Journal of Convex Analysis
Volume 21 (2014), No. 3, 765–783

On Quasi-Gamma Functions

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Dedicated to Juan-Enrique Martínez-Legaz on the occasion of his 60th birthday.

Received: March 07, 2012
Revised manuscript received: July 05, 2013
Accepted: July 12, 2013

We define the quasi-gamma functions as the functions $f :]0,\infty[\longrightarrow]0,\infty[$ such that $f(1) = 1$, $f(x+1) = xf(x)$ for every $x > 0$, and f is quasi-convex. The main example of quasi-gamma function is the gamma function defined by Euler. We study some properties of the quasi-gamma functions and of the class Q of these functions.

Keywords: Gamma function, quasi-gamma function, quasi-convex function

2010 Mathematics Subject Classification: 26A51, 33B15, 52A41

1. Introduction

The gamma function Γ appears in several branches of Mathematics and some papers about its properties are published frequently. It was defined by Euler and studied by many mathematicians such as Legendre, Gauss, Liouville, Weierstrass, Hermite... The gamma function Γ is defined in the following way:

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt \quad (x > 0).$$

*The first author is partially supported by grant of Ministerio de Ciencia e Innovacion, Spain, project MTM2011-26538.

ISSN 0944-6532 / $ 2.50

This function verifies some remarkable properties: $\Gamma(1) = 1$, $\Gamma(x+1) = x\Gamma(x)$, for every $x > 0$, and Γ is logarithmically convex (that is, the function $\log\Gamma$ is convex). From the first two properties it is obtained that Γ is a generalization to positive real values of the factorial function; more precisely,

$$\Gamma(n+1) = n! \quad (n = 1, 2, 3...).$$

Notice that the property $\log\Gamma$ convex is stronger than the condition Γ is convex. For details about the Gamma function, see the monograph of Artin [1] or the books of Bourbaki [2], Rudin [12] or Webster [14]. In the paper by Davis [4] we can read about the historical development of certain aspects of Γ.

The celebrated theorem of Bohr and Mollerup (also called Bohr-Mollerup-Artin theorem) affirms that Γ is the unique function satisfying those three properties.

Theorem 1.1. *If $f :]0,\infty[\longrightarrow]0,\infty[$ is a function such that*

(G1) $f(1) = 1$,
(G2) $f(x+1) = xf(x)$, *for every $x > 0$,*
(G) *f is logarithmically convex; that is, the function $\log f$ is convex,*

then $f = \Gamma$.

It is natural to consider a weaker condition than (G). The quasi-convexity is an important notion which generalizes the convexity. We are interested in functions $f :]0,\infty[\longrightarrow]0,\infty[$ which satisfy (G1) and (G2), and moreover that $\log f$ is quasi-convex. However, f is quasi-convex if and only if $\log f$ is quasi-convex. Consequently, we consider the functions $f :]0,\infty[\longrightarrow]0,\infty[$ which satisfy (G1) and (G2), and moreover that f is quasi-convex. We call those functions *quasi-gamma functions.*

This paper is focused on the study of these functions. In particular, we give attention to Q, the class formed by all quasi-gamma functions. We establish some properties. In particular:

In Section 2 we introduce the basic terminology and recall some well known results.

In Section 3 we give the definition of the quasi-gamma functions and also the first properties of those functions.

In Section 4 we prove that every quasi-gamma function is continuous.

In Section 5 we consider the pointwise order in the class Q and determine the functions $q_1 = \min Q$ and $q_0 = \max Q$. A surprising result is that the golden number Φ plays a central role in the definition of q_1.

In Section 6 we precise the conditions about the behavior of a function $f :]0,\infty[\longrightarrow]0,\infty[$ on the interval $[1,2]$ which assure that f is quasi-gamma. Basically the conditions are based on the Dini derivatives of f.

In Section 7 is proved that the class Q is not convex and in Section 8 we study the convergence of quasi-gama functions in the Fréchet space $C(]0,\infty[)$. We obtain that Q is closed under pointwise convergence, is topologically bounded and closed in the space $C(]0,\infty[)$. If a sequence of quasi-gamma functions converges uniformly to a function f on the interval $]a,a+1]$, for some $a>0$, then f is a quasi-gamma function.

Finally, in Section 9 consider the subclass Q_c of Q formed by all quasi-gamma functions f such that $f(c)\leq f(x)$ for any $x>0$; that is, c is a minimum point of f. We find the function $h_c=\min Q_c$ and obtain that $q_0=\max Q_c=\max Q$. The class Q_c is a convex and compact set of the space $C(]0,\infty[)$.

2. Preliminaries

Let C be a convex subset of $\mathbb{R}$ (that is, an interval). Recall that a function $f:C\longrightarrow\mathbb{R}$ is *convex* if

$$x,y\in C \text{ and } 0\leq t\leq 1 \Longrightarrow f(tx+(1-t)y)\leq tf(x)+(1-t)f(y).$$

An important generalization of the above notion is given in the following definition.

Definition 2.1. Let C be a convex subset of $\mathbb{R}$. The function $f:C\longrightarrow\mathbb{R}$ is said to be quasi-convex if it satisfies the following:

$$x,y\in C \text{ and } 0\leq t\leq 1 \Longrightarrow f(tx+(1-t)y)\leq \max\{f(x),f(y)\}.$$

In the works of Crouzeix [3], Greenberg and Pierskalla [6], López-Cerdá and Valls-Vallejo [7], and Frenk and Kassay [5] there are a lot of properties of the quasi-convex functions. Recently, Martínez-Legaz and Martinón have published two papers on the behavior of a quasi-convex function over the intersection of sets ([8], [9]).

Now we can give the next result, which is well known and will be used in many occasions. For the convenience of the reader, give a reference for Proposition 2.2.

Proposition 2.2. *Let C be a convex subset of $\mathbb{R}$ and $f:C\longrightarrow\mathbb{R}$ be a function. Then f is quasi-convex if and only if, either f is monotone or, for some $c\in C$,*

(*1*) *either f is decreasing on the left of c and f is increasing on the right of c including c,*
(*2*) *either f is decreasing on the left of c including c and f is increasing on the right of c.*

We obtain the following result from above proposition.

Proposition 2.3. *Let C be a convex subset of $\mathbb{R}$ and $f:C\longrightarrow\mathbb{R}$ be a continuous function. The following are equivalent:*

(*1*) *f is quasi-convex*
(*2*) *either f is monotone or, for some $c\in C$, is decreasing on the left of c and f is increasing on the right of c*

Of course, c in the above proposition is a minimum point of f.

Given $f : C \subset \mathbb{R} \longrightarrow]0,\infty[$, defined on a convex subset C, we consider the function $\log f : C \subset \mathbb{R} \longrightarrow \mathbb{R}$. We say that f is *logarithmically convex* if $\log f$ is convex. Notice that if f is logarithmically convex, then f is convex, but the converse is not true. For example, the function $f(x) = x$ $(x > 0)$ is convex, but $\log x$ $(x > 0)$ is not convex. Analogously, f is *logarithmically quasi-convex* if $\log f$ is quasi-convex. Really this notion is redundant since:

$$f \text{ is logarithmically quasi-convex} \iff f \text{ is quasi-convex}$$

because of the increasing property of the function log.

3. Quasi-gamma functions

Now we give the main notion of this paper.

Definition 3.1. The function $f :]0,\infty[\longrightarrow]0,\infty[$ is called quasi-gamma if it satisfies the following:

(G1) $f(1) = 1$
(G2) $f(x+1) = xf(x)$, for every $x > 0$
(Q) f is quasi-convex

Denote by Q the class of all quasi-gamma functions.

Of course, the gamma function Γ is an example of quasi-gamma function. Really, it is the unique quasi-gamma function that is logarithmically convex (theorem of Bohr and Mollerup).

Note that f is determined by its values on every interval $]a, a+1]$ $(a > 0)$, in particular on $]1, 2]$.

Proposition 3.2. *Let $f :]0,\infty[\longrightarrow]0,\infty[$ be a quasi-gamma function. Then*

(*1*) $f(1) = f(2) = 1$
(*2*) $f(n+1) = n!$ *($n = 0, 1, 2, 3...$)*
(*3*) $f \leq 1$ *on* $[1, 2]$
(*4*) f *is decreasing on* $]0, 1]$
(*5*) f *is increasing on* $[2,\infty[$.

Proof. (*1*) and (*2*) are deduced directly from (G1) and (G2).

(*3*). As f is quasi-convex and $f(1) = f(2) = 1$, we have that $f(x) \leq 1$ for any $1 \leq x \leq 2$.

(*4*) and (*5*). The function f has its infimum on $[1, 2]$. In fact, if $f(y) < 1$ for some $0 < y < 1$, then $f(y+1) = yf(y) < 1$, so $f(1) = 1 > \max\{f(y), f(y+1)\}$: contradiction with f quasi-convex. Analogously, if $f(z) < 1$ for some $2 < z$, so necessarily $z < 3$, then $f = 1$ on $[1, 2]$ and $f(z) = (z-1)f(z-1) = z-1 > 1$: contradiction. Consequently, by Proposition 2.2, f is decreasing on $]0, 1]$ and f is increasing on $[2,\infty[$. $\square$

We obtain from the above proposition that the number c in Propositions 2.2 and 2.3 can be taken on the interval $[1, 2]$.

By Propositions 2.2 and 3.2 we have that

Proposition 3.3. *The function* $f\ :]0,\infty[\longrightarrow]0,\infty[$ *is quasi-gamma if satisfies the following:*

(*1*) $f(1) = 1$
(*2*) $f(x+1) = xf(x)$, *for every* $x > 0$
(*3*) f *is quasi-convex on* $[1, 2]$
(*4*) $f \geq 1$ *and decreasing on* $]0, 1]$
(*5*) $f \geq 1$ *and increasing on* $[2, 3[$

4. Continuity of the quasi-gamma functions

It is well known that the gamma function is continuous. In the following result we obtain the same property for quasi-gamma functions.

Theorem 4.1. *The quasi-gamma functions are continuous.*

Proof. It is enough to prove that f is continuous on the interval $[1, 2]$.

Let a be a real number such that $1 < a < 2$. Consider $1 < x < a < y < 2$, hence $2 < x + 1 < a + 1 < y + 1$. As f is increasing on $[2,\infty[$ we obtain $1 \leq f(x + 1) \leq f(a + 1) \leq f(y + 1)$, so

$$xf(x) \leq af(a) \leq yf(y).$$

Taking $x \longrightarrow a^-$ and $y \longrightarrow a^+$ we obtain

$$\limsup_{x\longrightarrow a^-} f(x) \leq f(a) \leq \liminf_{y\longrightarrow a^+} f(y).$$

Therefore

$$\limsup_{x\longrightarrow a^-}(x-1)f(x-1) \leq (a-1)f(a-1) \leq \liminf_{y\longrightarrow a^+}(y-1)f(y-1)$$

and consequently

$$\limsup_{x\longrightarrow a^-} f(x-1) \leq f(a-1) \leq \liminf_{y\longrightarrow a^+} f(y-1).$$

But f is decreasing on $]0, 1]$, hence $f(x-1) \geq f(a-1) \geq f(y-1)$. Thus

$$\liminf_{x\longrightarrow a^-} f(x-1) \geq f(a-1) \geq \limsup_{y\longrightarrow a^+} f(y-1).$$

Therefore

$$\lim_{x\longrightarrow a} f(x) = f(a),$$

so f is continuous in a.

For $a = 1 < y < 2$, we have that $2 = a + 1 < y + 1$, hence $f(2) = 1 \leq f(y+1) = yf(y)$, so

$$1 \leq \liminf_{x \longrightarrow 1^+} f(y).$$

As $f(y) \leq 1$, we obtain

$$\limsup_{y \longrightarrow 1^+} f(y) \leq 1,$$

thus

$$\lim_{y \longrightarrow 1^+} f(y) = f(1).$$

Analogously, for $1 < x < a = 2$, we obtain $2 < x + 1 < 3 = a + 1$, thus $1 = f(2) \leq f(x+1) = xf(x) \leq f(a+1) = af(a) = 2$. Therefore,

$$\limsup_{x \longrightarrow 2^-} f(x) \leq f(a) = 1.$$

As $f(x-1) \geq 1$, we obtain

$$\liminf_{x \longrightarrow 2^-} f(x-1) = \liminf_{x \longrightarrow 2^-} \frac{f(x)}{x-1} = \liminf_{x \longrightarrow 2^-} f(x) \geq 1.$$

Consequently

$$\lim_{x \longrightarrow 2^-} f(x) = f(2).$$ □

Taking into account Proposition 2.3 and Theorem 4.1 we obtain the following result:

Proposition 4.2. *Every quasi-gamma function has a minimum point* $c \in]1,2[$, *is decreasing on* $]0,c]$ *and increasing on* $[c,\infty[$.

5. Order in Q

In the class Q of all quasi-gamma functions we consider the pointwise order: given $f, g \in Q$, we write $f \leq g$ if $f(x) \leq g(x)$ for every $x > 0$.

The first result affirms that Q is closed under pointwise suprema.

Proposition 5.1. *Let* $(f_i)_{i \in I}$ *be a family of quasi-gamma functions such that, for every* $x > 0$, $\sup_{i \in I} f_i(x)$ *is finite. Then the function* $f = \sup_{i \in I} f_i$ *is also a quasi-gamma function.*

Proof. The supremum f is quasi-gamma because the supremum of quasi-convex functions is a quasi-convex function and, moreover, $f(1) = 1$ and $f(x+1) = xf(x)$, for $x > 0$. □

Recall that the *golden number* Φ is defined by

$$\Phi := \frac{1+\sqrt{5}}{2} = 1.618034...$$

Hence

$$\Phi - 1 = \frac{1}{\Phi} = \frac{-1+\sqrt{5}}{2} \quad \text{and} \quad \Phi^2 - \Phi - 1 = 0.$$

Now we give an important family of quasi-gamma functions.

Proposition 5.2. *Let* $0 \leq \lambda \leq 1$. *The functions* $q_\lambda :]0,\infty[\longrightarrow]0,\infty[$ *defined on the interval* $]1,2]$ *by*

$$q_\lambda(x) := \begin{cases} x^{-\lambda} & \text{if } 1 < x \leq \Phi \\ (x-1)^\lambda & \text{if } \Phi < x \leq 2 \end{cases} \tag{1}$$

and extended to $]0,\infty[$ *by* (G2) $f(x+1) = xf(x)$, *for every* $x > 0$, *are quasi-gamma functions.*

Proof. We have that $q_0 = 1$ on the interval $[1,2]$. For $\lambda \neq 0$ it is clear that q_λ is strictly decreasing on $[1,\Phi]$ and strictly increasing on $[\Phi,2]$, so it has a minimum at $x = \Phi$.

By (1) and (G2) we have that

$$q_\lambda(x) := \begin{cases} (x+1)^{-\lambda}x^{-1} & \text{if } 0 < x \leq \Phi - 1 \\ x^{\lambda-1} & \text{if } \Phi - 1 < x \leq 1 \\ x^{-\lambda} & \text{if } 1 < x \leq \Phi \\ (x-1)^\lambda & \text{if } \Phi < x \leq 2 \\ (x-1)^{1-\lambda} & \text{if } 2 < x \leq \Phi + 1 \\ (x-2)^\lambda(x-1) & \text{if } \Phi + 1 < x \leq 3 \\ \dots\dots & \dots\dots\dots\dots \\ (x-1)(x-2)...(x-n+2)(x-n+1)^{1-\lambda} & \text{if } n < x \leq \Phi + n - 1 \\ (x-1)(x-2)...(x-n+2)(x-n+1)(x-n)^\lambda & \text{if } \Phi + n - 1 < x \leq n+1 \\ \dots\dots & \dots\dots\dots\dots \end{cases}$$

Then q_λ is ≥ 1 and decreasing on $]0,1]$. Moreover $q_\lambda(1) = 1$, hence (G1) holds. Also q_λ is ≥ 1 and increasing on $]2,3]$. So q_λ is ≥ 1 and increasing on $]2,\infty[$, since (G2). $\square$

The functions q_1 and q_0 have got a great importance in the study of the class Q of quasi-gamma functions, because they are the minimum and the maximum,

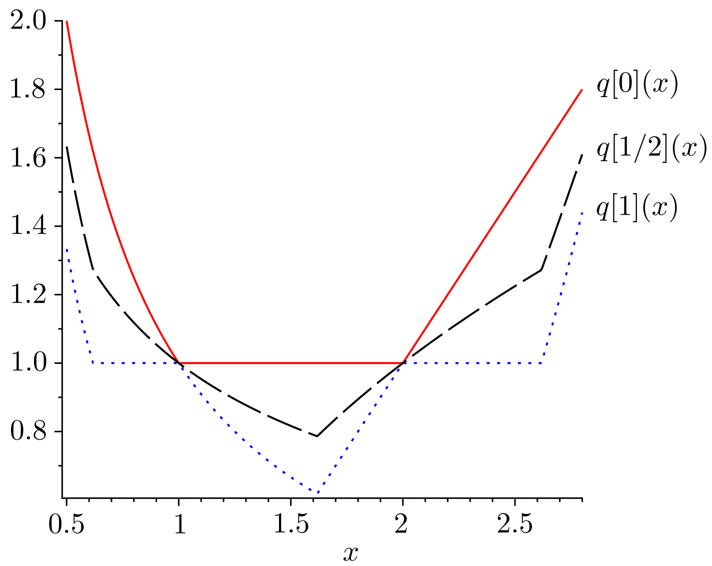

Figure 5.1: The functions q_λ

respectively, of that class (see Figure 5.1). They are defined in the following way:

$$q_1(x) := \begin{cases} x^{-1}(x+1)^{-1} & \text{if } 0 < x \le \Phi - 1 \\ 1 & \text{if } \Phi - 1 < x \le 1 \\ x^{-1} & \text{if } 1 < x \le \Phi \\ x-1 & \text{if } \Phi < x \le 2 \\ 1 & \text{if } 2 < x \le \Phi + 1 \\ (x-1)(x-2) & \text{if } \Phi + 1 < x \le 3 \\ \ldots & \ldots\ldots \\ (x-1)(x-2)\ldots(x-n+2) & \text{if } n < x \le \Phi + n - 1 \\ (x-1)(x-2)\ldots(x-n+2)(x-n+1)(x-n) & \text{if } \Phi + n - 1 < x \le n+1 \\ \ldots & \ldots\ldots \end{cases}$$

and

$$q_0(x) := \begin{cases} x^{-1} & \text{if } 0 < x \le 1 \\ 1 & \text{if } 1 < x \le 2 \\ x-1 & \text{if } 2 < x \le 3 \\ \ldots & \ldots\ldots \\ (x-1)(x-2)\ldots(x-n+1) & \text{if } n < x \le n+1 \\ \ldots & \ldots\ldots \end{cases}$$

Theorem 5.3. *Every quasi-gamma function f verifies*

$$q_1 \le f \le q_0 \,. \tag{2}$$

Consequently,

$$q_1 = \min \mathrm{Q} \quad \textit{and} \quad q_0 = \max \mathrm{Q}\,. \tag{3}$$

Proof. Let $1 \le x \le 2$. Then

(1) As f is quasi-convex and $f(1) = f(2) = 1$, we have $f(x) \le 1 = q_0(x)$.

(2) On the other hand, $xf(x) = f(x+1) \ge 1$, since $f \ge 1$ on $[2,3]$, hence $f(x) \ge x^{-1}$.

(3) Moreover, from $f \ge 1$ on $]0,1]$ we obtain $x - 1 \le (x-1)f(x-1) = f(x)$.

(4) From (2) and (3),

$$\max\{x-1, x^{-1}\} = q_1(x) \le f(x).$$

(5) By (1) and (4), we have that $q_1 \le f \le q_0$ on $[1,2]$. Consequently, $q_1 \le f \le q_0$ on $]0,\infty[$. □

Remark 5.4. The golden number Φ plays a central role in the definition of functions q_λ in Proposition 5.2. That is, if we consider the functions h_λ, defined as q_λ but we put $a \in]1,2[$ instead of Φ, we obtain

$$h_\lambda(x) := \begin{cases} x^{-\lambda} & \text{if } 1 < x \le a \\ (x-1)^\lambda & \text{if } a < x \le 2. \end{cases}$$

If

$$\alpha := \lim_{x \to a^-} h_\lambda(x) = a^{-\lambda} \quad \text{and} \quad \beta := \lim_{x \to a^+} h_\lambda(x) = (a-1)^\lambda,$$

we obtain that h_λ is continuous at a if and only if $\alpha = \beta$; equivalently, $a = \Phi$. Therefore, the only value of a making h_λ a quasi-gamma function is $a = \Phi$.

6. Extension of a function to a quasi-gamma function

We want to give conditions about the behavior of a function $f :]0,\infty[\longrightarrow]0,\infty[$ on the interval $[1,2]$ which assure that f is quasi-gamma.

The next example shows that it is possible that a function f verifies (G1), (G2) and $q_1 \le f \le q_0$, and moreover it is quasi-convex and continuous on $[1,2]$, but it is not a quasi-gamma function.

Example 6.1. Let $0 < \lambda \le \Phi - 1$. We consider the family of functions $f_\lambda :]0,\infty[\longrightarrow]0,\infty[$, defined on $]1,2]$ in the following way

$$f_\lambda(x) := \begin{cases} 1 & \text{if } 1 < x \le \Phi - \lambda \\ \lambda^{-1}(\Phi - 2)(x - \Phi) + \Phi - 1 & \text{if } \Phi - \lambda < x \le \Phi \\ x - 1 & \text{if } \Phi < x \le 2 \end{cases}$$

and extended to $]0,\infty[$ by (G2) $f(x+1) = xf(x)$ for all $x > 0$. These functions are quasi-convex and continuous on $[1,2]$, verify $q_1 \le f_\lambda \le q_0$ and moreover satisfies

(G1), that is $f_\lambda(1) = 1$. However the functions f_λ are not quasi-gamma. Indeed,

$$f_\lambda(x) = \begin{cases} x-1 & \text{if } 2 < x \leq \Phi + 1 - \lambda \\ (x-1)[\lambda^{-1}(\Phi-2)(x-1-\Phi) + \Phi - 1] & \text{if } \Phi + 1 - \lambda < x \leq \Phi + 1 \\ (x-1)(x-2) & \text{if } \Phi + 1 < x \leq 3 \end{cases}$$

satisfies $f_\lambda(\Phi+1) = 1 = f_\lambda(2)$, but $f_\lambda(\Phi+1-\lambda) = \Phi - \lambda > 1$, hence f is not quasi-convex.

In order to establish our result we need the notions of Dini derivatives. Given an open interval $I \subset \mathbb{R}$ and a function $f : I \longrightarrow \mathbb{R}$, the four Dini derivatives are the mappings $\overline{D}_+$, $\overline{D}_-$, $\underline{D}_+$ and $\underline{D}_-$ with values in $\mathbb{R} \cup \{\infty, -\infty\}$ and defined at $a \in I$ by the following formulas:

$$\overline{D}_+ f(a) := \limsup_{b \to a^+} \frac{f(b) - f(a)}{b - a},$$

$$\overline{D}_- f(a) := \limsup_{b \to a^-} \frac{f(b) - f(a)}{b - a},$$

$$\underline{D}_+ f(a) := \liminf_{b \to a^+} \frac{f(b) - f(a)}{b - a},$$

$$\underline{D}_- f(a) := \liminf_{b \to a^-} \frac{f(b) - f(a)}{b - a}.$$

It is clear that f is differentiable at a if and only if all Dini derivatives at a are equal and finite. Moreover, by definition we have that

$$\underline{D}_+ f \leq \overline{D}_+ f \quad \text{and} \quad \underline{D}_- f \leq \overline{D}_- f\,.$$

Every monotone function defined on a bounded closed interval I is differentiable almost everywhere on I (Lebesgue theorem; [10, Theorem 4.10]). Hence quasi-gamma functions are differentiable almost everywhere on $]0, \infty[$.

In the next theorem we give conditions for a function to be quasi-gamma based on its behavior on the interval $]1, 2]$. We begin with two lemmas.

Lemma 6.2. *Let* $f :]0, \infty[\longrightarrow]0, \infty[$ *be a continuous and quasi-convex function on* $[1, 2]$, *with minimum point at* $c \in]1, 2[$, *which verifies* (G1) *and* (G2). *The following assertions are equivalent:*

(*1*) f *is increasing on* $[2, 1 + c]$

(*2*) f *satisfies the "left slope condition" on* $[1, c]$*:*

$$-\frac{f(x)}{y} \leq \frac{f(y) - f(x)}{y - x} \leq 0 \quad (1 \leq x < y \leq c)$$

(*3*) f *satisfies the "left derivative condition" on* $[1, c]$*:*

(*a*)

$$-\frac{f(x)}{x} \leq \widetilde{D}f(x) \leq 0\,; \qquad \widetilde{D} \in \{\overline{D}_+, \overline{D}_-, \underline{D}_+, \underline{D}_-\} \quad (1 < x < c)$$

(*b*)

$$-1 = -\frac{f(1)}{1} \leq \widetilde{D}f(1) \leq 0\,; \qquad \widetilde{D} \in \{\overline{D}_+, \underline{D}_+\}$$

(*c*)

$$-\frac{f(c)}{c} \leq \widetilde{D}f(c) \leq 0\,; \qquad \widetilde{D} \in \{\overline{D}_-, \underline{D}_-\}$$

Proof. (*1*) $\Longrightarrow$ (*2*). Let $1 \leq x < y \leq c$. As f is increasing on $[2, 1+c]$ we have that

$$xf(x) = f(x+1) \leq f(y+1) = yf(y).$$

By the decreasing property on $[1, c]$ we obtain $f(y) \leq f(x)$, hence

$$xf(x) \leq yf(y) \leq yf(x),$$

so

$$\left(\frac{x}{y} - 1\right) f(x) \leq f(y) - f(x) \leq 0.$$

Furthermore

$$-\frac{f(x)}{y} \leq \frac{f(y) - f(x)}{y - x} \leq 0.$$

(*2*) $\Longrightarrow$ (*3*). Taking lim inf and lim sup as $y \longrightarrow x^+$ on the "left slope condition" we have that

$$-\frac{f(x)}{x} \leq \underline{D}_+ f(x) \leq \overline{D}_+ f(x) \leq 0 \quad (1 \leq x < c).$$

Analogously, for $x \longrightarrow y^-$ $(1 < y)$ we obtain

$$-\frac{f(y)}{y} \leq \underline{D}_- f(y) \leq \overline{D}_- f(y) \leq 0 \quad (1 < y \leq c).$$

(*3*) $\Longrightarrow$ (*1*). Given $1 \leq x < c$, we can to write

$$\begin{aligned}
\underline{D}_+ f(x+1) &= \liminf_{y \to x^+} \frac{f(y+1) - f(x+1)}{y - x} \\
&= \liminf_{y \to x^+} \frac{yf(y) - xf(y) + xf(y) - xf(x)}{y - x} \\
&\geq \liminf_{y \to x^+} f(y) + x \liminf_{y \to x^+} \frac{f(y) - f(x)}{y - x} \\
&= f(x) + x\underline{D}_+ f(x) \geq f(x) - x\frac{f(x)}{x} = 0.
\end{aligned}$$

Hence f is increasing [10, Exercise 15.C] $\square$

Lemma 6.3. *Let $f :]0,\infty[\longrightarrow]0,\infty[$ be a continuous and quasi-convex function on $[1,2]$, with minimum point at $c \in]1,2[$, which verifies* (G1) *and* (G2). *The following assertions are equivalent:*

(*1*) f *is decreasing on* $[c-1,1]$

(*2*) f *satisfies the "right slope condition" on* $[c,2]$*:*

$$0 \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(x)}{x-1} \quad (c \leq x < y \leq 2)$$

(*3*) f *satisfies the "right derivative condition" on* $[c,2]$*:*

(*a*)

$$0 \leq \widetilde{D}f(x) \leq \frac{f(x)}{x-1}\,; \qquad \widetilde{D} \in \{\overline{D}_+, \overline{D}_-, \underline{D}_+, \underline{D}_-\} \quad (c < x < 2)$$

(*b*)

$$0 \leq \widetilde{D}f(c) \leq \frac{f(c)}{c-1}\,; \qquad \widetilde{D} \in \{\overline{D}_+, \underline{D}_+\}$$

(*c*)

$$0 \leq \widetilde{D}f(2) \leq \frac{f(2)}{2-1} = 1\,; \qquad \widetilde{D} \in \{\overline{D}_-, \underline{D}_-\}$$

Proof. (*1*) $\Longrightarrow$ (*2*). Let $c \leq x < y \leq 2$. Since f is decreasing on $[c-1,1]$,

$$\frac{f(x)}{x-1} = f(x-1) \geq f(y-1) = \frac{f(y)}{y-1}.$$

As f is increasing on $[c,2]$, we have $f(x) \leq f(y)$. Consequently

$$\frac{f(x)}{y-1} \leq \frac{f(y)}{y-1} \leq \frac{f(x)}{x-1},$$

so

$$f(x) \leq f(y) \leq \frac{y-1}{x-1} f(x)\,,$$

hence

$$0 \leq f(y)-f(x) \leq \left(\frac{x-1}{y-1} - 1\right) f(x) = \frac{y-x}{x-1} f(x)\,.$$

Furthermore

$$0 \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(x)}{x-1}\,.$$

(*2*) $\Longrightarrow$ (*3*). Taking $y \longrightarrow x^+$ and $x \longrightarrow y^-$ we obtain

$$0 \leq \underline{D}_+ f(x) \leq \overline{D}_+ f(x) \leq \frac{f(x)}{x-1} \quad (c \leq x < 2)$$

$$0 \leq \underline{D}_{-}f(y) \leq \overline{D}_{-}f(y) \leq \frac{f(y)}{y-1} \quad (c < y \leq 2)$$

(3) $\Longrightarrow$ *(1)*. Given $c < x \leq 2$. Then

$$\begin{aligned}
\overline{D}_{+}f(x-1) = & \limsup_{y\to x^{+}} \frac{f(y-1)-f(x-1)}{y-x} \\
= & \limsup_{y\to x^{+}} \frac{\frac{f(y)}{y-1} - \frac{f(x)}{x-1}}{y-x} = \limsup_{y\to x^{+}} \frac{\frac{f(y)}{y-1} - \frac{f(y)}{x-1} + \frac{f(y)}{x-1} - \frac{f(x)}{x-1}}{y-x} \\
\leq & \limsup_{y\to x^{+}} \frac{\frac{x-y}{(x-1)(y-1)}}{y-x} f(y) + \frac{1}{x-1} \limsup_{y\to x^{+}} \frac{f(y)-f(x)}{y-x} \\
= & -\frac{f(x)}{(x-1)^2} + \frac{1}{x-1}\overline{D}_{+}f(x) \leq -\frac{f(x)}{(x-1)^2} + \frac{1}{(x-1)^2} f(x) = 0.
\end{aligned}$$

Therefore f is decreasing on $[c-1, 1]$ [10, Exercise 15.C] □

The above two lemmas allow us to give a characterization of quasi-gamma function, that we consider another main result.

Theorem 6.4. *Let* $f :]0, \infty[\longrightarrow]0, \infty[$ *be a continuous and quasi-convex function on* $[1, 2]$*, with minimum point at* $c \in]1, 2[$*, which verifies* (G1) *and* (G2)*. The following assertions are equivalent:*

(*1*) f *is a quasi-gamma function*

(*2*) f *is decreasing on* $[c-1, 1]$ *and increasing on* $[2, 1+c]$

(*3*) f *satisfies the "left slope condition" on* $[1, c]$ *and the "right slope condition" on* $[c, 2]$

(*4*) f *satisfies the "left derivative condition" on* $[1, c]$ *and the "right derivative condition" on* $[c, 2]$

Proof. By the property (G2), from f decreasing on $[1, c]$ and increasing on $[c, 2]$, we deduce that f is decreasing on $]0, c-1]$ and increasing on $[c+1, 3]$, respectively. Hence f is quasi-gamma if and only if f decreasing on $[c-1, 1]$ and increasing on $[2, c+1]$. The results follow from Lemmas 6.2 and 6.3. □

7. Non convexity of Q

An important negative result is that Q is not convex.

Theorem 7.1. *The class* Q *is not a convex set.*

Proof. Consider the functions $f, g \in Q$ defined on $[1, 2]$ in the following way (see Figure 7.1). We put $a = 1.7072$.

$$f(x) := \begin{cases} (2-a)(1-a)^{-1}(x-1)+1 & \text{if } 1 \leq x \leq a \\ x-1 & \text{if } a < x \leq 2 \end{cases}$$

and

$$g(x) := \begin{cases} x^{-1} & \text{if } 1 \le x \le 3-\Phi \\ \left(\frac{1}{3-\Phi} - \frac{4}{5}\right)\left(\frac{3}{2} - \Phi\right)^{-1}\left(x - \frac{3}{2}\right) + \frac{4}{5} & \text{if } 3-\Phi < x \le \frac{3}{2} \\ \frac{2}{5}x + \frac{1}{5} & \text{if } \frac{3}{2} < x \le 2 \end{cases}$$

The function f has got its minimum at a and belongs to Q. Indeed,

$$\begin{aligned} -\frac{f(x)}{x} &= \frac{1}{1-a}\left(\frac{1}{x} - (2-a)\right) \\ &\le \frac{1}{1-a}(2(2-a) - (2-a)) = \frac{2-a}{1-a} = f'(x) \le 0, \end{aligned}$$

since $x \le a \le 2^{-1}(2-a)^{-1}$; moreover, for $a < x < 2$,

$$0 \le f'(x) = 1 = \frac{f(x)}{x-1};$$

then we apply Theorem 6.4.

On the other hand, g has got its minimum at $3-\Phi$. Moreover, for $1 \le x \le 3-\Phi$,

$$-\frac{f(x)}{x} = -\frac{1}{x^2} = f'(x) \le 0;$$

on the the other hand, for $3 - \Phi < x \le 3/2$,

$$0 \le f'(x) = 0.645... \le \frac{f(x)}{x-1};$$

also for $3/2 < x \le 2$,

$$0 \le f'(x) = \frac{2}{5} < \frac{4}{5} \le \frac{f(x)}{x-1}.$$

By Theorem 6.4 we obtain that $g \in Q$.

The function

$$h = \frac{f+g}{2}$$

verifies

$$h(3-\Phi) = \frac{f(3-\Phi) + g(3-\Phi)}{2} = \frac{0.8418... + 0.7236...}{2} = 0.7827...$$

$$h\left(\frac{3}{2}\right) = f\left(\frac{3}{2}\right) = g\left(\frac{3}{2}\right) = \frac{4}{5} = 0.8$$

$$h(a) = \frac{f(a) + g(a)}{2} = \frac{0.7072... + 0.8828...}{2} = 0.7950...$$

In summary, we have that

$$h(3-\Phi) < h\left(\frac{3}{2}\right) > h(a),$$

so h is not a quasi-convex function. Hence the class Q is not convex. □

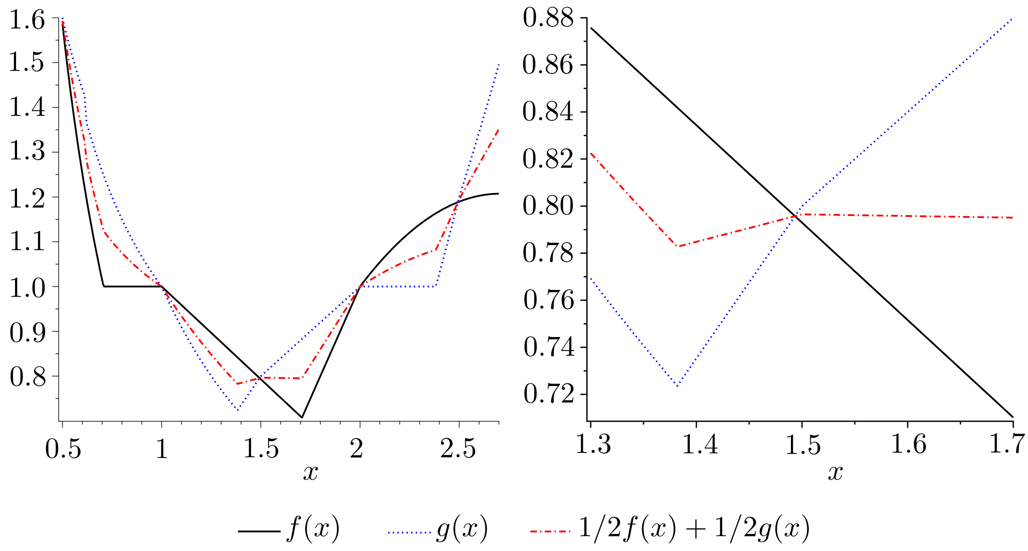

Figure 7.1: Q is not convex

8. Convergence of quasi-gama functions

The class Q is closed under pointwise convergence:

Proposition 8.1. *Let $(f_n)_{n\geq 1}$ be a sequence of quasi-gamma functions such that, for every $x > 0$, there is*

$$f(x) := \lim_{n\to\infty} f_n(x).$$

and $f(x)$ is finite. Then f is a quasi-gamma function.

Proof. The properties (G1) and (G2) are clear. Moreover, for $x, y > 0$ and $0 \leq t \leq 1$, we have

$$\begin{aligned} f(tx + (1-t)y) &= \lim_{n\to\infty} f_n(tx + (1-t)y) \leq \lim_{n\to\infty} \max\{f_n(x), f_n(y)\} \\ &= \max\{\lim_{n\to\infty} f_n(x), \lim_{n\to\infty} f_n(y)\} = \max\{f(x), f(y)\}. \end{aligned}$$ □

If the pointwise convergence holds on some interval $]a, a+1]$, then it holds on $]0, \infty[$:

Proposition 8.2. *Let $(f_n)_{n\geq 1}$ be a sequence of quasi-gamma functions which converges pointwise to a function f on $]a, a+1]$, for some $a > 0$. Then $(f_n)_{n\geq 1}$ converges pointwise on $]0, \infty]$.*

Proof. Let $x > 0$. There is an integer k such that $x - k \in]a, a+1]$. Then, for every positive integer n,

$$f_n(x) = (x-1)(x-2)...(x-k)f_n(x-k) \quad (x > a; k \geq 0),$$

$$f_n(x) = x^{-1}(x+1)^{-1}...(x-k-1)^{-1}f_n(x-k) \quad (x \leq a; k < 0).$$

Hence, as $n \longrightarrow \infty$,

$$f_n(x) \longrightarrow (x-1)(x-2)...(x-k)f(x-k) = f(x) \quad (x > a; k \geq 0),$$

$$f_n(x) \longrightarrow x^{-1}(x+1)^{-1}...(x-k-1)^{-1}f(x-k) = f(x) \quad (x \leq a; k < 0).$$

Thus $f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$, for every $x > 0$. □

Recall that the space $C(]0,\infty[)$ of all the continuous functions $g :]0,\infty[\longrightarrow \mathbb{R}$ is a Fréchet space [11, 1.44]. Its topology is defined by the distance

$$d(h,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(g-h)}{1+p_n(g-h)} \quad (h, g \in C(]0,\infty[)$$

where

$$p_n(g) = \max_{1/n \leq x \leq n} |g(x)| \quad (g \in C(]0,\infty[).$$

A subset E of $C(]0,\infty[)$ is topologically bounded if and only if $p_n(E)$ is bounded, for every $n = 1, 2, 3...$ [11, Theorem 1.37].

Proposition 8.3. *The class* Q *is topologically bounded in* $C(]0,\infty[)$.

Proof. For $n = 1, 2, 3...$ and $f \in Q$, we have

$$\sup_{1/n \leq x \leq n} f(x) = \max\{f(1/n), f(n)\} \leq \max\{n, (n-1)!\} \leq n!\ .$$

Hence $p_n(Q) \subset [0, n!]$, so it is bounded. □

The sequence $(f_n)_{n\geq 1}$ in $C(]0,\infty[)$ converges to $f \in C(]0,\infty[)$ in the topology of $C(]0,\infty[)$, that is $d(f_n, f) \longrightarrow 0$ as $n \longrightarrow \infty$, if and only if $(f_n)_{n\geq 1}$ converges uniformly to f on compact subsets.

Proposition 8.4. *Let $(f_n)_{n\geq 1}$ be a sequence of quasi-gamma functions which converges uniformly to the function f on the interval $]a, a+1]$, for some $a > 0$. Then f is a quasi-gamma function and $(f_n)_{n\geq 1}$ converges to f in the space $C(]0,\infty[)$.*

Proof. First we note that $(f_n)_{n\geq 1}$ converges uniformly to f on $]a+1, a+2]$; indeed,

$$\begin{aligned}\sup_{a+1<x\leq a+2} |f_n(x) - f(x)| &= \sup_{a+1<x\leq a+2} |(x-1)f_n(x-1) - (x-1)f(x-1)| \\ &= |x-1| \sup_{a<x-1\leq a+1} |f_n(x-1) - f(x-1)| \\ &\leq (a+1) \sup_{a<x\leq a+1} |f_n(x) - f(x)| \longrightarrow 0 \quad (n \longrightarrow \infty).\end{aligned}$$

Therefore, the uniform convergence of $(f_n)_{n\geq 1}$ to f is satisfied on $]m-1, m]$ for some positive integer m, thus for every positive integer m. Hence that convergence holds on every compact subset of $]0,\infty[$. □

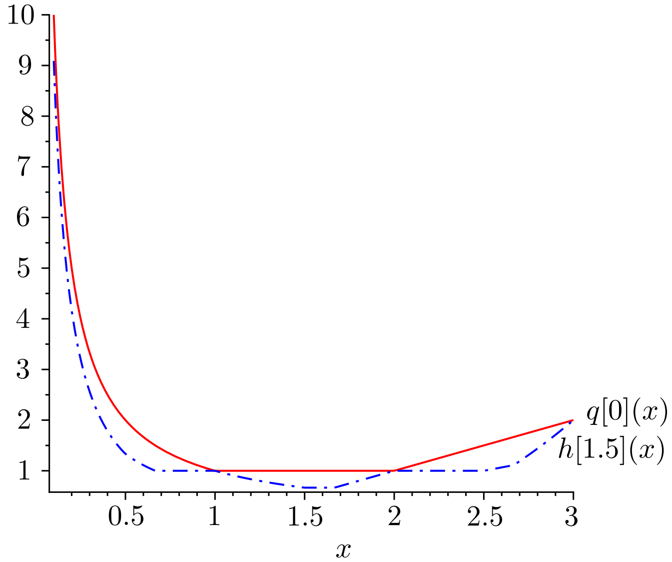

Figure 9.1: The function q_0 and $h_{1.5}$

We obtain the following from the above proposition

Theorem 8.5. *The class* Q *is closed in the space* $C(]0,\infty[)$.

9. The subclass $\mathcal{Q}_c$

Denote by $\mathcal{Q}_c$ $(1 < c < 2)$ the set of all quasi-gamma functions f such that $f(c) \le f(x)$ for any $x > 0$; that is, c is a minimum point of f. Note that q_0 is the unique function that belongs to $\mathcal{Q}_c$ for any c, hence the sets $\mathcal{Q}_c$ satisfy that

$$\bigcap_{1<c<2} Q_c = \{q_0\}.$$

Let $1 < c < 2$. If $1 < c \le \Phi$ we define h_c by

$$h_c(x) := \begin{cases} q_1(x) = x^{-1} & \text{if } 1 \le x \le c \\ c^{-1} & \text{if } c < x \le c^{-1}(1+c) \\ q_1(x) = x - 1 & \text{if } c^{-1}(1+c) < x \le 2 \end{cases}$$

If $\Phi < c < 2$, we define

$$h_c = h_{(c-1)^{-1}}\,.$$

The functions h_c and q_0 play an important role into Q_c (see Figure 9.1).

Theorem 9.1. *Let* $1 < c < 2$. *Every quasi-gamma function* $f \in Q_c$ *verifies*

$$h_c \le f \le q_0\,. \tag{4}$$

Consequently,

$$h_c = \min \mathrm{Q}_c \quad \textit{and} \quad q_0 = \max \mathrm{Q}_c\,. \tag{5}$$

Proof. Let $1 < c \leq \Phi$ and $f \in Q_c$. As $q_1 \leq f$ and $f(c) \leq f(x)$, for every $x > 0$, we have

$$c^{-1} = q_1(c) \leq f(c) \leq f(x),$$

for every $c \leq x \leq c^{-1} + 1$. Thus $h_c \leq f$. It is obvious that $f \leq q_0$. □

Proposition 9.2. *The class* Q_c *is a convex set for any* $1 < c < 2$.

Proof. Let $f, q \in Q_c$ and $0 \leq t \leq 1$. The function $h := tf + (1-t)g$ verifies the conditions (G1) and (G2). Moreover h is decreasing on $]0, c]$ and is increasing on $[c, \infty[$, so h is quasi-convex with minimum at c. Hence $h \in Q_c$. □

Now we give our last main result.

Theorem 9.3. *Let* $1 < c < 2$. *The class* Q_c *is a compact subset of the space* $C(]0, \infty[)$.

Proof. We divide the proof in five steps.

(1) We denote by $Q_c[1,2]$ the class of all functions $g : [1,2] \longrightarrow]0, \infty[$ such that there exists $f \in Q_c$ with $g = f_{[1,2]}$, the restriction of f to $[1,2]$. We affirm that $Q_c[1,2]$ is a compact subset of the Banach space $C[1,2]$ of all real continuous functions defined on $[1,2]$ endowed with the uniform norm

$$\|h\|_\infty = \sup_{1 \leq x \leq 2} |h(x)| \quad (h \in C[1,2]).$$

In order to prove it we show that $Q_c[1,2]$ is an uniformly bounded equicontinuous and closed subset of $C[1,2]$ (Arzelà-Ascoli Theorem; [13, p. 295]).

(2) $Q_c[1,2]$ is uniformly bounded. Indeed, for every $g \in Q_c[1,2]$, we have $0 \leq g \leq 1$, hence $\|g\|_\infty \leq 1$. Really, this result is contained in Proposition 8.3.

(3) $Q_c[1,2]$ is an equicontinuous subset of $C[1,2]$. Indeed, given $\varepsilon > 0$, we choose $\delta := \min\left\{\frac{\varepsilon}{2}, \frac{(c-1)\varepsilon}{2}\right\}$ and we apply Theorem 6.4: for every $g \in Q_c[1,2]$,

- if $1 \leq x, y \leq c$ and $|x - y| < \delta$, then we obtain

$$|g(y) - g(x)| \leq \frac{g(x)}{y}|x - y| < \frac{\delta g(x)}{y} \leq \delta < \varepsilon.$$

- if $c \leq x, y \leq 2$ and $|x - y| < \delta$, then

$$|g(y) - g(x)| \leq \frac{g(x)}{x-1}|x - y| \leq \frac{1}{c-1}\delta < \delta < \varepsilon.$$

- if $1 \leq x \leq c \leq y \leq 2$ and $|x - y| < \delta$, then $|x - c| < \delta$ and $|c - y| < \delta$, and we can to write

$$|g(y) - g(x)| \leq |g(y) - g(c)| + |g(c) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(4) $Q_c[1,2]$ is a closed subset of $C[1,2]$. Indeed, this assertion is contained in Proposition 8.4.

(5) Finally we prove that Q_c is a compact subset of $C(]0,\infty[)$. Let $(f_n)_{n\geq 1}$ be a sequence of functions of Q_c. Let $g_n = f_{n[1,2]}$ be the restriction of f_n to $[1,2]$. So $(g_n)_{n\geq 1}$ is a sequence in the class $Q_c[1,2]$, hence there is a subsequence $(g_{n_k})_{k\geq 1}$ of $(g_n)_{n\geq 1}$ which is convergent to a certain $g \in C[1,2]$, thus $g \in Q_c[1,2]$. From Proposition 8.4, we obtain that there exists $f \in Q$ such that $f_{[1,2]} = g$ and $(f_{n_k})_{k\geq 1}$ converges to f in the space $C(]0,\infty[)$. $\square$

Recall that the Krein-Milman Theorem affirms that as Q_c is a nonempty convex compact subset of $C(]0,\infty[)$, then it is the closed convex hull of the set of its extremal points [13, Theorem 11.3]. Note that $h_c = \min Q_c$ and $q_0 = \max Q_c$ are extremal points of Q_c.

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