Journal of Convex Analysis
Volume 21 (2014), No. 4, 925–950

# Asymptotic Order of the Parallel Volume Difference in Minkowski Spaces

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Received: June 8, 2012
Revised manuscript received: July 27, 2013
Accepted: August 30, 2013

In this paper we investigate the asymptotic behavior of the parallel volume of fixed non-convex bodies in Minkowski spaces as the distance $r$ tends to infinity. We will show that the difference of the parallel volume of the convex hull of a body and the parallel volume of the body itself, which is called parallel volume difference, can at most have order $r^{d-2}$ in a $d$-dimensional Minkowski space. Then we will show that in certain Minkowski spaces (and in particular in Euclidean spaces) this difference can at most have order $r^{d-3}$. We will characterize the 2-dimensional Minkowski spaces in which the parallel volume difference has always at most order $r^{-1}$. Finally we present applications concerning Brownian paths and Boolean models.

*Keywords:* Convex geometry, parallel volume, non-convex body, random body

*2000 Mathematics Subject Classification:* 52A20, 52A21, 52A22, 52A38

## 1. Introduction and main results

The *parallel volume* of a body $K \subseteq \mathbb{R}^d$ at distance $r \geq 0$ is the Lebesgue measure of the *parallel body*, i.e. of set of all points which have at most distance $r$ from $K$ w.r.t. the Euclidean norm, where by body we mean a non-empty compact subset of $\mathbb{R}^d$. Since Steiner [21] discovered in 1840 that the parallel volume of certain convex bodies is a polynomial (meanwhile it is known that this is true for all convex bodies), the parallel volume has been studied intensively. Essential concepts of convex geometry, like intrinsic volumes, mixed volumes and support measures, are usually defined with help of the parallel volume or local or relative versions of it. Moreover, the parallel volume has many applications, e.g. in stochastic geometry [20], geometric functional analysis [3] or statistics [6], [17]. While in many of these applications the parallel volume of arbitrary bodies is of interest, it has been mainly investigated in the special case of convex bodies.

However, there are some important results for the parallel volume of non-convex bodies. The Brunn-Minkowski-inequality, which gives an upper bound for the parallel volume, was proven by Lusternik [18] for arbitrary bodies. It implies the isoperimetric inequality and is closely related to many other inequalities in various

ISSN 0944-6532 / $ 2.50 

branches of mathematics and physics [2]. Kneser [16] and Sz.-Nagy [22] obtained inequalities saying that the parallel volume of a fixed body considered as function of the distance cannot be "too convex". Heveling, Hug and Last [5] showed that a planar body can only have polynomial parallel volume if it is convex (see also [9]). Motivated by applications in stereology and stochastic geometry, Kiderlen and Rataj [15] started an investigation of the asymptotic behavior of the parallel volume as the distance tends to infinity.

It is well-known that the parallel body $K_r$ of a body $K \subseteq \mathbb{R}^d$ at distance $r \geq 0$ equals $K + rB^d$, where $K + L := \{x + y \mid x \in K, y \in L\}$ is the Minkowski-sum of two bodies $K, L \subseteq \mathbb{R}^d$, $rK := \{rx \mid x \in K\}$ is the homothetic copy of a body $K \subseteq \mathbb{R}^d$ with scaling factor $r \geq 0$ and $B^d$ is the closed Euclidean unit ball.

In [11] we have shown

$$\lim_{r\to\infty} V_2((\operatorname{conv} K) + rB^2) - V_2(K + rB^2) = 0, \tag{1}$$

for an arbitrary body $K \subseteq \mathbb{R}^2$, where $V_d$ denotes the $d$-dimensional Lebesgue measure and $\operatorname{conv} K$ denotes the convex hull of a body $K \subseteq \mathbb{R}^d$. In the present paper we will show that the order of convergence in (1) is $1/r$. Indeed, we will show that in arbitrary dimensions $d \geq 2$ there is a constant $c$ depending on $d$ and $K$ such that

$$V_d((\operatorname{conv} K) + rB^d) - V_d(K + rB^d) < c \cdot r^{d-3} \tag{2}$$

for sufficiently large $r$. We will always assume $d \geq 2$ in this paper, since the left-hand side of (2) is obviously zero for all sufficiently large $r$ in the case $d = 1$.

Just like in [11] we will generalize (2) by replacing the Euclidean unit ball with other convex bodies $B \subseteq \mathbb{R}^d$. For a convex body $B \subseteq \mathbb{R}^d$ with $0 \in \operatorname{int} B$, where $\operatorname{int} B$ denotes the interior of a set $B \subseteq \mathbb{R}^d$, we put

$$d_B(A, x) = \inf\{r \geq 0 \mid x \in A + rB\}, \quad x \in \mathbb{R}^d,\ A \subseteq \mathbb{R}^d$$

and

$$d_B(y, x) = d_B(\{y\}, x), \quad x, y \in \mathbb{R}^d.$$

It is possible to reconstruct $B$ from the function $d_B$, since

$$B = \{x \in \mathbb{R}^d \mid d_B(0, x) \leq 1\}.$$

The body $B$ is called the *gauge body* of the function $d_B$. It is easy to see that the function $d_B : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is symmetric iff $B$ is symmetric and, moreover, that in this case $d_B$ is a metric which is induced by a norm. Every norm $\|\cdot\|$ in $\mathbb{R}^d$ can be obtained as a function $d_B$ by putting $B = \{x \in \mathbb{R}^d \mid \|x\| \leq 1\}$. Thus all definitions made for functionals $d_B$ apply in particular to norms.

Now the parallel body of a body $K \subseteq \mathbb{R}^d$ at distance $r \geq 0$ w.r.t. a function $d_B$ is defined as

$$K_r := \{x \in \mathbb{R}^d \mid \text{there is } y \in K \text{ with } d_B(y, x) \leq r\} = \{x \in \mathbb{R}^d \mid d_B(K, x) \leq r\}$$

and equals $K + rB$. Hence by replacing the Euclidean unit ball in (2) with other convex bodies, we generalize the result from Euclidean spaces to more general finite-dimensional normed spaces. Finite-dimensional normed spaces are called *Minkowski spaces*, but should not be confused with space-time-Minkowski spaces.

We will impose a smoothness assumption on the gauge body $B$. Namely, $B$ must contain a (Euclidean) ball (of positive radius) as summand, where a convex body $S$ is said to be a summand of a convex body $B$ if there is a convex body $L$ such that $B = L + S$. A weaker asymptotic bound for the *parallel volume difference* $V_d((\operatorname{conv} K)+rB)-V_d(K+rB)$ holds without any smoothness assumptions. The precise formulation is as follows:

**Theorem 1.1.** *Let $K \subseteq \mathbb{R}^d$ be a body and $B \subseteq \mathbb{R}^d$ a convex body with interior points.*

(*i*) *There is some constant $c$ (depending on $d$, $B$ and $K$) such that for all sufficiently large $r$ we have*

$$V_d((\operatorname{conv} K) + rB) - V_d(K + rB) < c \cdot r^{d-2}.$$

(*ii*) *If $B$ contains a ball as summand, then there is $c$ (depending on $d$, $B$ and $K$) such that for all sufficiently large $r$ we have*

$$V_d((\operatorname{conv} K) + rB) - V_d(K + rB) < c \cdot r^{d-3}.$$

The condition that $B$ has a ball as summand in part (*ii*) is indeed necessary. Corollary 4.3 says that in the planar case the fact that the order of $V_2((\operatorname{conv} K) + rB) - V_2(K + rB)$ is $1/r$ (with the same constant for all bodies $K \subseteq \mathbb{R}^2$ whose diameter is at most 1) characterizes gauge bodies $B$ that have a ball as summand. However, we conjecture that in higher dimensions $d \geq 3$ there are gauge bodies $B$ that do not have a ball as summand, but for which $V_d((\operatorname{conv} K)+rB)-V_d(K+rB)$ always is of order $r^{d-3}$. We will show in Example 4.1 and Example 4.2 that the orders in Theorem 1.1 are optimal.

For a norm $\|\cdot\|$ on $\mathbb{R}^d$ we define the dual norm $\|\cdot\|^*$ by

$$\|v\|^* := \max\{\langle x, v\rangle \mid \|x\| \leq 1\}, \quad v \in \mathbb{R}^d,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product. The Euclidean norm on $\mathbb{R}^d$ will be denoted by $\|\cdot\|_2$.

The following corollary rephrases Theorem 1.1 in terms of Minkowski spaces:

**Corollary 1.2.** *Let $K \subseteq \mathbb{R}^d$ be a body and $\|\cdot\|$ a norm on $\mathbb{R}^d$.*

(*i*) *There is some constant $c$ (depending on $d$, $\|\cdot\|$ and $K$) such that for all sufficiently large $r$ we have*

$$V_d((\operatorname{conv} K)_r) - V_d(K_r) < c \cdot r^{d-2}.$$

(*ii*) *If $v \mapsto \|v\|^* - R\|v\|_2$ is a norm for some $R > 0$, then there is $c$ (depending on $d$, $\|\cdot\|$ and $K$) such that for all sufficiently large $r$ we have*

$$V_d((\operatorname{conv} K)_r) - V_d(K_r) < c \cdot r^{d-3}.$$

This corollary is an immediate consequence of Theorem 1.1 and Lemma 2.3 below, where we will show that $v \mapsto \|v\|^* - R\|v\|_2$ is a norm for some $R > 0$ iff the gauge body of $\|\cdot\|$ has a ball as summand.

Now we will extend Theorem 1.1 to random bodies and discuss the case that the dimension (of the affine hull) of $K$ is larger than that of $B$.

A random closed set is a measurable map from some probability space to the space of all closed subsets of $\mathbb{R}^n$ equipped with the Borel-$\sigma$-algebra of the Fell-Matheron-topology; see e.g. [20] for details (since we will embed $\mathbb{R}^d$ into $\mathbb{R}^n$ for $n \geq d$, we have to use both letters for dimensions). A random body, random convex body, random $d$-dimensional convex body (in $\mathbb{R}^n$), random ($d$-dimensional) ball is a random closed set that a.s. takes values in the set of all bodies, convex bodies, $d$-dimensional convex bodies, ($d$-dimensional) balls.

We let $\rho_B$ denote the radius of the largest $d$-dimensional ball contained in a $d$-dimensional convex body $B \subseteq \mathbb{R}^n$, and by $\operatorname{diam} A$ we denote the diameter of a set $A \subseteq \mathbb{R}^n$. Moreover, we let $\hat{B}$ denote the affine hull of a body $B \subseteq \mathbb{R}^n$ and

$$B^\perp := \{v \in \mathbb{R}^d \mid \langle y - x, v\rangle = 0 \text{ for all } x, y \in B\}$$

the affine-orthogonal complement of $\hat{B}$. For two bodies $K, B \subseteq \mathbb{R}^n$ we call

$$K_B := \bigcup_{x \in B^\perp} \operatorname{conv}(K \cap (\hat{B} + x))$$

the *$\hat{B}$-convex hull* of $K$.

**Theorem 1.3.** *Let $X \subseteq \mathbb{R}^n$ be a random body and $Y \subseteq \mathbb{R}^n$ a random $d$-dimensional convex body, $1 < d \leq n$. Put $G := \max\{\operatorname{diam} X, 1\}$, $S := \max\{\operatorname{diam} Y, 1\}$. If*

$$c := d2^d \kappa_d \kappa_{n-d} \mathbb{E}\left[\frac{S^{d-1} \cdot G^n}{\rho_Y}\right] < \infty,$$

*then we have*

$$\mathbb{E}[V_n(X_Y + rY) - V_n(X + rY)] < c \cdot r^{d-2}, \quad r \geq 1.$$

**Theorem 1.4.** *Let $X \subseteq \mathbb{R}^n$ be a random body and $Y \subseteq \mathbb{R}^n$ a random $d$-dimensional convex body, $1 < d \leq n$, which a.s. contains a random $d$-dimensional ball of random radius $R > 0$ as summand. Put $G := \max\{\operatorname{diam} X, 1\}$ and $S := \max\{\operatorname{diam} Y, 1\}$. If*

$$c := d2^{d+2} \kappa_d \kappa_{n-d} \mathbb{E}\left[\frac{S^d \cdot G^{n+1}}{R^3}\right] < \infty,$$

*then we have*

$$\mathbb{E}[V_n(X_Y + rY) - V_n(X + rY)] < c \cdot r^{d-3}, \quad r \geq 1.$$

All expressions occurring in Theorem 1.3 and Theorem 1.4 will be shown to be measurable in Remark 3.1.

Theorem 1.1 is indeed a special case of Theorem 1.3 and Theorem 1.4: If $d = n$, then we have $X_Y = \operatorname{conv} X$, and if all sets involved are deterministic, the integrability conditions are fulfilled and the expectation signs can be dropped.

The proofs of Theorem 1.3 and 1.4 are organized as follows: First a special case of these theorems, that is similar to Theorem 1.1, will be shown by an analysis of the set difference $((\operatorname{conv} K) + rB) \setminus (K + rB)$. Here, the $B$-metric projection map that is defined in the next section will play a central role. In the case that $B$ contains a ball as summand, also a (new) asymptotic version of the Pythagorean Theorem will be used. The results are then generalized from the special case to the general case by approximation arguments and easy computations.

The line of research to which this paper belongs will be continued further: In [14] an optimal value for the constant $c$ in Theorem 1.1($ii$) is determined in the special case that $B$ is the Euclidean unit ball. In [12] the asymptotic behavior of the derivative of the parallel volume difference in planar Minkowski spaces is investigated.

This paper is organized as follows. In Section 2 we collect mathematical tools, mainly from geometry, that will be needed in later sections. Section 3 is devoted to the proofs of Theorem 1.3 and Theorem 1.4. In Section 4 we show that certain improvements of these theorems are not possible: We will show that the derived rates are optimal and we will prove that the gauge body of a planar Minkowski space in which the parallel volume difference has uniformly order $1/r$ must have a ball as summand. In Section 5 we examine the asymptotic behavior of the parallel volume of Brownian paths as the time tends to zero and derive results about the contact distribution of Boolean models.

## 2. Preparation

In this section we collect tools, especially from geometry, that we will need in later sections.

We start with a weakened notion of twice differentiability, that is due to Alexandrov.

**Definition 2.1.** Let $G \subseteq \mathbb{R}^d$ be an open convex set and $f : G \to \mathbb{R}$ a convex function.

(i) A function $\theta : G \to \mathbb{R}^d$ is called a choice of subgradients of $f$ if $\theta(x)$ is a subgradient of $f$ at $x$ for every $x \in G$.

(ii) Let $x \in G$. If all choices of subgradients of $f$ are differentiable at $x$ with the same derivative, then $f$ is said to be Alexandrov-twice-differentiable at $x$. The derivative of the choices of subgradients is called second derivative of $f$ at $x$.

By $\operatorname{bd} A$ we denote the boundary of a set $A \subseteq \mathbb{R}^d$.

**Lemma 2.2.** *Let $K \subseteq \mathbb{R}^d$ be a body and $x \in (\operatorname{bd}\operatorname{conv} K) \setminus K$. Then $x$ is contained in the relative interior of a face of positive dimension of* $\operatorname{conv} K$.

**Proof.** According to [19, Theorem 2.1.2] the point $x$ is contained in the relative interior of a face $F$ of $\operatorname{conv} K$. So all we have to show is $F \neq \{x\}$. Since $(\operatorname{conv} K) \setminus F$ is convex, we cannot have $K \subseteq (\operatorname{conv} K) \setminus F$ by the definition of the convex hull. So $K \cap F \neq \emptyset$ and $x \notin K$ implies $F \neq \{x\}$. $\square$

A convex body $S \subseteq \mathbb{R}^d$ is called *summand* of a convex body $K \subseteq \mathbb{R}^d$ if there is a convex body $L \subseteq \mathbb{R}^d$ such that $S + L = K$. For a more detailed introduction we refer to [19, Sections 3.1 and 3.2]. In particular, a convex body $S \subseteq \mathbb{R}^d$ is a summand of a convex body $K \subseteq \mathbb{R}^d$ iff for each point $p \in K$ there is a vector $m \in \mathbb{R}^d$ with

$$p \in m + S \subseteq K;$$

see [19, Theorem 3.2.2].

We denote the support function of a convex body $K \subseteq \mathbb{R}^d$ by $h_K$.

**Lemma 2.3.** *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ with gauge body $B \subseteq \mathbb{R}^d$ and let $R > 0$ be a number. Then $\|\cdot\|^* - R\|\cdot\|_2$ is a norm iff $B$ has a summand $RB^d$.*

**Proof.** Assume that $\|\cdot\|^* - R\|\cdot\|_2$ is a norm and denote the gauge body of its dual norm by $L$. Then

$$\|v\|^* - R\|v\|_2 = \max\{\langle x, v\rangle \mid x \in L\}, \quad v \in \mathbb{R}^d,$$

which is equivalent to

$$h_B(v) - h_{RB^d}(v) = h_L(v), \quad v \in \mathbb{R}^d,$$

which in turn is equivalent $B = RB^d + L$ due to [20, Theorem 1.7.5]. Thus $B$ has a summand $RB^d$. Since the order of the proof can be reverted, we have shown that equivalence holds. $\square$

Choose a convex body $B \subseteq \mathbb{R}^d$ with $0 \in \operatorname{int} B$ to be the gauge body. For a closed set $A \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we denote the set of all points of $A$ which are closest to $x$ by

$$\Pi_B(A, x) := \{y \in A \mid d_B(y, x) = d_B(A, x)\}.$$

The *$B$-exoskeleton* $\operatorname{exo}_B(A)$ of $A$ is the set of all points $x \in \mathbb{R}^d$ for which $\Pi_B(A, x)$ consists of more than one point. For $x \in \mathbb{R}^d \setminus \operatorname{exo}_B(A)$ we define the *$B$-metric projection* $p_B(A, x)$ of $x$ onto $A$ to be the unique point in $\Pi_B(A, x)$, and if moreover $x \notin A$, we denote the normalized vector pointing from $p_B(K, x)$ to $x$ by $u_B(A, x) := (x - p_B(A, x))/d_B(A, x)$. The *$B$-normal bundle* of $A$ is

$$\mathcal{N}_B(A) := \{(p_B(A, x), u_B(A, x)) \mid x \in \mathbb{R}^d \setminus A \setminus \operatorname{exo}_B(A)\}.$$

Now we will introduce the mixed volumes, which are the analogues to the intrinsic volumes in Minkowski spaces, and certain measures that provide information about the contribution of different parts of a body to its mixed volumes.

The *mixed volumes* $V(K[j], B[d-j])$, $j = 0, \dots, d$, of convex bodies $K, B \subseteq \mathbb{R}^d$ are the uniquely determined numbers such that

$$V_d(K + rB) = \sum_{j=0}^{d} r^{d-j} \binom{d}{j} V(K[j], B[d-j]), \quad r \geq 0.$$

For further information on mixed volumes, see e.g. [19, Section 5.1].

Now assume that $B$ is a strictly convex body satisfying $0 \in \operatorname{int} B$. It is known that for convex bodies $K \subseteq \mathbb{R}^d$ we have $\operatorname{exo}_B(K) = \emptyset$. Hence we can consider the local $B$-parallel set

$$\mu_r^B(K, \eta) := V_d\big(\{x \in (K + rB) \setminus K \mid (p_B(K,x), u_B(K,x)) \in \eta\}\big)$$

for $r \geq 0$ and Borel sets $\eta \subseteq \mathbb{R}^d \times \mathbb{R}^d$ (see e.g. [8]). There are uniquely determined measures $\Xi_j^B(K, \cdot)$, $j = 0, \dots, d-1$, called *relative support measures*, on $\mathbb{R}^d \times \mathbb{R}^d$ with

$$\mu_r^B(K, \eta) = \sum_{j=0}^{d-1} r^{d-j} \kappa_{d-j} \Xi_j^B(K, \eta), \quad r \geq 0, \, \eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d),$$

where $\kappa_j$ denotes the volume of the $j$-dimensional unit ball and $\mathcal{B}(M)$ is the Borel-$\sigma$-algebra of a metric space $M$. Their projections on the first component,

$$\Phi_j^B(K, \beta) := \Xi_j^B(K, \beta \times \mathbb{R}^d), \quad j = 0, \dots, d-1, \, \beta \in \mathcal{B}(\mathbb{R}^d),$$

are called *relative curvature measures*. Their projections on the second component,

$$\Psi_j^B(K, \gamma) := \Xi_j^B(K, \mathbb{R}^d \times \gamma), \quad j = 0, \dots, d-1, \, \gamma \in \mathcal{B}(\mathbb{R}^d),$$

are called *relative area measures*.

The total masses of these measures are, up to normalization, the mixed volumes. More precisely, for a strictly convex body $B \subseteq \mathbb{R}^d$ with $0 \in \operatorname{int} B$, a convex body $K \subseteq \mathbb{R}^d$ and $j \in \{0, \dots, d-1\}$ we have

$$\Phi_j^B(K, \mathbb{R}^d) = \Psi_j^B(K, \mathbb{R}^d) = \Xi_j^B(K, \mathbb{R}^d \times \mathbb{R}^d) = \frac{\binom{d}{j}}{\kappa_{d-j}} V(K[j], B[d-j]). \tag{3}$$

The Hausdorff measure $\mathcal{H}^j(A)$ of a subset $A \subseteq \mathbb{R}^d$ is the length of $A$ if $j = 1$, the area of $A$ if $j = 2$, the volume of $A$ if $j = 3$ etc., provided that $A$ is a $j$-dimensional set. For a precise introduction see e.g. [1].

For a convex body $K \subseteq \mathbb{R}^d$ the measure $S_{d-1}(K, \omega) := 2\Psi_{d-1}^{B^d}(K, \omega)$, $\omega \in \mathcal{B}(\mathbb{R}^d)$, which is concentrated on the unit sphere $S^{d-1}$, is called *surface area measure*. Its name derives from the fact (see e.g. [19, (4.2.24)]) that

$$S_{d-1}(K, \omega) = \mathcal{H}^{d-1}(\tau_K(\omega)), \quad \omega \in \mathcal{B}(S^{d-1}), \tag{4}$$

for convex bodies $K \subseteq \mathbb{R}^d$ with interior points, where the *reverse spherical image* $\tau_K(\omega)$ of $\omega$ is the set of all boundary points of $K$ having an exterior unit normal vector in $\omega$.

For convex bodies $K, B \subseteq \mathbb{R}^d$ the *mixed area measures* are defined to be the measures $S(K[j], B[d-j-1], \cdot)$, $j = 0, \ldots, d-1$, on $\mathbb{R}^d$ with

$$S_{d-1}(sK + rB, \omega) = \sum_{j=0}^{d-1} \binom{d-1}{j} s^j r^{d-j-1} S(K[j], B[d-j-1], \omega), \quad r, s \geq 0, \tag{5}$$

for any Borel set $\omega \subseteq \mathbb{R}^d$. For further information on mixed area measures we refer to [19, Section 5.1].

The mixed area measures are related to the relative area measures. In order to make this relationship precise, we define the *reverse spherical image map* of a strictly convex body $K \subseteq \mathbb{R}^d$ to be the map $\tau_K : S^{d-1} \to \operatorname{bd} K$ that assigns to a vector $u \in S^{d-1}$ the point $p \in \operatorname{bd} K$ with exterior unit normal vector $u$. Since for a strictly convex body $K \subseteq \mathbb{R}^d$ the image of a set $\omega$ under the reverse spherical image map is its reverse spherical image, we use the same symbol for both.

**Remark 2.4.** From [19, Corollary 1.7.3] we get that $K$ is strictly convex iff $h_K$ is differentiable on $\mathbb{R}^d \setminus \{0\}$, and in this case we have $\nabla h_K(u) = \tau_K(\frac{u}{\|u\|})$ for all $u \in \mathbb{R}^d \setminus \{0\}$.

The relationship between relative area measures and mixed area measures is given by Theorem 2.14 in [7], which says:

**Theorem 2.5.** *Let $K, B \subseteq \mathbb{R}^d$ be two convex bodies such that $0 \in \operatorname{int} B$ and $B$ is strictly convex. Then we have for $j \in \{0, \ldots, d-1\}$ and Borel sets $\gamma \subseteq \mathbb{R}^d$*

$$\Psi_j^B(K, \gamma) = \frac{\binom{d}{j}}{d\kappa_{d-j}} \int_{S^{d-1}} \mathbf{1}_\gamma(\nabla h_B(u)) h_B(u)\, S(K[j], B[d-j-1], du).$$

Now we will show that the 0-th relative curvature measure of a convex body is concentrated on the set of its extreme points. For the Euclidean curvature measure this is well-known (see e.g. [19, (4.6.1)]). We remark that the set $\operatorname{ext} K$ of extreme points of a convex body $K \subseteq \mathbb{R}^d$ is an intersection of countably many open sets by [19, p. 66] and hence measurable.

**Theorem 2.6.** *Let $B \subseteq \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} B$, whose support function $h_B$ is twice continuously differentiable on $\mathbb{R}^d \setminus \{0\}$. Let $K \subseteq \mathbb{R}^d$ be a convex body. Then the relative curvature measure $\Phi_0^B(K, \cdot)$ is concentrated on $\operatorname{ext} K$.*

The proof of Theorem 2.6 is based on the following lemma.

**Lemma 2.7.** *Let $B, K \subseteq \mathbb{R}^d$ be two convex bodies such that $0 \in \operatorname{int} B$ and $B$ is strictly convex. Then with $M := \max\{h_B(u) \mid u \in S^{d-1}\}$ we have for any Borel* *set* $\gamma \subseteq \operatorname{bd} B$

$$\Psi_0^B(K,\gamma) \le \frac{M}{d\kappa_d} \cdot \mathcal{H}^{d-1}(\gamma).$$

**Proof.** From Theorem 2.5 and Remark 2.4 we get, since $S(K[0], B[d-1], \cdot) = S_{d-1}(B, \cdot)$,

$$\begin{aligned}\Psi_0^B(K,\gamma) &= \frac{1}{d\kappa_d}\int_{S^{d-1}} \mathbf{1}_\gamma(\nabla h_B(u)) h_B(u) S_{d-1}(B, du)\\ &\le \frac{1}{d\kappa_d}\int_{S^{d-1}} \mathbf{1}_\gamma(\tau_B(u)) M S_{d-1}(B, du)\\ &= \frac{M}{d\kappa_d} S_{d-1}(B, \{u \in S^{d-1} \mid \tau_B(u) \in \gamma\}).\end{aligned}$$

Since $\tau_B : S^{d-1} \to \operatorname{bd} B$ is surjective, we derive from (4) that

$$S_{d-1}(B, \{u \in S^{d-1} \mid \tau_B(u) \in \gamma\}) = \mathcal{H}^{d-1}(\gamma),$$

which completes the proof of the lemma. $\square$

**Proof of Theorem 2.6.** Since $h_B$ is differentiable, $B$ must be strictly convex by Remark 2.4. Hence $\Phi_0^B(K, \cdot)$ is defined. Let $\omega \subseteq S^{d-1}$ be the set of all (Euclidean) exterior unit normal vectors of $K$ in points of $\operatorname{bd} K \setminus \operatorname{ext} K$. Since for every vector $u \in \omega$ there is more than one point in $\operatorname{bd} K$ having exterior normal vector $u$, [19, Theorem 2.2.9] implies $\mathcal{H}^{d-1}(\omega) = 0$. Put

$$\gamma := \{b \in \mathbb{R}^d \mid \text{there is } x \in \operatorname{bd} K \setminus \operatorname{ext} K \text{ with } (x,b) \in \mathcal{N}_B(K)\}.$$

By [8, Lemma 2.1] we have $\gamma = \{\nabla h_B(u) \mid u \in \omega\}$. Since $h_B$ is assumed to be twice continuously differentiable, the restriction of $\nabla h_B$ to $S^{d-1}$ is Lipschitz continuous with Lipschitz constant $L$, say. By Theorem 1 from [1, Section 2.4] this implies

$$\mathcal{H}^{d-1}(\gamma) \le L^{d-1} \cdot \mathcal{H}^{d-1}(\omega) = 0.$$

Because the relative support measure $\Xi_0^B(K, \cdot)$ is concentrated on the relative normal bundel $\mathcal{N}_B(K)$, we get from Lemma 2.7 that

$$\begin{aligned}\Phi_0^B(K, \operatorname{bd} K \setminus \operatorname{ext} K) &= \Xi_0^B(K, (\operatorname{bd} K \setminus \operatorname{ext} K) \times \mathbb{R}^d)\\ &= \Xi_0^B(K, (\operatorname{bd} K \setminus \operatorname{ext} K) \times \gamma)\\ &\le \Psi_0^B(K,\gamma)\\ &\le \frac{M}{d\kappa_d} \cdot \mathcal{H}^{d-1}(\gamma)\\ &= 0.\end{aligned}$$ $\square$

We finish this section with a continuity result.

We let $\mathcal{K}_0$ denote the set of all convex bodies with interior points. For an introduction of the Hausdorff metric see e.g. [19, Section 1.8].

**Lemma 2.8.** *For a fixed (convex or non-convex) body $K$ the map*

$$\mathcal{K}_0 \to [0,\infty),\ B \mapsto V_d(K+B)$$

*is continuous w.r.t. the Hausdorff metric.*

**Proof.** By [20, Theorem 12.3.5 and 12.3.6] this map is upper semicontinuous (in fact, even its extension to the set of all bodies is). In order to show that it is also lower semicontinuous, let $B \in \mathcal{K}_0$. Then $B$ contains a ball of radius $R > 0$, say, which has w.l.o.g. its center at the origin. Let $\epsilon > 0$. Since

$$[0,\infty) \to [0,\infty),\ r \mapsto V_d(K+rB)$$

is continuous according to [4, Lemma 3], there is $\delta \in (0,1)$ with $V_d(K+B)-\epsilon < V_d(K+(1-\delta)B)$. Let $\tilde{B} \in \mathcal{K}_0$ be a body whose Hausdorff distance from $B$ is less than $R\delta$. Then

$$B \subseteq \tilde{B} + R\delta B^d \subseteq \tilde{B} + \delta B,$$

which implies $(1-\delta)B \subseteq \tilde{B}$ by the cancelation law for Minkowski sums (see [19, p. 41]). This, however, implies

$$V_d(K+B) - \epsilon < V_d(K+\tilde{B}). \qquad \square$$

## 3. The main results

In this section we prove Theorem 1.3 and Theorem 1.4.

**Remark 3.1.** All expressions occurring in Theorem 1.3 and Theorem 1.4 are measurable.

The set $\mathcal{C}$ of non-empty compact sets and the set $\mathcal{K}$ of non-empty convex compact sets are measurable by [20, Lemma 2.1.2 and Theorem 2.4.2]. The set of $d$-dimensional bodies in $\mathbb{R}^n$, $d \leq n$, is measurable according to [11, Corollary 6.3]. The set of $d$-dimensional balls in $\mathbb{R}^n$ is obviously closed and hence measurable. The functions $\mathcal{K} \to \mathbb{R}$, $B \mapsto \rho_B$ and $\operatorname{diam} : \mathcal{C} \to \mathbb{R}$ are obviously continuous w.r.t. the Hausdorff metric and hence measurable due to [20, Theorem 12.3.2]. By [20, Theorem 12.3.5] the same holds for $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(K,L) \mapsto K+L$ and $\mathcal{C} \times [0,\infty) \to \mathcal{C}$, $(K,r) \mapsto rK$. The map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(K,B) \mapsto K_B$ is shown to be measurable in [11, Lemma 6.5]. The map $V_n : \mathcal{C} \to \mathbb{R}$ is upper semicontinuous by [20, Theorem 12.3.6] and hence measurable (see [20, p. 19]).

We first prove a lemma making the same statement as Theorem 1.3 under additional assumptions, in particular $n = d$. For this we need the function

$$\begin{aligned} w_B : {}& [0,\infty) \to [0,\infty), \\ & r \mapsto \min\{d_B(y,z) \mid y \in B^d,\ z \in \mathbb{R}^d,\ y \in \Pi_B(0y,z),\ d_B(0,z) = r\}, \end{aligned} \tag{6}$$

which is defined for convex bodies $B \subseteq \mathbb{R}^d$ with $0 \in \operatorname{int} B$, where $xy := \{\lambda x + (1-\lambda)y \mid \lambda \in \mathbb{R}\}$ is the line through $x$ and $y$ if $x \neq y$ and the singleton $\{x\}$ otherwise.

The geometric interpretation of $w_B(r)$ is the following: Consider a triangle with vertices $x, y, z \in \mathbb{R}^d$ which has a right angle at $y$ in the sense that $y \in \Pi_B(xy, z)$. Assume that the hypotenuse has $B$-length $r$, i.e. $d_B(x, z) = r$, and the cathetus between $x$ and $y$ has Euclidean length 1. The $B$-length of the other cathetus, i.e. $d_B(y, z)$, will depend on the choice of $x$, $y$ and $z$ in general. Its least possible value is $w_B(r)$ for sufficiently large $r$. For a more detailed introduction see [11].

**Lemma 3.2.** *Let $K \subseteq \mathbb{R}^d$ be a body and $B \subseteq \mathbb{R}^d$ be a convex body with interior points such that $h_B$ is twice differentiable on $\mathbb{R}^d \setminus \{0\}$. Put $G := \max\{\operatorname{diam} K, 1\}$, $S := \max\{\operatorname{diam} B, 1\}$ and*

$$C' := d2^d\kappa_d \frac{S^{d-1} \cdot G^d}{\rho_B}.$$

*Then we have for all $r \geq 1$*

$$V_d(\operatorname{conv} K + rB) - V_d(K + rB) < C' \cdot r^{d-2}.$$

**Proof.** By the translation-invariance of the Lebesgue measure we may assume $\rho_B B^d \subseteq B$. Moreover, $B$ is strictly convex by Remark 2.4.

We put $L := \operatorname{conv} K$. Then we get by [11, Lemma 3.5] and the definition of the relative support measures that

$$\begin{aligned}
&V_d(L + rB) - V_d(K + rB) \\
&= V_d((L + rB) \setminus (K + rB)) \\
&\leq V_d\big(\{x \in \mathbb{R}^d \setminus L \mid p_B(L, x) \in L \setminus \operatorname{ext} L,\ d_B(L, x) \in (Gw_B(\tfrac{r}{G}), r]\} \\
&\qquad \cup (L \setminus (K + rB))\big) \\
&= \sum_{i=1}^{d} \kappa_i \Phi^B_{d-i}(L, L \setminus \operatorname{ext} L)\big(r^i - (Gw_B(\tfrac{r}{G}))^i\big) + V_d(L \setminus (K + rB)). \qquad (7)
\end{aligned}$$

Now we will derive an upper bound for the right-hand side in (7). The equation

$$s - w_B(s) \leq \frac{1}{\rho_B}, \quad s \geq 0, \qquad (8)$$

is shown in [11, formula (8)] to be an easy corollary of the triangular inequality for $d_B$. Since obviously $w_B(s) \le s$ for all $s \ge 0$, we conclude

$$\begin{aligned} r^i - (Gw_B(\tfrac{r}{G}))^i &= (r - Gw_B(\tfrac{r}{G})) \sum_{j=0}^{i-1} r^j (Gw_B(\tfrac{r}{G}))^{i-1-j} \\ &= G \cdot (\tfrac{r}{G} - w_B(\tfrac{r}{G})) \sum_{j=0}^{i-1} r^j (Gw_B(\tfrac{r}{G}))^{i-1-j} \\ &\le \frac{G}{\rho_B} \sum_{j=0}^{i-1} r^j (G\tfrac{r}{G})^{i-1-j} \\ &= \frac{G}{\rho_B} \sum_{j=0}^{i-1} r^{i-1} \\ &= \frac{G}{\rho_B} i r^{i-1}. \end{aligned} \tag{9}$$

By (3) we have

$$\kappa_i \Phi^B_{d-i}(L, L \setminus \operatorname{ext} L) \le \binom{d}{i} V(L[d-i], B[i]).$$

Since $L$ is contained in the circumsphere of $K$, whose radius is smaller than $G$, and $B$ is contained in a ball of radius $S$, we get by elementary properties of mixed volumes ([19, p. 277 and (5.1.24)])

$$V(L[d-i], B[i]) \le V(GB^d[d-i], SB^d[i]) = G^{d-i} S^i \kappa_d.$$

Thus

$$\kappa_i \Phi^B_{d-i}(L, L \setminus \operatorname{ext} L) \le \binom{d}{i} G^{d-i} S^i \kappa_d. \tag{10}$$

Using (7), (10) and Theorem 2.6, which says $\Phi^B_0(L, L \setminus \operatorname{ext} L) = 0$, we obtain

$$\begin{aligned} & V_d(L + rB) - V_d(K + rB) \\ \le\ & \sum_{i=1}^{d} \kappa_i \Phi^B_{d-i}(L, L \setminus \operatorname{ext} L)\big(r^i - (Gw_B(\tfrac{r}{G}))^i\big) + V_d(L \setminus (K + rB)) && (11) \\ \le\ & \sum_{i=1}^{d-1} \binom{d}{i} G^{d-i} S^i \kappa_d \cdot \frac{G}{\rho_B} i r^{i-1} + \kappa_d G^d && (12) \\ \le\ & \sum_{i=1}^{d-1} (d-1) \binom{d}{i} \kappa_d \frac{S^{d-1} G^d}{\rho_B} r^{d-2} + \kappa_d G^d r^{d-2} && (13) \\ <\ & \left( (d-1) 2^d \kappa_d \frac{S^{d-1} G^d}{\rho_B} + \kappa_d G^d \right) r^{d-2} \\ <\ & \left( d 2^d \kappa_d \frac{S^{d-1} \cdot G^d}{\rho_B} \right) r^{d-2}. \end{aligned}$$

□

**Proof of Theorem 1.3.** Let $r \geq 1$. We want to prove

$$\mathbb{E}[V_n(X_Y + rY) - V_n(X + rY)] < c \cdot r^{d-2}.$$

Put $X^x := X \cap ((\hat{Y} - \hat{Y}) + x)$ and $Z^x := \operatorname{conv} X^x$ for $x \in Y^\perp$ (recall that $\hat{Y} - \hat{Y}$ is the linear subspace parallel to $\hat{Y}$).

As an easy consequence of [19, Theorem 3.3.1] there is a sequence $(Y_k)_{k\in\mathbb{N}}$ of random convex bodies lying a.s. in $\hat{Y}$ such that for all $k \in \mathbb{N}$ the support function of $Y_k$ is twice continuously differentiable on $(\hat{Y} - \hat{Y})\setminus\{0\}$, and that $\lim_{k\to\infty} Y_k = Y$ a.s. Due to Lemma 2.8 (applied with $\mathbb{R}^d = x + r\hat{Y}$) we have a.s. for all $x \in Y^\perp$, that

$$V_d(Z^x + rY) - V_d(X^x + rY) = \lim_{k\to\infty} V_d(Z^x + rY_k) - V_d(X^x + rY_k),$$

where $V_d(A)$ denotes the $d$-dimensional Lebesgue measure of $A$ in $\hat{A}$, when the latter space is $d$-dimensional. Putting

$$C' := d2^d\kappa_d \frac{S^{d-1} \cdot G^d}{\rho_Y},$$

it follows from Lemma 3.2 (applied with $\mathbb{R}^d = x + r\hat{Y}$) that a.s. for all $x \in Y^\perp$ we have

$$V_d(Z^x + rY) - V_d(X^x + rY) = \lim_{k\to\infty} V_d(Z^x + rY_k) - V_d(X^x + rY_k) \leq C' \cdot r^{d-2},$$

since diameter and $B \mapsto \rho_B$ are continuous when considered as functionals from the set of $d$-dimensional convex bodies to the real numbers. We denote the image of a set $A \subseteq \mathbb{R}^n$ under the orthogonal projection onto an affine subspace $E \subseteq \mathbb{R}^n$ by $A|E$. Now we get (we will comment on the measurability below)

$$\begin{aligned}
&\mathbb{E}[V_n(X_Y + rY) - V_n(X + rY)] \\
&= \mathbb{E}\left[\int_{X|Y^\perp} V_d(Z^x + rY) - V_d(X^x + rY)\,dx\right] \\
&\leq \mathbb{E}\left[\int_{X|Y^\perp} C' \cdot r^{d-2}\,dx\right] \\
&= \mathbb{E}\left[V_{n-d}(X|Y^\perp)C'\right] \cdot r^{d-2} \\
&\leq \mathbb{E}\left[\kappa_{n-d}G^{n-d}C'\right] \cdot r^{d-2} \\
&= c \cdot r^{d-2}.
\end{aligned}$$

It remains to show those expressions of the previous equation, whose measurability was not proven in Remark 3.1, are measurable, too. The map $\mathcal{C} \times \mathcal{K} \times \mathbb{R}^n \to \mathcal{C}$, $(X, Y, x) \mapsto X^x$ is measurable by [11, Lemma 6.2] and [20, Theorem 12.2.6]. The map $\operatorname{conv} : \mathcal{C} \to \mathcal{K}$ is measurable by [20, Theorem 12.3.5]. As an easy consequence of [20, Theorem 12.3.6] the map $K \mapsto V_d(K)$ on the set of all bodies of $\mathbb{R}^n$, whose affine hull has dimension $d$, is upper semicontinuous and hence measurable. In [11, Lemma 6.6] it is shown that for measurable maps $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$, where $\Omega$ denotes the probability space on which $Y$ is defined, $\int_{Y^\perp} f(\omega, x)\, dx$ is a random variable. □

Now we turn to the proof of Theorem 1.4.

The main reason, why we obtain a lower order now, is that (8) is improved by the following lemma. This lemma can be seen as an asymptotic version of the Pythagorean Theorem. Indeed, the Pythagorean Theorem implies $w_{B^d}(r) = \sqrt{r^2 - 1}$ and hence there is a constant $C > 0$ such that $r - w_{B^d}(r) < C/r$ for sufficiently large $r$. Now we will show this asymptotic result for more general gauge bodies.

**Lemma 3.3.** *Let $B \subseteq \mathbb{R}^d$ be a convex body which has a summand $RB^d$ for some $R > 0$ and let $S > 0$ be a number with $B \subseteq SB^d$. Assume that one largest ball contained in $B$ has its center at the origin. Then, with $C := \frac{4S}{R{\rho_B}^2}$, we have*

$$r - w_B(r) < \frac{C}{r}, \quad r > 0.$$

**Proof.** If $r \leq \frac{4S}{R\rho_B}$, we conclude from (8)

$$r - w_B(r) \leq \frac{1}{\rho_B} \leq \frac{4S}{rR{\rho_B}^2}.$$

So let $r > \frac{4S}{R\rho_B}$ from now on. By (6) there are points $z \in \mathbb{R}^d$ and $y \in B^d$ with $y \in \Pi_B(0y, z)$, $d_B(0, z) = r$ and $d_B(y, z) = d_B(0y, z) = w_B(r)$. Put $q := (z - y)/d_B(y, z)$. Then $q \in \operatorname{bd} B$ due to [11, Lemma 2.1]; see Figure 3.1. Moreover, we let $u$ denote the (Euclidean) exterior normal vector of $B$ in $q$, which is determined uniquely, since $RB^d$ is a summand of $B$, and we let $v$ denote an arbitrary unit vector perpendicular to $u$.

The idea of the proof is now the following: Since it is relatively easy to see that $r - w_B(r)$ is proportional to $\langle rq - z, u\rangle$ ($z$ depends on $r$ here; see (14)), it suffices to find a bound of order $r^{-1}$ for the latter expression. If the projection of $rq - z$ onto the hyperplane orthogonal to $u$ was independent of $r$, then the fact that $B$ contains a ball as summand would yield the desired bound in a rather direct way. Thus the statement that this projection does not dependent "too strongly" on $r$ has to be made precise.

For reasons that will become clear soon, we formulate several equations for $t \in [w_B(r), r]$, while we are mainly interested in the case $t = r$. By [11, Lemma 2.1] there is a common exterior unit normal vector of $B$ in $q$ and of $0y$ in $y$ and hence $\langle y, u\rangle = 0$. From $z = y + w_B(r)q$ we get on the one hand

$$\langle tq - z, u\rangle = \langle tq - w_B(r)q - y, u\rangle = (t - w_B(r))\langle q, u\rangle$$

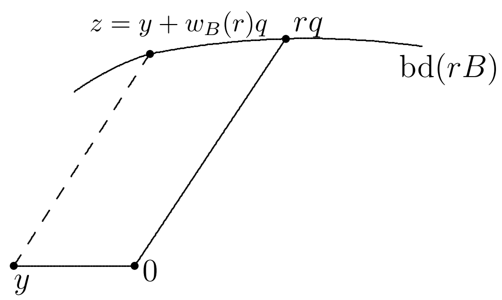


Figure 3.1: The points $y$, $z$ and $rq$

and thus

$$t - w_B(r) = \frac{\langle tq - z, u\rangle}{\langle q, u\rangle} \tag{14}$$

and on the other hand

$$|\langle tq - z, v\rangle| = |(t - w_B(r))\langle q, v\rangle - \langle y, v\rangle| \leq (t - w_B(r))|\langle q, v\rangle| + 1.$$

From the last two equations we get

$$|\langle tq - z, v\rangle| \leq \frac{\langle tq - z, u\rangle}{\langle q, u\rangle}|\langle q, v\rangle| + 1.$$

Since $q \in B \subseteq SB^d$ we get $|\langle q, v\rangle| \leq S$ and since $\rho_B u \in \rho_B B^d \subseteq B$ and $u$ is exterior normal vector of $B$ in $q$ we get $\langle q, u\rangle \geq \rho_B$. Thus

$$|\langle tq - z, v\rangle| \leq \frac{S}{\rho_B}\langle tq - z, u\rangle + 1. \tag{15}$$

We want to show

$$\langle tq - z, u\rangle < \frac{4}{Rr} \tag{16}$$

for all $t \in [w_B(r), r]$. Now the point has come, where we use that $t \in [w_B(r), r]$ is arbitrary: In order to show that $\langle rq - z, u\rangle$ is small it is helpful to show that $|\langle rq - z, v\rangle|$ is not to large, which – due to (15) – is shown if we can prove that $\langle rq - z, u\rangle$ is small. We escape the circle using the following trick: Since (16) is true for $t = w_B(r)$ and the left-hand side of (16) is obviously continuous in $t$, it suffices to show that there is no $t \in [w_B(r), r]$ for which equality holds in (16). So assume, there is $t \in [w_B(r), r]$ for which equality holds in (16). By (15) we get

$$|\langle tq - z, v\rangle| \leq \frac{4S}{\rho_B Rr} + 1. \tag{17}$$

Let $m \in \mathbb{R}^d$ be the point with $q \in m + RB^d \subseteq B$. Then by [11, Lemma 2.1], $u$ is also the exterior normal vector of $m + RB^d$ in $q$ and thus $q = m + Ru$. We

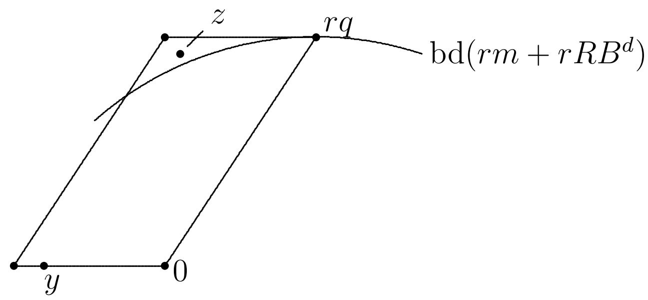


Figure 3.2: The point $z$ lies to the right of the line of all points for which equality holds in (15) (the left boundary of the parallelogram), but above the ball $rm + RB^d$. Hence it cannot be too far from the top boundary of the parallelogram.

denote the unit vector in the direction $z - tm - \langle z - tm, u\rangle u$ by $v_0$. The idea of the remaining proof can be seen in Figure 3.2. Since we have assumed equality in (16), we get, using (17),

$$\begin{aligned}\|z - tm\|^2 &= \langle z - tm, u\rangle^2 + \langle z - tm, v_0\rangle^2 \\ &= \langle z - tq + tRu, u\rangle^2 + \langle z - tq + tRu, v_0\rangle^2 \\ &= (tR - \langle tq - z, u\rangle)^2 + \langle tq - z, v_0\rangle^2 \\ &\le \left(tR - \frac{4}{Rr}\right)^2 + \left(\frac{4S}{\rho_B Rr} + 1\right)^2.\end{aligned}$$

Due to

$$r > \frac{4S}{\rho_B R} \ge \max\left\{\frac{4}{\rho_B}, \frac{4}{R}\right\}$$

and (8) we get

$$\frac{t}{r} \ge \frac{r - \frac{1}{\rho_B}}{r} = 1 - \frac{1}{r\rho_B} \ge \frac{3}{4}$$

and thus

$$(tR)^2 - \left(tR - \frac{4}{Rr}\right)^2 = 8\frac{t}{r} - \left(\frac{4}{Rr}\right)^2 \ge 6 - 1 > 2^2 > \left(\frac{4S}{\rho_B Rr} + 1\right)^2.$$

Hence

$$\|z - tm\|^2 \le \left(tR - \frac{4}{Rr}\right)^2 + \left(\frac{4S}{\rho_B Rr} + 1\right)^2 < (tR)^2.$$

So we have $z \in \mathrm{int}(tm + tRB^d)$ and therefore $z \in \mathrm{int}\, tB$. Hence there is $t' < t$ with $z \in t'B$ and so $d_B(0, z) < t$, which contradicts $t \le r = d_B(0, z)$. So we have proven inequality (16).

Now we use the equations (14) and (16) in the special case $t = r$ and we use again the inequality $\langle q, u\rangle \geq \rho_B$ we derived before (15) and get

$$r - w_B(r) = \frac{\langle rq - z, u\rangle}{\langle q, u\rangle} < \frac{\frac{4}{rR}}{\rho_B} = \frac{4}{rR\rho_B} \leq \frac{4S}{rR{\rho_B}^2}. \qquad \square$$

The following lemma is the counterpart to Lemma 3.2.

**Lemma 3.4.** *Let $K \subseteq \mathbb{R}^d$ be a body and $B \subseteq \mathbb{R}^d$ be a convex body with a summand $RB^d, R > 0$, such that $h_B$ is twice differentiable on $\mathbb{R}^d \setminus \{0\}$. Put $G := \max\{\operatorname{diam} K, 1\}$, $S := \max\{\operatorname{diam} B, 1\}$ and*

$$C' := d2^{d+2}\kappa_d \frac{S^d \cdot G^{d+1}}{R^3}.$$

*Then we have for all $r \geq 1$*

$$V_d(\operatorname{conv} K + rB) - V_d(K + rB) < C' \cdot r^{d-3}.$$

We omit the proof of this lemma, since it is too similar to the proof of Lemma 3.2. The main difference is that we concluded (9) from (8). We now get

$$r^i - (Gw_B(\tfrac{r}{G}))^i \leq CG^2 i r^{i-2}$$

from Lemma 3.3, where $C$ is the constant defined in that lemma. A further technicality is that we used

$$V_d(L \setminus (K + rB)) \leq \kappa_d G^d \leq \kappa_d G^d r^{d-2}$$

in (11)–(13) and here $r^{d-2}$ cannot be replaced by $r^{d-3}$. Instead one has to use that, since $L \subseteq K + rB$ if $rR \geq G$, we have

$$V_d(L \setminus (K + rB)) \leq V_d(L)\mathbf{1}_{\{rR<G\}} \leq \kappa_d G^d \frac{G}{rR} \leq \kappa_d \frac{G^{d+1}}{R} r^{d-3}.$$

We omit also the proof of Theorem 1.4, as it is completely analogue to the proof of Theorem 1.3.

## 4. Optimality of the main results

In this section we will show that certain improvements of our main results are not possible. First we will show that the order $r^{d-2}$ in Theorem 1.1($i$) and hence in Theorem 1.3 is optimal.

**Example 4.1.** Put $K := \{-e_1, e_1\}$ and let $B := \operatorname{conv}\{-e_1, e_1, \dots, -e_d, e_d\}$ be the unit ball of the $L_1$-norm. Because $V_{d-1}(\operatorname{conv}\{-e_2, e_2, \dots, -e_d, e_d\}) =$

$2^{d-1}/(d-1)!$, we have for $r \geq 1$

$$\begin{aligned}
&V_d(\operatorname{conv} K + rB) - V_d(K + rB)\\
&= \int_{-1}^{1} V_{d-1}(\{(x_1,\dots,x_d) \in (\operatorname{conv} K + rB) \setminus (K + rB) \mid x_1 = t\})\, dt\\
&= 2\int_0^1 r^{d-1}\frac{2^{d-1}}{(d-1)!} - (r-t)^{d-1}\frac{2^{d-1}}{(d-1)!}\, dt\\
&= 2^d \sum_{j=0}^{d-2} (-1)^{d-j} \frac{1}{j!(d-j)!} r^j.
\end{aligned}$$

The following example shows that even in the special case $B = B^d$ we cannot obtain a better order in Theorem 1.1$(ii)$. This implies that the order in Theorem 1.4 is optimal, too.

**Example 4.2.** Put $K := \{-e_1, e_1\}$ and

$$D_r := \left\{(x_1,\dots,x_d) \in \mathbb{R}^d \mid |x_1| \leq \tfrac{1}{2},\ \sqrt{r^2 - \tfrac{1}{4}} < \|(x_2,\dots,x_d)\| \leq r\right\}, \quad r > \tfrac{1}{2}.$$

So $D_r$ is a cylinder of length 1 and radius $r$, from which a cylinder of radius $\sqrt{r^2 - \frac{1}{4}}$ with same length and same symmetry axis has been removed. Since $D_r \subseteq \operatorname{conv} K + rB^d$ and $D_r \cap (K + rB^d) = \emptyset$ we have

$$V_d(\operatorname{conv} K + rB^d) - V_d(K + rB^d) \geq V_d(D_r) = \kappa_{d-1}\left(r^{d-1} - \sqrt{r^2 - \tfrac{1}{4}}^{\,d-1}\right).$$

A purely analytical computation shows that the latter expression is indeed of order $r^{d-3}$.

Now we will show that the assumption that the gauge body contains a ball as summand in Theorem 1.4 cannot be relaxed in the planar case.

**Corollary 4.3.** *Let $B \subseteq \mathbb{R}^2$ be a convex body. Then the following are equivalent:*

$(i)$ *There is a constant $c \geq 0$ such that*

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < \frac{c}{r}$$

*for all $r > 0$ and every body $K \subseteq \mathbb{R}^2$ with* $\operatorname{diam} K \leq 1$.

$(ii)$ *$B$ has a summand $RB^2$, $R > 0$.*

**Proof.** If $(ii)$ is fulfilled, then by Theorem 1.4 there is a constant $C$ such that

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < \frac{C}{r}$$

for all $r \geq 1$ and every body $K \subseteq \mathbb{R}^2$ with $\operatorname{diam} K \leq 1$. For $r \in (0,1]$ and a body $K \subseteq \mathbb{R}^2$ with $\operatorname{diam} K \leq 1$ we have

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < V_2(B^2 + B) \leq \frac{V_2(B^2 + B)}{r}.$$

Hence we have

$$V_2(\operatorname{conv} K + rB) - V_2(K + rB) < \frac{\max\{C, V_2(B^2 + B)\}}{r}$$

for all $r > 0$ and bodies $K \subseteq \mathbb{R}^2$ with $\operatorname{diam} K \leq 1$.

The idea behind the other direction of the proof is the following. We consider a boundary point $q$ of $B$ with exterior unit normal $\nu$. We choose $K$ to be a set consisting of two points such that the segment $\operatorname{conv} K$ is perpendicular to $\nu$. Now the boundaries of $\operatorname{conv} K + rB$ and $K + rB$ cannot deviate too far from each other, since we assume the parallel volume difference to be small. If these two boundaries do not deviate too far, then the "exterior normal vectors" of $K + rB$ in the boundary part between the two points of $K + rq$ cannot be too different from $\nu$. Thus the exterior normal vectors of $B$ in points near $q$ are not too different from $\nu$, which shows that $B$ has a ball as summand.

So assume that $(i)$ is fulfilled. Then [20, Corollary 3.10] implies that $B$ is smooth.

Since the support function $h_B$ is convex, it is a.e. Alexandrov-twice-differentiable (recall Def. 2.1 and see e.g. [19, Section 1.5, note 2] for this fact). In particular, the second derivative of $h_B$ in $\nu$ in direction orthogonal to $\nu$ exists for a.e. $\nu \in S^1$. We call it $R(B, \nu)$. According to a theorem of Weil ([23, Theorem 1]) it suffices to show that there is a constant $\tilde{c} > 0$ such that $R(B, \nu) \geq \tilde{c}$ holds, whenever $R(B, \nu)$ exists, in order to prove $(ii)$.

So let $\nu \in S^1$ be a point in which $h_B$ is Alexandrov-twice-differentiable and choose $\tau \in S^1$ perpendicular to $\nu$. Let $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a choice of subgradients of $h_B$. Then according to [19, Theorem 1.7.4] the point $q := \theta(\nu)$ lies in $\operatorname{bd} B$ and has exterior normal vector $\nu$.

There is $\epsilon > 0$ and $\xi \in \mathbb{R}^2$ with $B_\epsilon(\xi) \subseteq \operatorname{int} B$, since $B$ is smooth. Let $b_0 \in \operatorname{bd} B$ denote a point which satisfies $0 < \langle b_0 - q, \tau \rangle \leq \frac{\epsilon}{2}$ and has an exterior unit normal vector $u_0$ with $\langle u_0, \nu \rangle > 0$. Let $b_0^*$ and $u_0^*$ be defined in the same way with $q - b_0^*$ instead of $b_0 - q$.

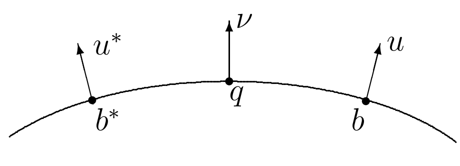


Figure 4.1: The points $q$, $b$ and $b^*$ and their normal vectors

Let $u \in S^1$ be a vector with $0 < \langle u, \tau \rangle \leq \min\{\langle u_0, \tau \rangle, -\langle u_0^*, \tau \rangle\}$ and $\langle \nu, u \rangle > 0$. Put $u^* := 2\langle u, \nu \rangle \nu - u$. Now $b := \theta(u)$ and $b^* := \theta(u^*)$ are points in $\operatorname{bd} B$ with exterior normal vectors $u$ resp. $u^*$, see Figure 4.1. We put $K := \{0, \tau\}$ and $r := \frac{1}{2\langle b-b^*, \tau\rangle}$.

Now the point $\frac{1}{2}\tau + r\xi$ is both in $\operatorname{int}(rB)$ and in $\operatorname{int}(\tau + rB)$. Indeed,

$$\langle u_0^*, \tau\rangle \leq \langle u^*, \tau\rangle \leq \langle u, \tau\rangle \leq \langle u_0, \tau\rangle$$

and thus

$$\epsilon r \geq \frac{\langle b_0 - b_0^*, \tau\rangle}{2\langle b - b^*, \tau\rangle} \geq \tfrac{1}{2}.$$

So $\frac{1}{2}\tau + r\xi \in rB_\epsilon(\xi) \subseteq \operatorname{int}(rB)$ and $\frac{1}{2}\tau + r\xi \in \tau + rB_\epsilon(\xi) \subseteq \operatorname{int}(\tau + rB)$.

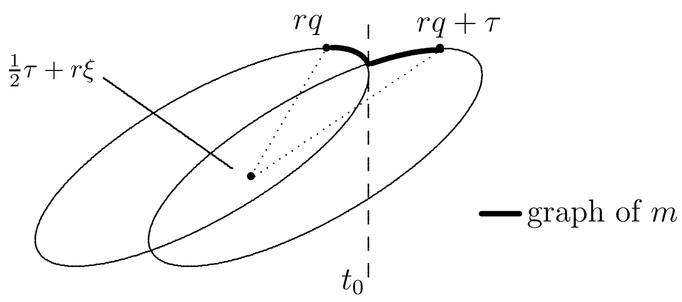


Figure 4.2: The set $K + rB$ is the union of the two convex bodies $rB$ and $\tau + rB$.

Now every line of the form $\{x \in \mathbb{R}^2 \mid \langle x, \tau\rangle = t\}$ with $t \in [t_1, t_1 + 1]$, where $t_1 := r\langle q, \tau\rangle$, intersects at least one of the segments $[rq, \frac{1}{2}\tau + r\xi]$ or $[\frac{1}{2}\tau + r\xi, \tau + rq]$. Hence the sets $\{x \in K + rB \mid \langle x, \tau\rangle = t\}$, $t \in [t_1, t_1 + 1]$, are not empty and the function

$$m : [t_1, t_1 + 1] \to \mathbb{R},\ t \mapsto \max\{\langle x, \nu\rangle \mid x \in K + rB,\ \langle x, \tau\rangle = t\}$$

is defined; see Figure 4.2. Now there is a number $t_0 \in [t_1, t_1 + 1]$ with

$$m(t) = \max\{\langle x, \nu\rangle \mid x \in rB,\ \langle x, \tau\rangle = t\}, \quad t < t_0,$$

and

$$m(t) = \max\{\langle x, \nu\rangle \mid x \in \tau + rB,\ \langle x, \tau\rangle = t\}, \quad t > t_0.$$

Since $\langle(\tau + rb^*) - rb, \tau\rangle = \frac{1}{2}$, we have either $t_0 - \langle rb, \tau\rangle \geq \frac{1}{4}$ or $\langle \tau + rb^*, \tau\rangle - t_0 \geq \frac{1}{4}$. We may, without loss of generality, assume that the first case holds.

Now we put (see Figure 4.3)

$$A := \{x \in \mathbb{R}^2 \mid \langle x - rb, \tau\rangle < \tfrac{1}{4},\ \langle x, u\rangle > \langle rb, u\rangle,\ \langle x, \nu\rangle \leq \langle rb, \nu\rangle\}.$$

It is easy to see that $A \subseteq \operatorname{conv} K + rB$, but $A \cap (K + rB) = \emptyset$. One cathetus of the rectangular triangle $A$ has length $\frac{1}{4}$. In order to compute the length of the other

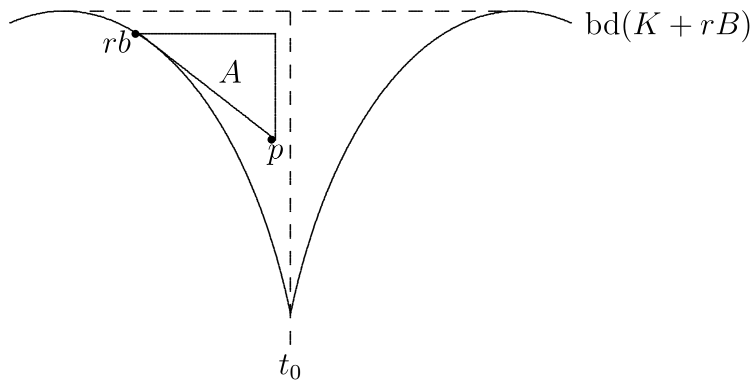


Figure 4.3: The set $A$

cathetus, we let $p$ denote the point with $\langle p, u\rangle = \langle rb, u\rangle$ und $\langle p, \tau\rangle = \langle rb, \tau\rangle + \frac{1}{4}$. Then

$$\begin{aligned} 0 &= \langle rb - p, u\rangle \\ &= \langle rb - p, \nu\rangle\langle u, \nu\rangle + \langle rb - p, \tau\rangle\langle u, \tau\rangle \\ &= \langle rb - p, \nu\rangle\langle u, \nu\rangle - \tfrac{1}{4}\langle u, \tau\rangle. \end{aligned}$$

Thus the length we looked for is

$$\langle rb, \nu\rangle - \langle p, \nu\rangle = \frac{\langle u, \tau\rangle}{4\langle u, \nu\rangle}.$$

Hence we get

$$\begin{aligned} \tfrac{1}{32}\langle u, \tau\rangle &\leq \tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \frac{\langle u, \tau\rangle}{4\langle u, \nu\rangle} \\ &= V_2(A) \\ &\leq V_2(\operatorname{conv} K + rB) - V_2(K + rB) \\ &< \frac{c}{r} \\ &= 2c\langle b - b^*, \tau\rangle \\ &= 2c\langle \theta(u) - \theta(u^*), \tau\rangle. \end{aligned}$$

This means

$$\langle \theta(u) - \theta(u^*), \tau\rangle \geq \frac{1}{64c}\langle u, \tau\rangle = \frac{1}{128c}\langle u - u^*, \tau\rangle.$$

Since we assumed $\theta$ to be differentiable in $\nu$ and any vector in $S^1$ sufficiently close to $\nu$ could be chosen to be $u$, this implies

$$R(B, \nu) = \tfrac{\partial}{\partial\lambda}\langle \theta(\nu + \lambda\tau), \tau\rangle_{|\lambda=0} \geq \frac{1}{128c}.$$

Now [23, Theorem 1] implies that $B$ has a summand of the form $RB^2$, $R > 0$. $\square$

**Conjecture 4.4.** *There are convex bodies $B \subseteq \mathbb{R}^d$, $d \geq 3$, which contain no ball as summand, but for which there is a constant $c \geq 0$ with*

$$V_d(\operatorname{conv} K + rB^d) - V_d(K + rB^d) < c \cdot r^{d-3} \tag{18}$$

*for all $r \geq 1$ and for all bodies $K \subseteq \mathbb{R}^d$ with* $\operatorname{diam} K \leq 1$.

**Reason.** Choose $B$ to be a convex body, for which there are numbers $S > 0$ and $\alpha \in (1, 2)$ and a convex body $\tilde{B}$ with a ball as summand such that

$$\begin{aligned} &\{(x_1, \dots, x_d) \in B \mid x_d \geq -S\} \\ &= \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid -S \leq x_d \leq -\|(x_1, \dots, x_{d-1})\|^\alpha\} \end{aligned}$$

and

$$\{(x_1, \dots, x_d) \in B \mid x_d \leq -S\} = \{(x_1, \dots, x_d) \in \tilde{B} \mid x_d \leq -S\}.$$

Now there is no $R > 0$ with $0 \in m + RB^d \subseteq B$ for any $m \in B$.

Geometric intuition tells us that it suffices to check (18) in the special case $K = \{-\frac{1}{2}e_1, \frac{1}{2}e_1\}$, where $e_1$ denotes the first unit vector. Checking (18) in this special case is, however, an easy computation. □

## 5. Stochastic applications

In this section we discuss the applications of Theorem 1.3 and Theorem 1.4 to Brownian paths and Boolean models.

The parallel body of a Brownian path is called Wiener sausage. While there are many papers dealing with the asymptotic behavior of the volume of the Wiener sausage as the time tends to infinity (see [10] and the literature cited therein), [13] seems to be the only one dealing with its asymptotics as the time tends to 0. There it was shown that

$$\mathbb{E}\, V_d(S_t + B) = V_d(B) + 2\sqrt{\tfrac{2}{\pi}} V_{d-1}(B)\sqrt{t} + o(\sqrt{t}), \quad t \to 0,$$

for convex bodies $B$ with interior points, where $S_t$ denotes a Brownian path up to time $t$. Even in the special case that $B$ is the Euclidean unit ball no stronger rate of convergence is known by now. We derive the following results.

**Theorem 5.1.** *Let $B$ be a convex body with interior points. Then*

$$\mathbb{E}\, V_d(S_t + B) = V_d(B) + 2\sqrt{\tfrac{2}{\pi}} V_{d-1}(B)\sqrt{t} + O(t), \quad t \to 0.$$

**Theorem 5.2.** *Let $B$ be a convex body which contains a ball as summand. Then*

$$\mathbb{E}\, V_d(S_t + B) = V_d(B) + 2\sqrt{\tfrac{2}{\pi}} V_{d-1}(B)\sqrt{t} + \frac{\pi}{2} V_{d-2}(B) t + O(t^{3/2}), \quad t \to 0.$$

Using the fact that

$$V_j(rB^d) = r^j \binom{d}{j} \frac{\kappa_d}{\kappa_{d-j}} = r^j \binom{d}{j} \frac{\pi^{j/2}\Gamma(\frac{d-j}{2}+1)}{\Gamma(\frac{d}{2}+1)}, \quad r \geq 0,$$

we obtain the following corollary.

**Corollary 5.3.** *Let $r \geq 0$. Then, as $t \to 0$,*

$$\begin{aligned}&\mathbb{E}\, V_d(S_t + rB^d)\\ &= \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} r^d + \frac{2\sqrt{2}\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})} r^{d-1}\sqrt{t} + \frac{(d-1)\pi^{d/2}}{2\Gamma(\frac{d}{2})} r^{d-2} t + O(t^{3/2}).\end{aligned}$$

The proof of Theorem 5.2 relies on the following lemma.

**Lemma 5.4.** *Let $B$ be a convex body. Then*

$$\mathbb{E}\, V_d(\operatorname{conv} S_t + B) = \sum_{k=0}^{d} \frac{k!(d-k)!}{d!} \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{k}{2}+1)\Gamma(\frac{d-k}{2}+1)} \mathbb{E}\, V_{d-k}(\operatorname{conv} S_t) V_k(B).$$

**Proof.** Let $R$ be a random rotation distributed according to the Haar measure on the group of orientation-preserving rotations of $\mathbb{R}^d$. By a well-known integral-geometric formula (see [20, Theorem 6.1.1]) we get

$$\begin{aligned}\mathbb{E}\, V_d(\operatorname{conv} S_t + B) &= \mathbb{E}\, V_d(R(\operatorname{conv} S_t) + B)\\ &= \sum_{k=0}^{d} \frac{k!\kappa_k (d-k)!\kappa_{d-k}}{d!\kappa_d} \mathbb{E}\, V_{d-k}(\operatorname{conv} S_t) V_k(B),\end{aligned}$$

where $R(K) := \{R(x) \mid x \in K\}$ for a body $K \subseteq \mathbb{R}^d$. Now the assertion follows from the fact $\kappa_j = \pi^{j/2}/\Gamma(\frac{j}{2}+1)$. □

**Proof of Theorem 5.2.** By Theorem 1.4 there is a constant $c > 0$ such that

$$\begin{aligned}&\mathbb{E}\, V_d(\operatorname{conv} S_t + B) - \mathbb{E}\, V_d(S_t + B)\\ &= \mathbb{E}\, V_d(\sqrt{t} \operatorname{conv} S_1 + B) - \mathbb{E}\, V_d(\sqrt{t} S_1 + B)\\ &= t^{d/2}\big(\mathbb{E}\, V_d(\operatorname{conv} S_1 + \tfrac{1}{\sqrt{t}} B) - \mathbb{E}\, V_d(S_1 + \tfrac{1}{\sqrt{t}} B)\big)\\ &< t^{d/2} \cdot c \cdot (\tfrac{1}{\sqrt{t}})^{d-3}\\ &= c \cdot t^{3/2}\end{aligned}$$

for all sufficiently small $t$. Since $\mathbb{E}\, V_1(\operatorname{conv} S_1) = 2\sqrt{2}\Gamma(\frac{d+1}{2})/\Gamma(\frac{d}{2})$ and $\mathbb{E}\, V_2(\operatorname{conv} S_1)$

$= (d-1)\pi/2$, see [13], we conclude from Lemma 5.4

$$\begin{aligned}
&\mathbb{E}\, V_d(\operatorname{conv} S_t + B)\\
&= V_d(B) + \frac{1}{d}\cdot\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d+1}{2})\cdot\frac{\sqrt{\pi}}{2}}\cdot\frac{2\sqrt{2}\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}V_{d-1}(B)\sqrt{t}\\
&\qquad + \frac{2}{d(d-1)}\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{2})}\cdot\frac{(d-1)\pi}{2}V_{d-2}(B)t + O(t^{3/2})\\
&= V_d(B) + 2\sqrt{\tfrac{2}{\pi}}V_{d-1}(B)\sqrt{t} + \frac{\pi}{2}V_{d-2}(B)t + O(t^{3/2}).
\end{aligned}$$

This shows, however, the assertion. $\square$

The proof of Theorem 5.1 is analogue to the proof Theorem 5.2.

Now we turn to the contact distributions of Boolean models. For the introduction of the notions of a Boolean model and the contact distribution, see [20, Sections 4.3 and 2.4]. Here we consider only stationary Boolean models and assume that their grain distributions are defined on the set $\mathcal{C}_0$ of centered bodies (see [20, Section 4.1]). The contact distribution of a stationary Boolean model $Z$ in $\mathbb{R}^d$ with intensity $\gamma$ and grain distribution $\mathbb{Q}$ is

$$H_B^Z(r) = 1 - \exp\left(-\gamma\int_{\mathcal{C}_0} V_d(A + rB^*) - V_d(A)\, d\mathbb{Q}(A)\right), \quad r \geq 0, \tag{19}$$

(see [20, Theorem 9.1.1]), where $B^* := \{-x \mid x \in B\}$ for $B \subseteq \mathbb{R}^d$. We will now derive a limit theorem including a statement about the speed of convergence for the contact distribution as the intensity of the Boolean model tends to 0.

**Theorem 5.5.** *Let $Z^{(r)}$, $r > 0$, be stationary Boolean models in $\mathbb{R}^d$ with intensity $r^d$ and a typical grain $Z_0$ that is independent of $r$ and fulfills $\mathbb{E}\,(\operatorname{diam} Z_0)^d < \infty$. Let $B \subseteq \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} B$. Let $D$ be an $[0,\infty)$-valued random variable with distribution function $1 - \exp(-t^d V_d(B))$. Then we have*

$$r \cdot d_B(Z^{(r)}, 0) \overset{r\to 0}{\longrightarrow} D \tag{20}$$

*in distribution. More precisely, for $t \geq 0$ we have*

$$\begin{aligned}
&\mathbb{P}(r\cdot d_B(Z^{(r)},0) \leq t)\\
&= 1 - \exp(-t^d V_d(B))\\
&\qquad + dt^{d-1}\exp(-t^d V_d(B))\mathbb{E}\, V(\operatorname{conv} Z_0[1], B^*[d-1])r + O(r^2).
\end{aligned}\tag{21}$$

*If $B$ has moreover a ball as summand, $d \geq 3$ and $\mathbb{E}(\operatorname{diam} Z_0)^{d+1} < \infty$, then*

$$\begin{aligned}
&\mathbb{P}(r\cdot d_B(Z^{(r)},0) \leq t)\\
&= 1 - \exp(-t^d V_d(B)) + dt^{d-1}\exp(-t^d V_d(B))\mathbb{E}\, V(\operatorname{conv} Z_0[1], B^*[d-1])r\\
&\qquad - \tfrac{1}{2}\exp(-t^d V_d(B))\big(d^2 t^{2d-2}(\mathbb{E}\, V(\operatorname{conv} Z_0[1], B^*[d-1]))^2\\
&\qquad - d(d-1)t^{d-2}\mathbb{E}\, V(\operatorname{conv} Z_0[2], B^*[d-2])\big)r^2 + O(r^3).
\end{aligned}\tag{22}$$

*If $B$ has a ball as summand, $d = 2$ and $\mathbb{E}(\operatorname{diam} Z_0)^3 < \infty$, then*

$$\begin{aligned}
&\mathbb{P}(r \cdot d_B(Z^{(r)}, 0) \le t) \\
&= 1 - \exp(-t^2 V_2(B)) + 2t \exp(-t^2 V_2(B)) \mathbb{E}\, V(\operatorname{conv} Z_0[1], B^*[1]) r \\
&\qquad - \exp(-t^2 V_2(B)) \big(2t^2 (\mathbb{E}\, V(\operatorname{conv} Z_0[1], B^*[1]))^2 \\
&\qquad - \mathbb{E}\, V_2(\operatorname{conv} Z_0) - \mathbb{E}\, V_2(Z_0)\big) r^2 + O(r^3).
\end{aligned} \tag{23}$$

The proof of the theorem relies on the following analytical lemma.

**Lemma 5.6.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = ax + bx^2 + O(x^3)$ for some $a, b \in \mathbb{R}$ as $x \to 0$. Then*

$$\exp(f(x)) = 1 + ax + (\tfrac{1}{2}a^2 + b)x^2 + O(x^3).$$

**Proof of Theorem 5.5.** We only proof equation (22), since the proofs of (21) and (23) are analogue and (20) is a trivial consequence of the other equations. By (19), Theorem 1.4 and Lemma 5.6 we get, if $d \ge 3$,

$$\begin{aligned}
&\mathbb{P}(r \cdot d_B(Z^{(r)}, 0) \le t) \\
&= 1 - \exp(-r^d \cdot \mathbb{E}[V_d(Z_0 + \tfrac{t}{r} B^*) - V_d(Z_0)]) \\
&= 1 - \exp\big( - t^d V_d(B^*) - drt^{d-1} \mathbb{E}\, V(\operatorname{conv} Z_0[1], B^*[d-1]) \\
&\qquad - \tbinom{d}{2} r^2 t^{d-2} \mathbb{E}\, V(\operatorname{conv} Z_0[2], B^*[d-2]) + O(r^3)\big) \\
&= 1 - \exp(-t^d V_d(B)) \cdot \big[1 - dt^{d-1} \mathbb{E}\, V(\operatorname{conv} Z_0[1], B^*[d-1]) r \\
&\qquad + \big(\tfrac{1}{2} d^2 t^{2d-2} (\mathbb{E}\, V(\operatorname{conv} Z_0[1], B^*[d-1]))^2 \\
&\qquad - \tbinom{d}{2} t^{d-2} \mathbb{E}\, V(\operatorname{conv} Z_0[2], B^*[d-2])\big) r^2 + O(r^3)\big] \\
&= 1 - \exp(-t^d V_d(B)) + dt^{d-1} \exp(-t^d V_d(B)) \mathbb{E}\, V(\operatorname{conv} Z_0[1], B^*[d-1]) r \\
&\qquad - \tfrac{1}{2} \exp(-t^d V_d(B)) \big(d^2 t^{2d-2} (\mathbb{E}\, V(\operatorname{conv} Z_0[1], B^*[d-1]))^2 \\
&\qquad - d(d-1) t^{d-2} \mathbb{E}\, V(\operatorname{conv} Z_0[2], B^*[d-2])\big) r^2 + O(r^3).
\end{aligned}$$

The difference between (22) and (23) is that in (23) we have an additional $-V_2(Z_0)$ in the coefficient of $r^2$. This is due to the fact that the term $r^d V_d(Z_0)$, which appears in $1 - \exp(-r^d \cdot \mathbb{E}[V_d(Z_0 + \frac{t}{r} B^*) - V_d(Z_0)])$, is in $O(r^3)$ for $d \ge 3$, but not for $d = 2$. □

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