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Sufficient Conditions for an Existence of a Solution to a Differential Inclusion

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We formulate geometric conditions induced by the compact set $K \subset \mathcal{R}^{m\times n}$, which imply existence of a Lipschitz solution u to the differential inclusion $Du \in K$. The solutions are obtained using the convex integration method. We illustrate our result for the known example $K = SO(2) \cup SO(2)B$, where B is a 2×2 diagonal matrix with $\det B = 1$.

1. Introduction

In the present paper we deal with a differential inclusion $Du \in K$, where $K \subset \mathbb{R}^{m\times n}$ and u is a Lipschitz mapping with affine boundary values defined on an open, bounded set $\Omega \subset \mathbb{R}^n$. One of the methods to solve this inclusion is to construct a sequence of piecewise affine mappings u_k having special structure and approximating the solution u. This special structure guaranties the strong convergence of the gradients Du_k in $L^1(\Omega)$ on one hand, and one the other allows to conclude the final property $Du \in K$. The main step obtaining these properties can be done using one of the several, parallel approaches: the convex integration method (developed by M. Gromov [4, 5] and extended later by S. Müller and V. Šverák [13, 14, 15] for Lipschitz mappings, see also [11, 8]), the Baire category theorem (used by B. Dacorogna and P. Marcellini [2, 3], B. Kirchheim [6, 7] and C. De Lellis, L. Székelyhidi [9]) or the martingale convergence theorem (such a possibility is mentionned in papers by M. Sychev [17, 18] and in the author's paper [16]). All these approches are well-known to the specialists.

Conditions implying existence of such a sequence u_k are usually formulated in a very general form. They are stated in terms of an existence of a sequence $U_k \subset \mathbb{R}^{m\times n}$ approximating the set K in some sense. (In the convex integration method such a sequence is called an *in-approximation* of the set K.) These conditions are general, but it is very often hard to verify, if they are satisfied for many particular sets K. Therefore it seems to be important to propose sufficient geometric conditions induced by the set K, which would guarantee existence of an in-approximation and consequently existence of solutions to $Du \in K$.

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Some theorems of this type have already been discovered by S. Müller and M. Sychev [12, Theorem 1.6] and also by B. Kirchheim [7, Theorem 3.22]. However, the former result assumes that the approximation process takes place in a *convex* subset of $\mathbb{R}^{m\times n}$. This already excludes many applications, since it is clear that the approximation should take place in the polyconvex hull of the set K, which in general is not convex. The latter result, however, replaces the convexity by the star-shape condition, which still excludes the non-linear constraints like $\det Du = 1$.

In this paper we present another results of that type: Theorems 2.1, 2.2 below. Our approximation process takes place in the space of minors and we use local convexity there, which correspondes to local polyconvexity in the space of matrices. Therefore the corresponding approximation process in the space of matrices does not necessarily take place in a convex or in a star-shaped set. Our theorem can be applied to the known example: the constrained version of the two-well problem in dimension 2, which we demonstrate at the end of the paper.

2. Statement of the results

Let Ω be an open, bounded set in $\mathbb{R}^n$. A Lipschitz mapping $u\colon \Omega \to \mathbb{R}^m$ is called *piecewise affine*, if there exist at most countably many disjoint open subsets $\Omega_1, \Omega_2, \dots$ of Ω with $|\Omega_1 \cup \Omega_2 \cup \dots| = |\Omega|$, such that Du is constant on each Ω_i.

We shall also use the following notation. For a matrix $F \in \mathbb{R}^{m\times n}$ we denote by $\mathbf{M}(F)$ the vector (of length d) of all the minors of F. Hence

$$d = \sum_{r=1}^{\min(m,n)} \binom{m}{r}\cdot\binom{n}{r} = \binom{m+n}{n} - 1\,. \tag{1}$$

Then $\mathcal{M} = \{\mathbf{M}(F) \ : \ F \in \mathbb{R}^{m\times n}\}$ is a closed $(m{\times}n)$-dimensional C^1-manifold in $\mathbb{R}^d$. The function $t \mapsto \mathbf{M}(X + tY)$ is linear if and only if $\operatorname{rank}(Y) \leq 1$. Therefore the line connecting the points $\mathbf{M}(A)$ and $\mathbf{M}(B)$ is contained in $\mathcal{M} = \{\mathbf{M}(F) \ : \ F \in \mathbb{R}^{m\times n}\}$ if and only if $\operatorname{rank}(A - B) = 1$.

Let $\mathcal{M}$ be an arbitrary closed C^1-manifold in $\mathbb{R}^k$. Assume that $\mathcal{K}$ is a compact subset of $\mathbb{R}^k$, whose interior (relative to $\mathbb{R}^k$) is nonempty and intersects $\mathcal{M}$.

We will call a subset $\mathcal{N}$ of $\partial\mathcal{K}$ a *regular rank-one part of $\partial\mathcal{K}$ with respect to $\mathcal{M}$*, if the following conditions are satisfied:

(A0) For any $P \in \mathcal{N}$ there exists a ball $B \subset \mathbb{R}^k$ centered at P such that $B \cap \mathcal{K}$ is convex (i.e. $\mathcal{K}$ is convex in a neighbourhood of $\mathcal{N}$);

(A1) $\mathcal{N}$ is a C^1-manifold of dimension $k{-}1$ and for each $P \in \mathcal{M}\cap\mathcal{N}$ the normal vector to $\mathcal{N}$ at P is not contained in the normal space of $\mathcal{M}$ at P (i.e. $\mathcal{M}$ and $\mathcal{N}$ intersect transversally);

(A2) For each $P \in \mathcal{M}\cap\mathcal{N}$ there exists an $\varepsilon > 0$ and a neighbourhood U in $\mathcal{M}\cap\mathcal{N}$ of P such that each point $X \in U$ is a midpoint of a line segment of length at least ε contained in $\mathcal{M}\cap\mathcal{N}$ (i.e. $\mathcal{M}\cap\mathcal{N}$ is covered "uniformly" by line segments).

The following theorems are the main results of the present paper.

Theorem 2.1. *Consider the manifold*

$$\mathcal{M} = \{\mathbf{M}(X) \ : \ X \in \mathbb{R}^{m\times n}\}\,, \tag{2}$$

contained in $W = \mathbb{R}^d$, *where* d *is defined by* (1). *Let* $\mathcal{K}$ *be a compact subset in* W *and let* $F \in \mathbb{R}^{m\times n}$ *be such that* $\mathbf{M}(F) \in \operatorname{int}_W \mathcal{K}$. *Finally, assume that* $\mathcal{N}$ *is a regular rank-one part of* $\partial\mathcal{K}$ *with respect to* $\mathcal{M}$.

Then there exists a Lipschitz mapping $u\colon \Omega \to \mathbb{R}^m$ *such that*

$$\mathbf{M}(Du) \in \partial\mathcal{K}\setminus\mathcal{N} \quad \textit{a.e. on } \Omega \quad \textit{and} \quad u(x) = Fx \quad \textit{on } \partial\Omega\,.$$

An analogous result, with almost the same proof, holds for the constrained case $\det F = 1$.

Theorem 2.2. *Consider the manifold*

$$\mathcal{M} = \{\mathbf{M}(X) \ : \ X \in \mathbb{R}^{n\times n} \textit{ and } \det X = 1\}\,, \tag{3}$$

contained in the affine space $W = \{(x_1, x_2, \ldots, x_d) \ : \ x_d = 1\}$, *where* d *is defined by* (1) *and* $m = n$. *Let* $\mathcal{K}$ *be a compact subset in* W *and let* $F \in \mathbb{R}^{n\times n}$ *be such that* $\mathbf{M}(F) \in \operatorname{int}_W \mathcal{K}$. *Finally, assume that* $\mathcal{N}$ *is a regular rank-one part of* $\partial\mathcal{K}$ *with respect to* $\mathcal{M}$.

Then there exists a Lipschitz mapping $u\colon \Omega \to \mathbb{R}^n$ *such that*

$$\mathbf{M}(Du) \in \partial\mathcal{K}\setminus\mathcal{N} \quad \textit{a.e. on } \Omega \quad \textit{and} \quad u(x) = Fx \quad \textit{on } \partial\Omega\,.$$

3. Proof of Theorems 2.1 and Theorem 2.2

The proof is based on the following lemmas. Everything up to statement of Lemma 3.4 is used only in the case of Theorem 2.2 (so it's not needed in the proof of Theorem 2.1).

Lemma 3.1. *Assume* $b > 0$ *and let* $D = \{X \in \mathbb{R}^{n\times n} \ : \ \det X = 1 \textit{ and } |X| < b\}$. *Let moreover* $R = \{Y \in \mathbb{R}^{n\times n} \ : \ \operatorname{rank} Y = 1 \textit{ and } |Y| = 1\}$. *Assume also that* $P \in D$ *and* $S \in R$ *are such that* $\det(P + tS) = 1$ *for all* $t \in \mathbb{R}$. *Then there exists a Lipschitz mapping* $Y\colon\ D \to R$ *with the following properties:*

(*1*) *For each* $X \in D$ *we have* $\det(X + tY(X)) = 1$;
(*2*) $Y(P + tS) = S$, *whenever* $P + tS \in D$ *and* $t \in \mathbb{R}$;
(*3*) $Y(X + tY(X)) = Y(X)$, *whenever* $X + tY(X) \in D$ *and* $t \in \mathbb{R}$;
(*4*) *The Lipschitz constant of the mapping* Y *depends only on* b *and the dimension* n *(in particular, it does not depend on* P *and* S*).*

Proof. For any matrix $X \in \mathbb{R}^{n\times n}$ we denote by x_i the i-th row of X, i.e. $x_i = (x_{i1}, x_{i2}, \dots, x_{in})$. Then for any $X \in D$ we have

$$1 = |\det X| \le |x_1|\cdot|x_2|\cdot \ldots \cdot|x_n| < |x_i| \cdot b^{n-1} ,$$

which gives $b^{1-n} < |x_i| < b$. Therefore for each $X \in D$ we obtain

$$b^{1-n}\sqrt{n} < |X| < b . \tag{4}$$

Assume first that the matrix S satisfies $s_i = 0$ for $i = 2, 3, \dots, n$. Since $\det P = 1 = \det(P + S)$, the vector s_1 is a linear combination of the vectors $p_2, p_3, \dots, p_n$, i.e.

$$s_1 = \alpha_2 p_2 + \alpha_3 p_3 + \ldots + \alpha_n p_n \quad (\alpha_2, \alpha_3, \dots, \alpha_n \in \mathbb{R}) .$$

For the matrix $X \in D$ define $Y(X)$ by the formulas

$$y_1(X) = \frac{\alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_n x_n}{|\alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_n x_n|}$$

and $y_i(X) = 0$ for $i \ge 2$. It is clear that *(1)*, *(2)* and *(3)* are satisfied. So it remains to show that Y is a Lipschitz mapping on D and that *(4)* is satisfied.

If $e_1, e_2, \dots, e_n$ denotes the standard orthonormal basis in $\mathbb{R}^n$, then we have $(P^T)^{-1} p_i = e_i$ for $i = 1, 2, \dots, n$, and consequently $(P^T)^{-1} s_1 = \alpha_2 e_2 + \alpha_3 e_3 + \ldots + \alpha_n e_n$. If we denote by $\langle x, y\rangle$ the standard scalar product in $\mathbb{R}^n$, then we obtain

$$|\alpha_i| = |\langle (P^T)^{-1} s_1, e_i\rangle| \le |(P^T)^{-1} s_1| \le c \quad (i = 2, 3, \dots, n) ,$$

where c is a constant dependent only on the norm of the matrix $(P^T)^{-1}$, and since $\det P = 1$, the constant c depends only on b and n.

Let L be a Lipschitz constant of the mapping $p(X) = X/|X|$ defined on the set of all matrices $X \in \mathbb{R}^{n\times n}$ satisfying (4). Then for all $X, Z \in D$ we have

$$\begin{aligned} |Y(X) - Y(Z)| &= |y_1(X) - y_1(Z)| \\ &= \left| p\left(\sum_{i=2}^{n} \alpha_i x_i\right) - p\left(\sum_{i=2}^{n} \alpha_i z_i\right)\right| \\ &\le L\left|\sum_{i=2}^{n} \alpha_i (x_i - z_i)\right| \le Lc(n-1)\,|X - Z| . \end{aligned}$$

Now let $S \in R$ be arbitrary and write $S = a \otimes b$ with $|a| = |b| = 1$. Let $T \in R$ be such that $t_1 = a$ and $t_i = 0$ for $i = 2, 3, \dots, n$. Let moreover $O \in SO(n)$ be a matrix, whose first column is equal to b. Then we have $S = OT$. Write $P = OQ$, where $Q \in D$ and define a Lipschitz mapping $Y_1\colon D \to R$ like above and such that $Y_1(Q + tT) = T$. Finally, let $Y(X) = OY_1(O^{-1}X)$. Then Y is a Lipschitz mapping defined on D, which satisfies *(1)* – *(4)*. Indeed:

$$\det(X + tY(X)) = \det(X + tOY_1(O^{-1}X)) = \det(O^{-1}X + tY_1(O^{-1}X)) = 1 ,$$

which gives (*1*). To see (*2*) note that

$$Y(P+tS) = OY_1(O^{-1}(P+tS)) = OY_1(Q+tT) = OT = S\,.$$

Condition (*3*) follows from

$$\begin{aligned} Y(X+tY(X)) &= OY_1(O^{-1}(X+t+Y(X))) = OY_1(O^{-1}X + tO^{-1}Y(X)) \\ &= OY_1(O^{-1}X + tY_1(O^{-1}X)) = OY_1(O^{-1}X) = Y(X)\,. \end{aligned}$$

Finally, to see (*4*) note that

$$\begin{aligned} |Y(X)-Y(Z)| &= |Y_1(O^{-1}X) - Y_1(O^{-1}Z)| \le Lc(n{-}1)\,|O^{-1}(X-Z)| \\ &= Lc(n{-}1)\,|X-Z|\,, \end{aligned}$$

which completes the proof. □

Corollary 3.2. *Assume* $b>0$ *and let* $D = \{X \in \mathbb{R}^{n\times n} \ :\ \det X = 1$ *and* $|X| < b\}$. *Let moreover* $A, B \in D$ *and* $\det(tA + (1-t)B) = 1$ *for all* $t \in \mathbb{R}$. *Then there exists a constant* $c > 0$, *dependent only on* b *and* n, *such that the following is satisfied:*

For each $X \in D$, *there exists a line segment* $[A(X), B(X)]$ *of length* $|A-B|$, *containing* X, *satisfying* $\det((1-t)A(X) + tB(X)) = 1$ *for all* $t \in \mathbb{R}$ *and*

$$|A(X)-A| + |B(X)-B| \le c\,\mathrm{dist}\,(X,[A,B])\,.$$

Moreover, $\mathrm{dist}\,(Y,[A,B]) \le c\,\mathrm{dist}\,(X,[A,B])$ *for all* $Y \in [A(X), B(X)]$.

Proof. Consider the Lipschitz function $Y\colon\ D \to R$ such that (*1*)–(*4*) are satisfied and $Y(A+t(B-A)) = (B-A)/|B-A|$. Let moreover $P\colon\ D \to [A,B]$ be the projection onto the line segment $[A,B]$. Next define

$$A(X) = X - Y(X)|P(X)-A| \quad \text{and} \quad B(X) = X + Y(X)|P(X)-B| \quad (X \in D)\,.$$

We will show that the mappings $A(X)$ and $B(X)$, defined on D, satisfy our requirements.

Using property (1) of Lemma 3.1 we infer that

$$\det((1-t)A(X) + tB(X)) = \det(X + sY(X)) = 1\,,$$

where $s = t|P(X)-B| - (1-t)|P(X)-A|$ is a real number.

Note that for each $X \in D$ we have $A(P(X)) = A$ and $B(P(X)) = B$. Thus

$$\begin{aligned} |A(X)-A| &= |A(X) - A(P(X))| \\ &\le |X-P(X)| + |Y(X) - Y(P(X))|\cdot|P(X)-A| \\ &\le |X-P(X)|\cdot(1 + c\,|P(X)-A|)\,, \end{aligned}$$

where $c > 0$ is the Lipschitz constant of the mapping Y (thus c depends only on b and n). Analogously, we obtain

$$|B(X) - B| \leq |X - P(X)| \cdot (1 + c\,|P(X) - B|)\,.$$

Adding the above inequalities we get

$$\begin{aligned}|A(X) - A| + |B(X) - B| &\leq |X - P(X)| \cdot (1 + c\,|A{-}B|)\\ &= (1 + c\,|A{-}B|)\,\mathrm{dist}\,(X, [A, B])\,.\end{aligned}$$

Moreover, if $Y = (1 - t)A(X) + tB(X)$ with $t \in [0, 1]$, then we have

$$\begin{aligned}\mathrm{dist}\,(Y, [A, B]) &\leq (1 - t)\,\mathrm{dist}\,(A(X), [A, B]) + t\,\mathrm{dist}\,(B(X), [A, B])\\ &\leq |A(X) - A| + |B(X) - B|\\ &\leq (1 + c\,|A{-}B|)\,\mathrm{dist}\,(X, [A, B])\,.\end{aligned}$$ □

The following lemma was obtained by a direct construction by S. Conti [1]. The unconstrained version can be found in [13] or [11].

Lemma 3.3. *Let $A, B \in \mathbb{R}^{n\times n}$ be such that $\det((1 - t)A + tB) = 1$ for all $t \in \mathbb{R}$. Let moreover $F = \lambda A + (1 - \lambda)B$, where $\lambda \in (0, 1)$. Then for each $\varepsilon > 0$ there exists a piecewise affine mapping u defined on Ω and having the following properties*

(*1*) $u(x) = Fx$ *on* $\partial\Omega$*;*
(*2*) $\det Du(x) = 1$ *a.e. on* Ω*;*
(*3*) $\mathrm{dist}\,(Du(x), \{A, B\}) < \varepsilon$ *a.e. on* Ω*.*

We shall also need the following lemma:

Lemma 3.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be two C^1-manifolds in $\mathbb{R}^d$ and let $P \in \mathcal{M} \cap \mathcal{N}$. Assume $\mathcal{N}$ is $(d{-}1)$-dimensional and the normal vector to $\mathcal{N}$ at P is not contained in the normal space to $\mathcal{M}$ at P. Then there exists a neighbourhood U of P in $\mathbb{R}^d$ and a constant $c > 0$ such that for all $X \in \mathcal{M} \cap U$ we have*

$$\mathrm{dist}\,(X, \mathcal{M} \cap \mathcal{N}) \leq c\,\mathrm{dist}\,(X, \mathcal{N})\,.$$

Proof. Denote by M and N the tangent spaces at P to $\mathcal{M}$ and $\mathcal{N}$, respectively. Since the sum of the codimensions of M and N does not exceed d and since $N^\perp$ is not contained in $M^\perp$, $\mathcal{M} \cap \mathcal{N}$ is itself a manifold and the intersection $M \cap N$ is the tangent space to $\mathcal{M} \cap \mathcal{N}$ at P.

Each $X \in \mathcal{M} \cap \mathcal{N}$ satisfies the desired inequality. So suppose that there exists a sequence $X_n \in \mathcal{M} \setminus \mathcal{N}$ converging to P such that

$$\frac{\mathrm{dist}\,(X_n, \mathcal{N})}{\mathrm{dist}\,(X_n, \mathcal{M} \cap \mathcal{N})} \to 0\,. \tag{5}$$

Let X_n' and X_n'' denote the orthogonal projections of X_n onto $\mathcal{N}$ and $\mathcal{M}\cap\mathcal{N}$, respectively. Then the convergence (5) is equivalent to

$$\frac{X_n - X_n''}{|X_n - X_n''|} - \frac{X_n' - X_n''}{|X_n - X_n''|} \to 0\,.$$

Passing to a subsequence if necessary, we may assume that

$$v_n = \frac{X_n - X_n''}{|X_n - X_n''|} \to v \quad \text{and} \quad w_n = \frac{X_n' - X_n''}{|X_n - X_n''|} \to v\,. \tag{6}$$

The vector v_n is orthogonal to $\mathcal{M}\cap\mathcal{N}$, so since $\mathcal{M}\cap\mathcal{N}$ is a C^1-manifold (in a neighbourhood of P), we get $v \in (M\cap N)^{\perp}$. On the other hand $X_n, X_n'' \in \mathcal{M}$, which gives $v \in M$ and also $X_n', X_n'' \in \mathcal{N}$, which in turn implies that $v \in N$. Hence $v = 0$, and this is impossible, since by (6) we get $|v| = 1$. A contradiction. □

Now we turn to the proof of Theorems 2.1 and 2.2.

We will use the following notation. Consider a line l passing through $\mathbf{M}(F)$ and contained in $\mathcal{M}$. Next denote by $\mathbf{M}(G)$ and $\mathbf{M}(H)$ the points of intersection of l with $\partial\mathcal{K}$ and define

$$\rho(\mathbf{M}(F), l) = \min(|\mathbf{M}(F) - \mathbf{M}(G)|, |\mathbf{M}(F) - \mathbf{M}(H)|)\,. \tag{7}$$

Let moreover

$$\rho(\mathbf{M}(F)) = \sup \rho(\mathbf{M}(F), l)\,,$$

where the supremum is taken over all lines l contained in $\mathcal{M}$ and passing through $\mathbf{M}(F)$.

Now using Lemma 3.3 (and its unconstrained version, in case of Theorem 2.1) we construct inductively the sequence $u_n(x)$ of piecewise affine mappings such that $\mathbf{M}(Du_n(x))$ lies in $\operatorname{int}_W \mathcal{K}$ as follows.

Let $u_1(x) = Fx$ and assume we have already defined the piecewise affine mapping u_n. Let Du_n be equal to F_{ni} on the set Ω_{ni} $(i = 1, 2, \ldots)$ and assume each $\mathbf{M}(F_{ni})$ lies in $\operatorname{int}_W \mathcal{K}$. To construct the mapping u_{n+1}, choose a line l_{ni} passing through $\mathbf{M}(F_{ni})$ and contained in $\mathcal{M}$ such that

$$\rho(\mathbf{M}(F_{ni}), l_{ni}) \geq \frac{n}{n+1}\,\rho(\mathbf{M}(F_{ni})) \quad (i = 1, 2, \ldots)\,.$$

Assume the line l_{ni} intersects $\partial\mathcal{K}$ at $\mathbf{M}(G_{ni})$ and $\mathbf{M}(H_{ni})$ and set

$$A_{ni} = \frac{1}{n}\,F_{ni} + \left(1 - \frac{1}{n}\right) G_{ni} \quad \text{and} \quad B_{ni} = \left(1 - \frac{1}{n}\right) H_{ni} + \frac{1}{n} F_{ni}\,. \tag{8}$$

Since $\mathbf{M}(A_{ni})$ and $\mathbf{M}(B_{ni})$ lie in $\operatorname{int}_W \mathcal{K}$, we use Lemma 3.3 to infer that there exists a piecewise affine mapping φ_{ni}, defined on Ω_{ni}, such that $\mathbf{M}(D\varphi_{ni}) \in \operatorname{int}_W \mathcal{K}$,

$$\varphi_{ni}(x) = u_{ni} \;\text{ on } \partial\Omega_{ni} \quad \text{and} \quad \operatorname{dist}(D\varphi_{ni}, \{A_{ni}, B_{ni}\}) < \frac{1}{n}\,. \tag{9}$$

Now on each Ω_{ni} replace the function u_n by the piecewise affine mapping φ_{ni}. Having done so on each Ω_{ni} we obtain a new piecewise linear mapping u_{n+1}, which agrees with u_n on $\partial\Omega$ and satisfies $\mathbf{M}(Du_{n+1}) \in \mathrm{int}_W\,\mathcal{K}$.

If we denote by $\mathcal{F}_n$ the σ-algebra generated by the sets $\Omega_{n1}, \Omega_{n2}, \ldots$, then we easily see that $(\mathbf{M}(Du_n), \mathcal{F}_n)$ is a martingale (see [10] for necessary definitions). Indeed, since u_n and u_{n+1} agree on $\partial\Omega_{ni}$ and since the minors are Null-Lagrangians, we have

$$\int_{\Omega_{ni}} \mathbf{M}(Du_{n+1}) = \int_{\Omega_{ni}} \mathbf{M}(Du_n) \quad \text{for } i = 1, 2, \ldots,$$

and therefore for each $A \in \mathcal{F}_n$ we obtain

$$\int_A \mathbf{M}(Du_{n+1}) = \int_A \mathbf{M}(Du_n)\,.$$

The Martingale Convergence Theorem [10, Chapter V, Theorem T17] states that $\mathbf{M}(Du_n)$ converges in $L^1(\Omega)$ and almost everywhere on Ω, provided the values $\mathbf{M}(Du_n(x))$ lie in a fixed bounded set. Since the convergence of $\mathbf{M}(Du_n)$ takes place in $L^1(\Omega)$ and since all the functions u_n agree on $\partial\Omega$, the pointwise limit of $\mathbf{M}(Du_n(x))$ equals $\mathbf{M}(Du(x))$, where u is a Lipschitz mapping with $u(x) = Fx$ on $\partial\Omega$. Hence to complete the proof, it remains to show that $\mathbf{M}(Du) \in \partial\mathcal{K} \setminus \mathcal{N}$.

Assume for some $x \in \Omega$ the sequence $P_n = Du_n(x)$ converges to a point P. It follows from (8) and (9) that $\mathbf{M}(P) \in \partial\mathcal{K}$. Suppose, to the contrary, that $\mathbf{M}(P) \in \mathcal{N}$.

Identify the affine space W with $\mathbb{R}^l$. Using (A0) we infer that there exists a neighbourhood U of $\mathbf{M}(P)$ in $\mathbb{R}^l$ such that the set $\mathcal{N} \cap U$ is a graph of a convex, Lipschitz function $\varphi\colon\ V \to \mathbb{R}$ such that

$$\mathcal{N} \cap U = \{(X, \varphi(X)) \in \mathbb{R}^{l-1} \times \mathbb{R} \ :\ X \in V\} \quad \text{and} \quad \mathbf{M}(P) = (0, \varphi(0))$$

Here V is a neighbourhood of 0 in $\mathbb{R}^{l-1}$. Moreover, for each $(X, Y) \in U$ we have $Y \geq \varphi(X)$ if and only if $(X, Y) \in \mathcal{K}$.

Scaling if necessary and using assumption (A2), we may assume that there is a neighbourhood V_0 of 0 lying in V such that each point of the set $\mathcal{M} \cap \mathcal{N}_0$, where $\mathcal{N}_0 = (V_0, \varphi(V_0))$ is a midpoint of a closed line segment of length 2 contained in $\mathcal{M} \cap \mathcal{N} \cap U$.

Let $\mathbf{M}(P_n) = (X_n, Y_n)$, where $Y_n > \varphi(X_n)$ and set $\varepsilon_n = Y_n - \varphi(X_n)$. Since P_n converges to P, we may assume that $X_n \in V_0$. Let $\mathbf{M}(P_n')$ be the projection of the point $\mathbf{M}(P_n)$ onto $\partial\mathcal{K} \cap \mathcal{M}$. Then for all sufficiently large values of n, $\mathbf{M}(P_n') \in \mathcal{N}_0$. Using Lemma 3.4 and assumption (A1) we infer that there is a constant $c > 0$ (independent of n) such that

$$|\mathbf{M}(P_n) - \mathbf{M}(P_n')| \leq c\,\mathrm{dist}\,(\mathbf{M}(P_n), \mathcal{N}) \leq c\varepsilon_n\,. \tag{10}$$

Since the point $\mathbf{M}(P_n')$ belongs to $\mathcal{M} \cap \mathcal{N}_0$, there exists a line segment $[\mathbf{M}(T_n'), \mathbf{M}(W_n')]$ of length 2 contained in $\mathcal{M} \cap \mathcal{N} \cap U$, whose midpoint is $\mathbf{M}(P_n')$. Using Corollary 3.2 with $A = T_n'$ and $B = W_n'$ we construct a line segment

$[\mathbf{M}(T_n), \mathbf{M}(W_n)]$ of length 2 contained in $\mathcal{M}$, whose midpoint is $\mathbf{M}(P_n)$ and such that

$$|T_n' - T_n| + |W_n' - W_n| \leq c \operatorname{dist}(P_n, [T_n', W_n']) \leq c\,|P_n' - P_n|\,,$$

where $c > 0$ is a constant independent of n. (In case of Theorem 2.1, we take $[T_n, W_n]$ parallel to $[T_n', W_n']$.)

This together with (10) and the fact that the mapping $X \mapsto \mathbf{M}(X)$ is locally Lipschitz, implies that there exists a constant $c_1 > 0$, independent of n, such that

$$|\mathbf{M}(T_n) - \mathbf{M}(T_n')| \leq c_1\varepsilon_n \quad \text{and} \quad |\mathbf{M}(W_n) - \mathbf{M}(W_n')| \leq c_1\varepsilon_n\,. \tag{11}$$

Write $\mathbf{M}(T_n) = (C_n, D_n)$ and $\mathbf{M}(T_n') = (C_n', D_n') = (C_n', \varphi(C_n'))$ and consider the function p_n defined on the interval $[-1, 1]$ by the formula

$$p_n(t) = \varphi((1-t)X_n + tC_n) - (1-t)Y_n - tD_n\,.$$

(In other words, the graph of the function $p_n(t)$ is obtained as an intersection of the surface $\mathcal{N}$ with the two-dimensional plane passing through the points $\mathbf{M}(P_n)$, $\mathbf{M}(T_n)$ and $(X_n, \varphi(X_n))$.) Then $p_n(0) = -\varepsilon_n$ and $p_n(1) = \varphi(C_n) - D_n$. Moreover, the function φ is Lipschitz on V, so using (11) we have

$$\begin{aligned} |p_n(1)| &\leq |\varphi(C_n) - \varphi(C_n')| + |\varphi(C_n') - D_n| \leq c_2|C_n - C_n'| + |D_n - D_n'| \\ &\leq c_3|\mathbf{M}(T_n) - \mathbf{M}(T_n')| \leq c_4\varepsilon_n\,, \end{aligned}$$

where $c_4 > 0$ is a constant independent of n. Moreover, $\mathbf{M}(W_n) = (2X_n - C_n, 2Y_n - D_n)$, since $\mathbf{M}(P_n)$ is a midpoint of the line segment $[\mathbf{M}(T_n), \mathbf{M}(W_n)]$. Therefore analogously like above, using the second inequality (11), we show that $|p_n(-1)| \leq c_4\varepsilon_n$.

The function p_n is convex on the interval $[-1, 1]$ and $p_0 = -\varepsilon_n < 0$, so $p_n(t) \leq 0$ for all t with

$$|t| \leq \frac{\varepsilon_n}{\varepsilon_n + c_4\varepsilon_n} = \frac{1}{1 + c_4}\,. \tag{12}$$

This means that the point $(1-t)\mathbf{M}(P_n) + t\,\mathbf{M}(T_n)$ belongs to $\mathcal{K} \cap \mathcal{M}$ for all real numbers t satisfying (12). Hence there is a line ℓ_n, passing through $\mathbf{M}(P_n)$ and contained in $\mathcal{M}$ for which $\rho(\mathbf{M}(P_n), \ell_n) \geq 1/(1+c_4)$, where $\rho(\mathbf{M}(P), \ell)$ is defined by (7). Therefore for all sufficiently large values of n,

$$|\mathbf{M}(P_{n+1}) - \mathbf{M}(P_n)| \geq \frac{1}{2(1 + c_4)}\,,$$

which cannot be satisfied if P_n converges. This contradiction yields $\mathbf{M}(Du) \in \partial\mathcal{K} \setminus \mathcal{N}$, which finishes the proof.

4. Application (the two well problem in dimension 2)

Here we apply Theorem 2.2 to prove existence of a solution to the two well problem in dimension 2. This problem was solved earlier by S. Müller and Šverák [14] using the convex integration method and by B. Dacorogna and P. Marcellini [2, 3] using the Baire category theorem. Let

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

where $\lambda\mu = 1$ and $\lambda, \mu > 0$. We prove that there exists a Lipschitz mapping u such that

$$Du \in SO(2) \cup SO(2)B$$

and $u(x) = Fx$ on $\partial\Omega$ for some $F \notin SO(2) \cup SO(2)B$.

Consider the manifold $\{\mathbf{M}(F) \in \mathbb{R}^{2\times 2}\times\mathbb{R} \,:\, \det F = 1\}$ in $W = \{(x_1, x_2, \ldots, x_5) \in \mathbb{R}^5 \,:\, x_5 = 1\}$. This manifold can be in a natural way identified with $\mathcal{M} = \{F \in \mathbb{R}^{2\times 2} \,:\, \det F = 1\}$ in $\mathbb{R}^4$. Moreover let $y = (y_1, y_2) \in \mathbb{R}^2$, $z = (z_1, z_2) \in \mathbb{R}^2$ and assume that

$$\mathcal{K} = \left\{ F(y,z) = \begin{pmatrix} y_1 & -y_2 \\ y_2 & y_1 \end{pmatrix} + \begin{pmatrix} \lambda z_1 & -\mu z_2 \\ \lambda z_2 & \mu z_1 \end{pmatrix} \,:\, |y| + |z| \le 1 \right\}.$$

Then $\mathcal{K}$ is a convex and compact subset of $\mathbb{R}^4$ and the boundary $\partial\mathcal{K}$ splits into two parts:

$$\partial\mathcal{K} = \Big(SO(2) \cup SO(2)B\Big) \cup \Big\{F(y,z) \in \mathcal{K} \,:\, |y| + |z| = 1 \text{ and } y, z \neq 0\Big\}.$$

We prove that $\mathcal{N} = \{F(y,z) \in \mathcal{K} \,:\, |y| + |z| = 1 \text{ and } y, z \neq 0\}$ is a regular rank-one part of $\partial\mathcal{K}$ with respect to $\mathcal{M}$. To see that (A2) is satisfied, we use the following argument used already by many authors to find the laminate convex hull of $SO(2) \cup SO(2)B$.

Fix $F = F(y,z) \in \mathcal{M} \cap \mathcal{N}$ and consider the line segment

$$G(t) = \frac{t}{|y|} \begin{pmatrix} y_1 & -y_2 \\ y_2 & y_1 \end{pmatrix} + \frac{1-t}{|z|} \begin{pmatrix} \lambda z_1 & -\mu z_2 \\ \lambda z_2 & \mu z_1 \end{pmatrix}.$$

It is contained in $\mathcal{N}$ and passes through $F = G(|y|)$. The function $f(t) = \det(G(t))$ is quadratic and $f(0) = f(1) = 1 = f(|y|)$, so $f(t) = 1$ for each $t \in \mathbb{R}$. Therefore $F = G(|y|)$ is the midpoint of a line segment contained in $\mathcal{M}\cap\mathcal{N}$, whose length is at least $2\cdot\min(|y|, |z|)\cdot\operatorname{dist}(SO(2), SO(2)B)$. This proves property (A2).

Now we show that (A1) is satisfied. Fix $P = F(y,z) \in \mathcal{M} \cap \mathcal{N}$. Since $\mathcal{N}$ and $\mathcal{M}$ are invariant under the left multiplication of an $SO(2)$ matrix, we may without loss of generality assume that $P = F(y,z)$ is such that $z_2 = 0$. Then $z_1 \neq 0$.

After a suitable linear change of variables the set $\mathcal{N}$ is transformed into

$$\mathcal{N}' = \{(y,z) \in \mathbb{R}^4 \,:\, |y| + |z| = 1 \text{ and } y, z \neq 0\}.$$

Computing the deteminant of F in terms of the variables (y, z) we see that $\mathcal{M}$ is transformed to

$$\mathcal{M}' = \{(y, z) \in \mathbb{R}^4 \ : \ |y|^2 + |z|^2 + (\lambda + \mu)(y_1 z_1 + y_2 z_2) = 1\} .$$

Let $G(y, z) = (|y| + |z| \ , \ |y|^2 + |z|^2 + (\lambda + \mu)(y_1 z_1 + y_2 z_2))$. To prove the property given in (A1) it suffieces to show that the matrix $DG(y, z)$ has maximal rank (equal to 2), whenever $(y, z) \in \mathcal{N}'$ and $z_2 = 0$.

We obtain

$$\begin{aligned} &DG(y, z) \\ &= \begin{pmatrix} y_1/|y| & y_2/|y| & z_1/|z| & z_2/|z| \\ 2y_1 + (\lambda + \mu)z_1 & 2y_2 + (\lambda + \mu)z_2 & 2z_1 + (\lambda + \mu)y_1 & 2z_2 + (\lambda + \mu)y_2 \end{pmatrix} \end{aligned}$$

If the determinant of the 2×2 minor composed of the first two columns of $DG(y, z)$ is equal to 0, then we have $y_2 z_1 = 0$ (since $z_2 = 0$), which gives $y_2 = 0$. Thus

$$DG(y_1, 0, z_1, 0) = \begin{pmatrix} y_1/|y_1| & 0 & z_1/|z_1| & 0 \\ 2y_1 + (\lambda + \mu)z_1 & 0 & 2z_1 + (\lambda + \mu)y_1 & 0 \end{pmatrix} .$$

If the matrix $DG(y_1, 0, z_1, 0)$ has rank less than 2, then $|2y_1 + (\lambda + \mu)z_1| = |2z_1 + (\lambda + \mu)y_1|$. This gives two possiblities $2y_1 + (\lambda + \mu)z_1 = 2z_1 + (\lambda + \mu)y_1$, or $2y_1 + (\lambda + \mu)z_1 + 2z_1 + (\lambda + \mu)y_1 = 0$, giving $y_1 = z_1$, or $y_1 = -z_1$, respectively (because $\lambda + \mu > 2$).

Now since $(y, z) \in \mathcal{N}'$ we have $|y| + |z| = 1$ giving $(y, z) = (\pm\frac{1}{2}, 0, \pm\frac{1}{2}, 0)$. But none of these four points belongs to $\mathcal{M}'$. This completes the proof of property (A1).

Noting that

$$\begin{aligned} &\mathcal{M} \cap (\operatorname{int}_W \mathcal{K}) \\ &= \left\{ F = \begin{pmatrix} y_1 & -y_2 \\ y_2 & y_1 \end{pmatrix} + \begin{pmatrix} \lambda z_1 & -\mu z_2 \\ \lambda z_2 & \mu z_1 \end{pmatrix} \ : \ |y| + |z| < 1 \text{ and } \det F = 1 \right\} \end{aligned}$$

we use Theorem 2.2 to conclude that if $F \in \mathcal{M} \cap (\operatorname{int}_W \mathcal{K})$, then there exists a Lipschitz mapping u defined on Ω such that $Du \in SO(2) \cup SO(2)B$ and $u(x) = Fx$ on $\partial\Omega$.

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