Journal of Convex Analysis
Volume 21 (2014), No. 3, 857–886

Proper Approximate Solutions and ε-Subdifferentials in Vector Optimization: Moreau-Rockafellar Type Theorems*

C. Gutiérrez
Departamento de Matemática Aplicada, E.T.S. de Ingenieros de Telecomunicación, Universidad de Valladolid, Paseo de Belén 15, Campus Miguel Delibes, 47011 Valladolid, Spain
cesargv@mat.uva.es

L. Huerga†
Departamento de Matemática Aplicada, E.T.S.I. Industriales, UNED, C/ Juan del Rosal 12, Ciudad Universitaria, 28040 Madrid, Spain
lhuerga@bec.uned.es

B. Jiménez
Departamento de Matemática Aplicada, E.T.S.I. Industriales, UNED, C/ Juan del Rosal 12, Ciudad Universitaria, 28040 Madrid, Spain
bjimenez@ind.uned.es

V. Novo
Departamento de Matemática Aplicada, E.T.S.I. Industriales, UNED, C/ Juan del Rosal 12, Ciudad Universitaria, 28040 Madrid, Spain
vnovo@ind.uned.es

Received: November 22, 2012
Revised manuscript received: July 24, 2013
Accepted: July 26, 2013

In this work, we provide Moreau-Rockafellar type theorems for a kind of proper ε-subdifferential of vector-valued mappings. For this aim, we introduce and study a new notion of approximate strong solution of a vector optimization problem, a new strong ε-subdifferential based on this type of approximate strong solutions, and a new regularity condition. Finally, as a consequence of the obtained exact sum rules, the gap between the strong and the proper ε-subdifferentials is provided.

Keywords: Vector optimization, proper ε-efficiency, proper ε-subdifferential, nearly subconvexlikeness, strong ε-efficiency, strong ε-subdifferential, linear scalarization, ε-subdifferential

2010 Mathematics Subject Classification: 90C48, 90C25, 90C29, 49J52, 49K27

*This work was partially supported by Ministerio de Economía y Competitividad (Spain) under project MTM2012-30942.
†Researcher of Spanish FPI fellowship Programme (BES-2010-033742).

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

1. Introduction

In [7] we introduced and studied a proper ε-subdifferential in the sense of Benson for extended vector-valued mappings, which is defined by means of a generalization of a Benson proper ε-efficiency notion due to Gao, Yang and Teo [5], based on an ε-efficiency concept introduced by Gutiérrez, Jiménez and Novo (see [9, 10]). This proper ε-subdifferential, called Benson (C, ε)-proper subdifferential, generalizes the proper ε-subdifferential defined by El Maghri [14] and Tuan [20] and improves them, in the sense that it has a better limit behaviour when the error ε tends to zero (see Sections 3 and 4 in [7]).

The main aim of this work is to obtain Moreau-Rockafellar type theorems, i.e., some exact rules to calculate the Benson (C, ε)-proper subdifferential of the sum of two mappings. These results are obtained by scalarization, under certain convexity assumptions, via well-known properties of the Brøndsted-Rockafellar ε-subdifferential of an extended real-valued convex mapping.

For this aim, we introduce and study a strong ε-subdifferential related to a new concept of approximate strong solution of vector optimization problems, and a regularity condition that extends the well-known regularity condition due to Raffin [18] (see also [14, 15, 21, 25]) to the Benson (C, ε)-proper subdifferential.

Several results of this work extend other similar in [14, 15] referred to proper subdifferentials and ε-subdifferentials of extended vector-valued mappings, since they are based on a more general proper ε-efficiency concept, and also because they have been obtained under weaker convexity assumptions.

This work is structured as follows. In Section 2, we fix the framework, the notations and the main concepts that are going to be used along the paper. In Section 3, we define and study a new approximate strong solution concept of vector optimization problems, which generalizes the approximate strong solution concept due to Kutateladze (see [13]). These solutions are characterized through linear scalarizations without assuming any convexity assumption.

In Section 4, we introduce and study a strong ε-subdifferential for extended vector-valued mappings in connection with the approximate strong solution notion defined in Section 3. In particular, we characterize it through ε-subgradients of extended real-valued mappings without assuming any convexity assumption, and we obtain a simple formula for it when the objective space of the mapping is finite-dimensional and the ordering cone is the nonnegative orthant.

This strong ε-subdifferential, called (C, ε)-strong subdifferential, motivates a regularity condition that is defined and studied in Section 5. It is more general than the regularity condition introduced by El Maghri [15]. However, we show some conditions under which both coincide.

Finally, in Section 6, we prove two Moreau-Rockafellar type theorems for Benson (C, ε)-proper subdifferentials. The first one is based on the regularity condition due to El Maghri [15] and involves strong ε-subgradients associated with approximate strong solutions in the sense of Kutateladze. The second one considers the regularity condition defined in Section 5 and involves (C,ε)-strong subgradients. Both sum rules are exact and from them we state the gap between the Benson (C,ε)-proper subdifferential and the (C,ε)-strong subdifferential.

2. Preliminaries

Let X be a topological vector space and Y be an ordered Hausdorff locally convex topological vector space with ordering cone D. We consider that D is nontrivial ($D \neq \{0\}$), pointed ($D \cap (-D) = \{0\}$), convex and closed. As usual, the order in Y is given by D as follows:

$$y_1, y_2 \in Y,\ y_1 \leq_D y_2 \Longleftrightarrow y_2 - y_1 \in D.$$

The topological dual spaces of X and Y are denoted by X^* and Y^*, respectively and the duality pairing in X (resp. Y) is denoted by $\langle x^*, x\rangle := x^*(x)$, $x^* \in X^*$, $x \in X$ (resp. $\langle y^*, y\rangle := y^*(y)$, $y^* \in Y^*$, $y \in Y$). We consider a locally convex topology $\mathcal{T}$ on Y^* compatible with the dual pair, i.e., $(Y^*, \mathcal{T})^* = Y$ (see [16, 17]). Recall that the Mackey topology of Y^* is the finest topology with this property. The set of continuous linear mappings from X to Y is denoted by $\mathcal{L}(X, Y)$.

For a set $F \subset Y$, we denote by $\operatorname{int} F$, $\operatorname{cl} F$, $\operatorname{co} F$ and $\operatorname{cone} F$ the topological interior, the closure, the convex hull and the cone generated by F, respectively. Moreover, it is said that F is solid if $\operatorname{int} F \neq \emptyset$ and F is coradiant if $\alpha F \subset F$, $\forall\, \alpha \geq 1$. We denote the coradiant set generated by F by $\operatorname{shw} F$ ("shadow" of F, see [23]), i.e.,

$$\operatorname{shw} F = \bigcup_{\alpha \geq 1} \alpha F.$$

We adjoint to Y an abstract element $+\infty_Y$ so that extending the ordering $\leq_D$ to $\overline{Y} := Y \cup \{+\infty_Y\}$ by setting $y \leq_D +\infty_Y$ for all $y \in Y$, the element $+\infty_Y$ appears as the greatest one of the ordered set $(\overline{Y}, \leq_D)$. The usual algebraic operations are extended as follows:

$$+\infty_Y + y = y + (+\infty_Y) = +\infty_Y, \quad \alpha \cdot (+\infty_Y) = +\infty_Y, \quad \forall\, y \in Y,\ \forall\, \alpha > 0.$$

We assume that $0 \cdot (+\infty_Y) = 0$ and we denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, where $+\infty := +\infty_{\mathbb{R}}$.

We denote the positive and the strict positive polar cones of D by D^+ and D^{s+}, respectively, i.e.,

$$D^+ = \{\mu \in Y^* : \langle \mu, d\rangle \geq 0, \forall\, d \in D\},$$
$$D^{s+} = \{\mu \in Y^* : \langle \mu, d\rangle > 0, \forall\, d \in D \backslash \{0\}\}.$$

The nonnegative orthant of $\mathbb{R}^p$ is denoted by $\mathbb{R}^p_+$ and $\mathbb{R}_+ := \mathbb{R}^1_+$.

The cone D is said to be based if there exists a nonempty convex set $B \subset D$, which is called a base of D, such that $0 \notin \operatorname{cl} B$ and $\operatorname{cone} B = D$ (see [6, Definiton 2.2.14(i)]).

Given an arbitrary mapping $f : X \to \overline{Y}$, we denote the effective domain of f by $\operatorname{dom} f$, i.e., $\operatorname{dom} f = \{x \in X : f(x) \in Y\}$, and we say that f is proper if $\operatorname{dom} f \neq \emptyset$. For each $y^* \in D^+\backslash\{0\}$ we assume that $\langle y^*, f(x)\rangle = +\infty$, $\forall\, x \in X, x \notin \operatorname{dom} f$ (see [14, 15]). Thus, $y^* \circ f : X \to \overline{\mathbb{R}}$ and $\operatorname{dom}(y^* \circ f) = \operatorname{dom} f$.

We study the following vector optimization problem:

$$\text{Minimize } f(x) \text{ subject to } x \in S, \tag{$\mathcal{P}_\mathrm{S}$}$$

where f is proper and $S \subset X$ is nonempty. We assume that $S_0 := S \cap \operatorname{dom} f \neq \emptyset$ in order to avoid trivial problems. We say that $(\mathcal{P}_\mathrm{S})$ is a Pareto problem when $Y = \mathbb{R}^p$ and $D = \mathbb{R}^p_+$. In this case we write f as $f = (f_1, f_2, \dots, f_p)$.

Let us recall that f is D-convex on S if S is convex and

$$f(\alpha x_1 + (1-\alpha)x_2) \leq_D \alpha f(x_1) + (1-\alpha) f(x_2), \quad \forall\, x_1, x_2 \in S,\ \forall\, \alpha \in [0,1].$$

In particular, if f is D-convex on S, then S_0 is convex.

Now, we recall two classical solution notions of problem $(\mathcal{P}_\mathrm{S})$. It is said, (see, for instance, [6, 11, 19]) that a feasible point $x_0 \in S$ is a strong solution of problem $(\mathcal{P}_\mathrm{S})$, denoted by $x_0 \in \mathrm{SE}(f,S)$, if $f(x_0) \leq_D f(x)$, for all $x \in S$. Moreover, a point $x_0 \in S_0$ is a Benson proper efficient solution of problem $(\mathcal{P}_\mathrm{S})$, denoted by $x_0 \in \mathrm{Be}(f,S)$, if

$$\operatorname{cl}\operatorname{cone}(f(S_0) - f(x_0) + D) \cap (-D) = \{0\}.$$

It is clear that $\mathrm{SE}(f,S) \subset \mathrm{Be}(f,S) \subset \operatorname{dom} f$.

In the sequel we show some well-known approximate solution notions of problem $(\mathcal{P}_\mathrm{S})$, which will be used in the paper. The first one was introduced by Kutateladze (see [13]) and generalizes the concept of strong solution from an approximate point of view. The second one was introduced by Tuan [20] and is an approximate version of the Benson proper efficient solution notion of $(\mathcal{P}_\mathrm{S})$ (see [1]).

Definition 2.1. Let $q \in D$. It is said that $x_0 \in S$ is a strong q-efficient solution of $(\mathcal{P}_\mathrm{S})$, denoted by $\mathrm{SE}(f,S,q)$, if

$$f(x_0) - q \leq_D f(x), \quad \forall\, x \in S.$$

Definition 2.2. Consider $q \in D$ and $\varepsilon \geq 0$. A point $x_0 \in S_0$ is a Benson ε-proper solution of problem $(\mathcal{P}_\mathrm{S})$ with respect to q, denoted by $x_0 \in \mathrm{Be}(f,S,q,\varepsilon)$, if

$$\operatorname{cl}\operatorname{cone}(f(S_0) + \varepsilon q - f(x_0) + D) \cap (-D) = \{0\}.$$

Clearly, $\mathrm{SE}(f,S,0) = \mathrm{SE}(f,S)$, $\mathrm{Be}(f,S,q,0) = \mathrm{Be}(f,S)$ and

$$\mathrm{SE}(f,S,\varepsilon q) \subset \mathrm{Be}(f,S,q,\varepsilon) \subset \operatorname{dom} f, \quad \forall\, q \in D.$$

For each nonempty set $C \subset Y$, we define the set-valued mapping $C : \mathbb{R}_+ \to 2^Y$ as follows:

$$C(\varepsilon) = \begin{cases} \varepsilon C & \text{if } \varepsilon > 0 \\ \operatorname{cone} C & \text{if } \varepsilon = 0. \end{cases}$$

The following notion of approximate efficiency was introduced by Gutiérrez, Jiménez and Novo (see [9, 10]) and generalizes the most important approximate efficiency notions in the literature.

Definition 2.3. Let $C \subset Y$ be nonempty and $\varepsilon \geq 0$. We say that $x_0 \in S_0$ is a (C, ε)-efficient solution of problem $(\mathcal{P}_\mathrm{S})$, denoted by $x_0 \in \mathrm{AE}(f, S, C, \varepsilon)$, if

$$(f(S_0) - f(x_0)) \cap (-C(\varepsilon)\backslash\{0\}) = \emptyset.$$

Obviously, if $\operatorname{cone} C = D$, then $\mathrm{AE}(f, S, C, 0) = \mathrm{E}(f, S)$, where $\mathrm{E}(f, S)$ denotes the set of exact efficient solutions of $(\mathcal{P}_\mathrm{S})$.

Next, we recall a meaningful generalization of the proper ε-efficiency notion due to Gao, Yang and Teo [5] (see Remark 2.4 in [7]). It can be considered as a proper version in the sense of Benson of the ε-efficiency notion due to Gutiérrez, Jiménez and Novo [9, 10].

Definition 2.4 ([7, Definition 2.3]). Consider $\varepsilon \geq 0$ and a nonempty set $C \subset Y$ such that $\operatorname{cl} C \cap (-D\backslash\{0\}) = \emptyset$. A point $x_0 \in S_0$ is a Benson (C, ε)-proper solution of $(\mathcal{P}_\mathrm{S})$, denoted by $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$, if

$$\operatorname{cl}\operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-D) = \{0\}.$$

It follows that

$$\mathrm{Be}(f, S, q, \varepsilon) = \mathrm{Be}(f, S, C_q, \varepsilon) = \mathrm{Be}(f, S, C_q^0, \varepsilon), \quad \forall\, q \in D,\ \forall\, \varepsilon \geq 0,$$

where $C_q := q + D$ and $C_q^0 := q + D\backslash\{0\}$.

In the following definition, we recall the Brønsted-Rockafellar ε-subdifferential [4].

Definition 2.5. Let $h : X \to \overline{\mathbb{R}}$ be a proper mapping and $\varepsilon \geq 0$. The ε-subdifferential of h at a point $x_0 \in \operatorname{dom} h$ is defined as follows:

$$\partial_\varepsilon h(x_0) = \{x^* \in X^* : h(x) \geq h(x_0) - \varepsilon + \langle x^*, x - x_0\rangle, \forall\, x \in X\}.$$

The elements of $\partial_\varepsilon h(x_0)$ are called ε-subgradients of h at x_0.

Given a proper mapping $h : X \to \overline{\mathbb{R}}$, a nonempty set $M \subset X$ such that $M_0 := M \cap \operatorname{dom} h \neq \emptyset$ and $\varepsilon \geq 0$, we denote

$$\varepsilon\text{-argmin}_M h = \{x \in M : h(x) - \varepsilon \leq h(z), \forall\, z \in M\} \subset \operatorname{dom} h$$

and $\operatorname{argmin}_M h := 0\text{-argmin}_M h$. The elements of $\varepsilon\text{-argmin}_M h$ are the suboptimal solutions with error ε of the following scalar optimization problem:

$$\text{Minimize } h(x) \text{ subject to } x \in M.$$

For $\varepsilon \geq 0$ and $x_0 \in \operatorname{dom} h$, we have that $x^* \in \partial_\varepsilon h(x_0)$ if and only if $x_0 \in \varepsilon\text{-}\operatorname{argmin}_X(h - x^*)$. In particular, given $x_0 \in \operatorname{dom} f$ and two arbitrary mappings $y^* \in D^+ \backslash \{0\}$ and $A \in \mathcal{L}(X, Y)$, it follows that

$$x_0 \in \varepsilon\text{-}\operatorname{argmin}_X(y^* \circ (f - A)) \iff y^* \circ A \in \partial_\varepsilon(y^* \circ f)(x_0). \tag{1}$$

In the following definition we recall a generalized convexity notion introduced by Gutiérrez, Huerga and Novo [8]. It is an "approximate" version of the well-known nearly subconvexlikeness concept due to Yang, Li and Wang [22]. Theorem 2.7 in [7] provides sufficient conditions for checking the nearly (C, ε)-subconvexlikeness of a map $f : X \to \overline{Y}$ on M.

Definition 2.6. Consider $\varepsilon \geq 0$, two nonempty sets $M \subset X$ and $C \subset Y$ and a proper vector-valued mapping $f : X \to \overline{Y}$ such that $M_0 \neq \emptyset$. It is said that f is nearly (C, ε)-subconvexlike on M if $\operatorname{cl} \operatorname{cone}(f(M_0) + C(\varepsilon))$ is convex.

Finally, for a nonempty set $C \subset Y$, we define the mapping $\tau_C : Y^* \to \mathbb{R} \cup \{-\infty\}$ as follows:

$$\tau_C(y^*) = \inf_{y \in C}\{\langle y^*, y\rangle\}, \quad \forall\, y^* \in Y^*.$$

We also denote

$$C^{\tau+} := \{y^* \in Y^* : \tau_C(y^*) \geq 0\},$$
$$\mathcal{F} := \{C \subset Y : C \neq \emptyset, D^{s+} \cap C^{\tau+} \neq \emptyset\}.$$

Given two nonempty sets $C_1, C_2 \subset Y$ and $\alpha_1, \alpha_2 \in \mathbb{R}_+$, observe that

$$\tau_{C_1(\alpha_1)+C_2(\alpha_2)}(y^*) = \alpha_1 \tau_{C_1}(y^*) + \alpha_2 \tau_{C_2}(y^*), \quad \forall\, y^* \in C_1^{\tau+} \cap C_2^{\tau+}. \tag{2}$$

It is clear that

$$D^{s+} \cap C^{\tau+} \neq \emptyset \Rightarrow \operatorname{cl}(C(0)) \cap (-D\backslash\{0\}) = \emptyset.$$

Moreover, if C is convex and $\operatorname{int} D^+ \neq \emptyset$, by [3, Proposition 2] the reciprocal implication is also true.

3. (C, ε)-strong efficiency

The following new concept of approximate strong solution of problem $(\mathcal{P}_\mathrm{S})$ is motivated by the (C, ε)-efficiency notion due to Gutiérrez, Jiménez and Novo (see Definition 2.3) and it generalizes the strong q-efficiency notion (see Definition 2.1).

Definition 3.1. Let $\varepsilon \geq 0$ and a nonempty set $C \subset D$. A point $x_0 \in S$ is a (C, ε)-strong efficient solution of $(\mathcal{P}_\mathrm{S})$, denoted by $x_0 \in \mathrm{SE}(f, S, C, \varepsilon)$, if

$$f(x_0) - q \leq_D f(x), \quad \forall\, q \in C(\varepsilon),\ \forall\, x \in S. \tag{3}$$

It is clear from Definitions 2.1 and 3.1 that x_0 is a (C,ε)-strong efficient solution of $(\mathcal{P}_\mathrm{S})$ if and only if x_0 is a strong q-efficient solution of $(\mathcal{P}_\mathrm{S})$ for each $q \in C(\varepsilon)$, i.e.,

$$\begin{aligned} x_0 \in \mathrm{SE}(f,S,C,\varepsilon) \quad &\Longleftrightarrow \quad f(S_0) \subset \bigcap_{q\in C(\varepsilon)} (f(x_0) - q + D) \\ &\Longleftrightarrow \quad f(S_0) - f(x_0) + C(\varepsilon) \subset D. \end{aligned} \tag{4}$$

Moreover, $\mathrm{SE}(f,S,C,\varepsilon) \subset \operatorname{dom} f$ and then statement (3) can be checked only for points $x \in S_0$. In the following proposition we show the main properties of this new kind of approximate strong solution of problem $(\mathcal{P}_\mathrm{S})$. Several of them are consequence of the following previous lemma.

Lemma 3.2. *Consider $\emptyset \neq F \subset Y$ and $\emptyset \neq C_1, C_2 \subset Y$. If $C_2 \subset \mathrm{cl}(C_1 + D)$, then*

$$F + C_1 \subset D \Rightarrow F + \operatorname{co} C_2 \subset D,$$

and if $C_2 \subset D$, then

$$F + C_2 \subset D \Rightarrow F + \mathrm{shw}(\operatorname{co} C_2) \subset D.$$

Proof. Suppose that $C_2 \subset \mathrm{cl}(C_1 + D)$ and let $y \in F$ and $q \in C_2$. There exist two nets $(q_i) \subset C_1$ and $(d_i) \subset D$ such that $q_i + d_i \to q$. As $F + C_1 \subset D$ and D is closed we deduce that

$$y + q = \lim((y + q_i) + d_i) \in \mathrm{cl}(D + D) = D$$

and then we have

$$F + C_2 \subset D. \tag{5}$$

Consider $q \in \operatorname{co} C_2$. There exist $q_j \in C_2$ and $\alpha_j \in [0,1]$, $1 \leq j \leq k$, $k \in \mathbb{N}$, $k \geq 1$, such that $\sum_{j=1}^k \alpha_j = 1$ and $q = \sum_{j=1}^k \alpha_j q_j$. Thus, by (5) we have

$$F + q \subset \sum_{j=1}^k \alpha_j (F + q_j) \subset D$$

and then we deduce that

$$F + C_2 \subset D \Rightarrow F + \operatorname{co} C_2 \subset D. \tag{6}$$

Suppose that $C_2 \subset D$, $F + C_2 \subset D$ and let $q \in \mathrm{shw}(\operatorname{co} C_2)$. There exist $\alpha \geq 1$ and $d \in \operatorname{co} C_2 \subset D$ such that $q = \alpha d$ and by (6) it follows that

$$F + q = F + d + (\alpha - 1)d \subset D,$$

i.e., $F + \mathrm{shw}(\operatorname{co} C_2) \subset D$, which finishes the proof. □

Recall that an element $\bar{y} \in Y$ is the infimum of a nonempty set $F \subset Y$ if the following two properties hold:

(a) $\bar{y} \leq_D y$, for all $y \in F$.
(b) If for some $y' \in Y$ one has $y' \leq_D y$ for all $y \in F$, then $y' \leq_D \bar{y}$.

If there exists an infimum of F, it is unique and we denote it by $\mathrm{Inf}_D F$.

Proposition 3.3. *Consider $\varepsilon \geq 0$ and a nonempty set $C \subset D$. It follows that*

(a) $\mathrm{SE}(f,S) \subset \mathrm{SE}(f,S,C,\varepsilon) = \bigcap_{\delta>\varepsilon} \mathrm{SE}(f,S,C,\delta)$.
(b) $\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S)$, *if* $0 \in \mathrm{cl}\, C(\varepsilon)$.
(c) *Consider a nonempty set $G \subset D$ and $\delta \geq 0$ satisfying $G(\delta) \subset \mathrm{cl}(C(\varepsilon)+D)$. Then $\mathrm{SE}(f,S,C,\varepsilon) \subset \mathrm{SE}(f,S,G,\delta)$. In particular,*

$$\mathrm{SE}(f,S,C,\varepsilon) \subset \mathrm{SE}(f,S,C+C',\varepsilon), \quad \forall\, C' \subset D, C' \neq \emptyset, \tag{7}$$

$$\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S,C+C',\varepsilon), \quad \forall\, C' \subset D,\, 0 \in \mathrm{cl}\, C'. \tag{8}$$

(d) $\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S,\mathrm{co}\, C,\varepsilon) = \mathrm{SE}(f,S,\mathrm{shw}\, C,\varepsilon) = \mathrm{SE}(f,S,\mathrm{cl}\, C,\varepsilon)$.
(e) *If $\bar{q} = \mathrm{Inf}_D C$, then* $\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S,\varepsilon\bar{q})$.
(f) *If $\bigcap_{c\in C}(-c+D) = D$, then* $\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S)$.

Proof. (a) Let $x_0 \in \mathrm{SE}(f,S)$. By applying Lemma 3.2 to $F = f(S_0) - f(x_0)$, $C_1 = \{0\}$ and $C_2 = C(\varepsilon)$ we deduce $f(S_0) - f(x_0) + C(\varepsilon) \subset D$, i.e., $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$ by (4).

Analogously, let $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$ and $\delta > \varepsilon$. It is clear that $C(\delta) \subset C(\varepsilon) + D$. Then, by applying Lemma 3.2 to $F = f(S_0) - f(x_0)$, $C_1 = C(\varepsilon)$ and $C_2 = C(\delta)$ we deduce that $x_0 \in \mathrm{SE}(f,S,C,\delta)$.

Reciprocally, let $x_0 \in \bigcap_{\delta>\varepsilon} \mathrm{SE}(f,S,C,\delta)$. As

$$C(\varepsilon) \subset \mathrm{cl}\left(\bigcup_{\delta>\varepsilon} C(\delta)\right),$$

by applying Lemma 3.2 to $F = f(S_0) - f(x_0)$, $C_1 = \bigcup_{\delta>\varepsilon} C(\delta)$ and $C_2 = C(\varepsilon)$ we deduce that $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$ by (4).

(b) Suppose that $0 \in \mathrm{cl}\, C(\varepsilon)$ and consider $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$. Then by applying Lemma 3.2 to $F = f(S_0) - f(x_0)$, $C_1 = C(\varepsilon)$ and $C_2 = \{0\}$ we deduce that $f(S_0) - f(x_0) \subset D$, i.e., $x_0 \in \mathrm{SE}(f,S)$. The reciprocal inclusion follows by part (a).

(c) The inclusion

$$\mathrm{SE}(f,S,C,\varepsilon) \subset \mathrm{SE}(f,S,G,\delta) \tag{9}$$

is a direct consequence of Lemma 3.2 and (4). Moreover, statement (7) follows by applying (9) to $G = C + C'$ and $\delta = \varepsilon$, and statement (8) follows from (7) and by applying (9) to $G = C$, $C + C'$ instead of C and $\delta = \varepsilon$.

(d) By part (c) we see that

$$\begin{aligned}\mathrm{SE}(f,S,\operatorname{co}C,\varepsilon)&\subset \mathrm{SE}(f,S,C,\varepsilon),\\ \mathrm{SE}(f,S,\operatorname{shw}C,\varepsilon)&\subset \mathrm{SE}(f,S,C,\varepsilon),\\ \mathrm{SE}(f,S,\operatorname{cl}C,\varepsilon)&\subset \mathrm{SE}(f,S,C,\varepsilon).\end{aligned}$$

The reciprocal inclusions are consequence of applying Lemma 3.2 to $F = f(S_0) - f(x_0)$ and $C_1 = C_2 = C(\varepsilon)$ (for the first two inclusions) and $C_1 = C(\varepsilon)$, $C_2 = (\operatorname{cl} C)(\varepsilon) = \operatorname{cl} C(\varepsilon)$ (for the last inclusion).

(e) The result is clear for $\varepsilon = 0$ by part (b). Suppose that $\varepsilon > 0$ and let $x_0 \in \mathrm{SE}(f,S,\varepsilon\bar{q})$. Then $x_0 \in S_0$ and $f(x_0) - \varepsilon\bar{q} \leq_D f(x)$, for all $x \in S_0$. Since $\bar{q} = \mathrm{Inf}_D C$, it follows that

$$f(x_0) - f(x) \leq_D \varepsilon\bar{q} \leq_D \varepsilon c, \quad \forall\, c \in C,\ \forall\, x \in S_0.$$

Hence, $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$. Now, take $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$. Then, $x_0 \in S_0$ and

$$f(x_0) - f(x) \leq_D \varepsilon c, \quad \forall\, x \in S_0,\ \forall\, c \in C.$$

Therefore, as $\bar{q} = \mathrm{Inf}_D C$, for each fixed $x \in S_0$ it follows that $f(x_0) - f(x) \leq_D \varepsilon\bar{q}$. This inequality holds for every $x \in S_0$ and so $x_0 \in \mathrm{SE}(f,S,\varepsilon\bar{q})$.

(f) The result is clear for $\varepsilon = 0$ by part (b). Suppose that $\varepsilon > 0$ and let $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$. By (4) we have that

$$f(S_0) - f(x_0) \subset \bigcap_{q\in C(\varepsilon)} (-q + D) = \bigcap_{c\in C}(-\varepsilon c + D) = \varepsilon \bigcap_{c\in C}(-c + D) = D.$$

Hence, $x_0 \in \mathrm{SE}(f,S)$ and so $\mathrm{SE}(f,S,C,\varepsilon) \subset \mathrm{SE}(f,S)$. The reciprocal inclusion follows by part (a) and the proof is complete. □

Remark 3.4. By Proposition 3.3 (c), (d) it follows that

$$\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S,\operatorname{co} C + D,\varepsilon)$$

i.e., we can deal with (C,ε)-strong efficient solutions of problem $(\mathcal{P}_{\mathrm{S}})$ by assuming that C is convex and $C = C + D$, which implies that C is coradiant.

Next we characterize the (C,ε)-strong efficient solutions of problem $(\mathcal{P}_{\mathrm{S}})$ through approximate solutions of associated scalar optimization problems.

Theorem 3.5. *Consider $\varepsilon \geq 0$ and a nonempty set $C \subset D$. Then,*

$$\mathrm{SE}(f,S,C,\varepsilon) = \bigcap_{\mu\in D^+\setminus\{0\}} \varepsilon\tau_C(\mu)\text{-}\mathrm{argmin}_S(\mu\circ f).$$

Proof. By the bipolar theorem (see for instance [14, Proposition 2.1(1)]) we deduce that

$$
\begin{aligned}
& x_0 \in \bigcap_{\mu \in D^+\backslash\{0\}} \varepsilon\tau_C(\mu)\text{-argmin}_S(\mu \circ f) \\
\iff\ & \langle \mu, f(x_0)\rangle \leq \langle \mu, f(x)\rangle + \tau_{C(\varepsilon)}(\mu), \quad \forall\, x \in S_0, \forall\, \mu \in D^+\backslash\{0\} \\
\iff\ & \langle \mu, f(x_0)\rangle \leq \langle \mu, f(x)\rangle + \langle \mu, q\rangle, \quad \forall\, q \in C(\varepsilon), \forall\, x \in S_0, \forall\, \mu \in D^+\backslash\{0\} \\
\iff\ & \langle \mu, f(x) - f(x_0) + q\rangle \geq 0, \quad \forall\, q \in C(\varepsilon), \forall\, x \in S_0, \forall\, \mu \in D^+\backslash\{0\} \\
\iff\ & f(x) - f(x_0) + q \in D, \quad \forall\, q \in C(\varepsilon), \forall\, x \in S_0 \\
\iff\ & x_0 \in \mathrm{SE}(f, S, C, \varepsilon)
\end{aligned}
$$

and the proof is complete. □

4. (C, ε)-strong subdifferential

In this section, we introduce and study a new notion of strong ε-subdifferential for extended vector-valued mappings. After that, we characterize it through ε-subgradients of associated scalar mappings.

Definition 4.1. Let $\varepsilon \geq 0$ and let $C \subset D$ be a nonempty set. The (C, ε)-strong subdifferential of f at a point $x_0 \in \operatorname{dom} f$ is defined as follows:

$$\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) := \{T \in \mathcal{L}(X, Y) : x_0 \in \mathrm{SE}(f - T, X, C, \varepsilon)\}.$$

In the sequel, the usual (exact) strong subdifferential of f at $x_0 \in \operatorname{dom} f$ is denoted by $\partial^{\mathrm{s}} f(x_0)$, i.e. (see, for instance, [15]),

$$
\begin{aligned}
\partial^{\mathrm{s}} f(x_0) &= \{T \in \mathcal{L}(X, Y) : f(x) \geq_D f(x_0) + T(x - x_0), \forall\, x \in X\} \\
&= \{T \in \mathcal{L}(X, Y) : x_0 \in \mathrm{SE}(f - T, X)\}.
\end{aligned}
$$

Remark 4.2. Definition 4.1 reduces to the ε-subdifferential notion introduced by Kutatelazde [13] by taking a singleton $C = \{q\}$, $q \in D$ (see also [14]).

As a direct consequence of Proposition 3.3, we obtain the following properties.

Proposition 4.3. *Let $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ and a nonempty set $C \subset D$. It follows that:*

(a) $\partial^{\mathrm{s}} f(x_0) \subset \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \bigcap_{\delta > \varepsilon} \partial^{\mathrm{s}}_{C,\delta} f(x_0)$.

(b) $\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{s}} f(x_0)$, *if* $0 \in \operatorname{cl} C(\varepsilon)$.

(c) *Consider a nonempty set $G \subset D$ and $\delta \geq 0$ satisfying $G(\delta) \subset \operatorname{cl}(C(\varepsilon) + D)$. Then $\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) \subset \partial^{\mathrm{s}}_{G,\delta} f(x_0)$. In particular,*

$$
\begin{aligned}
\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) &\subset \partial^{\mathrm{s}}_{C+C',\varepsilon} f(x_0), \quad \forall\, C' \subset D, C' \neq \emptyset, \\
\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) &= \partial^{\mathrm{s}}_{C+C',\varepsilon} f(x_0), \quad \forall\, C' \subset D,\, 0 \in \operatorname{cl} C'.
\end{aligned}
$$

(d) $\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{\operatorname{co} C,\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{\operatorname{shw} C,\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{\operatorname{cl} C,\varepsilon} f(x_0)$.

(*e*) *If* $\bar{q} = \mathrm{Inf}_D C$, *then* $\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{\{\bar{q}\},\varepsilon} f(x_0)$.

(*f*) *If* $\bigcap_{c\in C}(-c + D) = D$, *then* $\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{s}} f(x_0)$.

In the following result we characterize the (C,ε)-strong subdifferential of f at a point $x_0 \in \mathrm{dom}\, f$ in terms of approximate subgradients of scalar mappings.

Theorem 4.4. *Let* $x_0 \in \mathrm{dom}\, f$, $\varepsilon \geq 0$ *and a nonempty set* $C \subset D$. *Then,*

$$\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \bigcap_{\mu\in D^+\backslash\{0\}} \{T \in \mathcal{L}(X,Y) : \mu\circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu\circ f)(x_0)\}.$$

Proof. Let $A \in \mathcal{L}(X,Y)$. From Definition 4.1 and Theorem 3.5 we have that $A \in \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0)$ if and only if

$$x_0 \in \bigcap_{\mu\in D^+\backslash\{0\}} \varepsilon\tau_C(\mu)\text{-}\mathrm{argmin}_X(\mu\circ(f-A)).$$

Hence, by applying (1) it follows that

$$A \in \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) \Longleftrightarrow A \in \bigcap_{\mu\in D^+\backslash\{0\}} \{T \in \mathcal{L}(X,Y) : \mu\circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu\circ f)(x_0)\}$$

and the proof is complete. □

Remark 4.5. (a) Let $C \subset D$ and $q \in D$. By Proposition 4.3(*b*) and Theorem 4.4 we have

$$\partial^{\mathrm{s}} f(x_0) = \partial^{\mathrm{s}}_{C,0} f(x_0) = \bigcap_{\mu\in D^+\backslash\{0\}} \{T \in \mathcal{L}(X,Y) : \mu\circ T \in \partial(\mu\circ f)(x_0)\},$$

$$\partial^{\mathrm{s}}_{\{q\},1} f(x_0) = \bigcap_{\mu\in D^+\backslash\{0\}} \{T \in \mathcal{L}(X,Y) : \mu\circ T \in \partial_{\langle\mu,q\rangle}(\mu\circ f)(x_0)\}.$$

Therefore, Theorem 4.4 generalizes the first parts of [14, Theorem 3.2] and [15, Theorem 3.2].

(b) Let $Y = \mathbb{R}$, $D = \mathbb{R}_+$, $x_0 \in \mathrm{dom}\, f$, $\varepsilon \geq 0$ and a nonempty set $C \subset \mathbb{R}_+$. It follows that

$$\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \partial_{\varepsilon \inf C} f(x_0), \tag{10}$$

where $\inf C := \mathrm{Inf}_{\mathbb{R}_+} C$ denotes the infimum of C. Indeed, given $T \in \mathcal{L}(X,Y)$, it follows that

$$\begin{aligned}
T \in \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) &\Longleftrightarrow x_0 \in \mathrm{SE}(f-T,X,C,\varepsilon) \text{ by Definition 4.1}\\
&\Longleftrightarrow x_0 \in \mathrm{SE}(f-T,X,\{\inf C\},\varepsilon) \text{ by Proposition 3.3}(e)\\
&\Longleftrightarrow x_0 \in \varepsilon \inf C\text{-}\mathrm{argmin}_X(f-T) \text{ by Definition 3.1}\\
&\Longleftrightarrow T \in \partial_{\varepsilon \inf C} f(x_0), \text{ by (1)}.
\end{aligned}$$

By statement (10) we see that the (C,ε)-strong subdifferential reduces to the ε-subdifferential of a scalar mapping when $\inf C = 1$.

Next theorem provides an easy calculus rule to obtain the (C,ε)-strong subdifferential in the Pareto case.

Theorem 4.6. *Let $Y=\mathbb{R}^p$, $D=\mathbb{R}^p_+$, $x_0\in\operatorname{dom} f$, $\varepsilon\geq 0$ and a nonempty set $C\subset\mathbb{R}^p_+$. It follows that*

$$\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0)=\prod_{i=1}^{p}\partial_{\varepsilon\tau_C(\bar{e}_i)} f_i(x_0),$$

where $\{\bar{e}_i\}_{1\leq i\leq p}$ is the canonical base of $\mathbb{R}^p$.

Proof. Let $T=(T_1,T_2,\ldots,T_p)\in\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0)$. From Theorem 4.4 we have that

$$\mu\circ T\in\partial_{\varepsilon\tau_C(\mu)}(\mu\circ f)(x_0),\quad\forall\,\mu\in\mathbb{R}^p_+\backslash\{0\}.$$

Then, for $\mu=\bar{e}_i$ it follows that

$$T_i\in\partial_{\varepsilon\tau_C(\bar{e}_i)} f_i(x_0),\quad\forall\, i=1,2,\ldots,p,$$

and so $T\in\prod_{i=1}^{p}\partial_{\varepsilon\tau_C(\bar{e}_i)} f_i(x_0)$.

For checking the reciprocal inclusion, let us consider $T\in\prod_{i=1}^{p}\partial_{\varepsilon\tau_C(\bar{e}_i)} f_i(x_0)$ and $\mu\in\mathbb{R}^p_+\backslash\{0\}$. Then, for each $i=1,2,\ldots,p$, $T_i\in\partial_{\varepsilon\tau_C(\bar{e}_i)} f_i(x_0)$ and $\mu_i\geq 0$, and so

$$\mu_i f_i(x)\geq\mu_i f_i(x_0)-\mu_i\varepsilon\tau_C(\bar{e}_i)+\mu_i T_i(x-x_0),\quad\forall\, x\in X,\ i=1,2,\ldots,p. \tag{11}$$

By adding in (11) we obtain that

$$\langle\mu,f(x)\rangle\geq\langle\mu,f(x_0)\rangle-\varepsilon\sum_{i=1}^{p}\mu_i\tau_C(\bar{e}_i)+\langle\mu,T(x-x_0)\rangle,\quad\forall\, x\in X. \tag{12}$$

Moreover,

$$\sum_{i=1}^{p}\mu_i\tau_C(\bar{e}_i)=\inf_{c\in C}\{\mu_1 c_1\}+\cdots+\inf_{c\in C}\{\mu_p c_p\}\leq\inf_{c\in C}\{\langle\mu,c\rangle\}=\tau_C(\mu). \tag{13}$$

Hence, from (12) and (13) we deduce that

$$\langle\mu,f(x)\rangle\geq\langle\mu,f(x_0)\rangle-\varepsilon\tau_C(\mu)+\langle\mu,T(x-x_0)\rangle,\quad\forall\, x\in X,$$

so $\mu\circ T\in\partial_{\varepsilon\tau_C(\mu)}(\mu\circ f)(x_0)$. Applying Theorem 4.4 we conclude that $T\in\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0)$, and the proof is finished. $\square$

Theorem 4.6 reduces to [14, statement (6)] and [15, statement (3.6)] by considering $\varepsilon=1$ and the sets $C=\{q\}$ and $C=D$, respectively.

5. Regularity condition

In this section, we introduce and study a new regularity condition for extended vector-valued mappings, which is defined in terms of (C,ε)-strong subdifferentials. For this aim we assume that C is a nonempty subset of D and we denote

$$C^{s\tau+} := \{y^* \in Y^* : \tau_C(y^*) > 0\}.$$

Definition 5.1. Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. We say that f is regularly (C,ε)-subdifferentiable at x_0 if $\varepsilon = 0$ and

$$\partial(\mu \circ f)(x_0) = \mu \circ \partial^{\mathrm{s}} f(x_0), \quad \forall\, \mu \in D^{s+}, \tag{14}$$

or $\varepsilon > 0$ and

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0), \quad \forall\, \mu \in D^{s+} \cap C^{s\tau+}.$$

When statement (14) holds, we say that f is p-regularly subdifferentiable at x_0.

Remark 5.2. For all $C' \subset D$, $\varepsilon \geq 0$ and for all $A \in \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0)$, by Theorem 4.4 we have that

$$\mu \circ A \in \partial_{\varepsilon\tau_{C'}(\mu)}(\mu \circ f)(x_0), \quad \forall\, \mu \in D^+\backslash\{0\}. \tag{15}$$

Hence, f is regularly (C,ε)-subdifferentiable at x_0 if and only if $\varepsilon = 0$ and

$$\partial(\mu \circ f)(x_0) \subset \mu \circ \partial^{\mathrm{s}} f(x_0), \quad \forall\, \mu \in D^{s+},$$

or $\varepsilon > 0$ and

$$\partial_\varepsilon(\mu \circ f)(x_0) \subset \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0), \quad \forall\, \mu \in D^{s+} \cap C^{s\tau+}.$$

For each $\mu \in D^{s+} \cap C^{s\tau+}$ and $\varepsilon > 0$, the set

$$\bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0)$$

can be rewritten in different ways and from them, several equivalent formulations of the regularly (C,ε)-subdifferentiablity are obtained, as it is showed in the following results.

Theorem 5.3. *Let $x_0 \in \operatorname{dom} f$ and $\varepsilon > 0$. The following statements are equivalent:*

(*a*) f *is regularly (C,ε)-subdifferentiable at x_0.*

(*b*) $\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=\tau_C(\mu)}} \mu \circ \partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}} f(x_0)$, $\forall\, \mu \in D^{s+} \cap C^{s\tau+}$.

(*c*) $\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=\varepsilon}} \mu \circ \partial^{\mathrm{s}}_{C',1} f(x_0)$, $\forall\, \mu \in D^{s+} \cap C^{s\tau+}$.

Proof. The result is a consequence of the equalities

$$\bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=\tau_C(\mu)}} \mu\circ\partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}} f(x_0) = \bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=\varepsilon}} \mu\circ\partial^{\mathrm{s}}_{C',1} f(x_0) \tag{16}$$

$$= \bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=1}} \mu\circ\partial^{\mathrm{s}}_{C',\varepsilon} f(x_0), \tag{17}$$

where μ is an arbitrary element of $D^{s+}\cap C^{s\tau+}$. Let us prove the equalities (16) and (17).

Consider $C'\subset D$ such that $\tau_{C'}(\mu)=\tau_C(\mu)$. It is obvious that

$$\partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}} f(x_0) = \partial^{\mathrm{s}}_{C'\left(\frac{\varepsilon}{\tau_C(\mu)}\right),1} f(x_0)$$

and by (2), $C'':=C'\left(\frac{\varepsilon}{\tau_C(\mu)}\right)\subset D$ satisfies $\tau_{C''}(\mu)=\frac{\varepsilon}{\tau_C(\mu)}\tau_{C'}(\mu)=\varepsilon$. Therefore,

$$\bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=\tau_C(\mu)}} \mu\circ\partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}} f(x_0) \subset \bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=\varepsilon}} \mu\circ\partial^{\mathrm{s}}_{C',1} f(x_0).$$

Consider $C'\subset D$ such that $\tau_{C'}(\mu)=\varepsilon$. It is clear that

$$\partial^{\mathrm{s}}_{C',1} f(x_0) = \partial^{\mathrm{s}}_{C'(1/\varepsilon),\varepsilon} f(x_0)$$

and by (2), $C'':=C'(1/\varepsilon)\subset D$ satisfies $\tau_{C''}(\mu)=(1/\varepsilon)\tau_{C'}(\mu)=1$. Thus, we see that

$$\bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=\varepsilon}} \mu\circ\partial^{\mathrm{s}}_{C',1} f(x_0) \subset \bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=1}} \mu\circ\partial^{\mathrm{s}}_{C',\varepsilon} f(x_0).$$

Finally, let $C'\subset D$ such that $\tau_{C'}(\mu)=1$. It follows that

$$\partial^{\mathrm{s}}_{C',\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{C'(\tau_C(\mu)),\frac{\varepsilon}{\tau_C(\mu)}} f(x_0)$$

and by (2), $C'':=C'(\tau_C(\mu))\subset D$ satisfies $\tau_{C''}(\mu)=\tau_C(\mu)\tau_{C'}(\mu)=\tau_C(\mu)$. Therefore,

$$\bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=1}} \mu\circ\partial^{\mathrm{s}}_{C',\varepsilon} f(x_0) \subset \bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=\tau_C(\mu)}} \mu\circ\partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}} f(x_0)$$

and the proof is complete. □

For certain vector optimization problems it is possible to give a unified formulation of the regularly (C,ε)-subdifferentiability notion, as it is showed in the following corollary.

Corollary 5.4. *Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. Assume that*

$$0 \notin \operatorname{cl} C, \quad \operatorname{argmin}_{\operatorname{cl} C} \mu \neq \emptyset, \ \forall \mu \in D^{s+}. \tag{18}$$

Then $D^{s+} \subset C^{s\tau+}$ and f is regularly (C, ε)-subdifferentiable at x_0 if and only if

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0), \quad \forall \mu \in D^{s+}. \tag{19}$$

Proof. If $\varepsilon = 0$, by Proposition 4.3(b) we deduce that $\partial^{\mathrm{s}}_{C',0} f(x_0) = \partial^s f(x_0)$, for all $C' \subset D$. Then statement (19) says that f is p-regularly subdifferentiable at x_0.

On the other hand, if $\varepsilon > 0$ then statement (19) reduces to the notion of regularly (C, ε)-subdifferentiability of f at x_0, since $D^{s+} \subset C^{s\tau+}$. Indeed, for each $\mu \in D^{s+}$, as $\operatorname{argmin}_{\operatorname{cl} C} \mu \neq \emptyset$, there exists $q \in \operatorname{cl} C$ such that

$$\tau_C(\mu) = \inf\{\langle \mu, d \rangle : d \in C\} = \inf\{\langle \mu, d \rangle : d \in \operatorname{cl} C\} = \langle \mu, q \rangle > 0, \tag{20}$$

since $q \in \operatorname{cl} C \subset D \backslash \{0\}$ and $\mu \in D^{s+}$. $\square$

Remark 5.5. In order to apply Corollary 5.4, let us observe that the assumption

$$\operatorname{argmin}_{\operatorname{cl} C} \mu \neq \emptyset, \quad \forall \mu \in D^{s+}$$

is satisfied if some of the following conditions holds (for part (e), see for instance [12, Theorem 7.3.7], where the coercivity condition can be checked via [2, Lemma 2.7]):

(a) C is compact.
(b) $C = P + Q$, with $P \subset D$ compact and $Q \subset D$.
(c) D has a compact base.
(d) Y is finite dimensional.
(e) Y is reflexive, C is convex and $\operatorname{int} D^+ \neq \emptyset$.

In the following important result, we show that if $\varepsilon > 0$, then the regularly (C, ε)-subdifferentiability condition can be checked by considering convex sets $C' \subset D$ such that $C' = C' + D$. This class of sets will be denoted in the sequel by Λ, i.e.,

$$\Lambda := \{C' \subset D : C' \neq \emptyset, C' = C' + D, C' \text{ is convex}\}.$$

Let us observe that each $C' \in \Lambda$ is a coradiant set.

Theorem 5.6. *Let $x_0 \in \operatorname{dom} f$ and $\varepsilon > 0$. The following statements are equivalent:*

(a) *f is regularly (C, ε)-subdifferentiable at x_0.*
(b) $\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \in \Lambda \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0)$, $\forall \mu \in D^{s+} \cap C^{s\tau+}$.
(c) $\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \in \Lambda \\ \tau_{C'}(\mu)=\tau_C(\mu)}} \mu \circ \partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}} f(x_0)$, $\forall \mu \in D^{s+} \cap C^{s\tau+}$.

(d) $\partial_\varepsilon(\mu\circ f)(x_0) = \bigcup_{\substack{C'\in\Lambda\\ \tau_{C'}(\mu)=\varepsilon}} \mu\circ\partial^{\mathrm{s}}_{C',1}f(x_0)$, $\forall\,\mu\in D^{s+}\cap C^{s\tau+}$.

Proof. Fix $\mu\in D^{s+}\cap C^{s\tau+}$. Clearly,

$$\bigcup_{\substack{C'\in\Lambda\\ \tau_{C'}(\mu)=1}} \mu\circ\partial^{\mathrm{s}}_{C',\varepsilon}f(x_0) \subset \bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=1}} \mu\circ\partial^{\mathrm{s}}_{C',\varepsilon}f(x_0).$$

Now, let $C'\subset D$ such that $\tau_{C'}(\mu)=1$. By applying Proposition 4.3(c), (d), we know that

$$\partial^{\mathrm{s}}_{C',\varepsilon}f(x_0) = \partial^{\mathrm{s}}_{\mathrm{co}\,C',\varepsilon}f(x_0) = \partial^{\mathrm{s}}_{\mathrm{co}\,C'+D,\varepsilon}f(x_0). \tag{21}$$

Observe that $\mathrm{co}\,C'+D\in\Lambda$ and $\tau_{C'}(\mu) = -\sigma_{-C'}(\mu)$, where $\sigma_{-C'}: Y^*\to\overline{\mathbb{R}}$ denotes the support function of $-C'$, i.e.,

$$\sigma_{-C'}(\mu) = \sup_{y\in -C'}\langle\mu, y\rangle.$$

Hence, by the well-known properties of the support function (see for instance [24]) and by (2) it follows that

$$\tau_{\mathrm{co}\,C'+D}(\mu) = \tau_{\mathrm{co}\,C'}(\mu) = -\sigma_{-\,\mathrm{co}\,C'}(\mu) = -\sigma_{-C'}(\mu) = \tau_{C'}(\mu) = 1. \tag{22}$$

From the equalities (21) and (22) we deduce that

$$\bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=1}} \mu\circ\partial^{\mathrm{s}}_{C',\varepsilon}f(x_0) \subset \bigcup_{\substack{C'\in\Lambda\\ \tau_{C'}(\mu)=1}} \mu\circ\partial^{\mathrm{s}}_{C',\varepsilon}f(x_0)$$

and so parts (a) and (b) are equivalent.

Moreover, for each $C\in\Lambda$ and $\alpha>0$ it is obvious that $\alpha C\in\Lambda$. Then, by using the same reasonings as in the proof of Theorem 5.3 it is easy to check that

$$\begin{aligned}\bigcup_{\substack{C'\in\Lambda\\ \tau_{C'}(\mu)=1}} \mu\circ\partial^{\mathrm{s}}_{C',\varepsilon}f(x_0) &= \bigcup_{\substack{C'\in\Lambda\\ \tau_{C'}(\mu)=\tau_C(\mu)}} \mu\circ\partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}}f(x_0)\\ &= \bigcup_{\substack{C'\in\Lambda\\ \tau_{C'}(\mu)=\varepsilon}} \mu\circ\partial^{\mathrm{s}}_{C',1}f(x_0),\quad \forall\,\mu\in D^{s+}\cap C^{s\tau+}\end{aligned}$$

and so parts (b), (c) and (d) are equivalent, which finishes the proof. □

It is obvious that the regularly (C,ε)-subdifferentiability condition reduces to the p-regularly subdifferentiability due to El Maghri and Laghdir (see [15, pg. 1977]) when $\varepsilon=0$. In the sequel, we relate the notion of p-regularly ε-subdifferentiability introduced by El Maghri in [14] with the regularly (C,ε)-subdifferentiability condition.

Definition 5.7 ([14, Definition 3.1(ii) and $\sigma = p$]). Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. It is said that f is p-regularly ε-subdifferentiable at x_0 if $\varepsilon = 0$ and

$$\partial(\mu \circ f)(x_0) = \mu \circ \partial^s f(x_0), \quad \forall\, \mu \in D^{s+}$$

or $\varepsilon > 0$ and

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{q \in D \\ \langle \mu, q\rangle = \varepsilon}} \mu \circ \partial^{\mathrm{s}}_{\{q\},1} f(x_0), \quad \forall\, \mu \in D^{s+}.$$

Proposition 5.8. *Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. If f is p-regularly ε-subdifferentiable at x_0 then f is regularly (C, ε)-subdifferentiable at x_0.*

Proof. Suppose that $\varepsilon > 0$, since the result is obvious when $\varepsilon = 0$. Then,

$$\begin{aligned} \partial_\varepsilon(\mu \circ f)(x_0) &= \bigcup_{\substack{q \in D \\ \langle \mu, q\rangle = \varepsilon}} \mu \circ \partial^{\mathrm{s}}_{\{q\},1} f(x_0) \\ &\subset \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu) = \varepsilon}} \mu \circ \partial^{\mathrm{s}}_{C',1} f(x_0), \quad \forall\, \mu \in D^{s+} \end{aligned}$$

and by (16)–(17) and Remark 5.2 we deduce that f is regularly (C, ε)-subdifferentiable at x_0. □

By Proposition 5.8 we see that, in general, the concept of regularly (C, ε)-subdifferentiability is weaker than the notion of p-regularly ε-subdifferentiability. Next we show that both notions are the same if

$$\operatorname{argmin}_{\operatorname{cl} C'} \mu \neq \emptyset, \quad \forall\, \mu \in D^{s+},\ \forall\, C' \in \Lambda,\ 0 \notin \operatorname{cl} C'. \tag{23}$$

Proposition 5.9. *Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. Assume that conditions* (18) *and* (23) *are fulfilled. If f is regularly (C, ε)-subdifferentiable at x_0 then f is p-regularly ε-subdifferentiable at x_0.*

Proof. Suppose that $\varepsilon > 0$, since the result is obvious for $\varepsilon = 0$. Then, by Corollary 5.4 and Theorem 5.6 we have that

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \in \Lambda \\ \tau_{C'}(\mu) = \varepsilon}} \mu \circ \partial^{\mathrm{s}}_{C',1} f(x_0), \quad \forall\, \mu \in D^{s+}. \tag{24}$$

Consider $C' \in \Lambda$ such that $\tau_{C'}(\mu) = \varepsilon$. Then $0 \notin \operatorname{cl} C'$ and by using the same reasoning as in (20) we deduce that there exists $q' \in \operatorname{cl} C'$ such that $\tau_{C'}(\mu) = \langle \mu, q'\rangle$. By Proposition 4.3($c$) we deduce that $\partial^{\mathrm{s}}_{C',1} f(x_0) \subset \partial^{\mathrm{s}}_{\{q'\},1} f(x_0)$ and by (24) we obtain that

$$\partial_\varepsilon(\mu \circ f)(x_0) \subset \bigcup_{\substack{q \in D \\ \langle \mu, q\rangle = \varepsilon}} \mu \circ \partial^{\mathrm{s}}_{\{q\},1} f(x_0), \quad \forall\, \mu \in D^{s+}.$$

By (15) we conclude that f is p-regularly ε-subdifferentiable at x_0, which finishes the proof. □

In the next result we show that the regularly (C,ε)-subdifferentiability notion reduces in the Pareto context to the well-known calculus rule on the ε-subdifferential of a sum of convex mappings.

Proposition 5.10. *Let $Y=\mathbb{R}^p$, $D=\mathbb{R}^p_+$, $x_0\in\operatorname{dom} f$, $\varepsilon\geq 0$ and suppose that $0\notin \operatorname{cl} C$. Then f is regularly (C,ε)-subdifferentiable at x_0 if and only if for all $\mu=(\mu_1,\mu_2,\dots,\mu_p)\in\operatorname{int}\mathbb{R}^p_+$ one has*

$$\partial_\varepsilon(\mu\circ f)(x_0)=\bigcup_{\substack{(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_p)\in\mathbb{R}^p_+\\ \varepsilon_1+\varepsilon_2+\cdots+\varepsilon_p=\varepsilon}}\sum_{i=1}^p\partial_{\varepsilon_i}(\mu_i f_i)(x_0). \tag{25}$$

Proof. The case $\varepsilon=0$ is clear by applying Proposition 4.3(b) and Theorem 4.6 to an arbitrary nonempty set $C\subset\mathbb{R}^p_+$ and $\varepsilon=0$. If $\varepsilon>0$, by Remark 5.5(d) and Propositions 5.8 and 5.9 we deduce that f is regularly (C,ε)-subdifferentiable at x_0 if and only if

$$\partial_\varepsilon(\mu\circ f)(x_0)=\bigcup_{\substack{q\in\mathbb{R}^p_+\\ \langle\mu,q\rangle=\varepsilon}}\mu\circ\partial^{\mathrm{s}}_{\{q\},1}f(x_0),\quad \forall\,\mu\in\operatorname{int}\mathbb{R}^p_+. \tag{26}$$

By applying Theorem 4.6 it follows that for all $\mu=(\mu_1,\mu_2,\dots,\mu_p)\in\operatorname{int}\mathbb{R}^p_+$

$$\mu\circ\partial^{\mathrm{s}}_{\{q\},1}f(x_0)=\sum_{i=1}^p\mu_i\partial_{q_i}f_i(x_0)=\sum_{i=1}^p\partial_{\varepsilon_i}(\mu_i f_i)(x_0),$$

where $\varepsilon_i=\mu_i q_i$ for all $i=1,2,\dots,p$. Therefore, by (26) and (27) we obtain that

$$\partial_\varepsilon(\mu\circ f)(x_0)=\bigcup_{\substack{(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_p)\in\mathbb{R}^p_+\\ \varepsilon_1+\varepsilon_2+\cdots+\varepsilon_p=\varepsilon}}\sum_{i=1}^p\partial_{\varepsilon_i}(\mu_i f_i)(x_0),\quad\forall\,\mu=(\mu_1,\mu_2,\dots,\mu_p)\in\operatorname{int}\mathbb{R}^p_+,$$

which finishes the proof. □

Remark 5.11. The previous proposition can be used, in connection with some constraint qualifications, to check the regularly (C,ε)-subdifferentiability of objective mappings of Pareto problems. For example, it is well-known (see for instance [24]) that calculus rule (25) is fulfilled under the well-known Moreau-Rockafellar constraint qualification (MRCQ in short form): X is a Hausdorff locally convex space, all objectives f_i are convex and there exists $\overline{x}\in\bigcap_{i=1}^p\operatorname{dom} f_i$ such that $p-1$ objectives f_i are continuous at $\overline{x}$. Then, (MRCQ) implies that the objective mappings of Pareto problems are regularly (C,ε)-subdifferentiable at every point in the effective domain.

6. Moreau-Rockafellar type theorems for Benson (C,ε)-proper subdifferentials

In this section, we provide some calculus rules for the Benson (C,ε)-subdifferential of the sum of two vector-valued mappings. In particular, we obtain two Moreau-Rockafellar type theorems, and as a consequence, two formulae that reveal the gap between the strong (C, ε)-subdifferential and the Benson (C, ε)-proper subdifferential. First, we recall the notion of Benson (C, ε)-proper subdifferential of f (Definition 4.1 in [7]) and some related results.

Definition 6.1. Let $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ and $C \in \mathcal{F}$. We define the (C, ε)-proper subdifferential of f at x_0 in the sense of Benson as follows:

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) := \{T \in \mathcal{L}(X, Y) : x_0 \in \mathrm{Be}(f - T, X, C, \varepsilon)\}.$$

Remark 6.2. If $C \in \mathcal{F}$ satisfies that $\operatorname{cl} \operatorname{cone} C = D$, from [7, Remark 2.4$(a)$] we have that

$$\partial^{\mathrm{Be}}_{C,0} f(x_0) := \{T \in \mathcal{L}(X, Y) : x_0 \in \mathrm{Be}(f - T, X)\}.$$

This exact proper subdifferential is denoted by $\partial^{\mathrm{Be}} f(x_0)$.

From now on, let $f_T := f - T$, where $T \in \mathcal{L}(X, Y)$. The following results were obtained in [7] and will be used in this section. They provide a characterization of the (C, ε)-proper subdifferential of f through approximate subgradients of associated scalar mappings.

Theorem 6.3 ([7, Theorem 4.4]). *Suppose that* $\operatorname{int} D^+ \neq \emptyset$. *Let* $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$, *and* $C \in \mathcal{F}$. *If* $T \in \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$ *and* $f_T - f_T(x_0)$ *is nearly* (C, ε)*-subconvexlike on* X, *then there exists* $\mu \in D^{s+} \cap C^{\tau+}$ *such that*

$$\mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0).$$

Theorem 6.4 ([7, Theorem 4.5]). *Let* $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ *and* $C \in \mathcal{F}$. *It follows that*

$$\bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \{T \in \mathcal{L}(X, Y) : \mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0)\} \subset \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0).$$

Corollary 6.5 ([7, Corollary 4.6]). *Let* $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ *and* $C \in \mathcal{F}$. *Suppose that* $\operatorname{int} D^+ \neq \emptyset$ *and* $f_T - f_T(x_0)$ *is nearly* (C, ε)*-subconvexlike on* X, *for all* $T \in \mathcal{L}(X, Y)$. *Then,*

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \{T \in \mathcal{L}(X, Y) : \mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0)\}.$$

The next result will be very useful in the following (see Theorem 2.7 and Remark 2.8 in [7]).

Theorem 6.6. *Let* $A \in \mathcal{L}(X, Y)$ *and* $y \in Y$. *If the mapping* f *is* D*-convex on* S, *then the mapping* $f + A + y : X \to \overline{Y}$ *is nearly* (C, ε)*-subconvexlike on* S, *for all* $\varepsilon \geq 0$ *and for all nonempty convex set* $C \subset Y$ *such that* $C + D = C$.

For an arbitrary set $C \in \mathcal{F}$ and $\varepsilon \geq 0$, we denote

$$S_{C(\varepsilon)} := \{(C_1, C_2) \in \mathcal{F} \times 2^D : C(\varepsilon) \subset C_1 + C_2\}$$

and $S_C := S_{C(1)}$.

Theorem 6.7. *Let $\varepsilon \geq 0$, $C \in \mathcal{F}$ and consider two proper mappings $f, g : X \to \overline{Y}$ and $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$. It follows that*

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0) \supset \bigcup_{\substack{(C_1,C_2)\in S_{C(\varepsilon)}\\ C_1+D\subset \operatorname{cl} C_1}} \{\partial^{\mathrm{Be}}_{C_1,1} f(x_0) + \partial^{\mathrm{s}}_{C_2,1} g(x_0)\}.$$

Moreover, if D has a compact base, then

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0) \supset \bigcup_{(C_1,C_2)\in S_{C(\varepsilon)}} \{\partial^{\mathrm{Be}}_{C_1,1} f(x_0) + \partial^{\mathrm{s}}_{C_2,1} g(x_0)\}.$$

Proof. Consider $(C_1, C_2) \in S_{C(\varepsilon)}$ and let $A \in \partial^{\mathrm{Be}}_{C_1,1} f(x_0)$, $B \in \partial^{s}_{C_2,1} g(x_0)$. Suppose by contradiction that $A + B \notin \partial^{\mathrm{Be}}_{C,\varepsilon}(f + g)(x_0)$. Then $x_0 \notin \mathrm{Be}(f + g - (A + B), X, C, \varepsilon)$, i.e., there exists $d_0 \in D\backslash\{0\}$ such that

$$-d_0 \in \operatorname{cl}\operatorname{cone}((f+g)_{A+B}(\operatorname{dom}(f+g)) + C(\varepsilon) - (f+g)_{A+B}(x_0)).$$

Hence, there exist nets $(\alpha_i) \subset \mathbb{R}_+$, $(x_i) \subset \operatorname{dom}(f + g)$, $(d_i) \subset C(\varepsilon)$ such that

$$\alpha_i((f-A)(x_i) + d_i - (f-A)(x_0) + (g-B)(x_i) - (g-B)(x_0)) \to -d_0.$$

As $C(\varepsilon) \subset C_1 + C_2$ there exist nets $(u_i) \subset C_1$ and $(v_i) \subset C_2$ such that $d_i = u_i + v_i$ for all i, and so

$$\alpha_i(f_A(x_i) + u_i - f_A(x_0) + g_B(x_i) + v_i - g_B(x_0)) \to -d_0. \tag{27}$$

Since $B \in \partial^{s}_{C_2,1} g(x_0)$, it follows that $x_0 \in \mathrm{SE}(g - B, X, C_2, 1)$ and then

$$(g-B)(x_i) + v_i - (g-B)(x_0) \in D, \quad \forall\, i.$$

By denoting $y_i := u_i + (g - B)(x_i) + v_i - (g - B)(x_0) \in C_1 + D$ we deduce from (27) that

$$\alpha_i((f-A)(x_i) + y_i - (f-A)(x_0)) \to -d_0. \tag{28}$$

If $C_1 + D \subset \operatorname{cl} C_1$, then $y_i \in \operatorname{cl} C_1$ for all i and (28) is a contradiction, since by [7, Proposition 4.3(b)] with $C' = \{0\}$ it follows that $A \in \partial^{\mathrm{Be}}_{C_1,1} f(x_0) = \partial^{\mathrm{Be}}_{\operatorname{cl} C_1,1} f(x_0)$ and so $x_0 \in \mathrm{Be}(f - A, X, \operatorname{cl} C_1, 1)$.

On the other hand, if D has a compact base, by [7, Proposition 3.1(e)] we have that

$$\mathrm{Be}(f-A, X, C_1 + D, 1) = \mathrm{Be}(f-A, X, C_1, 1).$$

Therefore, by (28) we see that $x_0 \notin \mathrm{Be}(f - A, X, C_1, 1)$, i.e, $A \notin \partial^{\mathrm{Be}}_{C_1,1} f(x_0)$, which is a contradiction. □

Remark 6.8. Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. Suppose that $\operatorname{int} D^+ \neq \emptyset$ and $f_T - f_T(x_0)$ is nearly (C_q, ε)-subconvexlike on X, for all $T \in \mathcal{L}(X, Y)$ and for all $q \in Y\backslash(-D\backslash\{0\})$. Let $q, q_1, q_2 \in Y$ be such that $q, q_1 \notin -D\backslash\{0\}$, $q_2 \in D$ and

$q = q_1 + q_2$. By applying the first part of Theorem 6.7 to $C = C_q$, $\varepsilon = 1$, $C_1 = C_{q_1}$ and $C_2 = \{q_2\}$ we obtain that

$$\partial^{\mathrm{Be}}_{C_{q_1},1} f(x_0) + \partial^{\mathrm{s}}_{\{q_2\},1} g(x_0) \subset \partial^{\mathrm{Be}}_{C_q,\varepsilon}(f+g)(x_0)$$

and by Remark 4.7(b) in [7] we see that the first part of Theorem 6.7 reduces to the first part of [14, Theorem 4.1, $\sigma = p$]. Analogously, by applying the first part of Theorem 6.7 to $C = C_1 = D$, $C_2 = \{0\}$ and $\varepsilon = 1$ it follows by the definitions that

$$\partial^{\mathrm{Be}} f(x_0) + \partial^{\mathrm{s}} g(x_0) \subset \partial^{\mathrm{Be}}(f+g)(x_0)$$

and then the first part of Theorem 6.7 reduces to the first part of [15, Theorem 4.1, $\sigma = p$].

The following notion is motivated by [14, Definition 2.1(ii)].

Definition 6.9. We say that f is star D-continuous at a point $x_0 \in \operatorname{dom} f$ if $\mu \circ f$ is continuous at x_0, for all $\mu \in D^{s+}$.

Theorem 6.10. *Assume that* $\operatorname{int} D^+ \neq \emptyset$. *Let* $\varepsilon \geq 0$, $C \in \mathcal{F}$, $f, g : X \to \overline{Y}$, $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$. *Suppose that* $(f+g)_T - (f+g)_T(x_0)$ *is nearly* (C,ε)-*subconvexlike on* X *for all* $T \in \mathcal{L}(X,Y)$, g *is* p-*regularly* ε'-*subdifferentiable at* x_0 *for all* $\varepsilon' \geq 0$, *and the following qualification condition* (QC) *holds:*

(QC) *f, g are D-convex on X and there exists $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} g$ such that f or g is star D-continuous at $\bar{x}$.*

Then,

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0) \subset \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \ \bigcup_{\substack{(C_1,q) \in \mathcal{F} \times D,\, C(\varepsilon) = C_1 + \{q\} \\ \tau_{C_1}(\mu) \geq 0}} \{\partial^{\mathrm{Be}}_{C_1,1} f(x_0) + \partial^{\mathrm{s}}_{\{q\},1} g(x_0)\}. \tag{29}$$

If additionally $C + D = C$ *or* D *has a compact base, then*

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0) = \bigcup_{\substack{(C_1,q) \in \mathcal{F} \times D \\ C(\varepsilon) = C_1 + \{q\}}} \{\partial^{\mathrm{Be}}_{C_1,1} f(x_0) + \partial^{\mathrm{s}}_{\{q\},1} g(x_0)\}.$$

Proof. In order to prove inclusion (29), let us consider $T \in \partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0)$. Since $(f+g)_T - (f+g)_T(x_0)$ is nearly (C,ε)-subconvexlike on X, by Theorem 6.3 there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that

$$\mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ (f+g))(x_0) = \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f + \mu \circ g)(x_0).$$

As condition (QC) holds, we can apply the well-known ε-subdifferential sum rule for proper extended real-valued convex mappings (see, e.g., [24, Theorem 2.8.7]) obtaining that

$$\partial_{\varepsilon\tau_C(\mu)}(\mu \circ f + \mu \circ g)(x_0) = \bigcup_{\substack{\varepsilon_1,\varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon\tau_C(\mu)}} \{\partial_{\varepsilon_1}(\mu \circ f)(x_0) + \partial_{\varepsilon_2}(\mu \circ g)(x_0)\}.$$

Hence, there exist $\bar{\varepsilon}_1, \bar{\varepsilon}_2 \geq 0$, with $\bar{\varepsilon}_1 + \bar{\varepsilon}_2 = \varepsilon\tau_C(\mu)$, $x_1^* \in \partial_{\bar{\varepsilon}_1}(\mu \circ f)(x_0)$ and $x_2^* \in \partial_{\bar{\varepsilon}_2}(\mu \circ g)(x_0)$ such that $\mu \circ T = x_1^* + x_2^*$.

Suppose that $\bar{\varepsilon}_2 > 0$. As g is p-regularly $\bar{\varepsilon}_2$-subdifferentiable at x_0, there exists $q \in D$ such that $\langle \mu, q \rangle = \bar{\varepsilon}_2$ and $x_2^* \in \mu \circ \partial^{\mathrm{s}}_{\{q\},1} g(x_0)$. Hence, there exists $B \in \partial^{\mathrm{s}}_{\{q\},1} g(x_0)$ such that $\mu \circ (T-B) = x_1^* \in \partial_{\bar{\varepsilon}_1}(\mu \circ f)(x_0)$. By defining $C_1 := C(\varepsilon) - \{q\}$ we obtain $C(\varepsilon) = C_1 + \{q\}$ and $\tau_{C_1}(\mu) = \varepsilon\tau_C(\mu) - \langle \mu, q \rangle = \bar{\varepsilon}_1$. Thus, $\mu \in D^{s+} \cap C_1^{\tau+}$ and $C_1 \in \mathcal{F}$. Moreover, by Theorem 6.4 we see that $T - B \in \partial^{\mathrm{Be}}_{C_1,1} f(x_0)$ and so

$$T = (T-B) + B \in \partial^{\mathrm{Be}}_{C_1,1} f(x_0) + \partial^{\mathrm{s}}_{\{q\},1} g(x_0).$$

If $\bar{\varepsilon}_2 = 0$, by applying the p-regularity condition and Proposition 4.3(b) we deduce that $x_2^* \in \mu \circ \partial^{\mathrm{s}} g(x_0) = \mu \circ \partial^{\mathrm{s}}_{\{q\},1} g(x_0)$, where $q = 0$. From here we obtain the same conclusions as in the case $\bar{\varepsilon}_2 > 0$ by following the same reasonings, and so the proof of inclusion (29) is complete.

On the other hand, it is clear that

$$\bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \; \bigcup_{\substack{(C_1,q) \in \mathcal{F} \times D,\, C(\varepsilon) = C_1 + \{q\} \\ \tau_{C_1}(\mu) \geq 0}} \{\partial^{\mathrm{Be}}_{C_1,1} f(x_0) + \partial^{\mathrm{s}}_{\{q\},1} g(x_0)\}$$

$$\subset \bigcup_{\substack{(C_1,q) \in \mathcal{F} \times D \\ C(\varepsilon) = C_1 + \{q\}}} \{\partial^{\mathrm{Be}}_{C_1,1} f(x_0) + \partial^{\mathrm{s}}_{\{q\},1} g(x_0)\}. \tag{30}$$

If $C(\varepsilon) = C_1 + \{q\}$ and $C + D = C$, then it is easy to prove that $C_1 + D \subset \operatorname{cl} C_1$. Therefore, by Theorem 6.7 we deduce that (30) is included in $\partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0)$, which finishes the proof. □

By using the same reasonings as in Remark 6.8 we see that Theorem 6.10 generalizes [14, Theorem 4.1, $\sigma = p$] and the second part of [15, Theorem 4.1, $\sigma = p$].

Next, we prove other Moreau-Rockafellar type results based on regularly (C,ε)-subdifferentiable mappings. For an arbitrary set $C \in \mathcal{F}$, $\varepsilon \geq 0$ and $x_0 \in \operatorname{dom} f$, we define the set-valued mapping $\Gamma_{C,\varepsilon} f(x_0) : D^{s+} \cap C^{\tau+} \to 2^{\mathcal{L}(X,Y)}$ as follows:

$$\Gamma_{C,\varepsilon} f(x_0)(\mu) := \{T \in \mathcal{L}(X,Y) : \mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0)\}.$$

Remark 6.11. By Theorem 6.4 it is clear that

$$\operatorname{Im} \Gamma_{C,\varepsilon} f(x_0) := \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \Gamma_{C,\varepsilon} f(x_0)(\mu) \subset \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$$

and by Corollary 6.5, if $\operatorname{int} D^+ \neq \emptyset$ and $f_T - f_T(x_0)$ is nearly (C,ε)-subconvexlike on X, for all $T \in \mathcal{L}(X,Y)$, then

$$\operatorname{Im} \Gamma_{C,\varepsilon} f(x_0) = \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0).$$

For each $C \in \mathcal{F}$ and $\mu \in D^{s+} \cap C^{\tau+}$ we denote

$$S_C(\mu) := \{(C_1, C_2) \in 2^Y \times 2^Y : \tau_{C_1}(\mu) \geq 0, \tau_C(\mu) \geq \tau_{C_1+C_2}(\mu)\}.$$

Proposition 6.12. *Let $f, g : X \to \overline{Y}$, $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$, $\varepsilon \geq 0$ and $C \in \mathcal{F}$. Then,*

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0) \supset \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0 \\ (C_1, C_2) \in \mathcal{F} \times 2^D \\ (C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}(\mu)}} \{\Gamma_{C_1,\varepsilon_1} f(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)\}.$$

Proof. Consider $\mu \in D^{s+} \cap C^{\tau+}$, $\varepsilon_1, \varepsilon_2 \geq 0$, $C_1 \in \mathcal{F}$ and $C_2 \subset D$ such that

$$(C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}(\mu), \tag{31}$$

and let $A \in \Gamma_{C_1,\varepsilon_1} f(x_0)(\mu)$ and $B \in \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)$. By Theorem 4.4 it follows that $\mu \circ B \in \partial_{\varepsilon_2 \tau_{C_2}(\mu)}(\mu \circ g)(x_0)$ and then we deduce that

$$\mu \circ (A+B) \in \partial_{\varepsilon_1 \tau_{C_1}(\mu) + \varepsilon_2 \tau_{C_2}(\mu)}(\mu \circ (f+g))(x_0). \tag{32}$$

Moreover, by (31) and (2) we deduce that

$$\varepsilon_1 \tau_{C_1}(\mu) + \varepsilon_2 \tau_{C_2}(\mu) = \tau_{C_1(\varepsilon_1) + C_2(\varepsilon_2)}(\mu) \leq \tau_{C(\varepsilon)}(\mu)$$

and by (32) we have that $\mu \circ (A+B) \in \partial_{\tau_{C(\varepsilon)}(\mu)}(\mu \circ (f+g))(x_0)$. Then, by Theorem 6.4 we deduce that $A + B \in \partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0)$ and the proof is complete. □

In the sequel, we denote

$$\mathcal{F}_D := \{C \in \mathcal{F} : C = C + D, C \text{ is convex}\}.$$

It is clear that $\Lambda = \mathcal{F}_D \cap 2^D$.

Theorem 6.13. *Suppose that* $\operatorname{int} D^+ \neq \emptyset$. *Let $\varepsilon \geq 0$, $C \in \mathcal{F}$, $f, g : X \to \overline{Y}$ and $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$. Suppose that f is D-convex on X. Then,*

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0) \supset \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0 \\ (C_1, C_2) \in \mathcal{F}_D \times 2^D \\ (C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}}} \{\partial^{\mathrm{Be}}_{C_1,\varepsilon_1} f(x_0) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)\}.$$

Proof. Let $\varepsilon_1, \varepsilon_2 \geq 0$, $C_1 \in \mathcal{F}_D$, $C_2 \subset D$ such that $(C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}$ and let $A \in \partial^{\mathrm{Be}}_{C_1,\varepsilon_1} f(x_0)$ and $B \in \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)$. Since $C_1 \in \mathcal{F}_D$ and f is D-convex on X, we know from Theorem 6.6 that $f_T - f_T(x_0)$ is nearly (C_1, ε_1)-subconvexlike on X, for all $T \in \mathcal{L}(X, Y)$. Then, from Theorem 6.3 there exists $\mu \in D^{s+} \cap C_1^{\tau+}$ such that $\mu \circ A \in \partial_{\varepsilon_1 \tau_{C_1}(\mu)}(\mu \circ f)(x_0)$ and so

$$A + B \in \Gamma_{C_1,\varepsilon_1} f(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0).$$

Since $\mu \in C_1^{\tau+}$ and $(C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}$ we see that

$$(C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}(\mu)$$

and the result follows by Proposition 6.12. □

Given a set $C \in \mathcal{F}$ and $\mu \in D^{s+} \cap C^{\tau+}$, we denote

$$C^\tau_\mu = \{y \in Y : \langle \mu, y\rangle \geq \tau_C(\mu)\}$$

and we define

$$\Lambda_C(\mu) := \{C' \in \Lambda : \tau_{C'}(\mu) = \tau_C(\mu)\},$$

$$\mathbb{F}_C(\mu) := \{C' \in \mathcal{F}_D : \tau_{C'}(\mu) = \tau_C(\mu)\}.$$

It is clear that $\mathbb{F}_C(\mu) \cap 2^D = \Lambda_C(\mu)$. Moreover, given $\varepsilon \geq 0$ and $x_0 \in \operatorname{dom} f$, for all $C', C'' \in \mathbb{F}_C(\mu)$ it follows that

$$\Gamma_{C',\varepsilon} f(x_0)(\mu) = \Gamma_{C'',\varepsilon} f(x_0)(\mu). \tag{33}$$

Theorem 6.14. *Assume that* $\operatorname{int} D^+ \neq \emptyset$. *Let* $\varepsilon \geq 0$, $C \in \mathcal{F}$, $f, g : X \to \overline{Y}$, $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$. *Suppose that* $(f+g)_T - (f+g)_T(x_0)$ *is nearly* (C,ε)*-subconvexlike on* X *for all* $T \in \mathcal{L}(X,Y)$, g *is regularly* (C,ε')*-subdifferentiable at* x_0 *for all* $\varepsilon' \geq 0$ *and the qualification condition* (QC) *holds. Then,*

$$\begin{aligned}
&\partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0)\\
&\subset \bigcup_{\substack{\mu \in D^{s+} \cap C^{\tau+}\\ \varepsilon_1,\varepsilon_2 \geq 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \bigcup_{C_2 \in \Lambda_C(\mu)} \bigcap_{C_1 \in \mathbb{F}_C(\mu)} \{\Gamma_{C_1,\varepsilon_1} f(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)\} && (34)\\
&= \bigcup_{\substack{\mu \in D^{s+} \cap C^{\tau+}\\ \varepsilon_1,\varepsilon_2 \geq 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \bigcup_{C_2 \in \Lambda_C(\mu)} \{\Gamma_{C^\tau_\mu,\varepsilon_1} f(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)\}. && (35)
\end{aligned}$$

Proof. Let $T \in \partial^{\mathrm{Be}}_{C,\varepsilon}(f+g)(x_0)$. By reasoning as in the proof of Theorem 6.10 we deduce that there exist $\mu \in D^{s+} \cap C^{\tau+}$, $\bar\varepsilon_1, \bar\varepsilon_2 \geq 0$, $x_1^* \in \partial_{\bar\varepsilon_1}(\mu \circ f)(x_0)$ and $x_2^* \in \partial_{\bar\varepsilon_2}(\mu \circ g)(x_0)$ such that $\bar\varepsilon_1 + \bar\varepsilon_2 = \varepsilon\tau_C(\mu)$ and $\mu \circ T = x_1^* + x_2^*$.

Suppose that $\mu \in C^{s\tau+}$. If $\bar\varepsilon_2 > 0$, since g is $(C, \bar\varepsilon_2)$-regularly subdifferentiable at x_0, by Theorem 5.6(c), there exists $C_2 \in \Lambda_C(\mu)$ satisfying

$$x_2^* \in \mu \circ \partial^{\mathrm{s}}_{C_2,\frac{\bar\varepsilon_2}{\tau_C(\mu)}} g(x_0).$$

Hence, there exists $B \in \partial^{\mathrm{s}}_{C_2,\frac{\bar\varepsilon_2}{\tau_C(\mu)}} g(x_0)$ such that $\mu \circ (T - B) = x_1^*$. Let $C_1 \in \mathbb{F}_C(\mu)$ be arbitrary. It follows that $T - B \in \Gamma_{C_1,\frac{\bar\varepsilon_1}{\tau_C(\mu)}} f(x_0)(\mu)$ and so

$$T = (T-B) + B \in \Gamma_{C_1,\varepsilon_1} f(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0),$$

where $\varepsilon_1 := \frac{\bar\varepsilon_1}{\tau_C(\mu)} \geq 0$ and $\varepsilon_2 := \frac{\bar\varepsilon_2}{\tau_C(\mu)} \geq 0$ satisfy $\varepsilon_1 + \varepsilon_2 = \varepsilon$, as we want to prove.

If $\bar\varepsilon_2 = 0$, by the regularly $(C,0)$-subdifferentiability property of g at x_0 we see that $x_2^* \in \mu \circ \partial^{\mathrm{s}} g(x_0)$. By Proposition 4.3($b$) it follows that $\mu \circ \partial^{\mathrm{s}} g(x_0) = \mu \circ \partial^{\mathrm{s}}_{C_2,0} g(x_0)$, where

$$C_2 = \{q \in D : \langle \mu, q\rangle \geq \tau_C(\mu)\} = C^\tau_\mu \cap D \in \Lambda_C(\mu).$$

Then, reasoning in the same way as above we deduce that

$$T \in \Gamma_{C_1,\varepsilon_1} f(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)$$

and $\varepsilon_1 = \frac{\bar{\varepsilon}_1}{\tau_C(\mu)} = \varepsilon$ and $\varepsilon_2 = 0$.

On the other hand, if $\tau_C(\mu) = 0$, then $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0$. By the regularly $(C, 0)$-subdifferentiability property of g at x_0 and Proposition 4.3(b) we obtain that $x_2^* \in \mu \circ \partial^{\mathrm{s}} g(x_0) = \mu \circ \partial^{\mathrm{s}}_{D,0} g(x_0)$. Thus, there exists $B \in \partial^{\mathrm{s}}_{D,0} g(x_0)$ such that $\mu \circ (T - B) = x_1^* \in \partial(\mu \circ f)(x_0)$ and we deduce that $T - B \in \Gamma_{C_1,\varepsilon} f(x_0)(\mu)$ for all $C_1 \in \mathbb{F}_C(\mu)$. Therefore,

$$T = (T - B) + B \in \Gamma_{C_1,\varepsilon_1} f(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0),$$

where $C_2 := D \in \Lambda_C(\mu)$, $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = 0$ and $C_1 \in \mathbb{F}_C(\mu)$ is arbitrary, and the proof of inclusion (34) is complete.

Finally, equality (35) is clear by statement (33), because $C^\tau_\mu \in \mathbb{F}_C(\mu)$, for all $\mu \in D^{s+} \cap C^{\tau+}$. □

Theorem 6.15. *Let $\varepsilon \geq 0$, $C \in \mathcal{F}$, $f, g : X \to \overline{Y}$ and $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$. Then,*

$$\bigcup_{\substack{\mu \in D^{s+} \cap C^{\tau+} \\ \varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 \leq \varepsilon}} \bigcup_{C_2 \in \Lambda_C(\mu)} \{\Gamma_{C^\tau_\mu, \varepsilon_1} f(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)\} \subset \partial^{\mathrm{Be}}_{C,\varepsilon}(f + g).$$

Proof. For each $\mu \in D^{s+} \cap C^{\tau+}$ it is easy to check that

$$(C^\tau_\mu(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}(\mu), \quad \forall\, \varepsilon_1, \varepsilon_2 \geq 0, \varepsilon_1 + \varepsilon_2 \leq \varepsilon, \ \forall\, C_2 \in \Lambda_C(\mu),$$

and so the result is a direct consequence of Proposition 6.12. □

By Theorems 6.14 and 6.15 we deduce the following Moreau-Rockafellar theorem for Benson (C, ε)-proper subdifferentials.

Corollary 6.16. *Assume that* $\operatorname{int} D^+ \neq \emptyset$. *Let $\varepsilon \geq 0$, $C \in \mathcal{F}$, $f, g : X \to \overline{Y}$, $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$. Suppose that $(f + g)_T - (f + g)_T(x_0)$ is nearly (C, ε)-subconvexlike on X for all $T \in \mathcal{L}(X, Y)$, g is regularly (C, ε')-subdifferentiable at x_0 for all $\varepsilon' \geq 0$ and the qualification condition* (QC) *holds. Then,*

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f + g)(x_0) = \bigcup_{\substack{\mu \in D^{s+} \cap C^{\tau+} \\ \varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 \leq \varepsilon}} \bigcup_{C_2 \in \Lambda_C(\mu)} \{\Gamma_{C^\tau_\mu, \varepsilon_1} f(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)\}.$$

Remark 6.17. If $C \in \mathcal{F}_D$ and f, g are D-convex, then by Theorem 6.6 we see that $(f + g)_T - (f + g)_T(x_0)$ is nearly (C, ε)-subconvexlike on X, for all $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ and for all $T \in \mathcal{L}(X, Y)$. Then the nearly (C, ε)-subconvexlikeness assumption of Theorems 6.10 and 6.14 and Corollary 6.16 on the mapping $(f + g)_T - (f + g)_T(x_0)$ can be removed when $C \in \mathcal{F}_D$.

In the sequel we focus on the gap between the (C,ε)-strong subdifferential and the Benson (C,ε)-proper subdifferential, which is obtained for regularly (C,ε)-subdifferentiable and p-regularly ε-subdifferentiable mappings. For $\mu \in D^{s+}$ and $C \in \mathcal{F}$, we denote

$$\begin{aligned} \mathrm{K}(\mu) &:= \{T \in \mathcal{L}(X,Y) : \mu \circ T = 0\}, \\ \mathrm{K}(C) &:= \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \mathrm{K}(\mu). \end{aligned}$$

Theorem 6.18. *Suppose that* $\operatorname{int} D^+ \neq \emptyset$*. Let* $\varepsilon \geq 0$*,* $C \in \mathcal{F}$*,* $g : X \to \overline{Y}$ *and* $x_0 \in \operatorname{dom} g$*. Assume that* $g_T - g_T(x_0)$ *is nearly* (C,ε)*-subconvexlike on* X*, for all* $T \in \mathcal{L}(X,Y)$ *and* g *is* D*-convex on* X*.*

(a) *If* g *is regularly* (C,ε')*-subdifferentiable at* x_0*, for all* $\varepsilon' \geq 0$*, then,*

$$\partial^{\mathrm{Be}}_{C,\varepsilon} g(x_0) = \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \left\{ \mathrm{K}(\mu) + \bigcup_{C' \in \Lambda_C(\mu)} \partial^{\mathrm{s}}_{C',\varepsilon} g(x_0) \right\}.$$

(b) *If* C *is convex,* $C + D = C$ *or* D *has a compact base, and* g *is* p*-regularly* ε'*-subdifferentiable at* x_0*, for all* $\varepsilon' \geq 0$*, then,*

$$\partial^{\mathrm{Be}}_{C,\varepsilon} g(x_0) = \bigcup_{\substack{q \in D \\ C(\varepsilon) - \{q\} \in \mathcal{F}}} \; \bigcup_{\substack{\mu \in D^{s+} \\ \langle \mu, q \rangle \leq \varepsilon \tau_C(\mu)}} \left\{ \mathrm{K}(\mu) + \partial^{\mathrm{s}}_{\{q\},1} g(x_0) \right\}. \tag{36}$$

Proof. (a) By applying Corollary 6.16 to the mappings $f \equiv 0$ and g we obtain that

$$\partial^{\mathrm{Be}}_{C,\varepsilon} g(x_0) = \bigcup_{\substack{\mu \in D^{s+} \cap C^{\tau+} \\ \varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 \leq \varepsilon}} \; \bigcup_{C_2 \in \Lambda_C(\mu)} \{\Gamma_{C^\tau_\mu, \varepsilon_1} 0(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)\}.$$

It is easy to see that $\Gamma_{C^\tau_\mu, \varepsilon_1} 0(x_0)(\mu) = \mathrm{K}(\mu)$, for all $\varepsilon_1 \geq 0$. Thus, it follows that

$$\begin{aligned} \partial^{\mathrm{Be}}_{C,\varepsilon} g(x_0) &= \bigcup_{\substack{\mu \in D^{s+} \cap C^{\tau+} \\ 0 \leq \varepsilon_2 \leq \varepsilon}} \; \bigcup_{C_2 \in \Lambda_C(\mu)} \{\mathrm{K}(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0)\} \\ &= \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \left\{ \mathrm{K}(\mu) + \bigcup_{\substack{0 \leq \varepsilon_2 \leq \varepsilon \\ C_2 \in \Lambda_C(\mu)}} \partial^{\mathrm{s}}_{C_2,\varepsilon_2} g(x_0) \right\} \\ &= \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \left\{ \mathrm{K}(\mu) + \bigcup_{C' \in \Lambda_C(\mu)} \partial^{\mathrm{s}}_{C',\varepsilon} g(x_0) \right\}, \end{aligned}$$

where the last equality follows by applying Proposition 4.3(a).

(b) For a nonempty and convex set $C' \in \mathcal{F}$, it is easy to check via Corollary 6.5 that

$$\partial^{\mathrm{Be}}_{C',\varepsilon} 0(x_0) = \mathrm{K}(C'), \quad \forall\, \varepsilon \geq 0.$$

Then, by applying Theorem 6.10 to $f \equiv 0$ and g it follows that

$$\begin{aligned}
\partial^{\mathrm{Be}}_{C,\varepsilon} g(x_0) &= \bigcup_{\substack{(C_1,q)\in\mathcal{F}\times D\\ C(\varepsilon)=C_1+\{q\}}} \left\{\mathrm{K}(C_1)+\partial^{\mathrm{s}}_{\{q\},1} g(x_0)\right\}\\
&= \bigcup_{\substack{q\in D\\ C(\varepsilon)-\{q\}\in\mathcal{F}}} \left\{\mathrm{K}(C(\varepsilon)-\{q\})+\partial^{\mathrm{s}}_{\{q\},1} g(x_0)\right\}\\
&= \bigcup_{\substack{q\in D\\ C(\varepsilon)-\{q\}\in\mathcal{F}}} \bigcup_{\substack{\mu\in D^{s+}\\ \langle\mu,q\rangle\le\varepsilon\tau_C(\mu)}} \left\{\mathrm{K}(\mu)+\partial^{\mathrm{s}}_{\{q\},1} g(x_0)\right\}. \qquad \square
\end{aligned}$$

Remark 6.19. Part (b) of Theorem 6.18 reduces to the following formula by considering $\bar{q}\in Y\backslash(-D\backslash\{0\})$, $C=C_{\bar{q}}$ and $\varepsilon=1$:

$$\partial^{\mathrm{Be}}_{C_{\bar{q}},1} g(x_0) = \bigcup_{q\in D\backslash(\bar{q}+D\backslash\{0\})} \bigcup_{\substack{\mu\in D^{s+}\\ \langle\mu,q\rangle\le\langle\mu,\bar{q}\rangle}} \left\{\mathrm{K}(\mu)+\partial^{\mathrm{s}}_{\{q\},1} g(x_0)\right\}. \tag{37}$$

Indeed, by Example 2.10 in [7] it is easy to check that

$$C_{\bar{q}}-\{q\}=C_{\bar{q}-q}\in\mathcal{F} \iff \bar{q}-q\notin -D\backslash\{0\}$$

and formula (37) is obtained by statement (36). In view of Remark 4.7(b) in [7], it can be checked that part (b) of Theorem 6.18 does not reduce to [14, Theorem 4.2] in this case. Next, we give a counterexample that shows that [14, Theorem 4.2] is not correct.

Example 6.20. Consider $X=\mathbb{R}$, $Y=\mathbb{R}^2$ and the mapping $f:\mathbb{R}\to\mathbb{R}^2$ defined by $f(t)=(0,t)$, for all $t\in\mathbb{R}$. Let $D=\mathbb{R}^2_+$, $\bar{q}=(1,-1)$ and $x_0=0$. Clearly, f is $\mathbb{R}^2_+$-convex and p-regularly ε-subdifferentiable at 0, for all $\varepsilon\ge 0$. Moreover, $-\bar{q}\notin\mathbb{R}^2_+\backslash\{(0,0)\}$. Hence, applying [14, Theorem 4.2] we obtain that

$$\partial^{\mathrm{Be}}_{C_{\bar{q}},1} f(0) = \bigcup_{\substack{q\in\mathbb{R}^2_+\\ q\notin\bar{q}+\mathbb{R}^2_+\backslash\{(0,0)\}}} \left\{\partial^{\mathrm{s}}_{\{q\},1} f(0)+Z_p(\mathbb{R},\mathbb{R}^2)\right\}, \tag{38}$$

where $Z_p(\mathbb{R},\mathbb{R}^2)=\{A\in\mathcal{L}(\mathbb{R},\mathbb{R}^2):\exists\mu\in\operatorname{int}\mathbb{R}^2_+, \mu\circ A=0\}$. By Theorem 4.6 we deduce that $\partial^{\mathrm{s}}_{\{q\},1} f(0)=\{(0,1)\}$ and with easy calculus we have that

$$Z_p(\mathbb{R},\mathbb{R}^2)=\{(a_1,a_2)\in\mathbb{R}^2: a_1a_2<0\}\cup\{(0,0)\}.$$

Therefore, by statement (38) it follows that

$$\partial^{\mathrm{Be}}_{C_{\bar{q}},1} f(0)=\{(a_1,a_2)\in\mathbb{R}^2: a_1(a_2-1)<0\}\cup\{(0,1)\}. \tag{39}$$

Now, we determinate $\partial^{\mathrm{Be}}_{C_{\bar{q}},1} f(0)$ applying formula (37). Denoting $\mu=(\mu_1,\mu_2)$ it is clear that

$$\mathrm{K}(\mu)=\{(a_1,a_2)\in\mathbb{R}^2: a_1\mu_1+a_2\mu_2=0\}.$$

Hence,

$$\begin{aligned}\partial^{\mathrm{Be}}_{C_{\bar q},1} f(0) &= \bigcup_{\substack{q_1\in[0,1)\\ q_2\in[0,+\infty)}} \bigcup_{\substack{\mu_1,\mu_2>0\\ \frac{\mu_2}{\mu_1}\le\frac{1-q_1}{1+q_2}}} \left\{(a_1,a_2)\in\mathbb{R}^2 : a_1\mu_1 + a_2(\mu_2-1)=0\right\}\\
&= \bigcup_{\substack{\mu_1,\mu_2>0\\ \mu_1\ge\mu_2}} \left\{(a_1,a_2)\in\mathbb{R}^2 : a_1\mu_1 + a_2(\mu_2-1)=0\right\}\\
&= \left\{(a_1,a_2)\in\mathbb{R}^2 : a_1<0, a_1+a_2\ge 1\right\}\\
&\quad \cup \left\{(a_1,a_2)\in\mathbb{R}^2 : a_1>0, a_1+a_2\le 1\right\} \cup \{(0,1)\}. \qquad (40)\end{aligned}$$

Since from Theorem 6.6, $f_T - f_T(0)$ is nearly $(C_{\bar q},1)$-subconvexlike on $\mathbb{R}$ for all $T\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)$, we can also apply Corollary 6.5 to calculate $\partial^{\mathrm{Be}}_{C_{\bar q},1} f(0)$, and it can be checked that we obtain the same result as in statement (40). As it is observed, statements (39) and (40) do not coincide, which proves that [14, Theorem 4.2] is incorrect.

Next, as another application of Theorem 6.18, we present an example in which we determinate the approximate Benson proper subdifferential of a map at a point, for more general sets $C\in\mathcal{F}$.

Example 6.21. Let $X = Y = \mathbb{R}^2$ and consider the map $f:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$f(x_1,x_2) = (x_1+x_2,(x_1-x_2)^2).$$

Let $D=\mathbb{R}^2_+$, $\varepsilon\ge 0$, $x_0=(0,0)$ and $C\in\mathcal{F}$ a convex set such that $C+\mathbb{R}^2_+ = C$ and $0\notin \operatorname{cl} C$. It is easy to check that f is $\mathbb{R}^2_+$-convex on $\mathbb{R}^2$ and by Theorem 6.6 we have that $f_T - f_T(0,0)$ is nearly (C,ε)-subconvexlike on $\mathbb{R}^2$, for all $T\in\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)$. Moreover, from Remarks 5.11 and 5.5 and Proposition 5.9 we deduce that f is p-regularly ε'-subdifferentiable at $(0,0)$, for all $\varepsilon'\ge 0$. Hence, by Theorem 6.18 (b) we know that

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(0,0) = \bigcup_{\substack{q\in\mathbb{R}^2_+\\ C(\varepsilon)-\{q\}\in\mathcal{F}}} \bigcup_{\substack{\mu\in\operatorname{int}\mathbb{R}^2_+\\ \langle\mu,q\rangle\le\varepsilon\tau_C(\mu)}} \left\{\mathrm{K}(\mu)+\partial^{\mathrm{s}}_{\{q\},1} f(0,0)\right\}.$$

For each $\mu=(\mu_1,\mu_2)\in\operatorname{int}\mathbb{R}^2_+$ we have that

$$\mathrm{K}(\mu) = \left\{\begin{pmatrix}\alpha & \beta\\ -\frac{\mu_1}{\mu_2}\alpha & -\frac{\mu_1}{\mu_2}\beta\end{pmatrix} : \alpha,\beta\in\mathbb{R}\right\}. \qquad (41)$$

Applying Theorem 4.6, for $q=(q_1,q_2)\in\mathbb{R}^2_+$ it follows that $\partial^{\mathrm{s}}_{\{q\},1} f(0,0) = \partial_{q_1} f_1(0,0)\times\partial_{q_2} f_2(0,0)$, where $f=(f_1,f_2)$. For $\delta\ge 0$, it is easy to check that

$$\partial_\delta f_1(0,0) = \{(1,1)\},$$
$$\partial_\delta f_2(0,0) = \{(w_1,w_2)\in\mathbb{R}^2 : w_1=-w_2, -2\sqrt{\delta}\le w_2\le 2\sqrt{\delta}\}.$$

Therefore,

$$\partial^{s}_{\{q\},1} f(0,0) = \left\{ \begin{pmatrix} 1 & 1 \\ w & -w \end{pmatrix} : -2\sqrt{q_2} \le w \le 2\sqrt{q_2} \right\}. \tag{42}$$

Taking into account statements (41) and (42) we obtain that

$$\begin{aligned}
&\partial^{\mathrm{Be}}_{C,\varepsilon} f(0,0) \\
&= \bigcup_{\substack{q\in\mathbb{R}^2_+ \\ C(\varepsilon)-\{q\}\in\mathcal{F}}} \ \bigcup_{\substack{\mu\in\operatorname{int}\mathbb{R}^2_+ \\ \langle\mu,q\rangle\le\varepsilon\tau_C(\mu)}} \left\{ \begin{pmatrix} \alpha+1 & \beta+1 \\ w-\frac{\mu_1}{\mu_2}\alpha & -w-\frac{\mu_1}{\mu_2}\beta \end{pmatrix} : \begin{array}{c} \alpha,\beta\in\mathbb{R} \\ -2\sqrt{q_2}\le w\le 2\sqrt{q_2} \end{array} \right\} \\
&= \bigcup_{\mu\in\operatorname{int}\mathbb{R}^2_+\cap C^{\tau+}} \left\{ \begin{pmatrix} \alpha+1 & \beta+1 \\ w-\frac{\mu_1}{\mu_2}\alpha & -w-\frac{\mu_1}{\mu_2}\beta \end{pmatrix} : \begin{array}{c} \alpha,\beta\in\mathbb{R} \\ -2\sqrt{\delta_\mu}\le w\le 2\sqrt{\delta_\mu} \end{array} \right\},
\end{aligned}$$

where $\delta_\mu := \frac{\varepsilon\tau_C(\mu)}{\mu_2}$. Inclusion "$\subset$" in last equality is obtained by taking into account that $q_2 \le \delta_\mu$, for all $\mu \in \operatorname{int}\mathbb{R}^2_+$ and for all $q \in \mathbb{R}^2_+$ such that $\langle\mu, q\rangle \le \varepsilon\tau_C(\mu)$. The reverse inclusion follows by considering for each $\mu \in \operatorname{int}\mathbb{R}^2_+ \cap C^{\tau+}$ the point $q_\mu = (0, \delta_\mu)$.

References

[1] H. P. Benson: An improved definition of proper efficiency for vector maximization with respect to cones, J. Math. Anal. Appl. 71 (1979) 232–241.

[2] S. Bolintinéanu: Vector variational principles; ε-efficiency and scalar stationarity, J. Convex Analyis 8 (2001) 71–85.

[3] J. Borwein: Proper efficient points for maximizations with respect to cones, SIAM J. Control Optim. 15 (1977) 57–63.

[4] A. Brøndsted, R. T. Rockafellar: On the subdifferentiability of convex functions, Proc. Amer. Math. Soc. 16 (1965) 605–611.

[5] Y. Gao, X. Yang, K. L. Teo: Optimality conditions for approximate solutions of vector optimization problems, J. Ind. Manag. Optim. 7 (2011) 483–496.

[6] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu: Variational Methods in Partially Ordered Spaces, Springer, New York (2003).

[7] C. Gutiérrez, L. Huerga, B. Jiménez, V. Novo: Proper approximate solutions and ε-subdifferentials in vector optimization: Basic properties and limit behaviour, Nonlinear Anal. 79 (2013) 52–67.

[8] C. Gutiérrez, L. Huerga, V. Novo: Scalarization and saddle points of approximate proper solutions in nearly subconvexlike vector optimization problems, J. Math. Anal. Appl. 389 (2012) 1046–1058.

[9] C. Gutiérrez, B. Jiménez, V. Novo: On approximate efficiency in multiobjective programming, Math. Methods Oper. Res. 64 (2006) 165–185.

[10] C. Gutiérrez, B. Jiménez, V. Novo: A unified approach and optimality conditions for approximate solutions of vector optimization problems, SIAM J. Optim. 17 (2006) 688–710.

[11] J. Jahn: Vector Optimization. Theory, Applications, and Extensions, Springer, Berlin (2004).

[12] A. J. Kurdila, M. Zabarankin: Convex Functional Analysis, Birkhäuser, Basel (2005).

[13] S. S. Kutateladze: Convex ε-programming, Sov. Math. Dokl. 20 (1979) 391–393.

[14] M. El Maghri: Pareto-Fenchel ε-subdifferential sum rule and ε-efficiency, Optim. Lett. 6 (2012) 763–781.

[15] M. El Maghri, M. Laghdir: Pareto subdifferential calculus for convex vector mappings and applications to vector optimization, SIAM J. Optim. 19 (2008) 1970–1994.

[16] J. H. Qiu: On solidness of polar cones, J. Optim. Theory Appl. 109 (2001) 199–214.

[17] J. H. Qiu: Dual characterization and scalarization for Benson proper efficiency, SIAM J. Optim. 19 (2008) 144–162.

[18] C. Raffin: Contribution à l'Étude des Programmes Convexes Définis dans des Espaces Vectoriels Topologiques, Thèse, Université Pierre et Marie Curie, Paris (1969).

[19] Y. Sawaragi, H. Nakayama, T. Tanino: Theory of Multiobjective Optimization, Academic Press, Orlando (1985).

[20] L. A. Tuan: ε-optimality conditions for vector optimization problems with set-valued maps, Numer. Funct. Anal. Optim. 31 (2010) 78–95.

[21] M. Valadier: Sous-différentiabilité de fonctions convexes à valeurs dans un espace vectoriel ordonné, Math. Scand. 30 (1972) 65–74.

[22] X. M. Yang, D. Li, S. Y. Wang: Near-subconvexlikeness in vector optimization with set-valued functions, J. Optim. Theory Appl. 110 (2001) 413–427.

[23] A. Zaffaroni: Superlinear separation for radiant and coradiant sets, Optimization 56 (2007) 267–285.

[24] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific, Singapore (2002).

[25] J. Zowe: Subdifferentiability of convex functions with values in an ordered vector space, Math. Scand. 34 (1974) 69–83.