

Gâteaux and Hadamard Differentiability via Directional Differentiability

Luděk Zajíček*

*Charles University, Faculty of Mathematics and Physics,
Sokolovská 83, 186 75 Praha 8-Karlín, Czech Republic
zajicek@karlin.mff.cuni.cz*

Received: November 12, 2012

Revised manuscript received: June 12, 2013

Accepted: June 22, 2013

Let X be a separable Banach space, Y a Banach space and $f : X \rightarrow Y$ an arbitrary mapping. Then the following implication holds at each point $x \in X$ except a σ -directionally porous set: If the one-sided Hadamard directional derivative $f'_{H+}(x, u)$ exists in all directions u from a set $S_x \subset X$ whose linear span is dense in X , then f is Hadamard differentiable at x . This theorem improves and generalizes a recent result of A. D. Ioffe, in which the linear span of S_x equals X and $Y = \mathbb{R}$. An analogous theorem, in which f is pointwise Lipschitz, and which deals with the usual one-sided derivatives and Gâteaux differentiability is also proved. It generalizes a result of D. Preiss and the author, in which f is supposed to be Lipschitz.

Keywords: Gâteaux differentiability, Hadamard differentiability, directional derivatives, Hadamard directional derivatives, σ -directionally porous set, pointwise Lipschitz mapping

2000 Mathematics Subject Classification: Primary 46G05; Secondary 26B05, 49J50

1. Introduction

The following result ([4, Theorem 5]) was used in the proof of the till now strongest version of Rademacher's theorem on Gâteaux differentiability of Lipschitz mappings on a separable Banach space.

Theorem PZ. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ a Lipschitz mapping. Then there exists a σ -directionally porous set $A \subset G$ such that for every $x \in G \setminus A$ the set U_x of those directions $u \in X$ in which the one-sided derivative $f'_+(x, u)$ exists is a closed linear subspace of X . Moreover, the mapping $u \mapsto f'_+(x, u)$ is linear on U_x .*

An immediate consequence of this result is the following.

Corollary PZ. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ a Lipschitz mapping. Then the following implication holds at each point $x \in G$ except a σ -directionally porous set:*

*The research was supported by the grant GAČR P201/12/0436.

If the one-sided directional derivative $f'_+(x, u)$ exists in all directions u from a set $S_x \subset X$ whose linear span is dense in X , then f is Gâteaux differentiable at x .

Note that each σ -directionally porous subset of a separable Banach spaces X is not only a first category set, but it is also “measure null”: it is Aronszajn (=Gauss) null, and so also Haar null, (see [1, p. 164 and Chap. 6]) and also Γ -null (see [3, Remark 5.2.4]).

It is an easy well-known fact that if $f : X \rightarrow Y$ is a Lipschitz mapping between Banach spaces, then the Hadamard (one-sided) directional derivatives coincide with the usual (one-sided) directional derivatives and the Hadamard derivative coincides with the Gâteaux derivative.

A. D. Ioffe in [2] recently observed that some known results dealing with the usual directional derivatives (and Gâteaux differentiability) of *Lipschitz* functions can be generalized to results dealing with the Hadamard directional derivatives (and Hadamard differentiability) of *arbitrary* functions. (Note that this Ioffe’s idea was followed in [6].)

Ioffe’s [2, Theorem 3.7(b)] can be reformulated (see Remark 3.10 below) in the following way.

Theorem I. *Let X be a separable Banach space, $G \subset X$ an open set, and $f : G \rightarrow \mathbb{R}$ an arbitrary function. Then the following implication holds at each point $x \in G$ except a σ -directionally porous set:*

If the one-sided Hadamard directional derivative $f'_{H+}(x, u)$ exists in all directions u from a set $S_x \subset X$ whose linear span equals to X , then f is Hadamard differentiable at x .

So, Theorem I is a partial generalization of Corollary PZ (since in Theorem I the assumption is stronger on S_x and $Y = \mathbb{R}$). We will prove (see Corollary 3.9 below) that the corresponding full generalization of Corollary PZ holds.

Moreover, Theorem 3.7 below on Hadamard derivatives generalizes Theorem PZ.

Further, we generalize Theorem PZ in another direction showing that, in this theorem, it is sufficient to suppose that f is pointwise Lipschitz (see Corollary 3.5 below).

Note that the methods of proofs from [4] and [2] cannot be easily used in the case when $f : X \rightarrow Y$ is not continuous and Y is nonseparable, since then $f(X)$ needs not be separable. Consequently, we use an alternative method based on a small trick in the proof of Lemma 3.1 (which shows that $f'_+(x, w_x)$ and $f'(y_k, w_{y_k})$ are “automatically” close to one another).

The main results are proved in Section 3. In Section 2 we recall basic definitions and known results we need.

2. Preliminaries

In the following, by a Banach space we mean a real Banach space. The symbol $B(x, r)$ will denote (in a metric space) the open ball with center x and radius r .

If X is a Banach space, we set $S_X := \{x \in X : \|x\| = 1\}$. Further, if $x \in X$, $v \in S_X$ and $\delta > 0$, then we define the open cone $C(x, v, \delta)$ as the set of all $y \neq x$ for which $\|v - \frac{y-x}{\|y-x\|}\| < \delta$.

Let X, Y be Banach spaces, $G \subset X$ an open set, and $f : G \rightarrow Y$ a mapping.

We say that f is *Lipschitz at* $x \in G$ if $\limsup_{y \rightarrow x} \frac{\|f(y) - f(x)\|}{\|y - x\|} < \infty$. We say that f is *pointwise Lipschitz* (on G) if f is Lipschitz at all points of G .

The directional and one-sided directional derivatives of f at $x \in G$ in the direction $v \in X$ are defined respectively by

$$f'(x, v) := \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t} \quad \text{and} \quad f'_+(x, v) := \lim_{t \rightarrow 0+} \frac{f(x + tv) - f(x)}{t}.$$

The *Hadamard directional and one-sided directional derivatives* of f at $x \in G$ in the direction $v \in X$ are defined respectively by

$$f'_H(x, v) := \lim_{z \rightarrow v, t \rightarrow 0} \frac{f(x + tz) - f(x)}{t}$$

$$\text{and} \quad f'_{H+}(x, v) := \lim_{z \rightarrow v, t \rightarrow 0+} \frac{f(x + tz) - f(x)}{t}.$$

The following facts are well-known and easy to prove.

Fact 2.1. *Let X, Y be Banach spaces, $G \subset X$ an open set, $x \in G$, $v \in X$, and $f : G \rightarrow Y$ a mapping. Then the following assertions hold.*

- (i) *The derivative $f'(x, v)$ (resp. $f'_H(x, v)$) exists if and only if $f'_+(x, -v) = -f'_+(x, v)$ (resp. $f'_{H+}(x, -v) = -f'_{H+}(x, v)$).*
- (ii) *If $f'_H(x, v)$ (resp. $f'_{H+}(x, v)$) exists, then $f'(x, v) = f'_H(x, v)$ (resp. $f'_+(x, v) = f'_{H+}(x, v)$).*
- (iii) *If f is locally Lipschitz on G , then $f'(x, v) = f'_H(x, v)$ (resp. $f'_+(x, v) = f'_{H+}(x, v)$) whenever one of these two derivatives exists.*
- (iv) *If $f'_+(x, v)$ (resp. $f'_{H+}(x, v)$) exists and $t > 0$, then $f'_+(x, tv) = tf'_+(x, v)$ (resp. $f'_{H+}(x, tv) = tf'_{H+}(x, v)$).*
- (v) *If f is Lipschitz at x with $\limsup_{y \rightarrow x} \frac{\|f(y) - f(x)\|}{\|y - x\|} \leq K < \infty$ and $f'_+(x, v)$ exists, then $\|f'_+(x, v)\| \leq K\|v\|$.*
- (vi) *If V_x is the set of all $u \in X$, for which $f'_{H+}(x, u)$ exists, then the mapping $u \mapsto f'_{H+}(x, u)$ is continuous on V_x .*
- (vii) *If $v \in S_X$ and $f'_{H+}(x, v)$ exists, then there exists a cone $C = C(x, v, \delta)$ such that $\limsup_{y \rightarrow x, y \in C} \frac{\|f(y) - f(x)\|}{\|y - x\|} < \infty$.*

We will need the following easy fact (see [7, Lemma 2.3]).

Lemma 2.2. *Let X be a Banach space, Y a Banach space, $G \subset X$ an open set, $a \in G$, and $f : G \rightarrow Y$ a mapping. Then the following are equivalent.*

- (i) $f'_{H+}(a, 0)$ exists,
- (ii) $f'_H(a, 0)$ exists,
- (iii) $f'_H(a, 0) = 0$,
- (iv) f is Lipschitz at a .

The usual modern definition of the Hadamard derivative is the following:

A continuous linear operator $L : X \rightarrow Y$ is said to be the *Hadamard derivative* of f at a point $x \in X$ if

$$\lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t} = L(v) \quad \text{for each } v \in X$$

and the limit is uniform with respect to $v \in C$, whenever $C \subset X$ is a compact set. In this case we set $f'_H(x) := L$.

The following fact is well-known (see [5]):

Lemma 2.3. *Let X, Y be Banach spaces, $\emptyset \neq G \subset X$ an open set, $x \in G$, $f : G \rightarrow Y$ a mapping and $L : X \rightarrow Y$ a continuous linear operator. Then the following conditions are equivalent:*

- (i) $f'_H(x) = L$,
- (ii) $f'_H(x, v) = L(v)$ for each $v \in X$,
- (iii) if $\varphi : [0, 1] \rightarrow X$ is such that $\varphi(0) = x$ and $\varphi'_+(0)$ exists, then $(f \circ \varphi)'_+(0) = L(\varphi'_+(0))$.

Definition 2.4. Let X be a Banach space. We say that $A \subset X$ is *directionally porous at a point* $x \in X$, if there exist $0 \neq v \in X$, $p > 0$ and a sequence $t_n \rightarrow 0$ of positive real numbers such that $B(x + t_n v, p t_n) \cap A = \emptyset$. (In this case we say that A is *porous at x in the direction v* .)

We say that $A \subset X$ is *directionally porous* if A is directionally porous at each point $x \in A$.

We say that $A \subset X$ is *σ -directionally porous* if it is a countable union of directionally porous sets.

We will need the obvious fact that A is not porous at x in the direction $v \neq 0$ if and only if

$$\text{for each } \omega > 0 \text{ there exists } \delta > 0 \text{ such that } B(x + tv, \omega t) \cap A \neq \emptyset \text{ for } 0 < t < \delta. \quad (1)$$

It is easy to see that if, for some $v \in S_X$, $\delta > 0$, $r > 0$,

$$C(x, v, \delta) \cap B(x, r) \cap A = \emptyset, \quad \text{then } A \text{ is porous at } x \text{ in direction } v. \quad (2)$$

Moreover, we will need the following results from [7].

Proposition 2.5 ([7, Proposition 3.2]). *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ a mapping. Let M be the set of all $x \in G$ at which f is Lipschitz and there exists $v \in X$ such that $f'_+(x, v)$ exists but $f'_{H+}(x, v)$ does not exist. Then M is σ -directionally porous.*

Proposition 2.6 ([7, Proposition 3.1]). *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ a mapping. Let A be the set of all points $x \in G$ for which there exists a cone $C = C(x, v, \delta)$ such that $\limsup_{y \rightarrow x, y \in C} \frac{\|f(y) - f(x)\|}{\|y - x\|} < \infty$ and f is not Lipschitz at x . Then A is a σ -directionally porous set.*

3. Main results

Lemma 3.1. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ a mapping. Let A be the set of all $x \in G$ at which f is Lipschitz, and for which there exist $v, w \in X$ such that $f'_+(x, v)$ and $f'_+(x, w)$ exist but either $f'_+(x, v - w)$ does not exist or $f'_+(x, v - w) \neq f'_+(x, v) - f'_+(x, w)$. Then A is a σ -directionally porous set.*

Proof. Let M be the σ -directionally porous set from Proposition 2.5. It is sufficient to prove that $A^* := A \setminus M$ is σ -directionally porous. For each $x \in A$, choose a corresponding pair $v = v_x, w = w_x$. Let $\{d_j : j \in \mathbb{N}\}$ be a dense subset of X .

Now consider an arbitrary $x \in A^*$. We can choose $n_x \in \mathbb{N}$ such that

$$\|f(y) - f(x)\| \leq n_x \|y - x\| \quad \text{whenever } \|y - x\| \leq \frac{1}{n_x}. \quad (3)$$

Further, we can choose $p_x \in \mathbb{N}$ and a sequence $1 > t_i^x \searrow 0$ such that, for each $i \in \mathbb{N}$,

$$\left\| \frac{f(x + t_i^x(v_x - w_x)) - f(x)}{t_i^x} - (f'_+(x, v_x) - f'_+(x, w_x)) \right\| > \frac{6}{p_x}. \quad (4)$$

Since $x \notin M$, we have $f'_+(x, w_x) = f'_{H+}(x, w_x)$ and so we can choose $k_x \in \mathbb{N}$ such that

$$\left\| \frac{f(x + tz) - f(x)}{t} - f'_+(x, w_x) \right\| < \frac{1}{p_x} \quad \text{whenever } \|z - w_x\| < \frac{1}{k_x} \quad \text{and } 0 < t < \frac{1}{k_x}, \quad (5)$$

and

$$\left\| \frac{f(x + tv_x) - f(x)}{t} - f'_+(x, v_x) \right\| < \frac{1}{p_x} \quad \text{whenever } 0 < t < \frac{1}{k_x}. \quad (6)$$

Finally choose $j_x \in \mathbb{N}$ such that $\|w_x - d_{j_x}\| < (4k_x)^{-1}$.

Now, for natural numbers n, p, k, j denote by $A_{n,p,k,j}^*$ the set of all $x \in A^*$ for which $n_x = n$, $p_x = p$, $k_x = k$, $j_x = j$. It is clearly sufficient to prove that, for any fixed quadruple n, p, k, j , the set $S := A_{n,p,k,j}^*$ is directionally porous.

So fix an arbitrary $x \in S$. Denote $t_i := t_i^x$ and choose η such that

$$0 < \eta < \min((pn)^{-1}, (2k)^{-1}). \quad (7)$$

We will prove that S is porous at x in the direction $v_x - w_x$.

To show this, it is sufficient to prove that there exists $i_0 \in \mathbb{N}$ such that

$$S \cap B(x + t_i(v_x - w_x), \eta t_i) = \emptyset \quad \text{whenever } i \geq i_0. \quad (8)$$

To this end, consider $i \in \mathbb{N}$ and $y_i \in S \cap B(x + t_i(v_x - w_x), \eta t_i)$.

First observe that $\|(x + t_i(v_x - w_x)) - y_i\| < \eta t_i < \eta < (pn)^{-1}$ and so (3) (with $x := y_i$ and $y := x + t_i(v_x - w_x)$) implies

$$\|f(x + t_i(v_x - w_x)) - f(y_i)\| \leq n\eta t_i < \frac{t_i}{p}. \quad (9)$$

Now we will show (and this is the main trick of the proof) that, if i is sufficiently large, then the difference of $f'_+(x, w_x)$ and $f'_+(y_i, w_{y_i})$ is “small” (see (12)). To this end, set $t^* := (2k)^{-1}$ and $\xi := x + t^*w_x$. By (5) we immediately obtain

$$\left\| \frac{f(\xi) - f(x)}{t^*} - f'_+(x, w_x) \right\| < \frac{1}{p}. \quad (10)$$

Further set $z^* := (t^*)^{-1}(\xi - y_i)$. Then

$$\|z^* - w_{y_i}\| \leq \|z^* - w_x\| + \|w_x - d_j\| + \|w_{y_i} - d_j\| \leq \|z^* - w_x\| + \frac{1}{2k}$$

and

$$\begin{aligned} \|z^* - w_x\| &= \|(t^*)^{-1}(\xi - y_i) - (t^*)^{-1}(\xi - x)\| \\ &= \|(t^*)^{-1}(x - y_i)\| \leq 2kt_i(\|v_x - w_x\| + \eta). \end{aligned}$$

So there exists $i_1 \in \mathbb{N}$ such that $i \geq i_1$ implies $\|z^* - w_{y_i}\| < 1/k$, and consequently also (5) with $x := y_i \in S$

$$\left\| \frac{f(\xi) - f(y_i)}{t^*} - f'_+(y_i, w_{y_i}) \right\| = \left\| \frac{f(y_i + t^*z^*) - f(y_i)}{t^*} - f'_+(y_i, w_{y_i}) \right\| < \frac{1}{p}. \quad (11)$$

Further there exists $i_1 < i_2 \in \mathbb{N}$ such that $i \geq i_2$ implies

$$\|x - y_i\| \leq t_i(\|v_x - w_x\| + \eta) < (2knp)^{-1} < n^{-1}$$

and consequently also (by (3))

$$\left\| \frac{f(\xi) - f(x)}{t^*} - \frac{f(\xi) - f(y_i)}{t^*} \right\| = 2k \|f(y_i) - f(x)\| \leq 2kn \|y_i - x\| < \frac{1}{p}.$$

Using this together with (10) and (11), we obtain that $i \geq i_2$ implies

$$\|f'_+(x, w_x) - f'_+(y_i, w_{y_i})\| < \frac{3}{p}. \quad (12)$$

Denote $a_i := x + t_i v_x$. There exists $i_0 > i_2$ such that $i \geq i_0$ implies $t_i < 1/k$, and consequently also (by (6))

$$\left\| \frac{f(a_i) - f(x)}{t_i} - f'_+(x, v_x) \right\| < \frac{1}{p}. \quad (13)$$

Set $z_i := (t_i)^{-1}(a_i - y_i) = (t_i)^{-1}(x + t_i v_x - y_i)$. Then

$$\|z_i - w_{y_i}\| \leq \|z_i - w_x\| + \|w_x - d_j\| + \|w_{y_i} - d_j\| \leq \|z_i - w_x\| + \frac{1}{2k}$$

and

$$\|z_i - w_x\| = \left\| \frac{x + t_i v_x - y_i}{t_i} - w_x \right\| = \left\| \frac{x + t_i(v_x - w_x) - y_i}{t_i} \right\| < \eta < \frac{1}{2k}.$$

So $\|z_i - w_{y_i}\| < 1/k$ and $i \geq i_0$ implies $t_i < 1/k$, and consequently (by (5) with $x := y_i$)

$$\left\| \frac{f(a_i) - f(y_i)}{t_i} - f'_+(y_i, w_{y_i}) \right\| = \left\| \frac{f(y_i + t_i z_i) - f(y_i)}{t_i} - f'_+(y_i, w_{y_i}) \right\| < \frac{1}{p}. \quad (14)$$

Thus, if $i \geq i_0$, then (9), (12), (13) and (14) imply

$$\begin{aligned} & \left\| \frac{f(x + t_i(v_x - w_x)) - f(x)}{t_i} - (f'_+(x, v_x) - f'_+(x, w_x)) \right\| \\ & \leq \left\| \frac{f(y_i) - f(x)}{t_i} - (f'_+(x, v_x) - f'_+(x, w_x)) \right\| + \frac{1}{p} \\ & \leq \left\| \frac{f(y_i) - f(x)}{t_i} - (f'_+(x, v_x) - f'_+(y_i, w_{y_i})) \right\| + \frac{4}{p} \\ & = \left\| \left(\frac{f(a_i) - f(x)}{t_i} - f'_+(x, v_x) \right) + \left(f'_+(y_i, w_{y_i}) - \frac{f(a_i) - f(y_i)}{t_i} \right) \right\| + \frac{4}{p} < \frac{6}{p}, \end{aligned}$$

which contradicts (4). So we have proved (8). $\square$

Lemma 3.2. *Let X, Y be Banach spaces, $G \subset X$ an open set, $f : G \rightarrow Y$ a mapping. For each $x \in G$ denote by U_x the set of all $v \in X$ such that $f'_+(x, v)$ exists. Then the set B of all $x \in G$ such that f is Lipschitz at x and U_x is not closed is σ -directionally porous.*

Proof. For each $k \in \mathbb{N}$, set

$$B_k := \{x \in B : \|f(y) - f(x)\| \leq k\|y - x\| \text{ whenever } \|y - x\| < 1/k\}.$$

It is clearly sufficient to prove that each B_k is directionally porous. So suppose that $k \in \mathbb{N}$ and $x \in B_k$ are given. Since U_x is not closed, it is easy to see (using Fact 2.1(iv)) that we can find $v \in S_X \setminus U_x$ and a sequence $v_n \rightarrow v$ with $v_n \in S_X \cap U_x$.

We will show that

$$B_k \text{ is porous at } x \text{ in the direction } v. \quad (15)$$

Suppose, to the contrary, that (15) does not hold. We will obtain a contradiction by proving that $v \in U_x$, i.e., that $\lim_{t \rightarrow 0+} t^{-1}(f(x + tv) - f(x))$ exists. Since Y is a complete space, it is sufficient to prove that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left\| \frac{f(x + t_1 v) - f(x)}{t_1} - \frac{f(x + t_2 v) - f(x)}{t_2} \right\| < \varepsilon \text{ whenever } 0 < t_1 < t_2 < \delta. \quad (16)$$

So, let $1 > \varepsilon > 0$ be given. Since (15) does not hold, by (1) we can find $\delta_1 > 0$ such that

$$B(x + tv, \varepsilon t/12k) \cap B_k \neq \emptyset \text{ whenever } 0 < t < \delta_1. \quad (17)$$

Now choose $n \in \mathbb{N}$ such that $\|v - v_n\| < \varepsilon/12k$. Since $v_n \in U_x$, we can find $\delta_2 > 0$ such that

$$\left\| \frac{f(x + t_1 v_n) - f(x)}{t_1} - \frac{f(x + t_2 v_n) - f(x)}{t_2} \right\| < \frac{\varepsilon}{3} \text{ whenever } 0 < t_1 < t_2 < \delta_2. \quad (18)$$

Put $\delta := \min(\delta_1, \delta_2)$. It is sufficient to prove (16). To this end, let arbitrary numbers $0 < t_1 < t_2 < \delta$ be given. By (18), we have

$$\left\| \frac{f(x + t_1 v_n) - f(x)}{t_1} - \frac{f(x + t_2 v_n) - f(x)}{t_2} \right\| < \frac{\varepsilon}{3}. \quad (19)$$

By (17), we can choose points $y_1 \in B(x + t_1 v, \varepsilon t_1/12k) \cap B_k$ and $y_2 \in B(x + t_2 v, \varepsilon t_2/12k) \cap B_k$. Observe that

$$\|y_1 - (x + t_1 v)\| < t_1 \varepsilon/12k < 1/k,$$

$$\|(x + t_1 v) - (x + t_1 v_n)\| = t_1 \|v - v_n\| < t_1 \varepsilon/12k,$$

and therefore $\|y_1 - (x + t_1 v_n)\| < t_1 \varepsilon/6k < 1/k$. Since $y_1 \in B_k$, we obtain

$$\|f(y_1) - f(x + t_1 v)\| \leq k t_1 \varepsilon/12k = t_1 \varepsilon/12,$$

$$\|f(y_1) - f(x + t_1 v_n)\| \leq k t_1 \varepsilon/6k = t_1 \varepsilon/6.$$

Consequently

$$\|f(x + t_1 v) - f(x + t_1 v_n)\| \leq t_1 \varepsilon / 3. \quad (20)$$

By the same way we obtain

$$\|f(x + t_2 v) - f(x + t_2 v_n)\| \leq t_2 \varepsilon / 3. \quad (21)$$

So, for $i \in \{1, 2\}$ we obtain

$$\left\| \frac{f(x + t_i v_n) - f(x)}{t_i} - \frac{f(x + t_i v) - f(x)}{t_i} \right\| \leq \frac{\varepsilon}{3}. \quad (22)$$

Using (19) and (22) we obtain

$$\left\| \frac{f(x + t_1 v) - f(x)}{t_1} - \frac{f(x + t_2 v) - f(x)}{t_2} \right\| < \varepsilon$$

which completes the proof of (16). $\square$

Theorem 3.3. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ a mapping. Then there exists a σ -directionally porous set $C \subset G$ such that if $x \in G \setminus C$ and f is Lipschitz at x , then the set U_x of those directions $u \in X$ in which the one-sided derivative $f'_+(x, u)$ exists is a closed linear subspace of X . Moreover, the mapping $u \mapsto f'_+(x, u)$ is linear on U_x .*

Proof. Let A be the set from Lemma 3.1 and B be the set from Lemma 3.2. We will show that it is sufficient to set $C := A \cup B$. Obviously, C is σ -directionally porous.

Now consider an arbitrary $x \in G \setminus C = G \setminus (A \cup B)$ at which f is Lipschitz. It is obvious that $0 \in U_x$, $f'_+(x, 0) = 0$ and (see Fact 2.1(iv))

$$\text{if } u \in U_x \text{ and } t \geq 0, \text{ then } tu \in U_x \text{ and } f'_+(x, tu) = t f'_+(x, u). \quad (23)$$

If $w \in U_x$, set $v := 0$ and obtain (since $x \notin A$)

$$f'_+(x, -w) = f'_+(x, 0) - f'_+(x, w) = -f'_+(x, w). \quad (24)$$

Obviously, (23) and (24) imply that

$$\text{if } u \in U_x \text{ and } t \in \mathbb{R}, \text{ then } tu \in U_x \text{ and } f'_+(x, tu) = t f'_+(x, u). \quad (25)$$

Further consider arbitrary vectors $v, u \in U_x$. Setting $w := -u$ and using $x \notin A$ and (24), we obtain

$$f'_+(x, v + u) = f'_+(x, v - w) = f'_+(x, v) - f'_+(x, w) = f'_+(x, v) + f'_+(x, u). \quad (26)$$

Thus (25) and (26) imply that U_x is a linear subspace of X and the mapping $u \mapsto f'_+(x, u)$ is linear on U_x . Since $x \notin B$, we have that U_x is closed. $\square$

Theorem 3.3 and Fact 2.1(i), (v) immediately imply the following corollaries.

Corollary 3.4. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ an arbitrary mapping. Then the following implication holds at each point $x \in G$ except a σ -directionally porous set:*

If f is Lipschitz at x and the one-sided directional derivative $f'_+(x, u)$ exists in all directions u from a set $S_x \subset X$ whose linear span is dense in X , then f is Gâteaux differentiable at x .

Corollary 3.5. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ a pointwise Lipschitz mapping. Then there exists a σ -directionally porous set $A \subset G$ such that for every $x \in G \setminus A$ the set U_x of those directions $u \in X$ in which the one-sided derivative $f'_+(x, u)$ exists is a closed linear subspace of X . Moreover, the mapping $u \mapsto f'_+(x, u)$ is linear on U_x .*

Corollary 3.6. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ a pointwise Lipschitz mapping. Then the following implication holds at each point $x \in G$ except a σ -directionally porous set:*

If the one-sided directional derivative $f'_+(x, u)$ exists in all directions u from a set $S_x \subset X$ whose linear span is dense in X , then f is Gâteaux differentiable at x .

Our main result on Hadamard derivatives is the following.

Theorem 3.7. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ a mapping. Then there exists a σ -directionally porous set $D \subset G$ such that if $x \in G \setminus D$, then either the set V_x of those directions $u \in X$ in which the one-sided Hadamard derivative $f'_{H+}(x, u)$ exists is an empty set or V_x is a closed linear subspace of X and the mapping $u \mapsto f'_{H+}(x, u)$ is linear on V_x .*

Proof. Let C be the set from Theorem 3.3, M the set from Proposition 2.5 and A the set from Proposition 2.6. Set $D := C \cup M \cup A$. Then $D \subset G$ is a σ -directionally porous set. Now suppose that a point $x \in G \setminus D$ is given and $V_x \neq \emptyset$.

Choose $w \in V_x$. If $w = 0$, then we obtain that f is Lipschitz at x by Lemma 2.2. If $w \neq 0$, then we set $v := w/\|w\|$ and apply Fact 2.1(iv), (vii). We obtain that there exists a cone $C := C(x, v, \delta)$ such that $\limsup_{y \rightarrow x, y \in C} \frac{\|f(y) - f(x)\|}{\|y - x\|} < \infty$. Since $x \notin A$, we obtain that f is Lipschitz at x also in this case.

Since $x \notin M$, we obtain that $f'_+(x, u) = f'_{H+}(x, u)$ whenever one of these two derivatives exists. So, since f is Lipschitz at x and $x \notin C$, we obtain our assertion. $\square$

Remark 3.8. If we have in our disposal Proposition 2.5, Proposition 2.6 and Lemma 2.2, then we can consider our two main theorems as “equivalent”. Indeed, we have already shown how Theorem 3.3 (and the above mentioned propositions)

easily implies Theorem 3.7. Further, Theorem 3.7 and Proposition 2.6 almost immediately imply Theorem 3.3.

Using Theorem 3.7, Fact 2.1(i), (vi) and Lemma 2.3, we immediately obtain the following result, which improves and generalizes [2, Theorem 3.7(b)] (see Remark 3.10 below).

Corollary 3.9. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \rightarrow Y$ an arbitrary mapping. Then the following implication holds at each point $x \in G$ except a σ -directionally porous set:*

If the one-sided Hadamard derivative $f'_{H+}(x, u)$ exists in all directions u from a set $S_x \subset X$ whose linear span is dense in X , then f is Hadamard differentiable at x .

Remark 3.10. Ioffe's [2, Theorem 3.7(b)] works with $f : X \rightarrow [-\infty, \infty]$ and $\text{dom}(f) := \{x \in X : |f(x)| < \infty\}$. Denote by B the set of all $x \in \text{dom}(f) \setminus \text{int}(\text{dom}(f))$ for which there exists $0 \neq v \in X$ such that $f'_{H+}(x, v) := \lim_{z \rightarrow v, t \rightarrow 0+} \frac{f(x+tz) - f(x)}{t}$ exists and is finite. If $x \in B$, then (cf. Fact 2.1(vii)) there exists a cone $C = C(x, v, \delta)$ such that $\limsup_{y \rightarrow x, y \in C} \frac{|f(y) - f(x)|}{\|y - x\|} < \infty$, which implies that $B \subset X \setminus \text{int}(\text{dom}(f))$ is directionally porous at x (see (2)). So B is a directionally porous set.

This simple observation shows that Corollary 3.9 remains true, if we work with $f : X \rightarrow [-\infty, \infty]$ (and $f'_{H+}(x, u)$ is, by definition, finite). In other words, the assumption of [2, Theorem 3.7(b)] that the linear span of S_x equals to X can be relaxed to the assumption that the linear span of S_x is dense in X .

References

- [1] Y. Benyamini, J. Lindenstrauss: Geometric Nonlinear Functional Analysis. Vol. 1, Colloquium Publications 48, American Mathematical Society, Providence (2000).
- [2] A. D. Ioffe: Typical convexity (concavity) of Dini-Hadamard upper (lower) directional derivatives of functions on separable Banach spaces, J. Convex Analysis 17 (2010) 1019–1032.
- [3] J. Lindenstrauss, D. Preiss, J. Tišer: Fréchet Differentiability of Lipschitz Functions and Porous Sets in Banach Spaces, Princeton University Press, Princeton (2012).
- [4] D. Preiss, L. Zajíček: Directional derivatives of Lipschitz functions, Israel J. Math. 125 (2001) 1–27.
- [5] A. Shapiro: On concepts of directional differentiability, J. Optim. Theory Appl. 66 (1990) 477–487.
- [6] L. Zajíček: Singular points of order k of Clarke regular and arbitrary functions, Comment. Math. Univ. Carol. 53 (2012) 51–63.
- [7] L. Zajíček: Hadamard differentiability via Gâteaux differentiability, preprint (2012), arXiv:1210.4715v1.