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Tangency, Paratangency and Four-Cones Coincidence Theorem in Carnot Groups

Francesco Bigolin[*]
Dipartimento di Matematica di Trento,
via Sommarive 14, 38100 Povo (Trento), Italy
bigolin@science.unitn.it

Artem Kozhevnikov
Université Paris Sud 11, bat. 425, 91405 Orsay Cedex, France
artem.kozhevnikov1@u-psud.fr

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We obtain an analogue of the classical "Four-Cones coincidence" theorem in Carnot groups, i.e. we characterize intrinsic sub-manifolds by the coincidence of its lower and upper paratangent (and tangent, as well) cones.

1. Introduction

The purpose of this paper is to give some notions and restate some results of classical convex analysis in the framework of Carnot groups. This study was motivated by historical arguments, in particular, because the tangent and paratangent cones have been introduced in the Euclidean setting by several authors and arise in many interesting applications (subdifferential in control theory, differential inclusions, ...).

The upper and lower tangent cones to a set $S \subset \mathbb{R}^n$ in a point $P \in S$, indicated respectively by $\operatorname{Tan}^+(S,P)$ and $\operatorname{Tan}^-(S,P)$, are defined by blow up as follows:

$$V \in \operatorname{Tan}^+(S,P) \quad \Longleftrightarrow \quad \liminf_{\lambda \to 0^+} \frac{1}{\lambda} \operatorname{dist}(P + \lambda V, S) = 0,$$

$$V \in \operatorname{Tan}^-(S,P) \quad \Longleftrightarrow \quad \lim_{\lambda \to 0^+} \frac{1}{\lambda} \operatorname{dist}(P + \lambda V, S) = 0.$$

These tangent cones were introduced by Peano [12] in 1897 to formulate necessary conditions in optimisation problems and recovered later by Bouligand [4] and Severi [13]. In particular, Bouligand called $\operatorname{Tan}^+(S,P)$ *'contingent cone'* to S in P. Similarly, the upper and lower paratangent cones to a set $S \subset \mathbb{R}^n$ in a

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point $P \in S$, that are denoted respectively by $\mathrm{pTan}^+(S,P)$ and $\mathrm{pTan}^-(S,P)$, are defined by blow up as follows:

$$V \in \mathrm{pTan}^+(S,P) \quad \iff \quad \liminf_{\substack{\lambda\to 0^+ \\ S\ni Q\to P}} \frac{1}{\lambda}\operatorname{dist}(Q+\lambda V, S) = 0,$$

$$V \in \mathrm{pTan}^-(S,P) \quad \iff \quad \lim_{\substack{\lambda\to 0^+ \\ S\ni Q\to P}} \frac{1}{\lambda}\operatorname{dist}(Q+\lambda V, S) = 0.$$

The upper paratangent cone was introduced by Bouligand [4] in 1932 and Severi [13] in 1935, while the lower paratangent cone was introduced by Clarke [5] in 1975. For more historical details concerning tangency and paratangency in $\mathbb{R}^n$, [2, 3, 7, 8] are recommended.

We refer Carnot group complete definitions to Section 2. At this point we only recall that a Carnot group $\mathbb{G}$, endowed with a left-invariant metric d_{cc} (called Carnot-Carathéodory metric), provides an example of a metric space that is not equivalent to the Euclidean, even locally. At the same time, it still has a sufficiently rich underlying structure (existence of family of left translations and dilations compatible with d_{cc}) to give raise to many notions of classical analysis.

In this paper we give definitions of $\mathbb{G}$-tangent and $\mathbb{G}$-paratangent cones in a Carnot group $\mathbb{G}$ in terms of sequences, inspired by the Euclidean ones, see Definition 2.4. The relationships between tangency, paratangency and intrinsic differentiability in $\mathbb{G}$ is a crucial point of the work. Indeed, we give definitions of differentiability and uniform differentiability and characterize them respectively in terms of tangency and paratangency in Lemmata 2.9 and 2.10. These Lemmata are the appropriate generalizations in Carnot groups of a well-known result in Euclidean setting, called in [7, 8] *Cyrenian lemma*. Coming now to the primary subject of this paper, let us recall the definition of intrinsic sub-manifold in $\mathbb{G}$.

Definition 1.1. A set $S \subset \mathbb{G}$ is called an *intrinsic sub-manifold* of codimension N if in a neighbourhood of every $P \in S$, S coincides with a level set of a differentiable map $F \in C^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^N)$ with $D_{\mathbb{G}}F(P)$ surjective.

We characterize intrinsic sub-manifolds in the main theorem, which is the analogue of the four-cones coincidence theorem in Carnot groups.

Theorem 1.2. *Let $S \subset \mathbb{G}$ be a closed connected set. The following conditions are equivalent:*

(*1*) *S is a regular manifold of codimension N;*

(*2*) *Tangent cones coincide at every point $P \in S$:*

$$\mathrm{pTan}^+_{\mathbb{G}}(S,P) = \mathrm{pTan}^-_{\mathbb{G}}(S,P), \quad \forall\, P \in S;$$

and there's some point P_0 such that $\mathrm{pTan}^+_{\mathbb{G}}(S,P_0)$ is a vertical subgroup of codimension N.

The paper is organised as follows. In Section 2 we recall some preliminary definitions and well-known facts in Carnot group, we present tangency and paratengency in $\mathbb{G}$ and we show their relationship with intrinsic differentiability. Section 3 is devoted to the proof of the main result. As an application, we present in Appendix A a generalization of the *Regula of Peano*, i.e. a necessary condition of optimality for intrinsic differentiable functions (Theorem A.1).

2. Preliminary results, Tangency and Paratangency in Carnot groups

A Carnot group is a connected, simply connected, nilpotent Lie group $(\mathbb{G}, \cdot)$ with stratified Lie algebra $\mathfrak{g}$ that is Lie generated by its first, so called *horizontal*, layer $\mathfrak{g}_1$:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k, \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \mathfrak{g}_k \neq \{0\}, \quad \mathfrak{g}_j = \{0\} \quad \text{if } j > k. \tag{1}$$

The integer k is called the *step* of the group. The unit element of $\mathbb{G}$ is 0 and the inverse P^{-1} of an element $P \in \mathbb{G}$ is $-P$. Thanks to the stratification, we define on the Lie algebra $\mathfrak{g}$ a one-parametric multiplicative group of automorphisms $\{\delta_\lambda\}_{\lambda>0}$, called *dilations*:

$$\delta_\lambda(P) = \lambda^i P \quad \text{for } P \in \mathfrak{g}_i.$$

Via the exponential map, we transfer the action of $\{\delta_\lambda\}_{\lambda>0}$ on $\mathbb{G}$ keeping for it the same notation (see [9], Ch. 1). The group $\mathbb{G}$ is also a homogeneous space of homogeneous dimension $\sum_{i=1}^k i \dim \mathfrak{g}_i$. A subgroup is called *homogeneous* if it's stable under the action of $\{\delta_\lambda\}_{\lambda>0}$. A closed (normal) subgroup $\mathbb{W} \subset \mathbb{G}$ is said to be *vertical* if $\exp(\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k) \subset \mathbb{W}$. Equipped with the natural topology, the family of vertical subgroups of given dimension form an open set inside all homogeneous subgroups of the same dimension.

Let $X_1, \ldots, X_l$ be a family of left invariant vector fields $\mathfrak{g}_1 \equiv \mathbb{R}^l$ such that $X_1(0) = \partial_{x_1}, \ldots, X_l(0) = \partial_{x_l}$. We refer to $\{X_1, \ldots, X_l\}$ as canonical generating vector fields of the group. It also turns out that these vector fields X_j have polynomial coefficients. The *horizontal bundle* $H\mathbb{G} \subset T\mathbb{G}$ is defined by the left translations of $\mathfrak{g}_1$: $H\mathbb{G}(P) = \text{span}\{X_1(P), \ldots, X_l(P)\}$ for $P \in \mathbb{G}$.

We say that an absolutely continuous curve $\gamma : [0, T] \to \mathbb{G}$ is a sub-unit curve (with respect to $X_1, \ldots, X_l$) if there exist real measurable functions $c_1(s), \ldots, c_l(s)$, $s \in [0, T]$, such that $\sum_j c_j^2 \leq 1$ and

$$\gamma'(s) = \sum\nolimits_j c_j(s) X_j(\gamma(s)), \quad \text{for a.e. } s \in [0, T].$$

If $P, Q \in \mathbb{G}$, their *Carnot-Carathéodory distance* $d_{cc}(P, Q)$ is given by

$$d_{cc}(P, Q) = \inf\{T > 0 \mid \gamma : [0, T] \to \mathbb{G} \text{ is sub-unit}, \gamma(0) = P, \ \gamma(T) = Q\}.$$

The fact, that under assumption (1), $d_{cc}(P, Q)$ is finite for any $P, Q \in \mathbb{G}$, is the content of Rashevsky-Chow theorem. We recall that the topology induced on $\mathbb{G}$

by d_{cc} coincides with the standard one. We denote by $B_{cc}(P,r) = \{Q \mid d_{cc}(P,Q) < r\}$ an open ball defined by metric d_{cc}. Being defined through left invariant vector fields, d_{cc} turns out to be a left-invariant and homogeneous metric on $\mathbb{G}$. It will be convenient to introduce the following notation: $\|P\| := d_{cc}(P,0)$, for $P \in \mathbb{G}$. Meanwhile, we keep the symbol $|\cdot|$ for the Euclidean norm in $\mathbb{R}^N$.

Definition 2.1. We call $\mathbb{G}$-linear map any continuous homogeneous homomorphism $L : \mathbb{G} \to \mathbb{R}^N$. We denote the space of all $\mathbb{G}$-linear maps equipped with the standard topology by $\mathrm{Hom}(\mathbb{G}, \mathbb{R}^N)$.

Definition 2.2. Let $\Omega \subset \mathbb{G}$ be an open set. Let $F : \Omega \longrightarrow \mathbb{R}^N$. We say that F is differentiable at $P_0 \in \Omega$ if there is a unique $\mathbb{G}$-linear map $L : \mathbb{G} \to \mathbb{R}^N$ such that

$$\lim_{P \to P_0} \frac{F(P) - F(P_0) - L\left\langle P_0^{-1} \cdot P \right\rangle}{d_{cc}(P, P_0)} = 0.$$

We say that F is uniformly differentiable at $P_0 \in \Omega$ if there is a unique $\mathbb{G}$-linear map $L : \mathbb{G} \to \mathbb{R}^N$ such that

$$\lim_{P,Q \to P_0} \frac{F(P) - F(Q) - L\left\langle Q^{-1} \cdot P \right\rangle}{d_{cc}(P, Q)} = 0.$$

We shall write $D_{\mathbb{G}}F(P_0) := L$. A function F from an open set $\Omega \subset \mathbb{G}$ to $\mathbb{R}^N$ is said to be of class $C^1_{\mathbb{G}}(\Omega, \mathbb{R}^N)$ if it's differentiable on Ω and $D_{\mathbb{G}}F$ is continuous. It is well know that if $F \in C^1_{\mathbb{G}}(\Omega, \mathbb{R}^N)$ then F is uniformly differentiable in Ω.

The key tool in the proof of Theorem 1.2 is the following

Theorem 2.3 (Whitney extension theorem; [15]). *Let $S \subset \mathbb{G}$ be a closed set. Let $F : S \to \mathbb{R}^N$ and $k : S \to \mathrm{Hom}(\mathbb{G}, \mathbb{R}^N)$ be continuous. For a subset $\tilde{S} \subset S$ we put*

$$\rho_{\tilde{S}}(\varepsilon) := \sup \left\{ \frac{|F(Q) - F(P) - k(P)\langle P^{-1} \cdot Q \rangle|}{d_{cc}(P,Q)} \mid P, Q \in \tilde{S},\ 0 < d_{cc}(P,Q) < \varepsilon \right\}.$$

If $\rho_{\tilde{S}}(\varepsilon) \to 0$ when $\varepsilon \to 0^+$ for any compact $\tilde{S} \subset S$, then there exists a map $\tilde{F} \in C^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^N)$ such that $\tilde{F}_{|S} = F$ and $D_{\mathbb{G}}\tilde{F}_{|S} = k$.

We can now give the notion of tangent and paratangent vectors in $\mathbb{G}$.

Definition 2.4. Let $S \subset \mathbb{G}$ and $P_0 \in \mathbb{G}$.

- $V \in \mathbb{G}$ belongs to $\mathrm{Tan}^+_{\mathbb{G}}(S, P_0)$ (an upper tangent vector to S in P_0) iff

$$\exists \{\lambda_m\}_m \subset \mathbb{R}_+ \text{ such that } \lim_m \lambda_m = 0,\ \exists \{P_m\}_m \subset S \text{ such that}$$
$$\lim_m d_{cc}(P_0, P_m) = 0, \qquad \lim_m \delta_{1/\lambda_m}(P_0^{-1} \cdot P_m) = V. \tag{2}$$

- $V \in \mathbb{G}$ belongs to $\mathrm{Tan}^-_{\mathbb{G}}(S, P_0)$ (a lower tangent vector to S in P_0) iff

$$\forall \{\lambda_m\}_m \subset \mathbb{R}_+ \text{ such that } \lim_m \lambda_m = 0,\ \exists \{P_m\}_m \subset S \text{ such that}$$
$$\lim_m d_{cc}(P_0, P_m) = 0, \qquad \lim_m \delta_{1/\lambda_m}(P_0^{-1} \cdot P_m) = V. \tag{3}$$

- $V \in \mathbb{G}$ belongs to $\mathrm{pTan}^+_{\mathbb{G}}(S, P_0)$ (an upper paratangent vector to S in P_0) iff

$$\exists\{\lambda_m\}_m \subset \mathbb{R}_+ \text{ such that } \lim_m \lambda_m = 0,\ \exists\{P_m\}_m, \{Q_m\}_m \subset S \text{ such that}$$
$$\lim_m d_{cc}(P_0, P_m) = 0, \qquad \lim_m \delta_{1/\lambda_m}(Q_m^{-1} \cdot P_m) = V. \tag{4}$$

- $V \in \mathbb{G}$ belongs to $\mathrm{pTan}^-_{\mathbb{G}}(S, P_0)$ (a lower paratangent vector to S in P_0) iff

$$\forall\{\lambda_m\}_m \subset \mathbb{R}_+ \text{ such that } \lim_m \lambda_m = 0,\ \forall\{P_m\}_m \subset S \text{ such that}$$
$$\lim_m d_{cc}(P_0, P_m) = 0,\ \exists\{Q_m\}_m \subset S \text{ such that } \lim_m \delta_{1/\lambda_m}(Q_m^{-1} \cdot P_m) = V. \tag{5}$$

Remark 2.5. The following inclusions follow from definitions

$$\mathrm{pTan}^-_{\mathbb{G}}(S, P_0) \subseteq \mathrm{Tan}^-_{\mathbb{G}}(S, P_0) \subseteq \mathrm{Tan}^+_{\mathbb{G}}(S, P_0) \subseteq \mathrm{pTan}^+_{\mathbb{G}}(S, P_0).$$

Finally, in order to define some properties of tangent and paratangent cones, let us introduce the so-called *Kuratowski limits* of sets. Let A_P be a subset of $\mathbb{G}$ indexed by points $P \in S \subset \mathbb{G}$. The *lower limit* $\mathrm{Li}_{S \ni P \to P_0} A_P$ and *upper limit* $\mathrm{Ls}_{S \ni P \to P_0} A_P$ are defined by

$$V \in \operatorname*{Li}_{S \ni P \to P_0} A_P \iff \begin{cases} \forall\{P_m\}_m \subset S \text{ with } P_m \to P_0, \exists\,\{Q_m\}_m \text{ with} \\ Q_m \in A_{P_m} \text{ eventually such that } \lim_m Q_m = V. \end{cases} \tag{6}$$

$$V \in \operatorname*{Ls}_{S \ni P \to P_0} A_P \iff \begin{cases} \exists\{P_m\}_m \subset S \text{ with } P_m \to P_0, \exists\,\{Q_m\}_m \text{ with} \\ Q_m \in A_{P_m} \text{ eventually such that } \lim_m Q_m = V. \end{cases} \tag{7}$$

Remark 2.6. Notice that, by definition, $\mathrm{Li}_{P \to P_0} A_P \subset A_{P_0}$ for every family of subset $A_P \subset \mathbb{G}$.

Adapting the proof of convexity (resp. bilaterality) of lower (resp. upper) paratangent cones known in the setting of vector spaces we obtain Proposition 2.7 and 2.8 below.

Proposition 2.7. *The lower paratangent cone* $\mathrm{pTan}^-_{\mathbb{G}}(S, P)$ *is* $\mathbb{G}$*-convex, i.e. satisfies*

$$V_1, V_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S, P) \Longrightarrow V_1 \cdot \delta_\lambda(V_1^{-1} \cdot V_2) \in \mathrm{pTan}^-_{\mathbb{G}}(S, P), \quad \lambda \in [0, 1].$$

Proof. *Step I*: $V_1, V_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S, P) \Longrightarrow V_1 \cdot V_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$.

Let $V_1, V_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$. Let us fix two sequences $\{P_m\}_m \subset S$, $\{\lambda_m\}_m \subset \mathbb{R}_+$ such that $\lim_m d_{cc}(P_0, P_m) = 0$ and $\lim_m \lambda_m = 0$. Since $V_1 \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$, there exists $\{Q_m\}_m \subset S$ such that

$$\lim_m \delta_{1/\lambda_m}(Q_m^{-1} \cdot P_m) = V_1.$$

Since $\lim_m d_{cc}(P_0, Q_m) = 0$ and $V_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$, there exists $\{R_m\}_m \subset S$ such that

$$\lim_m \delta_{1/\lambda_m}(R_m^{-1} \cdot Q_m) = V_2. \tag{8}$$

We can conclude that $V_1 \cdot V_2 \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$, because for all $\{P_m\}_m \subset S$, $\{\lambda_m\}_m \subset \mathbb{R}_+$ as above we can find a sequence $\{R_m\}_m \subset S$, chosen like in (8), such that

$$\lim_m \delta_{1/\lambda_m}(R_m^{-1} \cdot P_m) = \lim_m \delta_{1/\lambda_m}\Big((R_m^{-1} \cdot Q_m) \cdot (Q_m^{-1} \cdot P_m)\Big) = V_2 \cdot V_1.$$

Step II: $V \in \mathrm{pTan}^-_{\mathbb{G}}(S, P) \Longrightarrow \delta_\lambda V \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$, $\lambda > 0$.

Let $V \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$ and $\lambda > 0$. Let us fix two sequences $\{P_m\}_m \subset S$, $\{\lambda_m\}_m \subset \mathbb{R}_+$ such that $\lim_m d_{cc}(P_0, P_m) = 0$ and $\lim_m \lambda_m = 0$. Since $V \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$, there exists a sequence $\{Q_m\}_m \subset S$ such that

$$\lim_m \delta_{1/\lambda_m}(Q_m^{-1} \cdot P_m) = V.$$

Let us consider the sequence $\{\mu_m\}_m \subset \mathbb{R}_+$ defined by $\mu_m := \frac{\lambda_m}{\lambda}$ for every $m \in \mathbb{N}$. Since $V \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$ there exists a sequence $\{R_m\}_m \subset S$ such that

$$\lim_m \delta_{1/\mu_m}(R_m^{-1} \cdot P_m) = V. \tag{9}$$

We can conclude that $\delta_\lambda V \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$, because for all $\{P_m\}_m \subset S$, $\{\lambda_m\}_m \subset \mathbb{R}_+$ as above we can find a sequence $\{R_m\}_m \subset S$, chosen like in (9), such that

$$\lim_m \delta_{1/\lambda_m}(R_m^{-1} \cdot P_m) = \lim_m \delta_\lambda \delta_{1/\mu_m}\Big((R_m^{-1} \cdot Q_m) \cdot (Q_m^{-1} \cdot P_m)\Big) = \delta_\lambda V.$$

By Steps *I* and *II* follows $V_1 \cdot \delta_\lambda(V_1^{-1} \cdot V_2) = \delta_{1-\lambda}(V_1) \cdot \delta_\lambda(V_2) \in \mathrm{pTan}^-_{\mathbb{G}}(S, P)$. $\square$

Proposition 2.8. *The upper paratangent cone* $\mathrm{pTan}^+_{\mathbb{G}}(S, P)$ *is bilateral, i.e. satisfies*

$$V \in \mathrm{pTan}^+_{\mathbb{G}}(S, P) \quad \Longrightarrow \quad V_1^{-1} \in \mathrm{pTan}^+_{\mathbb{G}}(S, P).$$

Proof. Let $V \in \mathrm{pTan}^+_{\mathbb{G}}(S, P)$. By definition, there exist three sequences $\{P_m\}_m, \{Q_m\}_m \subset S$, $\{\lambda_m\}_m \subset \mathbb{R}_+$ such that

$$\lim_m d_{cc}(P_0, P_m) = 0, \qquad \lim_m \lambda_m = 0, \qquad \lim_m \delta_{1/\lambda_m}(Q_m^{-1} \cdot P_m) = V.$$

Notice that in this case $\lim_m \delta_{1/\lambda_m}(P_m^{-1} \cdot Q_m) = V^{-1}$, therefore $V^{-1} \in \mathrm{pTan}^+_{\mathbb{G}}(S, P)$. $\square$

We present now the Cyrenian Lemma in a Carnot group $\mathbb{G}$. It is an important characterization of differentiability and turns out very instrumental in effective calculus.

Lemma 2.9. *Let $\Omega \subset \mathbb{G}$ be an open set. Let $F : \Omega \to \mathbb{R}$ be a real function. The following properties are equivalent:*

(*i*) *F is differentiable in $P_0 \in \Omega$ and $L = D_{\mathbb{G}} F(P_0)$.*

(*ii*) *There exists a linear map $L : \mathbb{G} \to \mathbb{R}$ such that*

$$\lim_m \frac{F(P_m) - F(P_0)}{\lambda_m} = L\langle V \rangle \tag{10}$$

for each $V \in \mathbb{G}$ and for every $\{\lambda_m\}_m \subset \mathbb{R}_+$ and $\{P_m\}_m \subset \Omega$ as in (2).

Proof. $(i) \Rightarrow (ii)$. We consider two cases:

Case 1: there exists $\bar{m} \in \mathbb{N}$ such that $P_m = P_0$ for every $m > \bar{m}$. Observe that

$$V = \lim_m \delta_{1/\lambda_m}(P_0^{-1} \cdot P_m) = 0, \qquad \lim_m \frac{F(P_m) - F(P_0)}{\lambda_m} = 0$$

and hence (10) holds.

Case 2: there does not exist $\bar{m} \in \mathbb{N}$ such that $P_m = P_0$ for every $m > \bar{m}$. Observe that

$$\begin{aligned} \frac{F(P_m) - F(P_0)}{\lambda_m} = & \frac{F(P_m) - F(P_0) - D_{\mathbb{G}} F(P_0)\langle P_0^{-1} \cdot P_m\rangle}{d_{cc}(P, P_m)} \frac{d_{cc}(P_0, P_m)}{\lambda_m} \\ & + D_{\mathbb{G}} F(P_0)\langle \delta_{1/\lambda_m}(P_0^{-1} \cdot P_m)\rangle. \end{aligned}$$

Sending $m \to \infty$ we obtain (10), because $\lim_m \frac{d_{cc}(P_0, P_m)}{\lambda_m} < \infty$.

$(ii) \Rightarrow (i)$. By contradiction, let F be not differentiable and let $L \neq D_{\mathbb{G}} F(P_0)$. Then, by definition of P-differentiability, there exists $\epsilon > 0$ such that for all m large enough

$$\left| \frac{F(P_m) - F(P_0)}{d_{cc}(P_m, P_0)} - L\left\langle \delta_{1/d_{cc}(P_m, P_0)}\left(P_0^{-1} \cdot P_m\right)\right\rangle \right| > \epsilon. \tag{11}$$

Since $\left\|\delta_{1/d_{cc}(P_m, P_0)}(P_0^{-1} \cdot P_m)\right\| = 1$, there exists a subsequence of $\{P_m\}_m$, denoted by $\{P_m\}_m$ too, such that

$$\lim_m \delta_{1/d_{cc}(P_m, P_0)}(P_0^{-1} \cdot P_m) = V.$$

Let $\lambda_m := d_{cc}(P_m, P_0)$, by (10) we have

$$\lim_m \frac{F(P_m) - F(P_0)}{d_{cc}(P_m, P_0)} = L\langle V \rangle.$$

Sending $m \to \infty$ in (11), we obtain $\epsilon = 0$, a contraddiction. $\square$

In conclusion of this section, let us present a paratangent version of the Cyrenian Lemma. The proof is analogous to that of Lemma 2.9.

Lemma 2.10. *Let $f : \Omega \subset \mathbb{G} \to \mathbb{R}$ be a real function. Then the following properties are equivalent:*

(*i*) *f is uniformly differentiable in $P_0 \in \Omega$ and $L = D_{\mathbb{G}} f(P_0)$.*

(*ii*) *there exists a $\mathbb{G}$-linear application $L : \mathbb{G} \to \mathbb{R}$ such that*

$$\lim_m \frac{f(P_m) - f(Q_m)}{\lambda_m} = L\langle V\rangle$$

for each $V \in \mathbb{G}$ and for every $\{\lambda_m\}_m \subset \mathbb{R}_+$, $\{Q_m\}_m \subset \Omega$ and $\{P_m\}_m \subset \Omega$ such that (4)*.*

3. Four cones Theorem

The implication (*1*) $\Longrightarrow$ (*2*) in Theorem 1.2 follows immediately from

Theorem 3.1. *Let $F \in C^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^N)$. If $D_{\mathbb{G}} F(A)$ is surjective at a point $A \in \mathbb{G}$ (of course, this implies $N \leq \dim H\mathbb{G}$), then $S := F^{-1}(F(A))$ satisfies*

$$\mathrm{pTan}^-_{\mathbb{G}}(S, A) = \mathrm{pTan}^+_{\mathbb{G}}(S, A) = \operatorname{Ker} D_{\mathbb{G}} F(A).$$

Proof. *Step I*: $\mathrm{pTan}^+_{\mathbb{G}}(S, A) \subset \operatorname{Ker} D_{\mathbb{G}} F(A)$. For any two points $P, Q \in S$, one has

$$0 = F(Q) - F(P) = D_{\mathbb{G}} F(P)\langle P^{-1} \cdot Q\rangle + o(d_{cc}(P, Q)), \quad \text{when } P \to Q. \tag{12}$$

According to an analogue of the Lagrange theorem [9] for $C^1_{\mathbb{G}}$, the small "o" is uniform for P, Q from any compact subset of S.

Take $V \in \mathrm{pTan}^+_{\mathbb{G}}(S, A)$. By definition, V is the limit of $\delta_{\lambda_m^{-1}}(P_m^{-1} \cdot Q_m)$ with $\lambda_m \to 0$ and $S \ni P_m, Q_m \to A$. By homogeneity, (12) reads

$$D_{\mathbb{G}} F(P_m)\langle \delta_{\lambda_m^{-1}}(P_m^{-1} \cdot Q_m)\rangle = \frac{o(d_{cc}(P_m, Q_m)}{\lambda_m}, \quad \text{when } m \to 0.$$

That gives $D_{\mathbb{G}} F(A)\langle V\rangle = 0$ at the limit because the ratio $d_{cc}(P_m, Q_m)\lambda_m^{-1}$ should to stay bounded in order to have $\|V\| < \infty$.

Step II: projection of $F^{-1}(A)$ on $\operatorname{Ker} D_{\mathbb{G}} F(A)$ is surjective. Since $D_{\mathbb{G}} F(A) : \mathbb{G} \to \mathbb{R}^N$ is surjective, we can find $\{X_i \mid i = 1, \ldots, N\}$ horizontal vector fields such that

$$\operatorname{span}\{X_i F(A) \mid i = 1, \ldots, N\} = \mathbb{R}^N.$$

As a consequence, there exists $k > 0$ such that

$$\left| D_{\mathbb{G}} F(A) \left\langle \exp\left(\sum_{i=0}^N x_i X_i\right)(0)\right\rangle \right| \geq k|x| \quad \text{for all } x = (x_1, \ldots, x_N) \in \mathbb{R}^N. \tag{13}$$

By continuity of $D_{\mathbb{G}} F$, the last inequality (13) will also hold for all points A' in some neighborhood of A. Now we prove an auxiliary

Lemma 3.2. *There are $r, \varepsilon > 0$ such that for all $P \in B_{cc}(A, r) \cap (A \cdot \operatorname{Ker} D_{\mathbb{G}} F(A))$ we can find at least one (but, in general, not necessarily only one) point $x(P) \in \mathbb{R}^N$, $|x(P)| < \varepsilon$, such that $F\big(\exp(\sum_0^N x_i(P) X_i)(P)\big) = F(A)$. Furthermore, $x(P) \to 0$ when $P \to A$.*

Proof of Lemma 3.2. We note

$$F_P(x) := F\left(\exp\left(\sum_0^N x_i X_i\right)(P)\right), \quad F_P : \mathbb{R}^N \to \mathbb{R}^N, \quad P \in \mathbb{G};$$

$$D_\varepsilon := \left\{\exp\left(\sum_0^N x_i X_i\right)(0) \mid |x| < \varepsilon\right\}, \quad \varepsilon > 0.$$

We choose some small $\varepsilon > 0$ such that for the small "o" in the definition of differentiability of F at A one has $|o(d_{cc}(Q, 0))| < k/2$ whatever $Q \in D_\varepsilon$. If $|x| < \varepsilon$, by (13) we get

$$\begin{aligned}
&|F_A(x) - F_A(0)| \\
= &\left|F\left(\exp\left(\sum_0^N x_i X_i\right)(A)\right) - F(A)\right| \\
= &\left|D_{\mathbb{G}} F(A)\left\langle \exp\left(\sum_0^N x_i X_i\right)(0)\right\rangle + o\left(d_{cc}\left(\exp\left(\sum_0^N x_i X_i\right)(0), 0\right)\right)\right| \\
> &\, k|x| - k/2|x| = k/2|x|.
\end{aligned} \tag{14}$$

That implies the image of the boundary $F_A(\partial D_\varepsilon)$ does not meet $F(A)$. By continuity of F, the same will be true for all $P \in B_{cc}(A, r)$ where $r > 0$ is small enough: $F(A) \notin F_P(\partial D_\varepsilon)$. Each map $F_P \in C^0(D_\varepsilon, \mathbb{R}^n)$ is obviously homotopic to $F_A \in C^0(D_\varepsilon, \mathbb{R}^n)$. The topological degree [11] is homotopy invariant, i.e.

$$\begin{aligned}
&\deg\big(F_P, D_\varepsilon, F(A)\big) \\
= &\deg\big(F_A, D_\varepsilon, F(A)\big) \text{ for all } P \in B_{cc}(A, r) \cap (A \cdot \operatorname{Ker} D_{\mathbb{G}} F(A)).
\end{aligned}$$

Observe that $F_A^{-1}(F(A)) \cap D_\varepsilon = \{A\}$. Furthermore, F_A is homotopic (by action of dilation δ_t) to its differential at A, and thus,

$$\deg(F_A, D_\varepsilon, F(A)) = \operatorname{sign}(\det(\{X_i F(A) \mid i = 0, \dots N\})) \in \{1, -1\}.$$

Since $\deg\big(F_P, D_\varepsilon, F(A)\big) \neq 0$, for all $P \in B_{cc}(A, r) \cap (A \cdot \operatorname{Ker} D_{\mathbb{G}} F(A))$ there exists $x(P) \in D_\varepsilon$ such that $F_P(x(P)) = F\big(\exp(\sum_0^N x_i(P) X_i)(P)\big) = F(A)$.

The fact that $x(P) \to 0$ as $P \to A$ is an easy consequence of (14). □

For A' close enough to A, the differential $D_{\mathbb{G}} F(A')$ is also surjective, We apply Lemma 3.2 to A' instead of A. And it's important to notice that $\varepsilon, r > 0$ and the vector fields $\{X_i\}$ can be chosen in Lemma 3.2 once for all A' in some compact neighbourhood of A.

Step III: $\mathrm{Ker}\, D_{\mathbb{G}}F(A) \subset \mathrm{pTan}^-(S, A)$. We fix an element $V \in \mathrm{Ker}\, D_{\mathbb{G}}F(A)$ and two sequences $S \ni P_m \to A$, $\lambda_m \to 0$. We denote by $V(P)$ a closest to V (in term of d_{cc}) element of $\mathrm{Ker}\, D_{\mathbb{G}}F(P)$:

$$\min_{V' \in \mathrm{Ker}\, D_{\mathbb{G}}F(P)} d_{cc}(V, V') = d_{cc}(V, V(P)), \quad V' \in \mathrm{Ker}\, D_{\mathbb{G}}F(P).$$

The continuity of $D_{\mathbb{G}}F$ implies that $V(P_m) \to V$. Now consider $V_m := \delta_{\lambda_m}(V(P_m)) \in \mathrm{Ker}\, D_{\mathbb{G}}F(P_m)$. Using Lemma 3.2, for large m we can find a point $Q_m = \exp(\sum_0^N x_i(V_m)X_i)(P_m \cdot V_m)$ belonging to S. It's not hard to see that $\delta_{\lambda_m^{-1}}(P_m^{-1} \cdot Q_m) \to V$ if we prove $\lambda_m^{-1} d_{cc}(P_m \cdot V_m, Q_m) \to 0$. We write (12) for $P_m, Q_m \in S$

$$\begin{aligned} D_{\mathbb{G}}F(P_m)\langle P_m^{-1} \cdot Q_m\rangle &= D_{\mathbb{G}}F(P_m)\langle V_m\rangle + D_{\mathbb{G}}F(P_m)\langle (P_m \cdot V_m)^{-1} \cdot Q_m\rangle \\ &= D_{\mathbb{G}}F(P_m)\left\langle \exp\left(\sum_0^N x_i(V_m)X_i\right)(0)\right\rangle \\ &= o(d_{cc}(P_m, Q_m)). \end{aligned}$$

It follows then from (13) that $|x(V_m)| = o(d_{cc}(P_m, Q_m))$. The triangle inequality (applied on the points $P_m, P_m \cdot V_m, Q_m$) allows us to conclude that

$$d_{cc}(P_m \cdot V_m, Q_m) \simeq |x(V_m)| = o(d_{cc}(P_m, P_m \cdot V_m)) = o(d_{cc}(V_m, 0)) = o(\lambda_m). \qquad \square$$

The proof of the implication (*2*) $\Longrightarrow$ (*1*) in Theorem 1.2 requires several lemmata. In particular, Lemma 3.3 in Euclidean setting is due by B. Cornet [6].

Lemma 3.3. *Let $S \subset \mathbb{G}$ be a closed connected set. Then*

$$\mathrm{pTan}_{\mathbb{G}}^-(S, P_0) \subset \operatorname*{Li}_{S \ni P \to P_0} \mathrm{Tan}_{\mathbb{G}}^+(S, P).$$

Proof. We adapt here a classical argument from [1]. Let us consider $V \in \mathrm{pTan}_{\mathbb{G}}^-(S, P_0)$. Fix any sequence $\{P_m\}_{m \geq 0} \subset S$ converging to P. It can be shown (by a contradiction argument, for instance) that $V \in \mathrm{pTan}_{\mathbb{G}}^-(S, P_0)$ implies that for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ and $\beta > 0$ such that for any $h \in (0, \beta]$ and any $m \geq N$ we can find $P_m^h \in S$ such that

$$d_{cc}(P_m \cdot V, P_m^h) \leq h\epsilon.$$

By compactness, for any $m \geq N$ there's a limit point V_m of the sequence $\{P_m^{-1} \cdot P_m^h\}_{0<h\leq\beta}$ when $h \to 0$. By definition, V_m belongs to $\mathrm{Tan}_{\mathbb{G}}^+(S, P_m)$ and $d_{cc}(V_m, V) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we obtain the statement of the lemma. $\square$

Remark 3.4. By Remark 2.6, $\mathrm{Li}_{S \ni P \to P_0} \mathrm{Tan}_{\mathbb{G}}^+(S, P) \subset \mathrm{Tan}_{\mathbb{G}}^+(S, P_0)$ for $S \subset \mathbb{G}$.

By a direct application of the Cantor diagonal method we achieve the following

Lemma 3.5. *Let $S \subset \mathbb{G}$ be a closed connected set. Then*

$$\operatorname*{Ls}_{S \ni P \to P_0} \mathrm{pTan}^+_{\mathbb{G}}(S, P) \subset \mathrm{pTan}^+_{\mathbb{G}}(S, P_0).$$

Lemma 3.6. *Let $E, E' \subset \mathbb{G}$ be two closed subsets with $E \subset E'$. Assume that at some point $P \in E$*

$$\mathrm{pTan}^-_{\mathbb{G}}(P, E) = \mathrm{pTan}^+_{\mathbb{G}}(P, E')$$

holds. Then E coincides with E' in some neighbourhood of P.

Proof. By contradiction, assume that there is a sequence $\{P_n\} \subset E' \backslash E$ such that $P_n \to P$. Define $Q_n \in E$ as a point such that $d_{cc}(Q_n, P_n) = \min_{Q \in E} d_{cc}(Q, P_n)$. Up to take a subsequence, $\delta_{d_{cc}(P_n, Q_n)^{-1}}(Q_n^{-1} \cdot P_n)$ converges (by definition) to some $V \in \mathrm{pTan}^+(P, E')$. By hypothesis, $V \in \mathrm{pTan}^-_{\mathbb{G}}(P, E)$ that means we can find a sequence $E \ni \tilde{P}_n \to P$ such that

$$\delta_{d_{cc}(Q_n, P_n)^{-1}}(Q_n^{-1} \cdot \tilde{P}_n) = V.$$

Thus, $\delta_{d_{cc}(Q_n, P_n)^{-1}}(\tilde{P}_n^{-1} \cdot P_n) \to V \cdot V^{-1} = 0$, or, equivalently, $d_{cc}(\tilde{P}_n^{-1}, P_n) = o(d_{cc}(Q_n, P_n))$, that contradicts the minimality in the definition of Q_n. □

Proof of *(2)* $\Longrightarrow$ *(1)*. Since by assumption all four cones coincide, we denote them all merely by $\mathrm{Tan}_{\mathbb{G}}(S, P)$. Observe that by Propositions 2.7 and 2.8 $\mathrm{Tan}_{\mathbb{G}}(S, P)$ is a homogeneous subgroup of $\mathbb{G}$. Lemmata 3.3 and 3.5 and Remark 3.4 imply then the continuity of the cones on S:

$$\lim_{S \ni P \to Q} \mathrm{Tan}_{\mathbb{G}}(S, P) = \mathrm{Tan}_{\mathbb{G}}(S, Q), \quad \forall\, Q \in S.$$

Observe that the natural topology on the Grassmannian of vertical planes is equivalent to the topology induced by Kuratowski limit.

As $\mathrm{Tan}_{\mathbb{G}}(S, P_0)$ is a vertical subgroup and S is connected, $\mathrm{Tan}_{\mathbb{G}}(S, P)$ is a vertical subgroup of the same codimension N for all $P \in S$. Due to the continuity of $\mathrm{Tan}_{\mathbb{G}}(S, P)$, we can find a continuous map $k : S \to \mathrm{Hom}(\mathbb{G}, \mathbb{R}^N)$ such that $\mathrm{Ker}\, k(P) = \mathrm{Tan}_{\mathbb{G}}(S, P)$ for every $P \in S$.

Let us prove that $F : S \to \mathbb{R}^N$, $F \equiv 0$, and k satisfy the condition of Whitney extension theorem 2.3. By contradiction, we can assume the existence of sequences P_m and Q_m belonging to some compact part $\tilde{S} \subset S$ such that

- $P_m, Q_m \to P_0 \in \tilde{S}$ when $m \to \infty$,
- $|F(Q) - F(P) - k(P)\langle P^{-1} \cdot Q\rangle| = |k(P)\langle P^{-1} \cdot Q\rangle| > c d_{cc}(P, Q)$ with $c > 0$.

One can check from the very definition of $\mathrm{pTan}^+_{\mathbb{G}}$ that this is not compatible with $\mathrm{pTan}^+_{\mathbb{G}}(S, P_0) = \mathrm{Ker}\, k(P_0)$. Thus, $S \subset F^{-1}(0)$ for some $F \in C^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^N)$ such that $D_{\mathbb{G}} F$ is surjective on S. But we also know from Theorem 3.1 that $\mathrm{Tan}_{\mathbb{G}}(F^{-1}(0), P) = \mathrm{Ker}\, D_{\mathbb{G}} F(P) = \mathrm{Tan}_{\mathbb{G}}(S, P)$ for all $P \in S$. And, thus, Lemma 3.6 gives the conclusion. □

A. Appendix: application to optimization problems

In this appendix we present an application of our results to optimality problems. In particular, we will consider scalar function $G : \mathbb{G} \to \mathbb{R}$ and prove an intrinsic form of Peano's Regula for differentiable functions in the spirit of [7]. As a corollary, we obtain an intrinsic version of the Lagrange Multipliers theorem.

Theorem A.1 (Peano's Regula). *Let $G : \mathbb{G} \to \mathbb{R}$ be such that G is differentiable in $P_0 \in S$, where $S \subset \mathbb{G}$. If $G(P_0) = \max\{G(P),\ P \in S\}$ (resp. $G(P_0) = \min\{G(P),\ P \in S\}$), then*

$$D_{\mathbb{G}}G(P_0)\langle V\rangle \le 0, \quad (\textit{resp. } D_{\mathbb{G}}G(P_0)\langle V\rangle \ge 0,) \quad \forall V \in \mathrm{Tan}^+_{\mathbb{G}}(S, P_0). \tag{15}$$

Proof. Let $V \in \mathrm{Tan}^+_{\mathbb{G}}(S, P_0)$. By definition, there exist two sequences $\{P_m\}_m \subset S$ and $\{\lambda_m\}_m \subset \mathbb{R}_+$ such that (2) holds. As G is differentiable, by Lemma 2.9 we have

$$\lim_m \frac{G(P_m) - G(P_0)}{\lambda_m} = D_{\mathbb{G}}G(P_0)\langle V\rangle;$$

since $G(P_0) = \max_S G$ it follows that $G(P_m) - G(P_0) \le 0$ and thus we obtain (15). □

Theorem A.2 (Lagrange multipliers). *Let $1 \le N \le \dim H\mathbb{G}$ and let $F \in C^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^N)$ be such that $D_{\mathbb{G}}F$ is surjective at every point of $S := F^{-1}(0)$. Let $P_0 \in S$ be such that $G(P_0) = \max\{G(P),\ P \in S\}$ (resp. $G(P_0) = \min\{G(P),\ P \in S\}$) for a scalar function $G : \mathbb{G} \to \mathbb{R}$. If G is differentiable at P_0 then there exists $(\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ such that*

$$D_{\mathbb{G}}G(P_0) = \sum_{i=1}^{N} \alpha_i\ D_{\mathbb{G}}F_i(P_0), \qquad F = (F_1, \dots, F_N). \tag{16}$$

Proof. Theorem 3.1 implies that $\mathrm{Tan}^+_{\mathbb{G}}(S, P_0) = \mathrm{Ker}\, D_{\mathbb{G}}F(P_0)$. Therefore, by Theorem A.1 it follows

$$D_{\mathbb{G}}G(P_0)\langle V\rangle \le 0 \ \ \forall V \in \mathrm{Ker}\, D_{\mathbb{G}}F(P_0).$$

Since $\mathrm{Ker}\, D_{\mathbb{G}}F(P_0)$ is bilateral, this implies that

$$D_{\mathbb{G}}G(P_0)\langle V\rangle = 0 \ \ \forall V \in \mathrm{Ker}\, D_{\mathbb{G}}F(P_0),$$

providing (in a standard manner) the linear dependency condition (16). □

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