \bar{x})$ is not an isolated point of $X \times X$, then $||v|| > 0$.

Examples of optimal control problems satisfying (A1)–(A3) are given in [29, 30].

Denote by Card(A) the cardinality of a set A and suppose that the sum over empty set is zero.

The following theorem is our first main result.

Theorem 2.2. *Let* (A3) *hold,* $\epsilon \in (0, \bar{r})$ *and* $M > 0$. *Then there exist a positive number*

$$\delta < \min\{1, M\}$$

and a natural number L *such that the following assertions hold.*

1. Assume that an integer $T \geq L$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$ *and a program* $\{x_t\}_{t=0}^{T}$ *satisfy*

$$||u_t - v|| \leq \delta, \quad t = 0, \ldots, T-1, \tag{9}$$

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M. \tag{10}$$

Then the inequality

$$\mathrm{Card}(\{t \in \{0, \ldots, T\} :\ \rho(x_t, \bar{x}) > \epsilon\}) < L \tag{11}$$

holds.

2. Assume that an integer $T \geq 2L$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$ *and a program* $\{x_t\}_{t=0}^{T}$ *satisfy* (9), (10) *and*

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - \delta. \tag{12}$$

Then there exist integers $\tau_1 \in [0, L]$ *and* $\tau_2 \in [T - L, T]$ *such that*

$$\rho(x_t, \bar{x}) \leq \epsilon, \quad t = \tau_1, \ldots, \tau_2.$$

Moreover, if $\rho(x_0, \bar{x}) \leq \delta$, *then* $\tau_1 = 0$ *and if* $\rho(x_T, \bar{x}) \leq \delta$, *then* $\tau_2 = T$.

3. Assume that $\{u_t\}_{t=0}^{\infty} \subset \mathcal{M}$ *and a program* $\{x_t\}_{t=0}^{\infty}$ *satisfy*

$$||u_t - v|| \leq \delta \ \ \textit{for all integers}\ t \geq 0, \tag{13}$$

$$\limsup_{T\to\infty}\left[\sum_{t=0}^{T-1} u_t(x_t,x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1},0,T)\right] > -M. \tag{14}$$

Then

$$\mathrm{Card}(\{t \textit{ is a nonnegative integer such that } \rho(x_t,\bar{x}) > \epsilon\}) < L.$$

4. Assume that an integer $T > 0$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$, a program $\{x_t\}_{t=0}^{T}$ satisfy (9), (10), (12) *and integers T_1, T_2 satisfy $0 \le T_1 < T_2 \le T$. Then*

$$\sum_{t=T_1}^{T_2-1} u_t(x_t,x_{t+1}) \ge \sigma(\{u_t\}_{t=T_1}^{T_2-1},T_1,T_2) - (4L+2)(2||v||+2) - M - 1. \tag{15}$$

Assertions *1* and *2* establish turnpike properties for approximate solutions of the problem (P3) while Assertion *3* establish the turnpike property for approximate solutions of the corresponding infinite horizon problem. Moreover, they also show the stability of the turnpike phenomenon under small perturbations of the objective function v. Assertion *4* plays an important role in the proof of Theorem 3.1.

3. A generic result

We use the notation, definitions and assumptions introduced in Section 2.

Clearly, $(\mathcal{M}, ||\cdot||)$ is a Banach space. Denote by $\mathcal{M}_0$ the set of all upper semi-continuous functions $v \in \mathcal{M}$ for which there exist $x_v \in X$, $c_v > 0$ and $r_v \in (0,1)$ such that

$$\{(x,y) \in X\times X:\ \rho(x,x_v),\ \rho(y,x_v) \le r_v\} \subset \Omega, \tag{16}$$

$$v \text{ is continuous at } (x_v,x_v), \tag{17}$$

$$\sigma(v,T) \le Tv(x_v,x_v) + c_v \quad \text{for all integers } T \ge 1. \tag{18}$$

In other words, $\mathcal{M}_0$ is the set of all upper semicontinuous functions $v \in \mathcal{M}$ which satisfy assumptions (A1) and (A2).

Denote by $\mathcal{M}_{c,0}$ the set of all continuous functions $v \in \mathcal{M}_0$. Denote by $\bar{\mathcal{M}}_{c,0}$ and $\bar{\mathcal{M}}_0$ the closure of subspaces $\mathcal{M}_{c,0}$ and $\mathcal{M}_0$ in $\mathcal{M}$, respectively.

We equip the sets $\bar{\mathcal{M}}_{c,0}$ and $\bar{\mathcal{M}}_0$ with the metric d induced by the norm $||\cdot||$: $d(u_1,u_2) = ||u_1 - u_2||$, $u_1, u_2 \in \bar{\mathcal{M}}_0$.

For each $u \in \bar{\mathcal{M}}_0$ and each $r > 0$ set

$$B_d(u,r) = \{w \in \bar{\mathcal{M}}_0:\ ||u-w|| < r\}.$$

We associate with any $v \in \mathcal{M}_0$ the triplet (x_v,c_v,r_v) satisfying (16), (17) and (18).

Denote by $\mathcal{M}_*$ the set of all $v \in \mathcal{M}_0$ such that for any (v)-program $\{x_i\}_{i=0}^{\infty}$,

$$\lim_{i\to\infty} \rho(x_i, x_v) = 0.$$

Set

$$\mathcal{M}_{c*} = \mathcal{M}_* \cap \mathcal{M}_c.$$

In other words, $\mathcal{M}_*$ is the set of all upper semicontinuous functions $v \in \mathcal{M}$ which satisfy assumptions (A1), (A2) and (A3).

The following theorem is our second main result.

Theorem 3.1. *$\mathcal{M}_*$ contains a set which is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_0$ and $\mathcal{M}_{c*}$ contains a set which is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_{c,0}$.*

4. Auxiliary results for Theorem 2.2

In this section we suppose that all the assumptions made for Theorem 2.2 hold.

In order to prove our main results we need the following lemmas obtained in [30].

Lemma 4.1 (30, Lemma 3.3). *Let M_0, ϵ be positive numbers. Then there exists a natural number L_0 such that for each integer $T \geq L_0$, each program $\{x_t\}_{t=0}^{T}$ which satisfies*

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq Tv(\bar{x}, \bar{x}) - M_0$$

and each integer $S \in [0, T - L_0]$ the inequality

$$\min\{\rho(x_t, \bar{x}) : \ t = S+1, \dots, S+L_0\} \leq \epsilon$$

holds.

Lemma 4.2 (30, Lemma 3.2). *Let $\epsilon > 0$. Then there exists $\delta \in (0, \bar{r})$ such that for each integer $T \geq 1$ and each program $\{x_t\}_{t=0}^{T}$ satisfying*

$$\rho(x_0, \bar{x}),\ \rho(x_T, \bar{x}) \leq \delta,\ \ \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0, x_T) - \delta$$

the inequality $\rho(x_t, \bar{x}) \leq \epsilon$ holds for all $t = 0, \dots, T$.

Lemma 4.3. *Let $\epsilon \in (0, \bar{r})$ and $M_0 > 0$. Then there exist $\delta_0 \in (0, 1)$ and a natural number $L_0 > 4$ such that for each integer $T \geq L_0$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$, each program $\{x_t\}_{t=0}^{T}$ satisfying*

$$||u_t - v|| \leq \delta_0, \quad t = 0, \dots, T-1, \tag{19}$$

$$\min\{\rho(x_t, \bar{x}) : \ t = 1, \dots, T\} > \epsilon \tag{20}$$

and each $z_0, z_1 \in X$ satisfying

$$\rho(\bar{x}, z_i) \le \bar{r}, \quad i = 0, 1, \tag{21}$$

the following inequality holds:

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \le u_0(z_0, \bar{x}) + \sum_{t=1}^{T-2} u_t(\bar{x}, \bar{x}) + u_{T-1}(\bar{x}, z_1) - M_0.$$

Proof. Choose a number

$$M_1 > 8 + 4||v|| + M_0. \tag{22}$$

By Lemma 4.1 there exists a natural number $L_0 > 4$ such that the following property holds:

(P1) for each integer $T \ge L_0$, each program $\{x_t\}_{t=0}^{T}$ which satisfies

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \ge Tv(\bar{x}, \bar{x}) - M_1$$

and each integer $S \in [0, T - L_0]$ the inequality

$$\min\{\rho(x_t, \bar{x}) : \ t = S + 1, \dots, S + L_0\} \le \epsilon$$

holds.

Choose a positive number

$$\delta_0 < (4L_0)^{-1}. \tag{23}$$

Assume that an integer $T \ge L_0$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$, a program $\{x_t\}_{t=0}^{T}$ satisfy (19) and (20) and $z_0, z_1 \in X$ satisfy (21). There is a natural number k such that

$$kL_0 \le T < (k + 1)L_0. \tag{24}$$

By (21) the sequence $z_0, \bar{x}, \dots, \bar{x}$ $(T - 2$ times$), z_1$ is a program. By (20) and property (P1) for each integer i satisfying $0 < i \le k - 1$

$$\sum_{t=(i-1)L_0}^{iL_0-1} v(x_t, x_{t+1}) < L_0 v(\bar{x}, \bar{x}) - M_1, \tag{25}$$

$$\sum_{t=(k-1)L_0}^{T-1} v(x_t, x_{t+1}) < (T - (k - 1)L_0)v(\bar{x}, \bar{x}) - M_1. \tag{26}$$

By (19), (23), (25) and (26), for each integer i satisfying $0 < i \le k-1$,

$$\begin{aligned}\sum_{t=(i-1)L_0}^{iL_0-1} u_t(x_t, x_{t+1}) &\le \sum_{t=(i-1)L_0}^{iL_0-1} v(x_t, x_{t+1}) + \delta_0 L_0 \\ &\le L_0 v(\bar{x}, \bar{x}) - M_1 + \delta_0 L_0 \\ &\le \sum_{t=(i-1)L_0}^{iL_0-1} u_t(\bar{x}, \bar{x}) + \delta_0 L_0 - M_1 + \delta_0 L_0 \\ &\le \sum_{t=(i-1)L_0}^{iL_0-1} u_t(\bar{x}, \bar{x}) - M_1 + 1 \end{aligned} \tag{27}$$

and

$$\begin{aligned}\sum_{t=(k-1)L_0}^{T-1} u_t(x_t, x_{t+1}) &\le \sum_{t=(k-1)L_0}^{T-1} v(x_t, x_{t+1}) + 2\delta_0 L_0 \\ &\le (T-(k-1)L_0) v(\bar{x}, \bar{x}) - M_1 + 2\delta_0 L_0 \\ &\le \sum_{t=(k-1)L_0}^{T-1} u_t(\bar{x}, \bar{x}) + 2\delta_0 L_0 - M_1 + 2\delta_0 L_0 \\ &\le \sum_{t=(k-1)L_0}^{T-1} u_t(\bar{x}, \bar{x}) - M_1 + 1. \end{aligned} \tag{28}$$

By (19), (22), (24), (27) and (28),

$$\begin{aligned}&\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \le \sum_{t=0}^{T-1} u_t(\bar{x}, \bar{x}) - M_1 + 1 \\ &\le u_0(z_0, \bar{x}) + \sum_{t=1}^{T-2} u_t(\bar{x}, \bar{x}) + u_{T-1}(\bar{x}, z_1) + 2||u_0|| + 2||u_T|| - M_1 + 1 \\ &\le u_0(z_0, \bar{x}) + \sum_{t=1}^{T-2} u_t(\bar{x}, \bar{x}) + u_{T-1}(\bar{x}, z_1) - M_0. \end{aligned}$$

Lemma 4.3 is proved. □

Lemma 4.4. *Let $\epsilon \in (0, \bar{r})$, $M_0 > 0$ and let $\delta_0 \in (0,1)$ and a natural number $L_0 > 4$ be as guaranteed by Lemma 4.3.*

Assume that an integer $T \ge L_0$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$, a program $\{x_t\}_{t=0}^{T}$ satisfy

$$||u_t - v|| \le \delta_0, \quad t = 0, \dots, T-1 \tag{29}$$

and

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) > \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M_0 + 2||v|| + 2. \tag{30}$$

Then there exists a strictly increasing sequence of natural numbers $\{T_j\}_{j=1}^q$ *where* q *is a natural number such that*

$$T_1 \in [1, L_0], \qquad T_q \in [T - L_0 + 1, T], \tag{31}$$

$$T_{j+1} - T_j \le L_0 \ \textit{ for all natural numbers } j < q, \tag{32}$$

$$\rho(\bar{x}, x_{T_j}) \le \epsilon, \quad j =, 1, \dots, q. \tag{33}$$

Proof. First we show that there is an integer $T_1 \in [1, L_0]$ such that $\rho(\bar{x}, x_{T_1}) \le \epsilon$. Assume the contrary. Then

$$\rho(x_t, \bar{x}) > \epsilon, \quad t = 1, \dots, L_0. \tag{34}$$

There are two cases:

$$\rho(x_t, \bar{x}) > \epsilon, \quad t = 1, \dots, T; \tag{35}$$

$$\min\{\rho(x_t, \bar{x}) : \ t = 1, \dots, T\} \le \epsilon. \tag{36}$$

Assume that (35) holds. Then by (29), (35), the inequality $T \ge L_0$, the choice of δ_0, L_0 and Lemma 4.3

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \le \sum_{t=0}^{T-1} u_t(\bar{x}, \bar{x}) - M_0 \le \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M_0.$$

This contradicts (30). The contradiction we have reached proves (36). By (34) and (36) there is an integer

$$S \in (L_0, T] \tag{37}$$

such that

$$\rho(x_S, \bar{x}) \le \epsilon, \quad \rho(x_t, \bar{x}) > \epsilon, \quad t = 1, \dots, S - 1. \tag{38}$$

Define

$$z_t = \bar{x}, \quad t = 0, \dots, S - 1, \qquad z_t = x_t, \quad t = S, \dots, T. \tag{39}$$

By (38) and (39), $\{z_t\}_{t=0}^T$ is a program. It follows from (29), (37), (38), (39), the choice of δ_0 and L_0 and Lemma 4.3 (with $T = S - 1$) that

$$\sum_{t=0}^{S-2} u_t(x_t, x_{t+1}) \le \sum_{t=0}^{S-2} u_t(\bar{x}, \bar{x}) - M_0 = \sum_{t=0}^{S-2} u_t(z_t, z_{t+1}) - M_0. \tag{40}$$

In view of (39) and (40),

$$\begin{aligned}
&\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=0}^{T-1} u_t(z_t, z_{t+1}) \\
= &\sum_{t=0}^{S-2} u_t(x_t, x_{t+1}) + u_{S-1}(x_{S-1}, x_S) - \sum_{t=0}^{S-2} u_t(\bar{x}, \bar{x}) - u_{S-1}(z_{S-1}, z_S) \\
\leq & -M_0 + 2||u_{S-1}|| \leq -M_0 + 2||v|| + 2.
\end{aligned}$$

This implies that

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \leq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M_0 + 2||v|| + 2.$$

This contradicts (30). The contradiction we have reached proves that there is an integer

$$T_1 \in [1, L_0] \tag{41}$$

such that

$$\rho(\bar{x}, x_{T_1}) \leq \epsilon. \tag{42}$$

Assume now that k is a natural number and we defined a strictly increasing sequence of natural numbers $\{T_i\}_{i=1}^k$ such that

$$T_1 \in [1, L_0], \quad T_k \leq T, \tag{43}$$

$$\rho(\bar{x}, x_{T_i}) \leq \epsilon, \quad i =, 1, \dots, k, \tag{44}$$

for each natural number $i < k$,

$$T_{i+1} - T_i \leq L_0. \tag{45}$$

(Clearly, for $k = 1$ this assumption holds.)

If $T - T_k < L_0$, then the construction is completed. Assume that

$$T - T_k \geq L_0. \tag{46}$$

We show that there is an integer T_{k+1} such that

$$T_{k+1} - T_k \leq [1, L_0], \quad \rho(\bar{x}, x_{T_{k+1}}) \leq \epsilon. \tag{47}$$

Assume the contrary. Then

$$\rho(\bar{x}, x_t) > \epsilon, \quad t = T_k + 1, \dots, T_k + L_0. \tag{48}$$

There are two cases:

$$\rho(\bar{x}, x_t) > \epsilon, \quad t = T_k + 1, \dots, T; \tag{49}$$

$$\min\{\rho(\bar{x}, x_t) : \ t = T_k + 1, \dots, T\} \leq \epsilon. \tag{50}$$

Assume that (49) holds. Then by (29), (44), (46), (49), the choice of δ_0, L_0 and Lemma 4.3,

$$\sum_{t=T_k}^{T-1} u_t(x_t, x_{t+1}) \le u_{T_k}(x_{T_k}, \bar{x}) + \sum_{t=T_k+1}^{T-1} u_t(\bar{x}, \bar{x}) - M_0. \tag{51}$$

Set

$$z_t = x_t, \quad t = 0, \dots, T_k, \qquad z_t = \bar{x}, \quad t = T_k + 1, \dots, T. \tag{52}$$

By (52) and (44), $\{z_t\}_{t=0}^{T}$ is a program. It follows from (52) and (51) that

$$\begin{aligned}
&\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) \\
\le\ &\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=0}^{T-1} u_t(z_t, z_{t+1}) \\
=\ &\sum_{t=T_k}^{T-1} u_t(x_t, x_{t+1}) - u_{T_k}(x_{T_k}, \bar{x}) - \sum_{t=T_k+1}^{T-1} u_t(\bar{x}, \bar{x}) \le -M_0.
\end{aligned}$$

This contradicts (30). The contradiction we have reached proves (50). By (48) and (50) there is an integer S such that

$$T_k + L_0 < S \le T, \quad \rho(x_S, \bar{x}) \le \epsilon, \tag{53}$$

$$\rho(x_t, \bar{x}) > \epsilon, \quad t = T_k + 1, \dots, S - 1. \tag{54}$$

Define

$$\begin{gathered}
z_t = x_t, \quad t = 0, \dots, T_k, \qquad z_t = \bar{x}, \quad t = T_k + 1, \dots, S - 1, \\
z_t = x_t, \quad t = S, \dots, T.
\end{gathered} \tag{55}$$

By (44), (53) and (55), $\{z_t\}_{t=0}^{T}$ is a program. It follows from (19), (53), (54), the choice of δ_0 and L_0 and Lemma 4.3 that

$$\sum_{t=T_k}^{S-2} u_t(x_t, x_{t+1}) \le u_{T_k}(x_{T_k}, \bar{x}) + \sum_{t=T_k+1}^{S-2} u_t(\bar{x}, \bar{x}) - M_0 = \sum_{t=T_k}^{S-2} u_t(z_t, z_{t+1}) - M_0. \tag{56}$$

In view of (55) and (56),

$$\begin{aligned}
&\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) \\
\le\ &\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=0}^{T-1} u_t(z_t, z_{t+1}) = \sum_{t=T_k}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=T_k}^{T-1} u_t(z_t, z_{t+1}) \\
=\ &\sum_{t=T_k}^{S-1} u_t(x_t, x_{t+1}) - u_{T_k}(x_{T_k}, \bar{x}) - \sum_{t=T_k+1}^{S-2} u_t(\bar{x}, \bar{x}) - u_{S-1}(\bar{x}, x_S) \\
\le\ &- M_0 + 2||u_{S-1}|| \le -M_0 + 2||v|| + 2.
\end{aligned}$$

This contradicts (30). The contradiction we have reached proves that there is an integer T_{k+1} satisfying (47).

Clearly, the assumption made for k also holds for $k+1$. Thus by induction we construct the sequence T_i, $i = 1, \dots, q$ which is obtained by a finite number of steps and T_q is its last element. It follows from the construction that (31)–(33) hold. Lemma 4.4 is proved. □

5. Proof of Theorem 2.2

By Lemma 4.2 there exists a positive number

$$\delta_0 < \min\{\bar{r}, \epsilon, 1\}$$

such that the following property holds:

(P2) for each integer $T \geq 1$ and each program $\{x_t\}_{t=0}^T$ satisfying

$$\rho(x_0, \bar{x}),\ \rho(x_T, \bar{x}) \leq \delta_0,\ \ \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0, x_T) - \delta_0$$

the inequality $\rho(x_t, \bar{x}) \leq \epsilon$ holds for all $t = 0, \dots, T$.

By Lemma 4.4 there exist $\delta_1 \in (0, 1)$ and a natural number $L_1 > 4$ such that the following property holds:

(P3) for each integer $T \geq L_1$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$ and each program $\{x_t\}_{t=0}^T$ satisfying

$$||u_t - v|| \leq \delta_1, \quad t = 0, \dots, T-1,$$

and

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) > \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M - 4$$

there exists a strictly increasing sequence of natural numbers $\{T_j\}_{j=1}^q$ where q is a natural number such that

$$T_1 \in [1, L_1], \quad T_q \in [T - L_1 + 1, T], \tag{57}$$

$$T_{j+1} - T_j \leq L_1 \ \text{ for all natural numbers } j < q, \tag{58}$$

$$\rho(\bar{x}, x_{T_j}) < \delta_0, \quad j =, 1, \dots, q. \tag{59}$$

Choose a positive number δ and a natural number L such that

$$\delta < \min\{\delta_1, M, 8^{-1}L_1^{-1}\delta_0\}, \tag{60}$$

$$L > 8L_1 + 2L_1(M+1)\delta_0^{-1}.$$

Let us prove Assertion *1*. By (9), (10), (60) and (P3) there exists a strictly increasing sequence of natural numbers $\{T_j\}_{j=1}^q$ where q is a natural number such that (57), (58) and (59) hold. By (57), (58) and (60),

$$q > 2. \tag{61}$$

Let

$$E = \Big\{ j \in \{1, \dots, q-1\} : \\ \sum_{t=T_j}^{T_{j+1}-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=T_j}^{T_{j+1}-1}, T_j, T_{j+1}, x_{T_j}, x_{T_{j+1}}) - \delta_0/2 \Big\}. \quad (62)$$

By (10) and (62),

$$\begin{aligned} -M &\leq \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) \\ &\leq \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) \\ &\leq \sum \Big\{ \sum_{t=T_j}^{T_{j+1}-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=T_j}^{T_{j+1}-1}, T_j, T_{j+1}, x_{T_j}, x_{T_{j+1}}) : \\ &\qquad j \in \{1, \dots, q-1\} \setminus E \Big\} \\ &\leq (-\delta_0/2)\mathrm{Card}(\{1, \dots, q-1\} \setminus E\} \end{aligned}$$

and

$$\mathrm{Card}(\{1, \dots, q-1\} \setminus E\}) \leq 2M\delta_0^{-1}. \quad (63)$$

Let

$$j \in E. \quad (64)$$

By (9), (62), (64) and (P4),

$$\begin{aligned} \sum_{t=T_j}^{T_{j+1}-1} v(x_t, x_{t+1}) \geq & \sum_{t=T_j}^{T_{j+1}-1} u_t(x_t, x_{t+1}) - \delta L_1 \\ & \geq \sigma(\{u_t\}_{t=T_j}^{T_{j+1}-1}, T_j, T_{j+1}, x_{T_j}, x_{T_{j+1}}) - \delta_0/2 - \delta L_1 \\ & \geq \sigma(v, T_{j+1} - T_j, x_{T_j}, x_{T_{j+1}}) - \delta L_1 - \delta_0/2 - \delta L_1 \\ & \geq \sigma(v, T_{j+1} - T_j, x_{T_j}, x_{T_{j+1}}) - \delta_0. \end{aligned} \quad (65)$$

By (59), (65) and (P2),

$$\rho(x_t, \bar{x}) \leq \epsilon, \quad t = T_j, \dots, T_{j+1} \text{ and all } j \in E.$$

Together with (59) this implies that

$$\begin{aligned} &\{t \in \{0, \dots, T\} : \ \rho(x_t, \bar{x}) > \epsilon\} \\ &\subset \{0, \dots, T_1 - 1\} \cup \{T - L_1 + 1, \dots, T\} \\ &\cup \{\{t \text{ is an integer and } T_j < t < T_{j+1}\} : \ j \in \{1, \dots, q-1\} \setminus E\}. \end{aligned}$$

Together with (57), (58), (60) and (63) this implies that

$$\begin{aligned}&\mathrm{Card}(\{t \in \{0,\dots,T\}:\ \rho(x_t,\bar{x}) > \epsilon\})\\ &\le 2L_1 + L_1 \,\mathrm{Card}(\{1,\dots,q-1\} \setminus E\}) \le 2L_1 + 2ML_1\delta_0^{-1} < L.\end{aligned}$$

This completes the proof of Assertion *1*.

Let us prove Assertion *2*. By (9), (10), (60) and (P3), there exists a strictly increasing sequence of natural numbers $\{T_j\}_{j=1}^{q}$ where q is a natural number such that (58), (59) hold and

$$T_q \in [T - L_1 + 1, T], \quad T_1 \in [0, L_1].$$

Clearly, if $\rho(x_0,\bar{x}) \le \delta$, we may assume that $T_1 = 0$ and if $\rho(x_T,\bar{x}) \le \delta$ we may assume that $T_q = T$.

Let an integer $j \in \{1,\dots,q-1\}$. By (9), (12), (58) and (60),

$$\begin{aligned}\sum_{t=T_j}^{T_{j+1}-1} v(x_t,x_{t+1}) &\ge \sum_{t=T_j}^{T_{j+1}-1} u_t(x_t,x_{t+1}) - \delta L_1\\ &\ge \sigma(\{u_t\}_{t=T_j}^{T_{j+1}-1}, T_j, T_{j+1}, x_{T_j}, x_{T_{j+1}}) - \delta(L_1+1)\\ &\ge \sigma(v, T_{j+1}-T_j, x_{T_j}, x_{T_{j+1}}) - \delta L_1 - \delta(L_1+1)\\ &\ge \sigma(v, T_{j+1}-T_j, x_{T_j}, x_{T_{j+1}}) - \delta_0.\end{aligned}$$

Together with (59) and (P2) this implies that

$$\rho(x_t,\bar{x}) \le \epsilon \ \text{ for all integers } t \in [T_j, T_{j+1}] \text{ and all } j = 1,\dots,q-1.$$

This implies that

$$\rho(x_t,\bar{x}) \le \epsilon \ \text{ for all integers } t \in [T_1, T_q].$$

This completes the proof of Assertion *2*.

Assertion *3* easily follows from Assertion *1*.

Let us prove Assertion *4*. There are two cases: $T_2 - T_1 \le 4L$; $T_2 - T_1 > 4L$.

Clearly, if $T_2 - T_1 \le 4L$, then in view of (9), (15) holds. Assume that

$$T_2 - T_1 > 4L. \tag{66}$$

There exists a program $\{y_t\}_{t=T_1}^{T_2}$ such that

$$\sum_{t=T_1}^{T_2-1} u_t(y_t,y_{t+1}) \ge \sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) - \delta. \tag{67}$$

By (9), (10), (12), (66), (67) and Assertion *2*,

$$\rho(x_t,\bar{x}) \le \epsilon \ \text{ for all integers } t = L,\dots,T-L, \tag{68}$$

$$\rho(y_t, \bar{x}) \le \epsilon \text{ for all integers } t = T_1 + L, \dots, T_2 - L. \tag{69}$$

Set

$$z_t = x_t, \quad t = 0, \dots, T_1 + L, \qquad z_t = y_t, \quad t = T_1 + L + 1, \dots, T_2 - L - 1,$$
$$z_t = x_t, \ t = T_2 - L, \dots, T. \tag{70}$$

By (70), (68), (69) and (A1), $\{z_t\}_{t=0}^{T}$ is a program. By (9), (10) and (70),

$$\begin{aligned}
-M &\le \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) \le \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=0}^{T-1} u_t(z_t, z_{t+1}) \\
&= \sum_{t=T_1+L}^{T_2-L-1} u_t(x_t, x_{t+1}) - \sum_{t=T_1+L}^{T_2-L-1} u_t(z_t, z_{t+1}) \\
&\le \sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) + 2L(||v|| + 1) - \sum_{t=T_1+L}^{T_2-L-1} u_t(y_t, y_{t+1}) + 4(||v|| + 1) \\
&\le \sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) + 2(||v|| + 1)(L + 2) - \sum_{t=T_1}^{T_2-1} u_t(y_t, y_{t+1}) + 2L(||v|| + 1).
\end{aligned}$$

Combined with (67) this implies that

$$\begin{aligned}
\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) &\ge \ - M - 2(||v|| + 1)(2L + 2) + \sum_{t=T_1}^{T_2-1} u_t(y_t, y_{t+1}) \\
&\ge \ - M - 2(||v|| + 1)(2L + 2) + \sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) - 1.
\end{aligned}$$

This implies (15). Assertion *4* is proved. This completes the proof of Theorem 2.2.

6. Proof of Theorem 3.1

Lemma 6.1. *The set $\mathcal{M}_*$ is an everywhere dense subset of $\bar{\mathcal{M}}_0$ and $\mathcal{M}_{c*}$ is an everywhere dense subset of $\bar{\mathcal{M}}_{c,0}$.*

Proof. Let $v \in \mathcal{M}_0$ and $\gamma \in (0, 1)$. Set

$$v_\gamma(x, y) = v(x, y) - \gamma\rho(x, x_v) - \gamma\rho(y, x_v).$$

It is easy to see that $v_\gamma \in \mathcal{M}_*$, if $v \in \mathcal{M}_{c,0}$, then $v_\gamma \in \mathcal{M}_{c*}$ and $\lim_{\gamma \to 0+} v_\gamma = v$. Lemma 6.1 is proved. □

Let $v \in \mathcal{M}_*$ and n be a natural number. By (A1) and (17) there is

$$r_1(v, n) \in (0, 4^{-1} r_v) \tag{71}$$

such that

$$|v(x_v, x_v) - v(z_1, z_2)| < 8^{-1} r_v n^{-1} \tag{72}$$

for each $z_1, z_2 \in X$ satisfying $\rho(z_i, x_v) \le r_1(v, n)$, $i = 1, 2$.

By Theorem 2.2 there exist a positive number

$$\delta(v, n) < (4n)^{-1}$$

and a natural number $L(v, n)$ such that for each

$$u \in B(v, \delta(v, n)) \tag{73}$$

the following properties hold:

(P4) for each integer $T \ge L(v, n)$ and each program $\{x_t\}_{t=0}^{T}$ satisfying

$$\sum_{t=0}^{T-1} u(x_t, x_{t+1}) \ge \sigma(u, T) - 2n \tag{74}$$

the inequality

$$\mathrm{Card}(\{t \in \{0, \dots, T\} : \ \rho(x_t, x_v) > (4n)^{-1} r_1(v, n)\}) < L(v, n)$$

holds;

(P5) for each program $\{x_t\}_{t=0}^{\infty}$ satisfying

$$\limsup_{T \to \infty} \left[\sum_{t=0}^{T-1} u(x_t, x_{t+1}) - \sigma(u, T) \right] > -2n$$

we have

$$\mathrm{Card}(\{t \text{ is a nonnegative integer such that } \rho(x_t, x_v) > (4n)^{-1} r_1(v, n)\}) < L(v, n);$$

(P6) for each integer $T \ge 2L(v, n)$ and each program $\{x_t\}_{t=0}^{T}$ satisfying (74) and

$$\sum_{t=0}^{T-1} u(x_t, x_{t+1}) \ge \sigma(u, T, x_0, x_T) - \delta(v, n) \tag{74a}$$

we have

$$\rho(x_t, x_v) \le (4n)^{-1} r_1(v, n), \quad t = L(v, n), \dots, T - L(v, n);$$

(P7) for each integer $T > 0$, each program $\{x_t\}_{t=0}^{T}$ satisfying (74) and (74a) and each pair of integers T_1, T_2 satisfy $0 \le T_1 < T_2 \le T$ we have

$$\sum_{t=T_1}^{T_2-1} u(x_t, x_{t+1}) \ge \sigma(u, T_2 - T_1) - (4L(v, n) + 2)(2||v|| + 2) - 2n - 1.$$

Define

$$\mathcal{F} = \cap_{n=1}^{\infty} \cup \{B(v, \delta(v, n)) : \ v \in \mathcal{M}_*, \ n \text{ is a natural number}\}, \tag{75}$$

$$\mathcal{F}_c = \cap_{n=1}^{\infty} \cup \{B(v, \delta(v, n)) \cap \bar{\mathcal{M}}_{c,0} : \ v \in \mathcal{M}_{c*}, n \text{ is a natural number}\}. \tag{76}$$

By Lemma 6.1, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_0$ and $\mathcal{F}_c$ is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_{c,0}$. Let

$$u \in \mathcal{F}. \tag{77}$$

In order to complete the proof it is sufficient to show that $u \in \mathcal{M}_*$. Clearly, for each natural number T there exists a program $\{x_i^{(T)}\}_{i=0}^{T}$ such that

$$\sum_{t=0}^{T-1} u(x_t^{(T)}, x_{t+1}^{(T)}) = \sigma(u, T). \tag{78}$$

Extracting subsequences and using diagonalization process we obtain that there exists a strictly increasing sequence of natural numbers $\{T_k\}_{k=1}^{\infty}$ such that for each integer $t \geq 0$ there is

$$x_t^* = \lim_{k \to \infty} x_t^{(T_k)}. \tag{79}$$

By (75) and (77), for each natural number n there is

$$v_n \in \mathcal{M}_* \tag{80}$$

such that

$$u \in B(v_n, \delta(v_n, n)). \tag{81}$$

Let n be a natural number. By (78), (80), (81) and property (P6), for each integer $T \geq 2L(v_n, n)$,

$$\rho(x_t^{(T)}, x_{v_n}) \leq (4n)^{-1} r_1(v_n, n), \quad t = L(v_n, n), \dots, T - L(v_n, n). \tag{82}$$

By (79) and (82), for each integer $T \geq L(v_n, n)$,

$$\rho(x_t^*, x_{v_n}) \leq (4n)^{-1} r_1(v_n, n). \tag{83}$$

By (78), (80), (81) and property (P7), for each integer $T > 0$ and each pair of integers T_1, T_2 satisfying $0 \leq T_1 < T_2 \leq T$ we have

$$\sum_{t=T_1}^{T_2-1} u(x_t^{(T)}, x_{t+1}^{(T)}) \geq \sigma(u, T_2 - T_1) - (4L(v_n, n) + 2)(2||v_n|| + 2) - 2n - 1. \tag{84}$$

By (79) and (84), for each pair of integers $T_2 > T_1 \geq 0$,

$$\sum_{t=T_1}^{T_2-1} u(x_t^*, x_{t+1}^*) \geq \sigma(u, T_2 - T_1) - (4L(v_n, n) + 2)(2||v_n|| + 2) - 2n - 1. \tag{85}$$

Since n is any natural number it follows from (83) that $\{x_t^*\}_{t=0}^\infty$ is a Cauchy sequence and there is

$$x^* = \lim_{t\to\infty} x_t^*. \tag{86}$$

By (83) and (86), for any natural number n,

$$\rho(x^*, x_{v_n}) \le (4n)^{-1} r_1(v_n, n). \tag{87}$$

By (85) and (86), for any natural number n and any natural number T,

$$Tu(x^*, x^*) \ge \sigma(u, T) - (4L(v_n, n) + 2)(2||v_n|| + 4) - 2n - 1. \tag{88}$$

Let n be a natural number and assume that $y, z \in X$ satisfy

$$\rho(y, x^*),\ \rho(z, x^*) \le (4n)^{-1} r_1(v_n, n). \tag{89}$$

In view of (87) and (89),

$$\rho(y, x_{v_n}) \le \rho(y, x^*) + \rho(x^*, x_{v_n}) \le (2n)^{-1} r_1(v_n, n), \tag{90}$$

$$\rho(z, x_{v_n}) \le \rho(z, x^*) + \rho(x^*, x_{v_n}) \le (2n)^{-1} r_1(v_n, n). \tag{91}$$

By (71), (72), (80), (90) and (91),

$$(y, z) \in \Omega, \quad (x^*, x^*) \in \Omega. \tag{92}$$

By (71), (72), (81), (87), (90) and (91),

$$\begin{aligned}
&|u(y,z) - u(x^*, x^*)| \\
&\le |v_n(y,z) - v_n(x^*, x^*)| + 2\delta(v_n, n) \\
&\le |v_n(y,z) - v_n(x_{v_n}, x_{v_n})| + |v_n(x_{v_n}, x_{v_n}) - v_n(x^*, x^*)| + 2\delta(v_n, n) \\
&\le (4n)^{-1} r_{v_n} + 2n^{-1} < 1/n.
\end{aligned}$$

Thus for each $y, z \in X$ satisfying (89), $(y, z) \in \Omega$ and

$$|u(y,z) - u(x^*, x^*)| < n^{-1}.$$

Since n is any natural number we obtain that u is continuous at (x^*, x^*) and combined with (88) this implies that $u \in \mathcal{M}_0$.

In order to complete the proof it is sufficient to show that for any (u)-good program $\{x_i\}_{i=0}^\infty$,

$$\lim_{i\to\infty} \rho(x_i, x^*) = 0.$$

Assume that a program $\{x_i\}_{i=0}^\infty$ is (u)-good. Then by Proposition 2.1 there is a natural number n_0 such that for all natural numbers T,

$$\sum_{t=0}^{T-1} u(x_t, x_{t+1}) - \sigma(u, T) > -n_0. \tag{93}$$

Let a natural number $n > n_0$. By (80), (81), (93) and property (P5), there is a natural number $j(n)$ such that for each integer $t \geq j(n)$,

$$\rho(x_t, x_{v_n}) \leq (4n)^{-1} r_1(v_n, n).$$

Together with (87) this implies that for all integers $t \geq j(n)$,

$$\rho(x_t, x^*) \leq \rho(x_t, x_{v_n}) + \rho(x_{v_n}, x^*) \leq (2n)^{-1} r_1(v_n, n) \leq (2n)^{-1}.$$

Since n is any natural number we conclude that

$$\lim_{i \to \infty} \rho(x_t, x^*) = 0$$

and $u \in \mathcal{M}_*$. Theorem 3.1 is proved.

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