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Supremum Norms for 2-Homogeneous Polynomials on Circle Sectors

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We consider the Banach space of two homogeneous polynomials endowed with the supremum norm $\|\cdot\|_{D(\beta)}$ over circle sectors $D(\beta)$ of angle β for several values of $\beta \in [0, 2\pi]$. We provide an explicit formula for $\|\cdot\|_{D(\beta)}$, a full description of the extreme points of the corresponding unit balls, and a parametrization and a plot of their unit spheres. This work is an extension of a series of papers on the same topic published in the last decade and it has a number of applications to obtain polynomial-type inequalities.

Keywords: Bernstein and Markov inequalities, unconditional constants, polarization constants, polynomial inequalities, homogeneous polynomials, extreme points

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1. Preliminaries

Inequalities for polynomials on non-symmetric convex bodies have been studied several times in the past (see for instance the study of Bernstein and Markov type inequalities in [2, 15, 25, 27]). Recall that a convex body in a finite dimensional

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space is a compact convex set with nonempty interior. In [19] the authors use the geometry of the unit ball of the space $\mathcal{P}(^2\Delta)$ of 2-homogeneous polynomials on the simplex (the triangle Δ of vertices $(0,0)$, $(0,1)$ and $(1,0)$) in order to obtain sharp Bernstein and Markov type inequalities for such polynomials (see also [16, 17] for a study on the geometry of polynomials of degree at most 2 on Δ). Their results are based on a consequence of the Krein-Milman Theorem, namely, a convex function on the unit ball of $\mathcal{P}(^2\Delta)$ attains its maximum at the extreme points. The same technique is used in [6] to obtain several sharp polynomial inequalities in $\mathcal{P}(^2\square)$, the space of 2-homogeneous polynomials on the unit square (the square of vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$). The Krein-Milman approach has also been applied in order to obtain sharp polynomial inequalities for polynomials on Banach spaces (see for instance [13, 20, 21, 22, 23]). Very recently, this same approach has also been employed in order to obtain sharp estimates in the famous Bohnenblust-Hille inequality (see [18]).

The geometry of other polynomial spaces has been studied in depth many times in the past. For instance, Aron and Klimek studied quadratic polynomials [1], Muñoz-Fernández and Seoane-Sepúlveda extended Aron and Klimek's work to real trinomials [24], whereas Neuwirth considered complex trinomials [26]. Choi and Kim on the one hand, and Grecu on the other, treated 2-homogeneous real and complex polynomials on ℓ_p^2 for various values of p (we refer the interested reader to [4, 5, 7, 8, 9, 10, 11] for a thorough account on the previously mentioned works). Grecu, Muñoz-Fernández and Seoane-Sepúlveda also considered the case of 3-homogeneous polynomials on the complex ℓ_2^2 [12]. Also, Konheim and Rivlin provided a characterization of the extreme points of the unit ball in the space of real polynomials of degree at most n [14].

The purpose of this paper is to study the geometry of the space of homogeneous polynomials of degree 2 on a sector of the unit circle.

For each $\alpha, \beta \in [0, 2\pi]$ with $\alpha \leq \beta$ we define the sector $D(\alpha, \beta)$ as

$$D(\alpha, \beta) := \{re^{i\theta} : \ 0 \leq r \leq 1, \ \alpha \leq \theta \leq \beta\}.$$

When $\alpha = 0$, we replace $D(0, \beta)$ by $D(\beta)$ for simplicity. If P is a 2-homogeneous polynomial on $\mathbb{R}^2$, we define its norm $\|P\|_{D(\alpha,\beta)}$ by

$$\|P\|_{D(\alpha,\beta)} := \sup\{|P(\mathbf{x})| : \mathbf{x} \in D(\alpha, \beta)\}.$$

The space of 2-homogeneous polynomials on $\mathbb{R}^2$ endowed with the above norm will be represented by $\mathcal{P}(^2D(\alpha, \beta))$. The unit ball and the unit sphere of $\mathcal{P}(^2D(\alpha, \beta))$ will be represented by $\mathsf{B}_{D(\alpha,\beta)}$ and $\mathsf{S}_{D(\alpha,\beta)}$ respectively, whereas the extreme points of $\mathsf{B}_{D(\alpha,\beta)}$ will be denoted by $\mathrm{ext}(\mathsf{B}_{D(\alpha,\beta)})$. In order to visualize $\mathsf{S}_{D(\alpha,\beta)}$ and $\mathrm{ext}(\mathsf{B}_{D(\alpha,\beta)})$, we will often represent the 2-homogeneous polynomial $P(x, y) = ax^2 + by^2 + cxy$ by $(a, b, c) \in \mathbb{R}^3$, which allows us to identify $\mathcal{P}(^2D(\alpha, \beta))$ with $(\mathbb{R}^3, \|\cdot\|_{D(\alpha,\beta)})$, where $\|(a, b, c)\|_{D(\alpha,\beta)} = \|P\|_{D(\alpha,\beta)}$.

It is interesting to notice that the unit ball of $\mathcal{P}(^2D(\alpha, \alpha + \beta))$ can be inferred from the unit ball of $\mathcal{P}(^2D(0, \beta))$. Indeed, let

$$\Phi(x, y) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha),$$

for all $(x, y) \in \mathbb{R}^2$, i.e., Φ is the rotation of angle α in $\mathbb{R}^2$ with center at the origin. Notice that Φ transforms $D(0,\beta)$ onto $D(\alpha,\alpha+\beta)$, and hence

$$\|P\|_{D(\alpha,\alpha+\beta)} = \|P \circ \Phi\|_{D(0,\beta)},$$

for all 2-homogeneous polynomial P on $\mathbb{R}^2$. Moreover, the mapping

$$\begin{gathered} T : \mathcal{P}(^2D(\alpha,\alpha+\beta)) \to \mathcal{P}(^2D(\beta)) \\ P \mapsto P \circ \Phi \end{gathered}$$

defines an isometry between $\mathcal{P}(^2D(\alpha,\alpha+\beta))$ and $\mathcal{P}(^2D(\beta))$. The reader can check easily that T is defined by the matrix

$$\begin{pmatrix} \cos^2\alpha & \sin^2\alpha & \frac{\sin 2\alpha}{2} \\ \sin^2\alpha & \cos^2\alpha & -\frac{\sin 2\alpha}{2} \\ -\sin 2\alpha & \sin 2\alpha & \cos 2\alpha \end{pmatrix}.$$

This isometry allows us to restrict attention to the study of the geometry of $\mathsf{B}_{D(\beta)}$.

The case where $\beta \geq \pi$ is particularly easy to handle and will be treated in the first place in Section 2. When studying the case $0 < \beta < \pi$, one soon realizes that the four cases $0 < \beta \leq \frac{\pi}{4}$, $\frac{\pi}{4} < \beta \leq \frac{\pi}{2}$, $\frac{\pi}{2} < \beta \leq \frac{3\pi}{4}$ and $\frac{3\pi}{4} < \beta < \pi$ must be treated separately. In this paper we will study explicitly only the limiting cases $\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ due to the high complexity that arises when treating in depth the general case.

We provide a formula for $\|\cdot\|_{D(\beta)}$ for $\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ in Section 3, whereas the geometry of $\mathsf{B}_{D(\beta)}$ will be described in Sections 4, 5 and 6 for $\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ respectively.

2. The case $\beta \geq \pi$

Observe that for any homogeneous polynomial P on $\mathbb{R}^2$ we have that $|P(-x)| = |P(x)|$ for all $x \in \mathbb{R}^2$. A moment's thought reveals that in order to compute the supremum of $|P|$ over the unit disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ we just need to evaluate $|P|$ over one half (any) of the unit disk. Therefore, if we now let $\beta \geq \pi$ then $\mathsf{B}_{D(\beta)} = \mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$, where $\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$ stands for the closed unit ball of the space $\mathcal{P}(^2\ell_2^2)$ of 2-homogeneous polynomials on $\mathbb{R}^2$ endowed with the sup norm over the unit disk. The extreme points of $\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$ were described by Choi and Kim in [3]. We provide below an alternative approach:

Let us consider a 2-homogeneous polynomial $P(x, y) = ax^2 + by^2 + cxy$ on $\mathbb{R}^2$. By homogeneity, the supremum of $|P|$ over the closed unit disk is attained on the unit circle, and when restricted to the unit circle, P is given by

$$\begin{aligned} f(\theta) := P(\cos\theta, \sin\theta) &= a\cos^2\theta + b\sin^2\theta + c\sin\theta\cos\theta \\ &= a\frac{1+\cos 2\theta}{2} + b\frac{1-\cos 2\theta}{2} + c\frac{\sin 2\theta}{2} \\ &= \frac{1}{2}\left[a + b + (a-b)\cos 2\theta + c\sin 2\theta\right], \end{aligned}$$

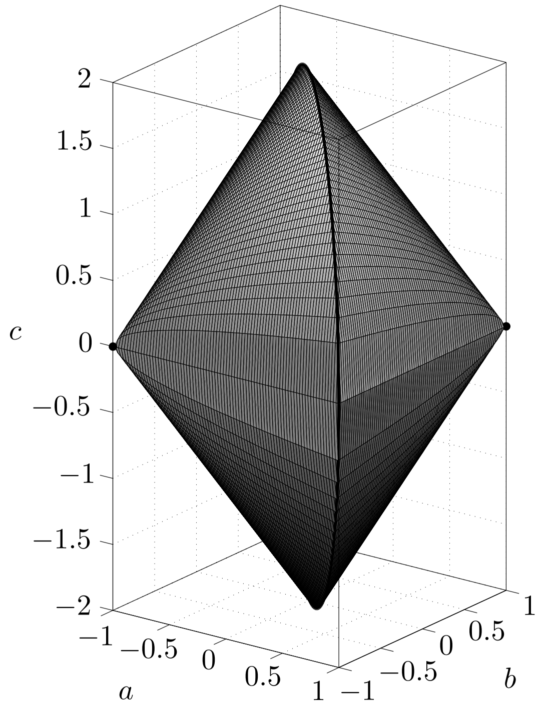

Figure 2.1: $\mathsf{S}_{\mathcal{P}(^2\ell_2^2)}$. The extreme points of $\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}$ are drawn with a thick line and dots

for $\theta \in [0, 2\pi]$. Hence, using an elementary duality argument

$$\begin{aligned} \|P\| &= \sup_{\theta\in[0,2\pi]} |f(\theta)| = \frac{1}{2}\left(|a+b| + \|(a-b,c)\|_2\right) \\ &= \frac{1}{2}\left(|a+b| + \sqrt{(a-b)^2+c^2}\right), \end{aligned}$$

where $\|\cdot\|_2$ stands for the Euclidean norm. The latter shows that the unit sphere of $\mathcal{P}(^2\ell_2^2)$ is given by

$$\mathsf{S}_{\mathcal{P}(^2\ell_2^2)} = \{(a,b,c) \in \mathbb{R}^3 : |a+b| + \sqrt{(a-b)^2+c^2} = 2\}.$$

Therefore, if we define

$$f(a,b) = 2\sqrt{1+ab-|a+b|},$$

for all $(a,b) \in [-1,1]^2$, then

$$\mathsf{S}_{\mathcal{P}(^2\ell_2^2)} = \mathrm{graph}(f) \cup \mathrm{graph}(-f).$$

The reader can find in Figure 2.1 a graph of $\mathsf{S}_{\mathcal{P}(^2\ell_2^2)}$.

Now let $a \in [-1,1]$ and consider the segment $\{(-1+\lambda(a+1), -1+\lambda(-a+1)) : \lambda \in [0,1]\}$ joining $(-1,-1)$ with the point $(a,-a)$. Then f (and therefore $-f$ too), restricted to that segment, namely

$$g(\lambda) := f(-1+\lambda(a+1), -1+\lambda(-a+1)) = 2\sqrt{1-a^2}\lambda,$$

with $\lambda \in [0,1]$, is linear. Similarly, f (and therefore $-f$ too) restricted to segments joining $(1,1)$ with points of the diagonal $b=-a$ of $[-1,1]^2$ are also linear. In other words, we have proved that graph(f) (and therefore graph($-f$) too) is the union of two ruled surfaces. This means that the extreme points of the closed unit ball of $\mathcal{P}(^2\ell_2^2)$, i.e., $\text{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_2^2)})$, must be contained in the set

$$\{(a,-a,\pm f(a,-a)) : a \in [-1,1]\} \cup \{\pm(1,1,0)\}.$$

It is easy to construct supporting planes to each of those points, revealing that they are indeed extreme. We conclude therefore that

$$\begin{aligned}\text{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_2^2)}) &= \{(a,-a,\pm\sqrt{1-a^2}) : a\in[-1,1]\} \cup \{\pm(1,1,0)\}\\ &= \{\pm(a,-a,\sqrt{1-a^2}) : a\in[-1,1]\} \cup \{\pm(1,1,0)\}.\end{aligned}$$

3. The supremum norm on $D(\beta)$ for $\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$.

We obtain first a formula for $\|ax^2+by^2+cxy\|_{D(\beta)}$ where $\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and $a,b,c \in \mathbb{R}$.

Theorem 3.1. *If $a,b,c\in\mathbb{R}$ and $P(x,y)=ax^2+by^2+cxy$ then $\|P\|_{D(\frac{\pi}{4})}$, $\|P\|_{D(\frac{\pi}{2})}$ and $\|P\|_{D(\frac{3\pi}{4})}$ are given, respectively, by*

$$\begin{cases}\max\left\{|a|, \frac{1}{2}|a+b+c|, \frac{1}{2}|a+b+\text{sign}(c)\sqrt{(a-b)^2+c^2}|\right\} & \text{if } c(a-b)\geq 0,\\ \max\{|a|, \frac{1}{2}|a+b+c|\} & \text{if } c(a-b)\leq 0,\end{cases}$$

$$\max\left\{|a|, |b|, \frac{1}{2}|a+b+\text{sign}(c)\sqrt{(a-b)^2+c^2}|\right\}, \quad \textit{and}$$

$$\begin{cases}\frac{1}{2}\left(|a+b|+\sqrt{(a-b)^2+c^2}\right) & \text{if } c(a-b)\geq 0,\\ \max\left\{|a|, \frac{1}{2}|a+b-c|, \frac{1}{2}|a+b+\text{sign}(c)\sqrt{(a-b)^2+c^2}|\right\} & \text{if } c(a-b)\leq 0.\end{cases}$$

Proof. The supremum of $|P|$ over $D(\beta)$ is obviously attained on the set $\{(\cos\theta, \sin\theta) : 0 \leq \theta \leq \beta\}$. When restricted to that set, as seen in the introduction, P is given by

$$f(\theta) = \frac{1}{2}\left[a+b+(a-b)\cos 2\theta + c\sin 2\theta\right],$$

for $\theta \in [0,\beta]$. Let

$$g(\theta) = \frac{1}{2}\left[a+b+(a-b)\cos\theta + c\sin\theta\right],$$

for $\theta\in[0,2\beta]$. Then obviously $\|P\|_{D(\beta)} = \sup_{\theta\in[0,2\beta]}|g(\theta)|$. Observe that the two lines appearing in the first and the third formulas coincide when $c(a-b)=0$. The study of this simpler case is left to the reader. If now $c(a-b)\neq 0$, then g' vanishes at the points θ_0 such that $\tan\theta_0 = \frac{c}{a-b}$.

Assume first that $\beta = \frac{\pi}{4}$. Then $\tan\theta = \frac{c}{a-b}$ has exactly one root $\theta_0 \in [0, \frac{\pi}{2}]$ if and only if $\frac{c}{a-b} > 0$, in which case

$$\cos\theta_0 = \frac{1}{\sqrt{1+\frac{c^2}{(a-b)^2}}} \quad \text{and} \quad \sin\theta_0 = \frac{\frac{c}{a-b}}{\sqrt{1+\frac{c^2}{(a-b)^2}}}.$$

Hence

$$\begin{aligned}
g(\theta_0) &= \frac{1}{2}\left[a+b+(a-b)\frac{1}{\sqrt{1+\frac{c^2}{(a-b)^2}}} + c\frac{\frac{c}{a-b}}{\sqrt{1+\frac{c^2}{(a-b)^2}}}\right] \\
&= \frac{1}{2}\left[a+b+\frac{(a-b)|a-b|}{\sqrt{c^2+(a-b)^2}} + \frac{c^2\operatorname{sign}(a-b)}{\sqrt{c^2+(a-b)^2}}\right] \\
&= \frac{1}{2}\left[a+b+\operatorname{sign}(a-b)\frac{(a-b)^2+c^2}{\sqrt{c^2+(a-b)^2}}\right] \\
&= \frac{1}{2}\left[a+b+\operatorname{sign}(c)\sqrt{c^2+(a-b)^2}\right].
\end{aligned}$$

This, together with the fact that $g(0) = a$ and $g(\frac{\pi}{2}) = \frac{1}{2}(a+b+c)$, prove the first formula.

Suppose that now $\beta = \frac{\pi}{2}$. Then $\tan\theta = \frac{c}{a-b}$ has exactly one solution $\theta_0 \in [0, \pi]$. If $c(a-b) > 0$ then $\theta_0 \in [0, \frac{\pi}{2}]$ and we have already seen that $g(\theta_0) = \frac{1}{2}\left[a+b+\operatorname{sign}(c)\sqrt{c^2+(a-b)^2}\right]$. If $c(a-b) < 0$ then $\theta_0 \in [\frac{\pi}{2}, \pi]$ and

$$\cos\theta_0 = -\frac{1}{\sqrt{1+\frac{c^2}{(a-b)^2}}} \quad \text{and} \quad \sin\theta_0 = -\frac{\frac{c}{a-b}}{\sqrt{1+\frac{c^2}{(a-b)^2}}}.$$

The reader can check, as above, that in this case we obtain again

$$g(\theta_0) = \frac{1}{2}\left[a+b+\operatorname{sign}(c)\sqrt{c^2+(a-b)^2}\right].$$

This, together with the fact that $g(0) = a$ and $g(\pi) = b$, prove the second formula.

Finally, assume that $\beta = \frac{3\pi}{4}$. If $c(a-b) > 0$ then $\tan\theta = \frac{c}{a-b}$ has exactly two roots in $[0, \frac{3\pi}{2}]$, one θ_1 in $[0, \frac{\pi}{2}]$ and the other one θ_2 in $[\pi, \frac{3\pi}{2}]$. Notice that $\theta_2 - \theta_1 = \pi$, from which

$$\begin{aligned}
\max\{|g(\theta_1)|, |g(\theta_2)|\} &= \max\left\{\frac{1}{2}\left|a+b+\epsilon\operatorname{sign}(c)\sqrt{c^2+(a-b)^2}\right| : \epsilon = \pm 1\right\} \\
&= \frac{1}{2}\left(|a+b| + \sqrt{c^2+(a-b)^2}\right).
\end{aligned}$$

Observe that $\frac{1}{2}\left(|a+b|+\sqrt{c^2+(a-b)^2}\right)$ is actually $\|P\|_{D(\pi)}$, as we saw in Section 2, and therefore $\|P\|_{D(\frac{3\pi}{4})} = \frac{1}{2}\left(|a+b|+\sqrt{c^2+(a-b)^2}\right)$ if $c(a-b) > 0$. On the other hand, if $c(a-b) < 0$, then $\tan\theta = \frac{c}{a-b}$ has exactly one solution $\theta_0 \in [0, \frac{3\pi}{2}]$. In fact $\theta_0 \in [\frac{\pi}{2}, \pi]$. As we did to prove the second formula,

$$g(\theta_0) = \frac{1}{2}\left[a+b+\operatorname{sign}(c)\sqrt{c^2+(a-b)^2}\right].$$

Considering that now $g(0) = a$ and $g(\frac{3\pi}{2}) = \frac{1}{2}(a+b-c)$, the third formula follows. □

4. The geometry of $\mathsf{B}_{D(\frac{\pi}{4})}$

The study of this case requires the following technical result:

Lemma 4.1. *Let us define P_1 and P_2 by*

$$P_1 := \{(a,b) \in \mathbb{R}^2 : \ a \geq -1 \ \textit{and}\ 4+a-4\sqrt{1+a} \leq b \leq 4+a+4\sqrt{1+a}\},$$
$$P_2 := \{(a,b) \in \mathbb{R}^2 : \ a \leq 1 \ \textit{and}\ -4+a-4\sqrt{1-a} \leq b \leq -4+a+4\sqrt{1-a}\}.$$

Then:

(*a*) *The inequality*

$$\frac{1}{2}\left|a+b-\sqrt{2\left[(a-1)^2+(b-1)^2\right]}\right| \leq 1, \tag{1}$$

holds if and only if $(a,b) \in P_1$.

(*b*) *The inequality*

$$\frac{1}{2}\left|a+b+2\sqrt{(1-a)(1-b)}\right| \leq 1, \tag{2}$$

holds if and only if $(a,b) \in P_2$.

Proof. In order to prove (*a*) let $h(a,b) = \frac{1}{2}\left|a+b-\sqrt{2\left[(a-1)^2+(b-1)^2\right]}\right|$ for $(a,b) \in \mathbb{R}^2$. Then $h(a,b) = 1$ holds either if

$$a+b-\sqrt{2\left[(a-1)^2+(b-1)^2\right]} = 2, \tag{3}$$

or

$$a+b-\sqrt{2\left[(a-1)^2+(b-1)^2\right]} = -2. \tag{4}$$

We have that (3) can only happen if $a+b-2 \geq 0$, in which case the reader can show easily by squaring the equality $a+b-2 = \sqrt{2\left[(a-1)^2+(b-1)^2\right]}$ that $a = b$. This shows that $h \equiv 1$ on the set $R := \{(a,a) : a \geq 1\}$. On the other hand, (4) occurs only if $a+b+2 \geq 0$, and in that case, by squaring the equality $a+b+2 = \sqrt{2\left[(a-1)^2+(b-1)^2\right]}$ we obtain easily that $b = 4+a\pm\sqrt{1+a}$ with

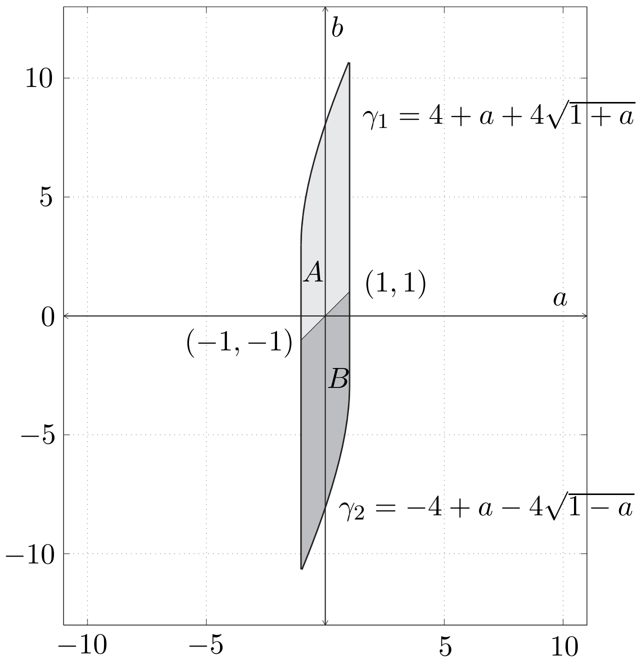

Figure 4.1: Projection of $\mathsf{S}_{D(\frac{\pi}{4})}$ onto the ab-plane

$a \geq -1$. Observe that the curve described by the equation $b = 4 + a \pm \sqrt{1+a}$ with $a \geq -1$ is in the halfplane $a + b + 2 \geq 0$. Hence we have proved so far that the set of zeros of $h - 1$ is

$$Z := R \cup \{(a, b) : a \geq -1,\ b = 4 + a \pm \sqrt{1+a}\}.$$

Now let

$$C_1 := \left\{(a, b) :\ a \geq -1,\, b > 4 + a + 4\sqrt{1+a},\, b < 4 + a - 4\sqrt{1+a}\right\}$$

and

$$C_2 := \left\{(a, b) :\ a \geq -1,\, 4 + a - 4\sqrt{1+a} < b < 4 + a + 4\sqrt{1+a}\right\} \setminus R$$

be the two connected components of $\mathbb{R}^2 \setminus Z$. By continuity, $h - 1$ preserves its sign on both C_1 and C_2. Since it is straightforward to check that $h - 1$ is positive on C_1 and negative on C_2, and $P_1 = C_2 \cup Z$, the result follows.

We proceed in exactly the same manner to prove (b). The calculations are left to the reader. □

Theorem 4.2. *Let A and B be as in Figure 4.1, namely*

$$A = \{(a, b) :\ a \in [-1, 1],\ a < b \leq \gamma_1(a)\},$$
$$B = \{(a, b) :\ a \in [-1, 1],\ \gamma_2(a) \leq b \leq a\},$$

where γ_1, γ_2 are defined by

$$\gamma_1(a) = 4 + a + 4\sqrt{1+a},$$
$$\gamma_2(a) = -\gamma_1(-a) = -4 + a - 4\sqrt{1-a},$$

for $a \in [-1, 1]$. Then, the projection of $\mathsf{S}_{D(\frac{\pi}{4})}$ over the ab-plane is

$$\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})}) = \{(a, b) : \ a \in [-1, 1], \ \gamma_2(a) \le b \le \gamma_1(a)\}.$$

Proof. Let $(a, b) \in A$ and set $c = 2 - a - b$. Suppose first that $2 - a - b > 0$. Then $c(a - b) < 0$, and therefore, according to the first formula in Theorem 3.1 we have

$$\|(a, b, 2 - a - b)\|_{D(\frac{\pi}{4})} = \max\left\{|a|, \frac{1}{2}|a + b + c|\right\} = \max\{|a|, 1\} = 1.$$

On the other hand, if $2 - a - b \le 0$, then $c(a - b) \ge 0$. Applying now the first formula in Theorem 3.1 again we obtain

$$\begin{aligned}\|(a, b, 2 - a - b)\|_{D(\frac{\pi}{4})} &= \max\left\{|a|, 1, \frac{1}{2}\left|a + b - \sqrt{(a - b)^2 + (2 - a - b)^2}\right|\right\}\\ &= \max\left\{|a|, 1, \frac{1}{2}\left|a + b - \sqrt{2[(a - 1)^2 + (b - 1)^2]}\right|\right\}.\end{aligned}$$

Recall that here (a, b) satisfies $|a| \le 1$ and $2 - a \le b \le 4 + a + 4\sqrt{1+a}$. Since $4 + a - 4\sqrt{1+a} \le 2 - a$ for $-1 \le a \le 1$, it follows from Lemma 4.1, part (a), that

$$\frac{1}{2}\left|a + b - \sqrt{2[(a - 1)^2 + (b - 1)^2]}\right| \le 1,$$

proving that $\|(a, b, 2 - a - b)\|_{D(\frac{\pi}{4})} = 1$.

Now assume that $(a, b) \in B$ and define $c = 2\sqrt{(1 - a)(1 - b)}$. Notice that if $(a, b) \in B$ then $2 - a - b \ge 0$, from which

$$\begin{aligned}\frac{1}{2}\left|a + b + \operatorname{sign}(c)\sqrt{(a - b)^2 + c^2}\right| &= \frac{1}{2}\left|a + b + \sqrt{(a - b)^2 + 4(1 - a)(1 - b)}\right|\\ &= \frac{1}{2}\left|a + b + \sqrt{(2 - a - b)^2}\right| = 1.\end{aligned}$$

In this case $c(a - b) \ge 0$, so using the first formula in Theorem 3.1 once again, we arrive at

$$\left\|\left(a, b, 2\sqrt{(1 - a)(1 - b)}\right)\right\|_{D(\frac{\pi}{4})} = \max\left\{|a|, \frac{1}{2}\left|a + b + 2\sqrt{(1 - a)(1 - b)}\right|, 1\right\}.$$

However, under the given conditions, Lemma 4.1, part (b) asserts that

$$\frac{1}{2}\left|a + b + 2\sqrt{(1 - a)(1 - b)}\right| \le 1,$$

which implies that $\left\|\left(a,b,2\sqrt{(1-a)(1-b)}\right)\right\|_{D(\frac{\pi}{4})}=1$.

We have proved so far that

$$\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})})\supset\{(a,b):\ a\in[-1,1],\ \gamma_2(a)\le b\le\gamma_1(a)\}.$$

On the other hand, if $(a,b)\notin\{(a,b):\ a\in[-1,1],\ \gamma_2(a)\le b\le\gamma_1(a)\}$ then either $|a|>1$ or one of the inequalities $\gamma_2(a)\le b\le\gamma_1(a)$ fail with $-1\le a\le 1$. In the first case we would have $\|(a,b,c)\|_{D(\frac{\pi}{4})}\ge|a|>1$ for all $c\in\mathbb{R}$, i.e., $(a,b)\notin\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})})$. Now if $|a|\le 1$ and $\gamma_1(a)<b$, then $a-b<0$. If $c\le 0$, from the first formula in Theorem 3.1 we would have

$$\|(a,b,c)\|_{D(\frac{\pi}{4})}\ge\max\left\{\frac{1}{2}|a+b+c|,\frac{1}{2}\left|a+b-\sqrt{(a-b)^2+c^2}\right|\right\}.$$

It can be easily checked that $\frac{1}{2}|a+b+c|>1$ if $0\ge c>2-a-b$ and $\frac{1}{2}\left|a+b-\sqrt{(a-b)^2+c^2}\right|>1$ if $c\le 2-a-b$. This last statement follows from the fact that $\frac{1}{2}\left|a+b-\sqrt{(a-b)^2+c^2}\right|>1$ is equivalent to the expression $c^2>4(1+a)(1+b)$ and, in this case, $c^2\ge(2-a-b)^2>4(1+a)(1+b)$.

Now if $c>0$, Theorem 3.1 allows us to show that

$$\|(a,b,c)\|_{D(\frac{\pi}{4})}\ge\frac{1}{2}\,|a+b+c|=\frac{1}{2}(a+b+c)>2+a+2\sqrt{1+a}\ge 1,$$

whenever $-1\le a\le 1$. In any case $(a,b)\notin\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})})$.

Finally, using the symmetry of the unit ball and the previous case, if $\gamma_2(a)>b$ with $-1\le a\le 1$ then $\|(a,b,c)\|_{D(\frac{\pi}{2})}>1$ for all $c\in\mathbb{R}$, from which $(a,b)\notin\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})})$. This concludes the proof. □

Remark 4.3. In the proof of the first part of the following result, it will be useful to observe that

$$\frac{1}{2}\,|b+c-1|\le 1\Leftrightarrow -1-b\le c\le 3-b.$$

Also, if $b\le -1$, then

$$\frac{1}{2}\left|b-1+\sqrt{(b+1)^2+c^2}\right|\le 1\Leftrightarrow |c|\le 3-b.$$

Theorem 4.4. *Let A and B be as in Theorem 4.2 (see also Figure 4.1) and define*

$$F_1(a,b)=\begin{cases}2-a-b & \text{if } (a,b)\in A,\\ 2\sqrt{(1-a)(1-b)} & \text{if } (a,b)\in B,\end{cases}$$

and $F_2(a,b)=-F_1(-a,-b)$ for all $(a,b)\in\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})})$. If

$$\Gamma=\{(\pm1,b,c)\in\mathbb{R}^2:\ (\pm1,b)\in\partial\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})}),\ F_2(\pm1,b)\le c\le F_1(\pm1,b)\},$$

then

(*a*) $\mathsf{S}_{D(\frac{\pi}{4})} = \mathrm{graph}(F_1) \cup \mathrm{graph}(F_2) \cup \Gamma$.

(*b*) *The set* $\mathrm{ext}(\mathsf{B}_{D(\frac{\pi}{4})})$ *consists of the elements*

$$\pm\left(t, 4+t+4\sqrt{1+t}, -2-2t-4\sqrt{1+t}\right) \quad \text{for } t \in [-1,1],$$

$$\pm\left(1, s, -2\sqrt{2(1+s)}\right) \quad \text{for } s \in [1, 5+4\sqrt{2}],$$

and

$$\pm(1,1,0).$$

Proof. As for the first part of the theorem, notice that $\mathrm{graph}(F_1) \subset \mathsf{S}_{D(\frac{\pi}{4})}$ as seen in the proof of Theorem 4.2. By symmetry we also have that $\mathrm{graph}(F_2) \subset \mathsf{S}_{D(\frac{\pi}{4})}$. Finally $\Gamma \subset \mathsf{S}_{D(\frac{\pi}{4})}$ too. Indeed, by symmetry we can focus on the study of the points of Γ of the form $(-1,b,c)$ with $-5-4\sqrt{2} \le b \le 3$ and $F_2(-1,b) \le c \le F_1(-1,b)$. Observe that

$$F_1(-1,b) = \begin{cases} 2\sqrt{2(1-b)} & \text{if } -5-4\sqrt{2} \le b \le -1, \\ 3-b & \text{if } -1 \le b \le 3, \end{cases}$$

and

$$F_2(-1,b) = \begin{cases} -1-b & \text{if } -5-4\sqrt{2} \le b \le -1, \\ 0 & \text{if } -1 \le b \le 3. \end{cases}$$

Hence $c \ge F_2(-1,b) \ge 0$ for all $b \in [-5-4\sqrt{2}, 3]$ and $c = 0$ can only be zero when $-3 \le b \le 1$. In that case, since $\frac{1}{2}|b-1| \le 1$, it follows from Theorem 3.1 that

$$\|(-1,b,0)\|_{D(\frac{\pi}{4})} = \max\left\{1, \frac{1}{2}|b-1|\right\} = 1.$$

Otherwise $c > 0$. First, if $-5-4\sqrt{2} \le b \le -1$ then $c(-1-b) \ge 0$. Therefore using Theorem 3.1 we have

$$\|(-1,b,c)\|_{D(\frac{\pi}{4})} = \max\left\{1, \frac{1}{2}|b+c-1|, \frac{1}{2}\left|b-1+\sqrt{(b+1)^2+c^2}\right|\right\}.$$

Since $-5-4\sqrt{2} \le b \le -1$ and $-3-b \le c \le 2\sqrt{2(1-b)} \le 3-b$, from Remark 4.3 it follows that $\|(-1,b,c)\|_{D(\frac{\pi}{4})} = 1$. Finally, if $-1 < b \le 3$ then $c(-1-b) < 0$, which implies, from Theorem 3.1 that

$$\|(-1,b,c)\|_{D(\frac{\pi}{4})} = \max\left\{1, \frac{1}{2}|b+c-1|\right\}.$$

Since now $-1 \le b \le 3$ and $-3-b \le 0 < c \le -1-b$, from Remark 4.3 we have that $\|(-1,b,c)\|_{D(\frac{\pi}{4})} = 1$ too.

We have proved so far that $\mathrm{graph}(F_1) \cup \mathrm{graph}(F_2) \cup \Gamma \subset \mathsf{S}_{D(\frac{\pi}{4})}$. On the other hand, suppose $(a,b,c) \notin \mathrm{graph}(F_1) \cup \mathrm{graph}(F_2) \cup \Gamma$. Obviously, $(0,0,0) \notin \mathsf{S}_{D(\frac{\pi}{4})}$,

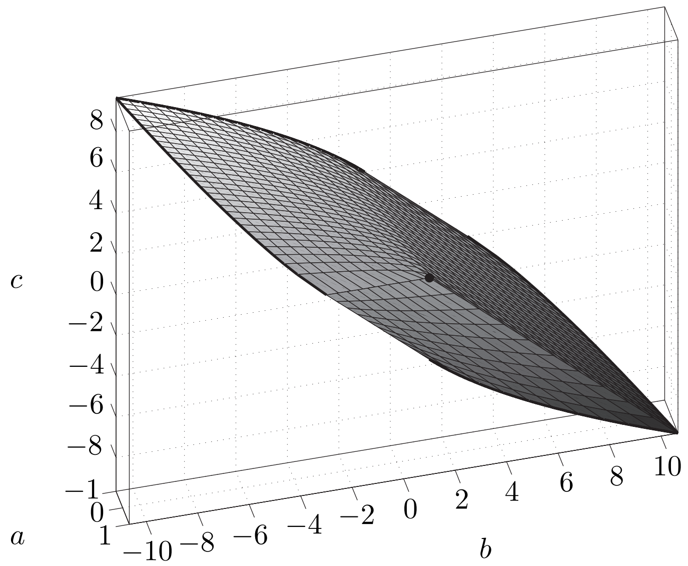

Figure 4.2: $\mathsf{S}_{D(\frac{\pi}{4})}$. The extreme points of $\mathsf{B}_{D(\frac{\pi}{4})}$ are drawn with a thicker line and dots

so we can also assume that $(a,b,c) \neq (0,0,0)$. The straight line $\{\lambda(a,b,c) : \lambda \in \mathbb{R}\}$ certainly meets the set graph$(F_1) \cup$ graph$(F_2) \cup \Gamma$ as the reader can easily verify. Put, $(a,b,c) = \lambda_0(a_0,b_0,c_0)$ with $\lambda_0 \neq 0,1$ and $(a_0,b_0,c_0) \in$ graph$(F_1) \cup$ graph$(F_2) \cup \Gamma$. Then $\|(a,b,c)\|_{D(\frac{\pi}{4})} = \lambda_0 \|(a_0,b_0,c_0)\|_{D(\frac{\pi}{4})} = \lambda_0 \neq 1$, and therefore $(a,b,c) \notin \mathsf{S}_{(\frac{\pi}{4})}$. This concludes part (a).

Part (b) is an easy consequence of the fact that the sets graph(F_1), graph(F_2) and Γ are ruled surfaces in $\mathbb{R}^3$. Actually Γ is contained in the two planes $a = \pm 1$. Hence the only possible extreme points are contained in the intersections graph$(F_1) \cap \Gamma$, graph$(F_2) \cap \Gamma$ and graph$(F_1) \cap$ graph(F_2). Once we have removed the interior of all straight lines contained in those intersections, we end up with the following candidates to be extreme points of $\mathsf{B}_{D(\frac{\pi}{4})}$:

$$\pm\left(t, 4 + t + 4\sqrt{1+t}, -2 - 2t - 4\sqrt{1+t}\right) \quad \text{for } t \in [-1,1],$$
$$\pm\left(1, s, -2\sqrt{2(1+s)}\right) \quad \text{for } s \in [1, 5 + 4\sqrt{2}],$$

and

$$\pm(1,1,0).$$

Constructing a supporting hyperplane to each of the previous points is a straightforward exercise that is left to the interested reader, which finishes the proof. $\square$

The reader can find a sketch of $\mathsf{S}_{D(\frac{\pi}{4})}$ in Figure 4.2.

5. The geometry of $\mathsf{B}_{D(\frac{\pi}{2})}$

Theorem 5.1. *The projection of* $\mathsf{S}_{D(\frac{\pi}{2})}$ *over the ab-plane is* $[-1,1]^2$.

Proof. Using the second formula in Theorem 3.1 we have that

$$\|(a,b,c)\|_{D(\frac{\pi}{2})} \geq \max\{|a|,|b|\},$$

for all $(a,b,c) \in \mathbb{R}^3$. This shows right away that $\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{2})}) \subset [-1,1]^2$. In order to prove that $[-1,1]^2 \subset \pi_{ab}(\mathsf{S}_{D(\frac{\pi}{2})})$ take $(a,b) \in \mathbb{R}^2$ so that $|a| \leq 1$ and $|b| \leq 1$. Then, using again the same formula, one shows easily that

$$\begin{aligned}
&\left\|\left(a,b,2\sqrt{(1-a)(1-b)}\right)\right\|_{D(\frac{\pi}{2})} \\
&= \max\left\{|a|,|b|,\frac{1}{2}\left|a+b+\sqrt{(a-b)^2+4(1-a)(1-b)}\right|\right\} \\
&= \max\left\{|a|,|b|,\frac{1}{2}\left|a+b+\sqrt{(2-a-b)^2}\right|\right\} \\
&= \max\{|a|,|b|,1\} = 1.
\end{aligned}$$

Observe that, similarly,

$$\left\|\left(a,b,-2\sqrt{(1+a)(1+b)}\right)\right\|_{D(\frac{\pi}{2})} = 1.$$

This concludes the proof. □

Theorem 5.2. *If we define the mappings*

$$G_1(a,b) = 2\sqrt{(1-a)(1-b)}$$

and

$$G_2(a,b) = -G_1(-a,-b) = -2\sqrt{(1+a)(1+b)},$$

for every $(a,b) \in [-1,1]^2$ *and the set*

$$\Upsilon = \{(a,b,c) \in \mathbb{R}^3 : (a,b) \in \partial[-1,1]^2 \text{ and } G_2(a,b) \leq c \leq G_1(a,b)\},$$

where $\partial[-1,1]^2$ *is the boundary of* $[-1,1]^2$*, then*

(*a*) $\mathsf{S}_{D(\frac{\pi}{2})} = \mathrm{graph}(G_1) \cup \mathrm{graph}(G_2) \cup \Upsilon$.

(*b*) *The set* $\mathrm{ext}(\mathsf{B}_{D(\frac{\pi}{2})})$ *consists of the elements*

$$\pm\left(1,t,-2\sqrt{2(1+t)}\right), \quad \pm\left(t,1,-2\sqrt{2(1+t)}\right) \quad \text{and} \quad \pm(1,1,0),$$

with $t \in [-1,1]$.

Proof. As for part (*a*), we have already seen in the proof of Theorem 5.1 that

$$\mathrm{graph}(G_1) \cup \mathrm{graph}(G_2) \subset \mathsf{S}_{(\frac{\pi}{2})}.$$

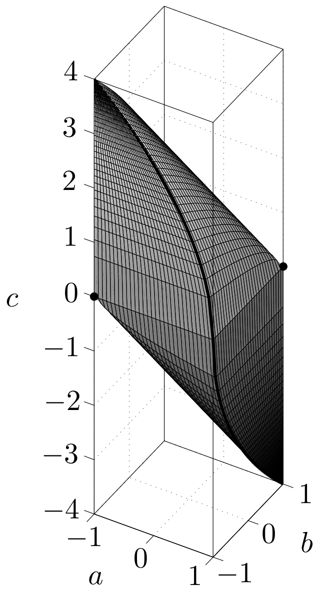

Figure 5.1: $\mathsf{S}_{D(\frac{\pi}{2})}$. The extreme points of $\mathsf{B}_{D(\frac{\pi}{2})}$ are drawn with a thick line and dots.

Also, if $(a,b,c) \in \Upsilon$, then $G_2(a,b) \leq c \leq G_1(a,b)$ with $\max\{|a|,|b|\} = 1$. Suppose $a = -1$. Then $0 \leq c \leq 2\sqrt{2(1-b)}$ with $|b| \leq 1$ and

$$\begin{aligned}\frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right| &\leq \frac{1}{2}\left|-1+b+\sqrt{(1+b)^2+8(1-b)}\right| \\ &= \frac{1}{2}\left|-1+b+\sqrt{(3-b)^2}\right| = 1.\end{aligned}$$

Hence using the second formula in Theorem 3.1 we see that $\|(-1,b,c)\|_{D(\frac{\pi}{2})} = 1$. It is left to the reader, as a simple exercise, to prove that the same holds if $a = 1$ or $|b| = 1$. Therefore

$$\operatorname{graph}(G_1) \cup \operatorname{graph}(G_2) \cup \Upsilon \subset \mathsf{S}_{(\frac{\pi}{2})}.$$

On the other hand, suppose $(a,b,c) \notin \operatorname{graph}(G_1) \cup \operatorname{graph}(G_2) \cup \Upsilon$. Obviously, $(0,0,0) \notin \mathsf{S}_{D(\frac{\pi}{2})}$, so we can also assume that $(a,b,c) \neq (0,0,0)$. The straight line $\{\lambda(a,b,c) : \lambda \in \mathbb{R}\}$ certainly meets the set $\operatorname{graph}(G_1) \cup \operatorname{graph}(G_2) \cup \Upsilon$ as the reader can easily verify. Put, $(a,b,c) = \lambda_0(a_0,b_0,c_0)$ with $\lambda_0 \neq 0,1$ and $(a_0,b_0,c_0) \in \operatorname{graph}(G_1) \cup \operatorname{graph}(G_2) \cup \Upsilon$. Then $\|(a,b,c)\|_{D(\frac{\pi}{2})} = \lambda_0\|(a_0,b_0,c_0)\|_{D(\frac{\pi}{2})} = \lambda_0 \neq 1$, and therefore $(a,b,c) \notin \mathsf{S}_{(\frac{\pi}{2})}$. This concludes part (a).

Part (b) is an easy consequence of the fact that the sets $\operatorname{graph}(G_1)$, $\operatorname{graph}(G_2)$ and Υ are ruled surfaces in $\mathbb{R}^3$. Actually Υ is contained in the four planes $a = \pm 1$ and $b = \pm 1$. Using this with the help of Figure 5.1 the reader will soon arrive at the conclusion that the only possible extreme points are contained in the intersections $\operatorname{graph}(G_1) \cap \Upsilon$, $\operatorname{graph}(G_2) \cap \Upsilon$ and $\operatorname{graph}(G_1) \cap \operatorname{graph}(G_2)$. The union of these three sets consists of the elements

$$\pm\left(1, t, -2\sqrt{2(1+t)}\right), \quad \pm\left(t, 1, -2\sqrt{2(1+t)}\right), \quad (\pm 1, t, 0), \quad \text{and} \quad (t, \pm 1, 0),$$

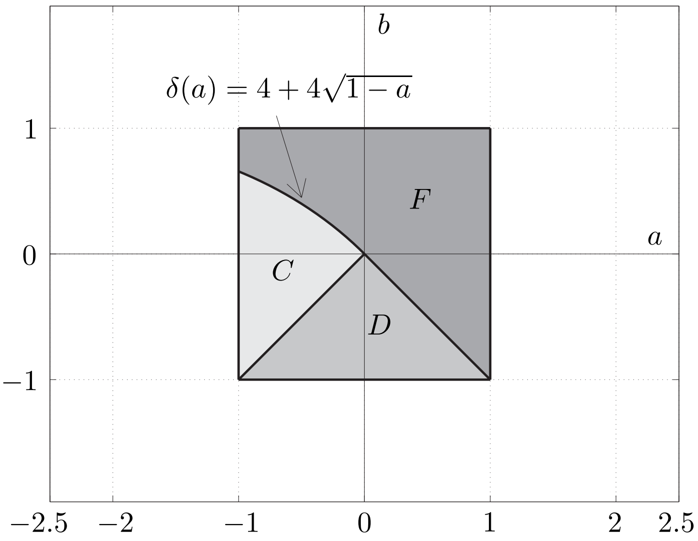

Figure 6.1: Projection of $\mathsf{S}_{D(\frac{3\pi}{4})}$ on the ab plane.

with $t \in [-1, 1]$. Observe that $(\pm 1, t, 0)$ and $(t, \pm 1, 0)$ with $|t| < 1$ cannot obviously be extreme and the point $\pm(1, -1, 0)$ are already considered in the curves $\left(1, t, -2\sqrt{2(1+t)}\right)$, $\pm\left(t, 1, -2\sqrt{2(1+t)}\right)$ with $t \in [-1, 1]$. Hence the candidates to be extreme points of $\mathsf{B}_{D(\frac{\pi}{2})}$ reduce to

$$\pm\left(1, t, -2\sqrt{2(1+t)}\right), \quad \pm\left(t, 1, -2\sqrt{2(1+t)}\right) \quad \text{and} \quad \pm(1, 1, 0),$$

with $t \in [-1, 1]$. Constructing a supporting hyperplane to each of the previous points is a straightforward exercise that is left to the interested reader, which finishes the proof. □

The reader can see a sketch of $\mathsf{S}_{D(\frac{\pi}{2})}$ in Figure 5.1.

6. The geometry of $\mathsf{B}_{D(\frac{3\pi}{4})}$

Let C, D and F as in Figure 6.1, namely

$$\begin{aligned} C &:= \{(a, b) : \ a \in [-1, 0], \ 0 < b \leq \ \delta(a)\}, \\ D &:= \{(a, b) : \ a \in [-1, 1], \ -1 \leq b \leq -|a|\}, \\ F &:= [-1, 1]^2 \backslash (C \cup D), \end{aligned}$$

where $\delta(a) = -4 + a + 4\sqrt{1-a}$ for $a \in [-1, 0]$. We show first that $\pi_{ab}(\mathsf{S}_{D(\frac{3\pi}{4})})$ is the union of C, D and F.

Theorem 6.1. *The projection of* $\mathsf{S}_{D(\frac{3\pi}{4})}$ *over the ab-plane is* $[-1, 1]^2$.

Proof. Let $(a,b) \in C$ and consider the point $(a,b,2+a+b)$. Then using Theorem 3.1 and the fact that $2+a+b > 0$ and $(a-b)(2+a+b) < 0$, we have

$$\|(a,b,2+a+b)\|_{D(\frac{3\pi}{4})} = \max\left\{|a|, 1, \frac{1}{2}\left|a+b+\sqrt{2[(1+a)^2+(1+b)^2]}\right|\right\}.$$

Now notice that $-C \subset P_1$, where P_1 was defined in Lemma 4.1. Hence

$$\frac{1}{2}\left|a+b+\sqrt{2[(1+a)^2+(1+b)^2]}\right| \leq 1,$$

which shows that $\|(a,b,2+a+b)\|_{D(\frac{3\pi}{4})} = 1$.

If $(a,b) \in D$ then we put $c = 2\sqrt{(1+a)(1+b)}$. Observe that $(a-b)c \geq 0$ and $a+b \leq 0$, so using again Theorem 3.1 we have

$$\begin{aligned}\|(a,b,c)\|_{D(\frac{3\pi}{4})} &= \frac{1}{2}\left(|a+b| + \sqrt{(a-b)^2+c^2}\right)\\ &= \frac{1}{2}\left(-a-b+\sqrt{(2+a+b)^2}\right) = 1.\end{aligned}$$

Finally, let $(a,b) \in F$ and put $c = 2\sqrt{(1-a)(1-b)}$. If $a-b \geq 0$ then Theorem 3.1 yields

$$\begin{aligned}\|(a,b,c)\|_{D(\frac{3\pi}{4})} &= \frac{1}{2}\left(|a+b| + \sqrt{(a-b)^2+c^2}\right)\\ &= \frac{1}{2}\left(a+b+\sqrt{(2-a-b)^2}\right) = 1.\end{aligned}$$

If, on the contrary, $a-b < 0$, then, using Theorem 3.1 once more yields

$$\|(a,b,c)\|_{D(\frac{3\pi}{4})} = \max\left\{|a|, \frac{1}{2}|a+b-c|, \frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right|\right\}.$$

However

$$\frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right| = \frac{1}{2}\left|a+b+\sqrt{(2-a-b)^2}\right| = 1$$

and

$$\frac{1}{2}|a+b-c| = \frac{1}{2}\left|a+b-2\sqrt{(1-a)(1-b)}\right| \leq 1,$$

proving that $\|(a,b,c)\|_{D(\frac{3\pi}{4})} = 1$.

We have shown so far that $[-1,1]^2 = C \cup D \cup F \subset \pi_{ab}(\mathsf{S}_{D(\frac{3\pi}{4})})$. On the other hand, if $(a,b) \notin [-1,1]^2$ then either $|a| > 1$, in which case

$$\|(a,b,c)\|_{D(\frac{3\pi}{4})} \geq |a| > 1,$$

for all $c \in \mathbb{R}$, or $|b| > 1$ and $|a| \leq 1$. In this latter situation, by symmetry, we just need to focus on the case $b < -1$ and $|a| \leq 1$. Since now $a - b > 0$, if $c \geq 0$, then by Theorem 3.1

$$\begin{aligned}\|(a,b,c)\|_{D(\frac{3\pi}{4})} &= \frac{1}{2}\left(|a+b| + \sqrt{(a-b)^2+c^2}\right)\\ &\geq \frac{1}{2}\left(|a+b| + |a-b|\right) = |b| > 1,\end{aligned}$$

and if $c < 1$, by Theorem 3.1 again

$$\begin{aligned}\|(a,b,c)\|_{D(\frac{3\pi}{4})} &\geq \frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right|\\ &= \frac{1}{2}\left(|a+b| + \sqrt{(a-b)^2+c^2}\right) \geq |b| > 1.\end{aligned}$$

Therefore if $(a,b) \notin [-1,1]^2$ then $(a,b) \notin \pi_{ab}(\mathsf{S}_{D(\frac{3\pi}{4})})$, concluding the proof. $\square$

Theorem 6.2. *Define*

$$H_1(a,b) = \begin{cases} 2+a+b & \text{if } (a,b) \in C,\\ 2\sqrt{(1+a)(1+b)} & \text{if } (a,b) \in D,\\ 2\sqrt{(1-a)(1-b)} & \text{if } (a,b) \in F,\end{cases}$$

where C, D and F were introduced at the beginning of Section 6, and $H_2(a,b) = -H_1(-a,-b)$ for all $(a,b) \in [-1,1]^2$. If

$$\Omega = \{\pm(-1,b,c) \in \mathbb{R}^3 : \ -1 \leq b \leq 1, \ 0 \leq c \leq H_1(-1,b)\},$$

then

(*a*) $\mathsf{S}_{D(\frac{3\pi}{4})} = \operatorname{graph}(H_1) \cup \operatorname{graph}(H_2) \cup \Omega$.

(*b*) *The set* $\operatorname{ext}(\mathsf{B}_{D(\frac{3\pi}{4})})$ *consists of the elements*

$$\begin{gathered}\pm\left(t, -4+t+4\sqrt{1-t}, -2+2t+4\sqrt{1-t}\right) \quad \text{for } t \in [-1,0],\\ \pm\left(s, -s, 2\sqrt{1-s^2}\right) \quad \text{for } s \in [0,1],\\ \pm\left(-1, r, 2\sqrt{2(1-r)}\right) \quad \text{for } r \in [-5+4\sqrt{2}, 1],\end{gathered}$$

and

$$\pm(1,1,0).$$

Proof. In the proof of Theorem 6.1 we saw already that

$$\operatorname{graph}(H_1) \cup \operatorname{graph}(H_2) \subset \mathsf{S}_{D(\frac{3\pi}{4})}.$$

On the other hand, let us consider a typical element of Ω, namely $\pm(-1, b, c)$ with $|b| \leq 1$ and $0 \leq c \leq H_1(-1, b)$. Since $-(1+b)c \leq 0$, using Theorem 3.1 we have

$$\| \pm (-1, b, c)\|_{D(\frac{3\pi}{4})} = \max \left\{ 1, \frac{1}{2} |1 - b + c| , \frac{1}{2} \left| 1 - b - \sqrt{(1+b)^2 + c^2} \right| \right\}.$$

Observe that

$$\begin{aligned} \frac{1}{2} |1 - b + c| &= \frac{1}{2}(1 - b + c) \leq \frac{1}{2}(1 - b + H_1(-1, b)) \\ &= \begin{cases} 1 & \text{if } -1 \leq b \leq -5 + 4\sqrt{2}, \\ \frac{1}{2}(1 - b + 2\sqrt{2(1-b)}) & \text{if } -5 + 4\sqrt{2} \leq b \leq 1, \end{cases} \end{aligned}$$

and that $\frac{1}{2}(1 - b + 2\sqrt{2(1-b)}) \leq 1$ for $-5 + 4\sqrt{2} \leq b \leq 1$. Also, notice that

$$\frac{1}{2} \left| 1 - b - \sqrt{(1+b)^2 + c^2} \right| \leq 1 \Leftrightarrow |c| \leq 2\sqrt{2(1-b)},$$

for $|b| \leq 1$. Since

$$0 \leq c \leq H_1(-1, b) = \begin{cases} 1 - b & \text{if } -1 \leq b \leq -5 + 4\sqrt{2}, \\ 2\sqrt{2(1-b)} & \text{if } -5 + 4\sqrt{2} \leq b \leq 1, \end{cases}$$

and $1 - b \leq 2\sqrt{2(1-b)}$ for $-1 \leq b \leq -5 + 4\sqrt{2}$, we conclude that

$$\frac{1}{2} \left| 1 - b - \sqrt{(1+b)^2 + c^2} \right| \leq 1.$$

Hence $\| \pm (-1, b, c)\|_{D(\frac{3\pi}{4})} = 1$.

This shows that $\mathrm{graph}(H_1) \cup \mathrm{graph}(H_2) \cup \Omega \subset \mathsf{S}_{D(\frac{3\pi}{4})}$. In order to prove that $\mathsf{S}_{D(\frac{3\pi}{4})} \subset \mathrm{graph}(H_1) \cup \mathrm{graph}(H_2) \cup \Omega$ we proceed as in the proof of Theorems 4.4 and 5.2 having in mind that any straight line passing through the origin meets the set $\mathrm{graph}(H_1) \cup \mathrm{graph}(H_2) \cup \Omega$.

As for the proof of the second part of the statement, notice that the set Ω and the surfaces defined by the formulas $c = 2\sqrt{1 - a - b + ab}$, $c = 2\sqrt{1 + a + b + ab}$ and $c = 2 + a + b$, are ruled sets of $\mathbb{R}^3$. Therefore the extreme points can only lie in the non straight intersections of the four surfaces Ω, $c = 2\sqrt{1 - a - b + ab}$, $c = 2\sqrt{1 + a + b + ab}$ and $c = 2 + a + b$ or, in other words, in the points described in the statement. As the reader can check easily by constructing a supporting hyperplane, all those points are indeed extreme, concluding the proof. $\square$

The reader can find a sketch of $\mathsf{S}_{D(\frac{3\pi}{4})}$ from different viewpoints in Figure 6.2.

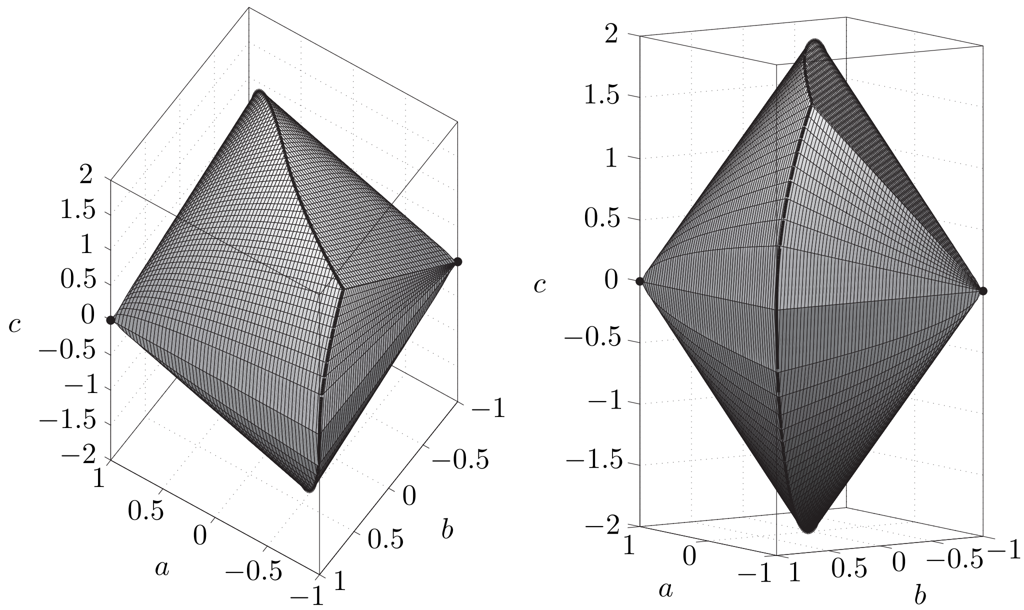

Figure 6.2: Two different perspectives of $\mathsf{S}_{D(\frac{3\pi}{4})}$. The extreme points of $\mathsf{B}_{D(\frac{3\pi}{4})}$ have been drawn with a thick line and dots

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