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Weak Concavity of the Antidistance Function

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We prove that the function which for a given point of a Banach space gives the largest distance to the points of a given convex closed bounded set (antidistance) is weakly concave on the complement to some neighborhood of the set if and only if the set is a summand of some ball of some radius. We obtain precise estimates for parameters of weak concavity via the size of the neighborhood and radius of the ball in the Hilbert space.

Keywords: Antidistance function, uniform convexity, uniform smoothness, weak concavity, modulus of weak concavity, proximal smoothness, P-supporting condition

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1. Introduction and main notations

We denote by E a Banach space, and by $\mathcal{H}$ a Hilbert space, over $\mathbb{R}$. For a point p from the topological dual space E^* and for $x \in E$ we define by $\langle p, x\rangle$ the value of the functional p at the point x. Also $\langle p, x\rangle$ will be the scalar product for vectors p and x from $\mathcal{H}$. Let $B_r(a) = \{x \in E \mid \|x - a\| \le r\}$. For a subset $A \subset E$ we denote by $\operatorname{bd} A$, $\operatorname{int} A$, $\operatorname{cl} A$ the boundary, the interior and the closure of the set A, respectively. The diameter of a set A is defined by $\operatorname{diam} A = \sup_{x,y\in A} \|x - y\|$.

Recall that Minkowskii sum and difference of two subsets $A, B \subset E$ are defined as follows

$$A + B = \bigcup_{b\in B}(A + b), \qquad A \overset{*}{-} B = \bigcap_{b\in B}(A - b) = \{x \in E \mid x + B \subset A\}.$$

Let $A \subset E$. The *distance function* from the point $x \in E$ to the set A is given by the formula

$$\mathrm{d}(x) = \mathrm{d}_A(x) = \inf_{a\in A} \|x - a\|.$$

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For points $x_0, x_1 \in E$, a number $\lambda \in [0,1]$ and a function $f : E \to \mathbb{R}$ we put

$$x_\lambda = (1-\lambda)x_0 + \lambda x_1, \qquad [x_0, x_1] = \{x_\lambda \mid \lambda \in [0,1]\},$$
$$f_\lambda = (1-\lambda)f(x_0) + \lambda f(x_1).$$

For a subset $A \subset E$ and a number $\delta > 0$ we define

$$U_A(\delta) = \{x \in E \mid 0 < \mathrm{d}_A(x) < \delta\}.$$

We denote by $s(p, A)$ the *supporting function* of the set $A \subset E$: $s(p, A) = \sup_{a \in A} \langle p, a\rangle$ for any $p \in E^*$.

Definition 1.1. Let $A \subset E$ be a closed convex bounded subset. Then the function $f_A : E \to [0, +\infty]$,

$$f_A(x) = \sup_{a \in A} \|x - a\|$$

is called the *antidistance* from the point x to the set A.

It is easy to see that the function f_A is convex as supremum of convex functions.

Definition 1.2. Let $A \subset E$ be a closed convex bounded subset, $r > 0$. Then the set

$$T_A(r) = \{x \in E \mid f_A(x) > r\}$$

is called the *antineighborhood of radius* r for the set A.

Note that equality $f_A(x) = r$ entails that $\|x - a\| \le r$ for any point $a \in A$. The latter means that $x - a \in B_r(0)$ for any $a \in A$ and hence $x + (-A) \subset B_r(0)$. Thus $x \in B_r(0) \overset{*}{-} (-A)$ and moreover, $x \in \mathrm{bd}\left(B_r(0) \overset{*}{-} (-A)\right)$. The last inclusion follows from the fact that the Minkowskii difference is continuous with respect to the radius r when $\mathrm{int}\left(B_r(0) \overset{*}{-} (-A)\right) \neq \emptyset$, see [21, §3] for details.

It is easy to see that the inclusion $x \in \mathrm{bd}\left(B_r(0) \overset{*}{-} (-A)\right)$ entails the equality $f_A(x) = r$. Indeed, if $f_A(x) = d < r$, then $x \in B_d(0) \overset{*}{-} (-A)$. We have

$$\begin{aligned} & B_r(0) \overset{*}{-} (-A) \\ = \ & (B_{r-d}(0) + B_d(0)) \overset{*}{-} (-A) \supset \left(B_d(0) \overset{*}{-} (-A)\right) + B_{r-d}(0) \supset B_{r-d}(x). \end{aligned}$$

The last inclusion contradicts the fact that x is boundary point of $B_r(0) \overset{*}{-} (-A)$.

Thus $\mathrm{bd}\, T_A(r) = \mathrm{bd}\left(B_r(0) \overset{*}{-} (-A)\right)$ and hence

$$T_A(r) = E \backslash \left(B_r(0) \overset{*}{-} (-A)\right) = (E \backslash B_r(0)) + A. \tag{1}$$

The last equality in the previous formula proved for example in [20, Proposition 1.1.1]. Formula (1) permits to calculate the set $T_A(r)$ for any subset $A \subset E$.

We use the term antidistance because the meaning of the function $f_A(x)$ is the value of the worst approximation for the point x by elements of the set A in the sense of the norm, instead of the best approximation in the case of the distance function.

Definition 1.3. We shall say that a closed convex subset $A \subset E$ is a *summand of the ball* $B_R(0) \subset E$ of radius R if there exists another closed convex subset $B \subset E$ with $A + B = B_R(0)$.

We shall say that a subset $A \subset E$ is *strongly convex of radius* R (or *R-strongly convex*) if it can be represented as intersection of closed balls of radius R.

In the real Hilbert space every nonempty R-strongly convex set, i.e. subset of the form $A = \cap_{x\in X} B_R(x) \neq \emptyset$ is a summand of the ball $B_R(0)$ [4, Theorems 2.6, 1.1].

If in a Banach space a set $A \subset E$ is a summand of the ball $B_R(0)$ then by Definition 1.3 $A + B = B_R(0)$ for some closed convex set B and hence $A = \cap_{x\in B} B_R(-x) = B_R(0) \overset{*}{-} B$. But a set of the form $A = \cap_{x\in X} B_R(x)$ is not necessary a summand of the ball $B_R(0)$. We mention some properties of summands of the ball in Appendix A.

In the case when a closed convex subset $A \subset E$ is a summand of the ball $B_R(0)$, this set satisfies some special supporting condition.

Proposition 1.4. *Let E be a reflexive strictly convex Banach space and A be a closed convex bounded subset of E. Then the following statements are equivalent:*

1) *The subset A is a summand of the ball $B_R(0)$.*
2) *For any point $a \in \operatorname{bd} A$ and for any unit vector $p \in E^*$ with $\langle p, a\rangle \geq s(p, A)$, we have*

$$A \subset B_R(a - Re),$$

where $e \in E$ with $\|e\| = 1 = \langle p, e\rangle$.

The proof of Proposition 1.4 can be given similarly with the proof of Theorem 1.1 from [4], but instead of the property of the ball to be a generating set in Theorem 1.1 one should use the property of the set A to be a summand of the ball.

Further we need some local variant of Proposition 1.4 in the Hilbert space. It was proved in a very interesting and admirable paper [26].

Proposition 1.5 [26, see Theorem 1.2, Remark after the proof of Theorem 1.2 and Proposition 3.3]. *Suppose that $A \subset \mathcal{H}$ is a convex closed bounded subset,* $\operatorname{int} A \neq \emptyset$*. Then the following properties are equivalent:*

1) *For any point $x \in \operatorname{bd} A$ there exist an open neighborhood U (U may be depends on x) of zero and some unit vector p with $\langle p, x\rangle \geq s(p, A)$ such that*

$$(x + U) \cap A \subset B_R(x - Rp).$$

2) *The set A is strongly convex of radius R.*

For a subset $A \subset E$ and $x \in E$ we define the *antiprojection* as follows

$$AP_A x = \{a \in A \mid \|x - a\| = f_A(x)\}.$$

Definition 1.6. Let $U \subset \mathcal{H}$ be an open subset. A function $f : U \to \mathbb{R}$ is called *weakly concave with the constant* $C > 0$ on the set U, if it is continuous on the set U and the function $f(x) - \frac{C}{2}\|x\|^2$ is concave on each convex subset $V \subset U$.

It is easy to see that the condition of concavity in Definition 1.6 is equivalent to the next inequality: for each pair of points $x_0, x_1 \in U$, such that $[x_0, x_1] \subset U$, and for any number $\lambda \in [0, 1]$, we have

$$f(x_\lambda) \geq f_\lambda - \frac{C}{2}\lambda(1-\lambda)\|x_0 - x_1\|^2. \qquad (2)$$

So, we shall use Definition 1.6 as well as Formula (2) when speaking about weakly concave function with the constant $C > 0$ in the space $\mathcal{H}$.

Definition 1.7. Let $U \subset E$ be an open subset. A function $f : U \to \mathbb{R}$ is called *weakly concave with the modulus* $\mu : [0, \operatorname{diam} U) \to [0, +\infty)$*,* $(\mu(t)$ increasing, $\mu(0) = 0)$ on the set U, if it is continuous on the set U and for any pair of points $x_0, x_1 \in U$, such that $[x_0, x_1] \subset U$, and for any number $\lambda \in [0, 1]$, we have

$$f(x_\lambda) \geq f_\lambda - \lambda(1-\lambda)\mu(\|x_0 - x_1\|). \qquad (3)$$

Note, that the function $f : U \to \mathbb{R}$ is weakly concave if and only if the function $-f : U \to \mathbb{R}$ is weakly convex, see [1, Definition 1.2].

To our knowledge the definition of weakly convex function appeared in the papers of S. Rolewicz [23, 24] as *strong* $\alpha(\cdot)$ *paraconvexity.* See the details in [24]. The works [25, 6, 27, 15] were also devoted to this concept.

It is easy to see that Definition 1.6 is some special case of Definition 1.7 for the modulus $\mu(t) = \frac{C}{2}t^2$ in the case $E = \mathcal{H}$.

Each function which is pointwise not less than

$$\mu_0(t) = \sup_{x_0, x_1 \in U:\ \|x_0 - x_1\| = t,\ \lambda \in (0,1)} \frac{f_\lambda - f(x_\lambda)}{\lambda(1-\lambda)}$$

can be taken in Definition 1.7 as the modulus of weak concavity μ.

We shall call the function μ_0 *the precise modulus of weak concavity.*

For nonconcave function the precise modulus of weak concavity can not be an arbitrary function. Using ideas from [1, Lemma 1.3] we shall prove the next result.

Lemma 1.8. *Let* $U \subset E$ *be an open set and* $f : U \to \mathbb{R}$ *be a weakly concave function with the precise modulus of weak concavity* μ_0*. Then for any number* $c > 1$ *we have the next inequality*

$$\mu_0(ct) \leq c^2 \mu_0(t) \qquad (4)$$

for all permissible $t > 0$. In particular, if the function f is not concave, then

$$t^2 = O(\mu_0(t)), \quad \text{as } t \to +0. \tag{5}$$

Proof. The proof follows from [1, Lemma 1.3] and the fact that the function $-f$ is weakly convex with the precise modulus of weak convexity μ_0. $\square$

The purpose of the present paper is to characterize convex closed bounded subsets $A \subset E$ with weakly concave antidistance function on some antineighborhood $T_A(r)$ of the set A, where $r > 0$.

The *modulus of convexity* of a Banach space E is the function $\delta_E : (0; 2] \to \mathbb{R}$, which is defined by the formula

$$\delta_E(\varepsilon) = \inf\Big\{1 - \frac{1}{2}\|x + y\| \mid \ x, y \in E, \ \|x\| \le 1, \ \|y\| \le 1, \ \|x - y\| \ge \varepsilon\Big\}.$$

A Banach space E is called *uniformly convex*, if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0; 2]$.

The *modulus of smoothness* of a Banach space E is the function $\varrho_E : [0; +\infty) \to \mathbb{R}$, which is defined by the formula

$$\varrho_E(\tau) = \sup\left\{\left\|\frac{x + y}{2}\right\| + \left\|\frac{x - y}{2}\right\| - 1 \mid \ \|x\| = 1, \ \|y\| = \tau\right\}.$$

A Banach space E is called *uniformly smooth*, if

$$\lim_{\tau \to +0} \frac{\varrho_E(\tau)}{\tau} = 0.$$

Remark 1.9. Any uniformly convex or uniformly smooth Banach space, as well as it dual Banach space, is reflexive [8, Chapter 2, §4, Theorem 2 and Corollary 2].

Proposition 1.10 ([2, 14]). *Let E be a uniformly convex and uniformly smooth Banach space. If a closed convex bounded set $A \subset E$ is a summand of the ball $B_R(0)$, then, for any $r > R$ and $x \in T_A(r)$, the antiprojection $AP_A x$ is a singleton and continuously depends on x.*

For the subset $A \subset E$ and the point $x \in E$ define $\pi_A x = \{a \in A \mid \|x - a\| = \mathrm{d}_A(x)\}$.

Definition 1.11 ([7]). A subset $A \subset E$ is called *proximally smooth* with the constant $R > 0$, if the distance function $x \to \mathrm{d}_A(x)$ is continuously differentiable on the set $U_A(R)$.

Definition 1.12. We say that a subset $A \subset E$ satisfies the *P-supporting condition* with the constant $R > 0$, if for any point $x \in U_A(R)$ such that $a \in \pi_A x$, we have the inequality

$$\mathrm{d}_A\left(a + \frac{R}{\|x - a\|}(x - a)\right) \ge R. \tag{6}$$

Proposition 1.13. *Let E be a uniformly convex and uniformly smooth Banach space. Let $A \subset E$ be a closed subset, $R > 0$. The next conditions are equivalent:*

(*1*) *The set A satisfies the P-supporting condition with the constant R.*
(*2*) *The set A is proximally smooth with the constant R.*
(*3*) *The projection mapping $x \to \pi_A x$ is a singleton and continuous on the set $U_A(R)$.*

Equivalences in Proposition 1.13 were established in the papers [9, 22, 25] in the finite dimension case. In the case of the real Hilbert space Proposition 1.13 was proved in the papers [7, 19]. In arbitrary uniformly convex and uniformly smooth Banach space Proposition 1.13 was established in [5].

Proposition 1.14 ([18, Lemma 4.1]). *Let E be a uniformly convex and uniformly smooth Banach space with the modulus of uniform smoothness ϱ_E. Suppose that a subset $A \subset E$ satisfies the P-supporting condition with the constant R. Let $x_0, x_1 \in A$ and $\lambda \in [0,1]$. Then*

$$\mathrm{d}_A(x_\lambda) \leq 8R\lambda(1-\lambda)\varrho_E\left(\frac{\|x_0 - x_1\|}{R}\right).$$

2. Weak concavity of the antidistance function in uniformly convex and uniformly smooth Banach spaces

Throughout this section E is a uniformly convex and uniformly smooth Banach space.

For points $x, y \in E$, $x \neq y$, define $l(x,y) = \{x + \lambda(y-x) \mid \lambda \geq 0\}$.

Theorem 2.1. *Let $A \subset E$ be a closed convex bounded subset, $r > 0$. Assume that the antidistance function $f_A(x)$ is weakly concave on the set $T_A(r)$ with the modulus of weak concavity $\mu(t)$, $\mu(t) = o(t)$, $t \to +0$. Then the set A is a summand of the ball $B_r(0)$.*

Proof. Put for simplicity $f := f_A$. For any segment $[x_0, x_1] \subset T_A(r)$ and $\lambda \in [0,1]$ we have

$$f(x_\lambda) \leq f_\lambda, \qquad (-f)(x_\lambda) \leq (-f)_\lambda + \lambda(1-\lambda)\mu(\|x_0 - x_1\|). \tag{7}$$

The first inequality in (7) follows from the convexity of the function f, the second follows from the weak concavity of f.

Thus the functions f and $-f$ are strongly $\mu(t)$-paraconvex (and even $\frac{\mu(t)}{t}$-semiconvex) on $T_A(r)$, see [11, §6]. These functions are also called weakly convex [1].

By [11, Theorem 6.1, part (a)] the function f continuously differentiable on the set $T_A(r)$. From results of G. Ivanov [14, Theorem 1 part (a), Theorem 2 part (h)] the set A is a summand of the ball $B_r(0)$. □

Theorem 2.2. *Let $A \subset E$ be summand of the ball $B_R(0)$, $r > R > 0$. Then the antidistance function f_A is weakly concave on the set $T_A(r)$ with the modulus*

$$\mu(t) = 8(r-R)\varrho_E\left(\frac{2t}{r-R}\right),$$

where ϱ_E is the modulus of smoothness for the space E.

Proof. Put $f := f_A$. Let $[x_0, x_1] \subset T_A(r)$, $\lambda \in (0,1)$. Let $a_\lambda = AP_A x_\lambda$, where the existence follows from Proposition 1.10. By Proposition 1.4 we have

$$A \subset B_R(z) =: B,$$

where $z = a_\lambda + R\frac{x_\lambda - a_\lambda}{\|x_\lambda - a_\lambda\|}$.

Suppose that $r < f(x_0) \le f(x_1)$. Let $\hat{x}_0 \in l(z, x_0)$ such that $\|\hat{x}_0 - z\| = \|x_1 - z\| = f_B(x_1) - R$; and let $\hat{x}_1 \in [z, x_1]$ such that $\|\hat{x}_1 - z\| = \|x_0 - z\| = f_B(x_0) - R$. Put $\overline{x}_0 = (1-\lambda)x_0 + \lambda\hat{x}_0$, $\overline{x}_1 = (1-\lambda)\hat{x}_1 + \lambda x_1$.

From the above definitions we have

$$\|\overline{x}_0 - z\| = \|\overline{x}_1 - z\| = f_B(\overline{x}_0) - R = f_B(\overline{x}_1) - R$$

and

$$\frac{\|x_0 - \overline{x}_0\|}{\|\overline{x}_0 - \hat{x}_0\|} = \frac{\lambda}{1-\lambda} = \frac{\|x_0 - x_\lambda\|}{\|x_\lambda - x_1\|} = \frac{\|\hat{x}_1 - \overline{x}_1\|}{\|\overline{x}_1 - x_1\|},$$

hence $x_\lambda \in [\overline{x}_0, \overline{x}_1]$. Let

$$w = l(a_\lambda, x_\lambda) \cap \operatorname{bd} B_{\|\overline{x}_0 - z\|}(z).$$

Then by Proposition 1.14

$$\|x_\lambda - w\| = \varrho_{\operatorname{bd} B_{\|\overline{x}_0 - z\|}(z)}(x_\lambda) \le 8\lambda(1-\lambda)\|\overline{x}_0 - z\|\varrho_E\left(\frac{\|\overline{x}_0 - \overline{x}_1\|}{\|\overline{x}_0 - z\|}\right).$$

From the equality $\|\overline{x}_0 - z\| = \|w - z\|$ and the inequality

$$\begin{aligned}\|x_\lambda - z\| &= \|(x_\lambda - w) - (z - w)\| \\ &\ge \|z - w\| - \|x_\lambda - w\| = (1-\lambda)\|x_0 - z\| + \lambda\|x_1 - z\| - \|x_\lambda - w\|\end{aligned}$$

we obtain that

$$\|x_\lambda - z\| \ge (1-\lambda)\|x_0 - z\| + \lambda\|x_1 - z\| - 8\lambda(1-\lambda)\|\overline{x}_0 - z\|\varrho_E\left(\frac{\|\overline{x}_0 - \overline{x}_1\|}{\|\overline{x}_0 - z\|}\right). \quad (8)$$

Note that

$$\|\overline{x}_0 - z\| = f_B(\overline{x}_0) - R \ge f(\overline{x}_0) - R > r - R. \quad (9)$$

Due to homothety with center z we get

$$\|\overline{x}_0 - \overline{x}_1\| \le \|\hat{x}_0 - x_1\|.$$

By the triangle inequality

$$\|\hat{x}_0 - x_1\| \le \|\hat{x}_0 - x_0\| + \|x_0 - x_1\|,$$
$$\|\hat{x}_0 - x_0\| = \varrho_{\mathrm{bd}\, B_{\|\hat{x}_0 - z\|}(z)}(x_0) \le \|x_0 - x_1\|,$$

and finally

$$\|\overline{x}_0 - \overline{x}_1\| \le 2\|x_0 - x_1\|. \tag{10}$$

Using Formulae (8, 9, 10) and the fact that ϱ_E is increasing function with the property $\lim_{\tau\to+0} \frac{\varrho_E(\tau)}{\tau} = 0$, we have the next estimates

$$\|x_\lambda - z\| + R \ge (1-\lambda)(\|x_0 - z\| + R) + \lambda(\|x_1 - z\| + R)$$
$$- 8\lambda(1-\lambda)(r-R)\varrho_E\left(\frac{2\|x_0 - x_1\|}{r-R}\right),$$

$$f(x_\lambda) = \|x_\lambda - a_\lambda\| = \|x_\lambda - z\| + R,$$

$$f(x_0) \le f_B(x_0) = \|x_0 - z\| + R, \qquad f(x_1) \le f_B(x_1) = \|x_1 - z\| + R,$$

and

$$f(x_\lambda) \ge (1-\lambda)f(x_0) + \lambda f(x_1) - 8\lambda(1-\lambda)(r-R)\varrho_E\left(\frac{2\|x_0 - x_1\|}{r-R}\right). \qquad \square$$

3. Weak concavity of the antidistance function in a Hilbert space

In the present section we consider the quite important case when $E = \mathcal{H}$. Our purpose is to give precise relationship between the next three parameters: radius R of strong convexity of the set A, radius r ($r > R$) of the antineighborhood $T_A(r)$ and the constant of weak concavity C of the antidistance function f_A on the set $T_A(r)$.

Theorem 3.1. *Let $A \subset \mathcal{H}$ be a strongly convex set of radius R (and hence a summand of the ball $B_R(0)$). Let $r > R$. Then the antidistance function $f_A(x)$ is weakly concave on the set $T_A(r)$ with the constant*

$$C = \frac{1}{r-R}.$$

Proof. Let $[x_0, x_1] \subset T_A(r)$. By Proposition 1.10 define $a_{\frac{1}{2}} = AP_A x_{\frac{1}{2}}$. By Proposition 1.4

$$A \subset B_R(z) =: B,$$

where $z = a_{\frac{1}{2}} + R\frac{x_{\frac{1}{2}} - a_{\frac{1}{2}}}{\|x_{\frac{1}{2}} - a_{\frac{1}{2}}\|}$.

Let $b_i = AP_{B_R(z)}x_i$, $f_i = f_A(x_i)$, $f_i^B = f_B(x_i)$, $i = 0, 1$, $f_{\frac{1}{2}} = \frac{1}{2}(f_A(x_0) + f_A(x_1))$. Then by the parallelogram equality we have

$$2\|x_0 - z\|^2 + 2\|x_1 - z\|^2 = \|x_0 - x_1\|^2 + 4\|x_{\frac{1}{2}} - z\|^2, \tag{11}$$

$$\|x_i - z\| = f_i^B - R, \; f_i^B \ge f_i, \quad i = 0, 1. \tag{12}$$

Note that $\|x_{\frac{1}{2}} - a_{\frac{1}{2}}\| = f_A(x_{\frac{1}{2}})$. By Formulae (11, 12)

$$2\left(f_0^B - R\right)^2 + 2\left(f_1^B - R\right)^2 = \|x_0 - x_1\|^2 + 4\left(f_A(x_{\frac{1}{2}}) - R\right)^2$$

and

$$2\left(f_0 - R\right)^2 + 2\left(f_1 - R\right)^2 \le 2\left(f_0^B - R\right)^2 + 2\left(f_1^B - R\right)^2.$$

It results that

$$\left(f_A(x_{\frac{1}{2}}) - R\right)^2 \ge \frac{1}{4}\left(2\left(f_0 - R\right)^2 + 2\left(f_1 - R\right)^2 - \|x_0 - x_1\|^2\right),$$

or equivalently

$$f_A(x_{\frac{1}{2}}) \ge \frac{1}{2}\left(f_0 + f_1 - 2R\right)\sqrt{1 - \frac{\|x_0 - x_1\|^2 - (f_0 - f_1)^2}{(f_0 + f_1 - 2R)^2}} + R.$$

From the asymptotic formula $\sqrt{1-\tau} - 1 \sim -\frac{\tau}{2}$, $\tau \to +0$, we obtain that for any $t > 1$ there exists $\delta > 0$ such that $\sqrt{1-\tau} - 1 \ge -\frac{t\tau}{2}$ for all $\tau \in (0, \delta)$. Thus for any $t > 1$ there exists $\delta > 0$ such that for any x_0, x_1 such that $[x_0, x_1] \subset T_A(r)$ and $\|x_0 - x_1\| < \delta$ we have

$$f_A(x_{\frac{1}{2}}) \ge f_{\frac{1}{2}} - \frac{t\|x_0 - x_1\|^2}{4\left(f_0 + f_1 - 2R\right)} \ge f_{\frac{1}{2}} - \frac{t\|x_0 - x_1\|^2}{8\left(r - R\right)}. \tag{13}$$

Formula (13) means that the function $g_t(x) = f_A(x) - \frac{t}{2(r-R)}\|x\|^2$ satisfies the inequality of concavity with $\lambda = \frac{1}{2}$ on each subset $T_A(r) \cap \operatorname{int} B_\delta(x)$, $x \in T_A(r)$. From continuity of the function g_t it is concave on each subset $T_A(r) \cap \operatorname{int} B_\delta(x)$. From the local concavity we obtain that the function g_t is concave on each convex subset of the antineighborhood $T_A(r)$. Passing the limit $t \to 1 + 0$, we obtain concavity of the function $f_A(x) - \frac{1}{2(r-R)}\|x\|^2$. Thus the function f_A is weakly concave with the constant $\frac{1}{r-R}$. □

Lemma 3.2. *Let $A, B \subset \mathcal{H}$ be closed strictly convex bounded subsets, $\|p\| = 1$. Let $a \in A$ and $b \in B$ be such that $\langle p, a\rangle = s(p, A)$, $\langle p, b\rangle = s(p, B)$. Let $t > 0$ and $U = \{x \in \mathcal{H} \mid \langle p, x\rangle > -t\}$. Let $x \in A$, $y \in B$ satisfy the inclusion $x + y \in (a + b + U) \cap (A + B)$. Then $x \in (a + U) \cap A$ and $y \in (b + U) \cap B$.*

Proof. Suppose that $x \in A\backslash(a + U)$, $y \in B$. Then $\langle p, x\rangle \le s(p, A) - t$, $\langle p, y\rangle \le s(p, B)$, $\langle p, x + y\rangle \le s(p, A + B) - t$. Hence $x + y \notin a + b + U$. □

Lemma 3.3. *Let $A, B, C \subset \mathcal{H}$ be closed strictly convex sets, $\|p\| = 1$. Let $a \in A \cap B$ be such that $\langle p, a\rangle = s(p, A) = s(p, B)$ and $c \in C$ such that $\langle p, c\rangle = s(p, C)$. Suppose that there exists $t > 0$ such that, for $U := \{x \in \mathcal{H} \mid \langle p, x\rangle > -t\}$*

$$(a + U) \cap B \subset (a + U) \cap A.$$

Then

$$(a + c + U) \cap (B + C) \subset (a + c + U) \cap (A + C).$$

Proof. Let $z \in (a+c+U) \cap (B+C)$. This implies that $z = x+y$, $x \in B$, $y \in C$. By Lemma 3.2 $x \in (a+U) \cap B$, $y \in (c+U) \cap C$, and from the condition of the present lemma we have $x \in (a+U) \cap A$. Thus $x \in A$. □

Theorem 3.4. *Let $A \subset \mathcal{H}$ be a closed convex bounded subset and $r > 0$. Suppose that the antidistance f_A is weakly concave on the set $T_A(r)$ with the constant $C > 0$. Then the set A is strongly convex of radius $r - \frac{1}{C}$.*

Proof. Let $R > 0$ be the minimal number for which the set A can be represented as intersection of balls of radius R. By Theorem 2.1 the set A is a summand of the ball $B_r(0)$, hence $R \le r$.

Let $B_t = B_t(0) \overset{*}{-} (-A)$, $t \ge R$. In the Hilbert space we have $B_t + (-A) = B_t(0)$ for any $t \ge R$ [4].

Fix $r_1 > r$ such that $\frac{r_1 - r}{2} < \frac{1}{C}$. Consider two points $x_0, x_1 \in \operatorname{cl} T_A(r_1)$ with $\|x_0 - x_1\| < \frac{r_1 - r}{2}$. Then $[x_0, x_1] \subset T_A(r)$ and

$$\frac{1}{2}\left(f_A(x_0) + f_A(x_1)\right) \ge f_A(x_{\frac{1}{2}}) \ge \frac{1}{2}\left(f_A(x_0) + f_A(x_1)\right) - \frac{C}{8}\|x_0 - x_1\|^2.$$

From Proposition 1.10 there exists $a_{\frac{1}{2}} = AP_A x_{\frac{1}{2}}$. Let $w = l(a_{\frac{1}{2}}, x_{\frac{1}{2}}) \cap \operatorname{bd} T_A(r_1)$. Then

$$f_A(x_{\frac{1}{2}}) = \|x_{\frac{1}{2}} - a_{\frac{1}{2}}\| \ge \frac{1}{2}\left(f_A(x_0) + f_A(x_1)\right) - \frac{C}{8}\|x_0 - x_1\|^2$$

and hence (in the case $x_{\frac{1}{2}} \notin \operatorname{cl} T_A(r_1)$)

$$\begin{aligned}
&\mathrm{d}_{\operatorname{cl} T_A(r_1)}(x_{\frac{1}{2}}) \\
&\le \mathrm{d}_{\operatorname{bd} B_{r_1}}(x_{\frac{1}{2}}) \le \|x_{\frac{1}{2}} - w\| = \|w - a_{\frac{1}{2}}\| - \|x_{\frac{1}{2}} - a_{\frac{1}{2}}\| \\
&\le \frac{1}{2}\left(f_A(x_0) + f_A(x_1)\right) - \left(\frac{1}{2}\left(f_A(x_0) + f_A(x_1)\right) - \frac{C}{8}\|x_0 - x_1\|^2\right) \\
&= \frac{C}{8}\|x_0 - x_1\|^2.
\end{aligned}$$

Thus for any points $x_0, x_1 \in \operatorname{cl} T_A(r_1)$, $\|x_0 - x_1\| < \frac{r_1 - r}{2}$, we have

$$\mathrm{d}_{\operatorname{cl} T_A(r_1)}(x_{\frac{1}{2}}) \le \frac{C}{8}\|x_0 - x_1\|^2.$$

From the convexity of the set B_r and the equality $B_{r_1} = B_r + B_{r_1 - r}(0)$ we obtain that $\operatorname{cl} T_A(r_1) = \mathcal{H} \setminus \operatorname{int} B_{r_1}$ is proximally smooth with the constant $r_1 - r$. By [3, Theorem 2.7] for any point $x \in \operatorname{cl} T_A(r_1)$ the set $B_{\frac{r_1 - r}{3}}(x) \cap \operatorname{cl} T_A(r_1)$ is connected. Repeating the proof of Theorem 2.1 [1] (note, that $\frac{r_1 - r}{2} < \frac{1}{C}$) we obtain that for any $x \in \operatorname{cl} T_A(r_1)$ the set $B_{\frac{r_1 - r}{3}}(x) \cap \operatorname{cl} T_A(r_1)$ is proximally smooth with the constant $\frac{1}{C}$. Using convexity of the set B_{r_1} and [13, Theorem 3] we obtain that $\operatorname{bd} B_{r_1}$ is a smooth manifold of codimension 1. By [12, Theorem 6.1] we can write the last fact as follows: for any $x \in \operatorname{bd} T_A(r_1) = \operatorname{bd} B_{r_1}$ and $\|p\| = 1$, $\langle p, x\rangle = s(p, B_{r_1})$,

$$B_{\frac{r_1-r}{3}}(x) \cap \operatorname{cl} T_A(r_1) \cap \operatorname{int} B_{\frac{1}{C}}\left(x - \frac{1}{C}p\right) = \emptyset. \tag{14}$$

Fix any such points x and p. By uniform convexity of the space $\mathcal{H}$ and Proposition 1.4 there exist $t > 0$ and $U = \{y \in \mathcal{H} \mid \langle p, y\rangle > -t\}$ such that

$$(x + U) \cap B_{r_1} \subset B_{\frac{r_1-r}{3}}(x) \cap B_{r_1},$$

$$(x + U) \cap \operatorname{int} B_{\frac{1}{C}}\left(x - \frac{1}{C}p\right) \subset B_{\frac{r_1-r}{3}}(x) \cap \operatorname{int} B_{\frac{1}{C}}\left(x - \frac{1}{C}p\right).$$

Using the last two formulae and Formula (14) for the complementary sets we get

$$(x + U) \cap B_{\frac{1}{C}}\left(x - \frac{1}{C}p\right) \subset (x + U) \cap B_{r_1}.$$

Note that by Theorem 2.1 the set A is strongly convex of radius r and hence of r_1. Let $a \in (-A)$ satisfying $\langle p, a\rangle = s(p, -A)$. By Lemma 3.3 we obtain that

$$(x + a + U) \cap \left(B_{\frac{1}{C}}\left(x - \frac{1}{C}p\right) + (-A)\right) \subset (x + a + U) \cap (B_{r_1} + (-A))$$
$$= (x + a + U) \cap B_{r_1}(0).$$

The last inclusion is valid for any $x \in \operatorname{bd} B_{r_1}$ and unit p with $\langle p, x\rangle = s(p, B_{r_1})$. By Proposition 1.5 the set $B_{\frac{1}{C}}(0) + (-A)$ is strongly convex of radius r_1 and (in the Hilbert space) summand of the ball $B_{r_1}(0)$ [4, Theorems 2.6, 1.1]. From this fact we obtain that there exists another closed convex set $D \subset \mathcal{H}$ with

$$B_{\frac{1}{C}}(0) + (-A) + D = B_{r_1}(0)$$

and hence

$$(-A) = B_{r_1 - \frac{1}{C}}(0) \overset{*}{-} D.$$

Thus the set A is strongly convex of radius $r_1 - \frac{1}{C}$ for any $r_1 \in \left(r, r + \frac{2}{C}\right)$. Taking the limit $r_1 \to r + 0$ we obtain that A is strongly convex of radius $r - \frac{1}{C}$. □

A. Appendix: about summands of the ball

In this small section we collect few important facts about summands of the balls.

Definition A.1 ([4]). Let E be a reflexive Banach space. A closed convex subset $M \subset E$ is called *generating* if *for any* nonempty subset $A = \bigcap_{x \in X}(M + x)$ $(X \subset E)$ there exists another closed and convex set $B \subset E$ with $A + B = M$.

It is obvious that if the ball in the space E is a generating set then each strongly convex subset $A \subset E$ of radius R is a summand of the ball $B_R(0)$.

Proposition A.2. *The ball in the real Hilbert space is generating set.*

Proposition A.2 was proved for the case of $\mathbb{R}^n$ in [10], for infinite dimension case in [4].

Properties of generating sets were studied in works [4, 20, 17].

Unfortunately, balls in most important cases of uniformly convex and smooth Banach spaces (except the Hilbert space) are not generating sets. For example, the ball in l_p, $p \in [1,2) \cup (2,+\infty)$ is not generating set. The idea for proving this is to consider a special vector $z \in l_p$ and the set $A = B_1(0) \cap B_1(z)$. The vector z can be taken such that Proposition 1.4 *2)* is false in some point $a \in \operatorname{bd} B_1(0) \cap \operatorname{bd} B_1(z) \subset \operatorname{bd} A$ at some unit vector $q \in l_p^*$ with $\langle q, a\rangle = s(q, A)$.

Nevertheless there exist summands of the ball from spaces l_p. Ivanov [16] proved the next proposition.

Proposition A.3. *Let E be a Banach space. Let $M \subset \mathbb{R}\times E$ be a set of the type*

$$M = \{(t,x) \mid t \in \operatorname{dom} f,\ x \in f(t)B\},$$

where the function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is concave with $f(t) \geq 0$ for all $t \in \operatorname{dom} f$ and the set $B \subset E$ is convex and $0 \in B$. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}\cup\{-\infty\}$ be concave functions (with $f_i(t) \geq 0$ for all $t \in \operatorname{dom} f_i$, $i = 1, 2$) such that

$$\operatorname{hypo} f = \operatorname{hypo} f_1 + \operatorname{hypo} f_2.$$

Then the set

$$A = \{(t,x) \mid t \in \operatorname{dom} f_1,\ x \in f_1(t)B\}$$

is summand of the set M, i.e. there exists another set $B \subset \mathbb{R}\times E$ with $A+B = M$.

Here $\operatorname{hypo} f$ is the hypograph of the function f, i.e. the set

$$\operatorname{hypo} f = \{(x,t) \in E \times \mathbb{R} \mid t \leq f(x),\ x \in \operatorname{dom} f\}.$$

With the help of Proposition A.3 Ivanov [16] proved that the set

$$\left\{ x = (x_1, x_2, x_3, \dots) \in l_p \;\middle|\; (|x_1| + \theta)^p + \sum_{k=2}^{\infty} |x_k|^p \leq 1 \right\}$$

is summand of the ball $B_1(0) \subset l_p$ for any $\theta \in (0,1)$ and $p \in (1,+\infty)$.

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