Journal of Convex Analysis
Volume 21 (2014), No. 3, 811–818

On Completely Continuous Integration Operators of a Vector Measure*

J. M. Calabuig
Instituto Universitario de Matemática Pura y Aplicada,
Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia, Spain
jmcalabu@mat.upv.es

J. Rodríguez
Departamento de Matemática Aplicada, Facultad de Informática,
Universidad de Murcia, 30100 Espinardo (Murcia), Spain
joserr@um.es

E. A. Sánchez-Pérez
Instituto Universitario de Matemática Pura y Aplicada,
Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia, Spain
easancpe@mat.upv.es

Received: January 17, 2013
Accepted: July 15, 2013

Let m be a vector measure taking values in a Banach space X. We prove that if the integration operator $I_m : L^1(m) \to X$, $I_m(f) = \int f\,dm$, is completely continuous and X is Asplund, then m has finite variation and $L^1(m) = L^1(|m|)$.

Keywords: Integration operator, vector measure, completely continuous operator, Asplund space

2000 Mathematics Subject Classification: 46E30, 46G10, 47B07

1. Introduction

Throughout this paper (Ω, Σ) is a measurable space and X is a Banach space. Let $m : \Sigma \to X$ be a vector measure (which we always assume to be countably additive). A Σ-measurable function $f : \Omega \to \mathbb{R}$ is said to be *m-integrable* if (i) it is integrable with respect to the composition $x^* \circ m : \Sigma \to \mathbb{R}$ for all $x^* \in X^*$, and (ii) for each $A \in \Sigma$ there is a vector $\int_A f\,dm \in X$ such that $x^*(\int_A f\,dm) = \int_A f\,d(x^* \circ m)$ for all $x^* \in X^*$. Let $L^1(m)$ be the Banach lattice of all equivalence classes of m-integrable functions, endowed with the norm $\|f\|_{L^1(m)} = \sup\{\int_\Omega |f|\,d|x^* \circ m| : x^* \in B_{X^*}\}$ and the $\|m\|$-a.e. order. Let

*Research supported by Ministerio de Economía y Competitividad and FEDER under projects MTM2011-23164 (J. M. Calabuig), MTM2011-25377 (J. Rodríguez) and MTM2012-36740-c02-02 (E. A. Sánchez-Pérez).

ISSN 0944-6532 / $ 2.50

$I_m : L^1(m) \to X$ be the operator defined by

$$I_m(f) := \int_\Omega f\, dm.$$

I_m is called the *integration operator* of m. It is known that any order continuous Banach lattice with weak unit is order isometric to the L^1 space of a suitable vector measure, [3, Theorem 8]. On the other hand, some properties of I_m might cause sound consequences on the structure of $L^1(m)$, like having the equality

$$L^1(m) = L^1(|m|), \tag{1}$$

which is equivalent to saying that $L^1(m)$ is order isomorphic to an AL-space, [4, Proposition 2]. Clearly, (1) implies that m has *finite variation*. The so-called "Operator Ideal Principle", [10, Proposition 1.1], asserts that if $\mathcal{A}$ is an operator ideal and *every* vector measure $m : \Sigma \to X$ with $I_m \in \mathcal{A}$ has finite variation, then (1) holds true for every vector measure $m : \Sigma \to X$ with $I_m \in \mathcal{A}$. The ideal $\mathcal{A}_c$ of compact operators has the property that every vector measure $m : \Sigma \to X$ with $I_m \in \mathcal{A}_c$ has finite variation, [9, Theorem 4] (cf. [11, Theorem 2.2]). The same is true for the ideal Π_p of p-summing operators (for $1 \leq p < \infty$), [10, Theorem 2.2]. In fact, equality (1) holds whenever I_m is positive p-summing (for $1 \leq p < \infty$), [1, Theorem 2.7] (cf. [10, Remark 2.5]).

For the ideal $\mathcal{A}_{cc}$ of completely continuous operators the situation is not so pleasant. Recall that an operator is called *completely continuous* if it maps weakly convergent sequences to norm convergent ones. Let us introduce the following

Definition 1.1. A Banach space X has *property* $(\mathfrak{D})$ if for every measurable space (Ω, Σ) and every vector measure $m : \Sigma \to X$ the following implication holds:

$$I_m \in \mathcal{A}_{cc} \Longrightarrow m \text{ has finite variation.}$$

As pointed out in [10, p. 217], if X has property $(\mathfrak{D})$, then X does not contain subspaces isomorphic to ℓ_1 (shortly $X \not\supseteq \ell_1$), since the integration operator of any ℓ_1-valued measure is completely continuous. Conversely, if $X \not\supseteq \ell_1$ and X *has an unconditional Schauder basis*, then X has property $(\mathfrak{D})$, [10, Theorem 1.2]. The following question was then raised in [10, p. 243]:

Problem 1.2. *Does X have property $(\mathfrak{D})$ whenever $X \not\supseteq \ell_1$?*

In this paper we address this question and show how the techniques of [10] can be pushed further to prove the following

Theorem 1.3. *If X is Asplund, then it has property $(\mathfrak{D})$.*

Recall that X is said to be *Asplund* if every separable subspace of X has separable dual or, equivalently, X^* has the Radon-Nikodým property, [5, p. 198]. It is known that a Banach lattice (e.g. a Banach space having an unconditional Schauder basis) is Asplund if and only if it does not contain subspaces isomorphic to ℓ_1, see [5, p. 95] and [8, Theorem 7]. Therefore, Theorem 1.3 extends [10, Theorem 1.2] and allows us to solve affirmatively Problem 1.2 within the setting of Banach lattices:

Corollary 1.4. *A Banach lattice X has property $(\mathfrak{D})$ if and only if $X \not\supseteq \ell_1$.*

We do not know, however, if the answer to Problem 1.2 is positive in full generality. In any case, Theorem 1.3 makes clear that if there is a counterexample, this should be based on some of the Banach spaces that draw the fine line between spaces not containing ℓ_1 and Asplund spaces, like the James tree space, see e.g. [7].

2. Preliminaries

All unexplained terminology can be found in our standard references [5, 6, 12]. Our Banach spaces are real. By an "operator" between Banach spaces we mean a linear continuous mapping. By a "subspace" of a Banach space we mean a closed linear subspace. The topological dual of X is denoted by X^*. We write B_X to denote the closed unit ball of X. The following lemma is a straightforward consequence of the Hahn-Banach separation theorem.

Lemma 2.1. *Let (x_n) be a sequence in X. The following statements are equivalent:*

(*i*) $0 \notin \overline{\mathrm{conv}}(x_n)$;
(*ii*) *there is $x^* \in X^*$ such that $x^*(x_n) \geq 1$ for every $n \in \mathbb{N}$;*
(*iii*) *there is a constant $C > 0$ such that*

$$\Big\| \sum_{n=1}^{N} a_n x_n \Big\| \geq C \sum_{n=1}^{N} a_n$$

for all $N \in \mathbb{N}$ and non-negative real numbers $a_1, \dots, a_N$.

A sequence satisfying the conditions of Lemma 2.1 will be called an ℓ_+*-sequence*. For basic sequences, this notion was considered in [13, Chapter II, §10]. To extend this concept to sequences of *sets*, we need some notation. Given $C \subseteq X$, we write

$$\|C\| := \sup\{\|x\| : x \in C\} \quad \text{and} \quad \delta^*(x^*, C) := \sup\{x^*(x) : x \in C\}$$

for every $x^* \in X^*$. The collection of all non-empty norm compact subsets of X is denoted by $k(X)$.

Definition 2.2. A sequence (K_n) of subsets of X is called an ℓ_+*-sequence* if there is a constant $C > 0$ such that

$$\Big\| \sum_{n=1}^{N} a_n K_n \Big\| \geq C \sum_{n=1}^{N} a_n$$

for all $N \in \mathbb{N}$ and non-negative real numbers $a_1, \dots, a_N$.

In the proof of Theorem 1.3 we shall use the next lemma. A slightly weaker version of it (see Remark 2.4 below) was proved in [10, Lemma 3.1]. Here we provide an alternative, simpler proof which uses tools from convex analysis.

Lemma 2.3. *Let (K_n) be an ℓ_+-sequence of subsets of X such that:*

(*i*) $\sup\{\|K_n\| : n \in \mathbb{N}\} < \infty$;
(*ii*) $K_n \in k(X)$ *for every* $n \in \mathbb{N}$.

Then there is $x^* \in X^*$ such that $\delta^*(x^*, K_n) \geq 1$ for infinitely many $n \in \mathbb{N}$.

Proof. Let B_{X^*} be equipped with the weak* topology and let $C(B_{X^*})$ be the Banach space of all real-valued continuous functions on B_{X^*}, equipped with the supremum norm $\|\cdot\|_\infty$. Note that the mapping $\Phi : k(X) \to C(B_{X^*})$ given by

$$\Phi(C)(x^*) := \delta^*(x^*, C)$$

is well-defined and satisfies the following properties:

- $\Phi(C + D) = \Phi(C) + \Phi(D)$ for every $C, D \in k(X)$;
- $\Phi(\lambda C) = \lambda\Phi(C)$ for every $C \in k(X)$ and every real number $\lambda \geq 0$;
- $\|\Phi(C)\|_\infty = \|C\|$ for every $C \in k(X)$;

see e.g. [2, Chapter II, §3].

Clearly, writing $h_n := \Phi(K_n)$ for all $n \in \mathbb{N}$, the above properties of Φ imply that (h_n) is an ℓ_+-sequence in $C(B_{X^*})$. In particular, (h_n) is not weakly null. Since (h_n) is bounded, it follows from Grothendieck's theorem (see e.g. [6, Corollary 3.138]) that there is $y^* \in B_{X^*}$ such that the sequence $(h_n(y^*))$ is not convergent to 0. Bearing in mind that $h_n(y^*) = \delta^*(y^*, K_n)$ for all $n \in \mathbb{N}$, it is clear that we can find $x^* \in X^*$ such that $\delta^*(x^*, K_n) \geq 1$ for infinitely many $n \in \mathbb{N}$. □

Remark 2.4. In [10, Lemma 3.1] the assumption that (K_n) is an ℓ_+-sequence is replaced by a stronger one, namely: there is a constant $C > 0$ such that

$$\Big\|\sum_{n=1}^{N} a_n K_n\Big\| \geq C \sum_{n=1}^{N} |a_n|$$

for all $N \in \mathbb{N}$ and *arbitrary* real numbers $a_1, \ldots, a_N$.

3. Proof of Theorem 1.3

As we mentioned in the introduction, our approach to Theorem 1.3 is based strongly on the ideas of [10]. In particular, we shall use the following two results, which appear in [10, Lemmas 3.3 and 3.4]. Note that the second one is stated in [10] under the additional assumption that the Schauder basis is unconditional. However, that assumption is not needed in the proof.

Lemma 3.1. *Let $m : \Sigma \to X$ be a vector measure. If $m(\Sigma)$ is relatively norm compact, then $I_m(fB_{L^\infty(m)}) \in k(X)$ for every $f \in L^1(m)$, where*

$$fB_{L^\infty(m)} := \{fg : g \in B_{L^\infty(m)}\}.$$

Lemma 3.2. *Suppose X has a Schauder basis and let (e_n) be a normalized monotone Schauder basis of X (for some equivalent norm $\|\cdot\|$). For each $n \in \mathbb{N}$, let $P_n : X \to X$ be the canonical projection onto* $\mathrm{span}(e_i)_{1\le i\le n}$ *and $Q_n := id - P_n$. Let $m : \Sigma \to X$ be a vector measure having infinite variation such that $m(\Sigma)$ is relatively norm compact. Then:*

(*i*) *There exist a strictly increasing sequence (k_j) in $\mathbb{N}$ and a sequence (f_j) in $L^1(m)$ with $\|f_j\|_{L^1(m)} = 1$ such that the sets*

$$K_j := I_m\big(f_j B_{L^\infty(m)}\big) \in k(X)$$

satisfy

$$\sup_{x\in K_{j+1}} \|P_{k_j}(x)\| \le \frac{1}{2^{j+1}} \quad \textit{and} \quad \sup_{x\in K_j} \|Q_{k_j}(x)\| \le \frac{1}{2^j} \quad \textit{for all } j \in \mathbb{N}. \tag{2}$$

(*ii*) *There is a sequence (ψ_j) in $B_{L^\infty(m)}$ with $\|I_m(f_j\psi_j)\| = 1$ such that*

$$\|I_m(f_i\psi_i) - I_m(f_j\psi_j)\| \ge 1/4 \quad \textit{whenever } i \ne j. \tag{3}$$

As an application of the previous lemma, we obtain yet another proof of [9, Theorem 4]:

Theorem 3.3. *Let $m : \Sigma \to X$ be a vector measure. If I_m is compact, then m has finite variation.*

Proof. Since I_m is compact, the set $m(\Sigma) = \{I_m(1_A) : A \in \Sigma\}$ is relatively norm compact and so $m(\Sigma) \subseteq I_m(L^1(m)) \subseteq Y$ for some separable subspace $Y \subseteq X$. Note that Y embeds isomorphically (even isometrically) into $C[0,1]$ (see e.g. [6, Theorem 5.8]), which has a Schauder basis. Let $T : Y \to C[0,1]$ be an isomorphic embedding and consider the vector measure $\tilde{m} := T \circ m : \Sigma \to C[0,1]$. The identity mapping $i : L^1(m) \to L^1(\tilde{m})$ is an isomorphism and $I_{\tilde{m}} \circ i = T \circ I_m$, hence $I_{\tilde{m}}$ is compact. Lemma 3.2(*ii*) ensures that $\tilde{m}$ has finite variation, and so does m. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $m : \Sigma \to X$ be a vector measure such that I_m is completely continuous. Then $m(\Sigma)$ is relatively norm compact (see e.g. [12, p. 153]) and therefore $m(\Sigma) \subseteq I_m(L^1(m)) \subseteq Y$ for some separable subspace $Y \subseteq X$. Since X is Asplund, Y^* is separable and so Y can be embedded isomorphically into a Banach space which has a shrinking Schauder basis, see [14]. The argument of the the proof of Theorem 3.3 makes clear that we can assume without loss of generality that X has a Schauder basis (e_n) which is shrinking, i.e. $X^* = \overline{\mathrm{span}}(e_n^*)$, where (e_n^*) is the sequence of biorthogonal functionals to (e_n). We can assume further that (e_n) is normalized and monotone for some equivalent norm $\|\cdot\|$ on X.

We shall prove that m has finite variation by contradiction. Suppose that m has infinite variation and then apply Lemma 3.2.

Step 1. The sequence $(f_j\psi_j)$ is not weakly null in $L^1(m)$, because I_m is completely continuous and $(I_m(f_j\psi_j))$ is not norm convergent in X (by (3)). Therefore, there is a strictly increasing sequence (j_n) in $\mathbb{N}$ such that $(f_{j_n}\psi_{j_n})$ is an ℓ_+-sequence in $L^1(m)$, that is, there is $C > 0$ such that

$$\Big\| \sum_{n=1}^{N} a_n f_{j_n}\psi_{j_n} \Big\|_{L^1(m)} \geq C \sum_{n=1}^{N} a_n \tag{4}$$

for every $N \in \mathbb{N}$ and non-negative real numbers $a_1, \ldots, a_N$.

Claim. (K_{j_n}) is an ℓ_+-sequence of subsets of X. Indeed, fix $N \in \mathbb{N}$ and non-negative real numbers $a_1, \ldots, a_N$. Define $f := \sum_{n=1}^{N} a_n f_{j_n}\psi_{j_n} \in L^1(m)$ and note that $\|f\|_{L^1(m)} = \|I_m(fB_{L^\infty(m)})\|$ (see e.g. [12, Lemma 3.11]). Since $m(\Sigma)$ is relatively norm compact, the set $I_m(fB_{L^\infty(m)})$ is norm compact (by Lemma 3.1) and so there is $g \in B_{L^\infty(m)}$ such that

$$\|f\|_{L^1(m)} = \|I_m(fB_{L^\infty(m)})\| = \|I_m(fg)\|.$$

Since $I_m(f_{j_n}\psi_{j_n}g) \in K_{j_n}$ for every $n \in \{1, \ldots, N\}$, we get

$$\begin{aligned} \Big\| \sum_{n=1}^{N} a_n K_{j_n} \Big\| &\geq \Big\| \sum_{n=1}^{N} a_n I_m(f_{j_n}\psi_{j_n}g) \Big\| = \|I_m(fg)\| \\ &= \Big\| \sum_{n=1}^{N} a_n f_{j_n}\psi_{j_n} \Big\|_{L^1(m)} \overset{(4)}{\geq} C \sum_{n=1}^{N} a_n. \end{aligned}$$

This proves the claim. Note also that $\|K_{j_n}\| = \|f_{j_n}\|_{L^1(m)} = 1$ for all $n \in \mathbb{N}$.

Step 2. By Lemma 2.3, there exist $x^* \in X^*$ and a sequence (x_n) in X such that $x_n \in K_{j_n}$ and $x^*(x_n) \geq 1$ for infinitely many $n \in \mathbb{N}$. Define a projection on X by $R_n := P_{k_{j_n}} - P_{k_{j_{n-1}}}$ and set $y_n := R_n(x_n)$ for every $n \geq 2$. Since $\|x_n\| \leq 1$ and $\|R_n\| \leq 2$, we have $\|y_n\| \leq 2$ for every $n \geq 2$.

Claim. We have $x^*(y_n) \geq 1/2$ for infinitely many $n \geq 2$. Indeed, observe that for every $n \geq 2$ we have

$$\|x_n - y_n\| = \|Q_{k_{j_n}}(x_n) + P_{k_{j_{n-1}}}(x_n)\| \leq \|Q_{k_{j_n}}(x_n)\| + \|P_{k_{j_{n-1}}}(x_n)\| \overset{(2)}{\leq} \frac{1}{2^{j_n-1}}$$

and therefore

$$x^*(y_n) = x^*(x_n) - x^*(x_n - y_n) \geq x^*(x_n) - \|x^*\| \cdot \|x_n - y_n\| \geq x^*(x_n) - \frac{\|x^*\|}{2^{j_n-1}}. \tag{5}$$

The fact that $x^*(x_n) \geq 1$ for infinitely many $n \in \mathbb{N}$ and (5) ensure the existence of infinitely many $n \geq 2$ for which $x^*(y_n) \geq 1/2$.

Step 3. Observe that for every $k \in \mathbb{N}$ and $n \geq 2$ satisfying $k_{j_{n-1}} \geq k$ we have

$$e_k^*(y_n) = e_k^*\left(P_{k_{j_n}}(x_n) - P_{k_{j_{n-1}}}(x_n)\right) = e_k^*\left(\sum_{i=k_{j_{n-1}}+1}^{k_{j_n}} e_i^*(x_n)e_i\right) = 0.$$

Thus, for any $y^* \in \mathrm{span}(e_k^*)$ there is $n(y^*) \geq 2$ such that $y^*(y_{n(y^*)}) = 0$ and $x^*(y_{n(y^*)}) \geq 1/2$, hence

$$2\|x^* - y^*\| \geq \|y_{n(y^*)}\| \cdot \|x^* - y^*\| \geq x^*(y_{n(y^*)}) - y^*(y_{n(y^*)}) \geq \frac{1}{2},$$

and so $\|x^* - y^*\| \geq 1/4$. This contradicts the fact that (e_n) is shrinking and the proof is over. $\square$

Remark 3.4. The proof of Theorem 1.3 does not involve Rosenthal's ℓ_1-theorem, [6, Theorem 5.37]. This result is used in [10, Proposition 3.6] to extract a subsequence $(f_{j_n}\psi_{j_n})$ which is an ℓ_1-sequence in $L^1(m)$.

Remark 3.5. It is easy to check that a bounded unconditional basic sequence is an ℓ_1-sequence if (and only if) it is an ℓ_+-sequence. In the proof of Theorem 1.3, (y_n) is a block basic sequence of (e_n). If in addition (e_n) is assumed to be *unconditional*, then so is (y_n). Any ℓ_+-subsequence of (y_n) (such subsequences exist by Step 2) is therefore an ℓ_1-*sequence*. This is the final argument in [10, Proposition 3.6].

References

[1] J. M. Calabuig, J. Rodríguez, E. A. Sánchez-Pérez: On the structure of L^1 of a vector measure via its integration operator, Integral Equations Oper. Theory 64(1) (2009) 21–33.

[2] C. Castaing, M. Valadier: Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics 580, Springer, Berlin (1977).

[3] G. P. Curbera: Operators into L^1 of a vector measure and applications to Banach lattices, Math. Ann. 293(2) (1992) 317–330.

[4] G. P. Curbera: When L^1 of a vector measure is an AL-space, Pacific J. Math. 162(2) (1994) 287–303.

[5] J. Diestel, J. J. Uhl, Jr.: Vector Measures, Mathematical Surveys 15, American Mathematical Society, Providence (1977).

[6] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler: Banach Space Theory, CMS Books in Mathematics, Springer, Berlin (2011).

[7] H. Fetter, B. Gamboa de Buen: The James Forest, London Mathematical Society Lecture Note Series 236, Cambridge University Press, Cambridge (1997).

[8] N. Ghoussoub, E. Saab: On the weak Radon-Nikodým property, Proc. Amer. Math. Soc. 81(1) (1981) 81–84.

[9] S. Okada, W. J. Ricker, L. Rodríguez-Piazza: Compactness of the integration operator associated with a vector measure, Stud. Math. 150(2) (2002) 133–149.

[10] S. Okada, W. J. Ricker, L. Rodríguez-Piazza: Operator ideal properties of vector measures with finite variation, Stud. Math. 205(3) (2011) 215–249.

[11] S. Okada, W. J. Ricker, L. Rodríguez-Piazza: Operator ideal properties of the integration map of a vector measure, Indag. Math., New Ser., to appear, doi:10.1016/j.indag.2012.11.008.

[12] S. Okada, W. J. Ricker, E. A. Sánchez Pérez: Optimal Domain and Integral Extension of Operators, Acting in Function Spaces, Operator Theory: Advances and Applications 180, Birkhäuser, Basel (2008).

[13] I. Singer: Bases in Banach Spaces. I, Springer, New York (1970).

[14] M. Zippin: Banach spaces with separable duals, Trans. Amer. Math. Soc. 310(1) (1988) 371–379.