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A Best Proximity Point Theorem for Generalized Weakly Contractive Non Self Mappings

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The purpose of this paper is to provide sufficient conditions for the existence of a unique best proximity point for generalized weakly contractive mappings in the context of complete metric spaces.

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1. Introduction

Let A and B nonempty subsets of a metric space (X, d). An operator $T\colon A \to B$ is said to be a k-contraction if there exists a $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for any $x, y \in A$. Banach's contraction principle states that, when A is a complete subset of X and T is a k-contraction which maps A into itself then T has a unique fixed point in A.

Due to its wide applications in various fields, a huge number of generalizations and extended versions of this principle appear in the literature (see [8, 9]).

The following notion of a weakly contractive mapping was introduced by Alber and Guerre-Delabriere in the context of Hilbert spaces in [2] and Rhoades proved that the result of [2] is also valid in complete metric spaces.

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Definition 1.1 ([10]). Let (X, d) be a metric space and $T\colon X \to X$ a mapping. T is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \phi\left(d(x, y)\right), \quad \text{for all } x, y \in X,$$

where $\phi\colon [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Notice that if $\phi(t) = kt$ with $k \in [0, 1)$ then T is a $(1 - k)$-contraction.

The following result is due to Rhoades [10].

Theorem 1.2 ([10]). *Let (X, d) be a complete metric space and $T\colon X \to X$ a weakly contractive mapping. Then T has a unique fixed point in X.*

In [4], Dutta and Choudhury present the following generalization of Theorem 1.2.

Theorem 1.3 ([4]). *Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping satisfying*

$$\varphi\left(d(Tx, Ty)\right) \leq \varphi\left(d(x, y)\right) - \phi\left(d(x, y)\right), \quad \textit{for all } x, y \in X,$$

where $\varphi, \phi\colon [0, \infty) \to [0, \infty)$ are continuous and nondecreasing functions such that $\phi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

In this paper, we present a best proximity point theorem for generalized weakly contractive non self mappings.

First, we present a brief discussion about best proximity points

Let A be a nonempty subset of a metric space (X, d) and $T\colon A \to X$ a mapping. The solutions of the equation $Tx = x$ are fixed points of T. Consequently, $T(A) \cap A \neq \emptyset$ is a necessary condition for the existence of a fixed point for the operator T. If this necessary condition does not hold then $d(Tx, x) > 0$ for any $x \in A$ and the mapping $T\colon A \to X$ does not have any fixed point. In this setting, our aim is to find an element $x \in A$ such that $d(x, Tx)$ is a minimum. Best approximation theory has been developed in this direction.

In our context we consider two nonempty subsets A and B of a complete metric space and a mapping $T\colon A \to B$. A natural question is whether one can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\left\{d(x, Tx)\colon x \in A.\right\}$. Since $d(x, Tx) \geq d(A, B)$ for any $x \in A$, the optimal solution to this problem will be the one for which the value $d(A, B)$ is attained by the real valued function $\varphi\colon A \to \mathbb{R}$ given by $\varphi(x) = d(x, Tx)$.

Some results about best proximity points can be found in [1, 5, 7, 11, 12].

2. Preliminaries

Let A and B be two nonempty subsets of a metric space (X,d). Denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A\colon d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B\colon d(x,y) = d(A,B) \text{ for some } x \in A\},$$

where $d(A,B) = \inf\{d(x,y)\colon x \in A \text{ and } y \in B\}$.

In [7], the authors present sufficient conditions which guarantee when the sets A_0 and B_0 are nonempty. Moreover, in [11] the authors proved that A_0 is contained in the boundary of A.

Now we present the notion of a generalized weakly contractive non self mapping.

Definition 2.1. Let A and B be two nonempty subsets of a metric space (X,d). A mapping $T\colon A \to B$ is said to be generalized weakly contractive if

$$\psi\left(d(Tx,Ty)\right) \leq \psi\left(M(x,y) - d(A,B)\right) - \phi\left(d(x,y)\right), \quad \text{for any } x,y \in A,$$

where $\psi\colon [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$ and $\phi\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$, and where

$$M(x,y) := \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left[d(x,Ty) + d(y,Tx)\right]\right\}.$$

Notice that in the case $A = B$ every operator $T\colon A \to A$ satisfying the contractive condition appearing in Theorem 1.3 is a generalized weakly contractive mapping, since in this case $d(A,A) = 0$ and

$$\varphi\left(d(Tx,Ty)\right) \leq \varphi\left(d(x,y)\right) - \phi\left(d(x,y)\right) \leq \varphi\left(M(x,y)\right) - \phi\left(d(x,y)\right).$$

In [13], the author introduced the following definition.

Definition 2.2 ([13]). Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$

$$\left.\begin{aligned} d(x_1,y_1) &= d(A,B) \\ d(x_2,y_2) &= d(A,B) \end{aligned}\right\} \Rightarrow d(x_1,x_2) = d(y_1,y_2).$$

It is easily seen that for any nonempty subset A of (X,d), the pair (A,A) has the P-property.

In [13], the author proves that any pair (A,B) of nonempty closed convex subsets of a real Hilbert space H satisfies the P-property.

3. Main results

We start this section presenting our main result.

Theorem 3.1. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$. Let $T: A \to B$ be a continuous generalized weakly contractive mapping satisfying $T(A_0) \subset B_0$. Suppose that the pair (A, B) has the P-property. Then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$.*

Proof. Since A_0 es nonempty, we take $x_0 \in A$. As $Tx_0 \in T(A_0) \subset B_0$, we can find $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Similarly, since $Tx_1 \in T(A_0) \subset B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Repeating this process, we obtain a sequence (x_n) in A_0 satisfying

$$d(x_{n+1}, Tx_n) = d(A, B), \text{ for any } n \in \mathbb{N}. \tag{1}$$

Since (A, B) has the P-property, we have that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}), \text{ for any } n \in \mathbb{N}.$$

Now, suppose that there exists an $n_0 \in \mathbb{N}$ with $d(x_{n_0+1}, x_{n_0}) = 0$. In this case we have

$$0 = d(x_{n_0+1}, x_{n_0}) = d(Tx_{n_0}, Tx_{n_0-1}),$$

and, consequently, $Tx_{n_0-1} = Tx_{n_0}$. Therefore,

$$d(A, B) = d(x_{n_0}, Tx_{n_0-1}) = d(x_{n_0}, Tx_{n_0}),$$

and the part of existence is proved.

On the contrary, suppose that $d(x_{n+1}, x_n) > 0$ for any $n \in \mathbb{N}$. Taking into account the contractive condition and (1) we have

$$\begin{aligned} &\psi\left(d(x_{n+1}, x_n)\right) \\ &= \psi\left(d(Tx_n, Tx_{n-1})\right) \le \psi\left(M(x_n, x_{n-1}) - d(A, B)\right) - \phi(d(x_n, x_{n-1})). \end{aligned} \tag{2}$$

We can distinguish four cases.

Case 1: Suppose that $M(x_n, x_{n-1}) = d(x_n, x_{n-1})$. In this case, from the last inequality and the fact that ψ is nondecreasing, we get

$$\begin{aligned} \psi\left(d(x_{n+1}, x_n)\right) &= \psi\left(d(Tx_n, Tx_{n-1})\right) \\ &\le \psi\left(d(x_n, x_{n-1}) - d(A, B)\right) - \phi\left(d(x_n, x_{n-1})\right) \\ &\le \psi\left(d(x_n, x_{n-1})\right) - \phi\left(d(x_n, x_{n-1})\right) \\ &\le \psi\left(d(x_n, x_{n-1})\right), \end{aligned} \tag{3}$$

and, again, as ψ is nondecreasing,

$$d(x_{n+1}, x_n) \le d(x_n, x_{n-1}).$$

Case 2: Suppose that $M(x_n, x_{n-1}) = d(x_n, Tx_n)$. Since $d(x_{n+1}, Tx_n) = d(A, B)$, we have

$$\begin{aligned} M(x_n, x_{n-1}) - d(A, B) &= d(x_n, Tx_n) - d(A, B) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) - d(A, B) \\ &= d(x_n, x_{n+1}). \end{aligned}$$

This and (2) give us

$$\begin{aligned} \psi\left(d(x_{n+1}, x_n)\right) &= \psi\left(d(Tx_n, Tx_{n-1})\right) \\ &\leq \psi\left(d(x_n, Tx_n) - d(A, B)\right) - \phi\left(d(x_n, x_{n-1})\right) \\ &\leq \psi\left(d(x_n, x_{n+1})\right) - \phi\left(d(x_n, x_{n-1})\right). \end{aligned}$$

Since $d(x_n, x_{n-1}) > 0$ and $\phi(t) = 0$ if and only if $t = 0$, the last inequality implies that

$$\psi\left(d(x_{n+1}, x_n)\right) \leq \psi\left(d(x_n, x_{n+1})\right) - \phi\left(d(x_n, x_{n-1})\right) < \psi\left(d(x_n, x_{n+1})\right),$$

which is a contradiction.

Case 3: Suppose that $M(x_n, x_{n-1}) = d(x_{n-1}, Tx_{n-1})$. In this case, by an argument similar to the one used in Case 2, we have

$$\begin{aligned} M(x_n, x_{n-1}) - d(A, B) &= d(x_{n-1}, Tx_{n-1}) - d(A, B) \\ &\leq d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) - d(A, B) = d(x_{n-1}, x_n), \end{aligned}$$

and, therefore,

$$\begin{aligned} \psi\left(d(x_{n+1}, x_n)\right) &= \psi\left(d(Tx_n, Tx_{n-1})\right) \\ &\leq \psi\left(d(x_{n-1}, Tx_{n-1}) - d(A, B)\right) - \phi\left(d(x_n, x_{n-1})\right) \\ &\leq \psi\left(d(x_{n-1}, x_n)\right) - \phi\left(d(x_n, x_{n-1})\right) \qquad (4) \\ &< \psi\left(d(x_{n-1}, x_n)\right). \end{aligned}$$

By using the fact that ψ is nondecreasing,

$$d(x_{n+1}, x_n) < d(x_{n-1}, x_n).$$

Case 4: Suppose that $M(x_n, x_{n-1}) = \frac{1}{2}\left[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)\right]$. In this case, we have

$$\begin{aligned} M(x_n, x_{n-1}) - d(A, B) &= \frac{1}{2}\left[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)\right] - d(A, B) \\ &\leq \frac{1}{2}\left[d(A, B) + d(x_{n-1}, x_{n+1}) + d(x_{n+1}, Tx_n)\right] - d(A, B) \\ &= \frac{1}{2}\left[d(A, B) + d(x_{n-1}, x_{n+1}) + d(A, B)\right] - d(A, B) \\ &= \frac{1}{2}\left[d(x_{n-1}, x_{n+1})\right] \\ &\leq \frac{1}{2}\left[d(x_n, x_{n-1}) + d(x_n, x_{n+1})\right], \end{aligned}$$

and, therefore,

$$\begin{aligned}\psi\left(d(x_{n+1},x_n)\right) &= \psi\left(d(Tx_n,Tx_{n-1})\right)\\ &\le \psi\left(M(x_n,x_{n-1})-d(A,B)\right)-\phi\left(d(x_n,x_{n-1})\right)\\ &\le \psi\left(\frac{1}{2}\left[d(x_n,x_{n-1})+d(x_n,x_{n+1})\right]\right)-\phi\left(d(x_n,x_{n-1})\right) \qquad (5)\\ &< \psi\left(\frac{1}{2}\left[d(x_n,x_{n-1})+d(x_n,x_{n+1})\right]\right).\end{aligned}$$

Since ψ is nondecreasing

$$d(x_{n+1},x_n) \le \frac{1}{2}\left[d(x_n,x_{n-1})+d(x_n,x_{n+1})\right],$$

and, consequently,

$$d(x_{n+1},x_n) \le d(x_n,x_{n-1}). \qquad (6)$$

From (5) and (6) we infer that

$$\psi\left(d(x_{n+1},x_n)\right) \le \psi\left(d(x_n,x_{n-1})\right)-\phi\left(d(x_n,x_{n-1})\right). \qquad (7)$$

Summarizing, in any case we get $d(x_{n+1},x_n) \le d(x_n,x_{n-1})$.

This means that $(d(x_{n+1},x_n))$ is a decreasing sequence of nonnegative real numbers and, consequently, there exists an $r \ge 0$ such that

$$\lim_{n\to\infty} d(x_{n+1},x_n) = r.$$

We shall now prove that $r = 0$.

Notice that the previously discussed Cases 1, 3 and 4 are the only ones possible.

In these cases we obtained (see (3), (4) and (7))

$$\begin{aligned}\psi\left(d(x_{n+1},x_n)\right) &\le \psi\left(d(x_n,x_{n-1})\right)-\phi\left(d(x_n,x_{n-1})\right)\\ &\le \psi\left(d(x_n,x_{n-1})\right) \quad \text{for any } n\in\mathbb{N}.\end{aligned}$$

Suppose that $r > 0$.

Taking $n \to \infty$ in the last inequality, and, by using the continuity of ψ, we infer that

$$\psi(r) \le \psi(r) - \lim_{n\to\infty} \phi\left(d(x_n,x_{n-1})\right) \le \psi(r),$$

which implies that

$$\lim_{n\to\infty} \phi\left(d(x_n,x_{n-1})\right) = 0. \qquad (8)$$

On the other hand, since $r > 0$, we have

$$0 < r \le d(x_n,x_{n-1}) \text{ for any } n\in\mathbb{N},$$

and, since ϕ is nondecreasing and $\phi(t) = 0$ if and only if $t = 0$, we get

$$0 < \phi(r) \leq \phi\left(d(x_n, x_{n-1})\right)) \text{ for any } n \in \mathbb{N}.$$

Therefore, $0 < \phi(r) \leq \lim_{n\to\infty} \phi\left(d(x_n, x_{n-1})\right)$ and this contradicts (8).

Consequently $r = 0$; i.e.,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{9}$$

We now prove that (x_n) is a Cauchy sequence

In the contrary case, by using Lemma 2.1 of [6], we can find an $\epsilon > 0$ and subsequences $(x_{n(k)})$ and $(x_{m(k)})$ de (x_n) satisfying:

(i) $n(k) > m(k) \geq k$ for $k \geq 0$.

(ii) $\epsilon \leq d(x_{n(k)}, x_{m(k)})$ and $d(x_{n(k)-1}, x_{m(k)}) < \epsilon$ for $k \geq 0$.

(iii) $$\begin{aligned}\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) &= \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \\ &= \lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon.\end{aligned}$$

Since, for any $k \geq 0$, $d(x_{n(k)+1}, Tx_{n(k)}) = d(A,B)$ and $d(x_{m(k)+1}, Tx_{m(k)} = d(A,B))$, and, as the pair (A,B) has the P-property, we infer that

$$d(x_{n(k)+1}, x_{m(k)+1}) = d(Tx_{n(k)}, Tx_{m(k)}) \text{ for any } k \geq 0.$$

Using the contractive condition, for any $k \geq 0$, we have

$$\begin{aligned}\psi\left(d(x_{n(k)+1}, x_{m(k)+1})\right) &= \psi\left(d(Tx_{n(k)}, Tx_{m(k)})\right) \\ &\leq \psi\left(M(x_{n(k)}, x_{m(k)}) - d(A,B)\right) - \phi\left(d(x_{n(k)}, x_{m(k)})\right).\end{aligned} \tag{10}$$

We distinguish four cases.

Case 1: Suppose that $M(x_{n(k)}, x_{m(k)}) = d(x_{n(k)}, x_{m(k)})$. In this case, from the last inequality and by using the fact that ψ is nondecreasing and ϕ is nonnegative, we obtain

$$\begin{aligned}\psi\left(d(x_{n(k)+1}, x_{m(k)+1})\right) &\leq \psi\left(d(x_{n(k)}, x_{m(k)}) - d(A,B)\right) - \phi\left(d(x_{n(k)}, x_{m(k)})\right) \\ &\leq \psi\left(d(x_{n(k)}, x_{m(k)})\right) - \phi\left(d(x_{n(k)}, x_{m(k)})\right) \qquad (11) \\ &\leq \psi\left(d(x_{n(k)}, x_{m(k)})\right).\end{aligned}$$

Case 2: Suppose that $M(x_{n(k)}, x_{m(k)}) = d(x_{n(k)}, Tx_{n(k)})$. In this case,

$$\begin{aligned}M(x_{n(k)}, x_{m(k)}) - d(A,B) &= d(x_{n(k)}, Tx_{n(k)}) - d(A,B) \\ &\leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, Tx_{n(k)}) - d(A,B) \\ &= d(x_{n(k)}, x_{n(k)+1}),\end{aligned}$$

and inequality (10) takes the form

$$\begin{aligned}\psi\left(d(x_{n(k)+1}, x_{m(k)+1})\right) &\leq \psi\left(d(x_{n(k)}, x_{n(k)+1})\right) - \phi\left(d(x_{n(k)}, x_{m(k)})\right) \\ &\leq \psi\left(d(x_{n(k)}, x_{n(k)+1})\right).\end{aligned} \tag{12}$$

Case 3: Suppose that $M(x_{n(k)}, x_{m(k)}) = d(x_{m(k)}, Tx_{m(k)})$. Using an argument similar to that in Case 2, we have

$$M(x_{n(k)}, x_{m(k)}) - d(A,B) \leq d(x_{m(k)}, x_{m(k)+1}),$$

and, from (10), we have

$$\begin{aligned}\psi\left(d(x_{n(k)+1}, x_{m(k)+1})\right) &\leq \psi\left(d(x_{m(k)}, x_{m(k)+1})\right) - \phi\left(d(x_{n(k)}, x_{m(k)})\right)\\ &\leq \psi\left(d(x_{m(k)}, x_{m(k)+1})\right). \end{aligned} \tag{13}$$

Case 4: Suppose that $M(x_{n(k)}, x_{m(k)}) = \frac{1}{2}\left[d(x_{n(k)}, Tx_{m(k)}) + d(x_{m(k)}, Tx_{n(k)})\right]$. In this case, we have that

$$\begin{aligned}&M(x_{n(k)}, x_{m(k)}) - d(A,B)\\ &= \frac{1}{2}\left[d(x_{n(k)}, Tx_{m(k)}) + d(x_{m(k)}, Tx_{n(k)})\right] - d(A,B)\\ &\leq \frac{1}{2}\left[d(x_{n(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, Tx_{m(k)})\right.\\ &\qquad \left. + d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)+1)}, Tx_{n(k)})\right] - d(A,B)\\ &= \frac{1}{2}\left[d(x_{n(k)}, x_{m(k)+1}) + d(A,B) + d(x_{m(k)}, x_{n(k)+1}) + d(A,B)\right] - d(A,B)\\ &= \frac{1}{2}\left[d(x_{n(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)+1})\right],\end{aligned}$$

and, consequently, from (10) we infer that

$$\begin{aligned}&\psi\left(d(x_{n(k)+1}, x_{m(k)+1})\right)\\ &\leq \psi\left(\frac{1}{2}\left[d(x_{n(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)+1})\right]\right) - \phi\left(d(x_{n(k)}, x_{m(k)})\right).\end{aligned} \tag{14}$$

Now, put

$$\begin{aligned} C &= \{k \geq 0\colon k \text{ satisfies Case } 2\},\\ D &= \{k \geq 0\colon k \text{ satisfies Case } 3\}, \quad \text{and}\\ E &= \{k \geq 0\colon k \text{ satisfies Case } 4\}.\end{aligned}$$

If $\operatorname{Card} C = \infty$ or $\operatorname{Card} D = \infty$ then, taking $k \to \infty$ in (12) or (13) and, taking into account the properties of the subsequences $(x_{n(k)})$ and $(x_{m(k)})$, (9) and the continuity of ψ, we get

$$0 < \psi(\epsilon) = \psi\left(\lim_{k\to\infty} d(x_{n(k)}, x_{n(k)+1})\right) = \psi(0) = 0$$

or

$$0 < \psi(\epsilon) = \psi\left(\lim_{k\to\infty} d(x_{m(k)}, x_{m(k)+1})\right) = \psi(0) = 0$$

which is a contradiction.

Therefore Card $C < \infty$ and Card $D < \infty$.

If Card $E = \infty$ then taking $k \to \infty$ in (14), we have

$$0 < \psi(\epsilon) \le \psi\left(\frac{1}{2}(\epsilon + \epsilon)\right) - \lim_{k\to\infty} \phi\big(d(x_{n(k)}, x_{m(k)}\big) \le \psi(\epsilon),$$

which implies that

$$\lim_{k\to\infty} \phi\big(d(x_{n(k)}, x_{m(k)})\big) = 0. \tag{15}$$

Since $0 < \epsilon \le d(x_{n(k)}, x_{m(k)})$ for any $k \ge 0$ and ϕ is nondecreasing, we get

$$0 < \phi(\epsilon) \le \phi\big(d(x_{n(k)}, x_{m(k)})\big), \quad \text{for any } k \ge 0.$$

Therefore Card $E < \infty$.
which contradicts (15).
This means that, for all $k \ge 0$ except for a finite number, Case 1 is satisfied.

Taking $k \to \infty$ in (11) we obtain

$$0 < \psi(\epsilon) \le \psi(\epsilon) - \lim_{k\to\infty} \phi\left(d(x_{n(k)}, x_{m(k)})\right) \le \psi(\epsilon),$$

which implies that

$$\lim_{k\to\infty} \phi\left(d(x_{n(k)}, x_{m(k)})\right) = 0. \tag{16}$$

But, since

$$0 < \epsilon \le d(x_{n(k)}, x_{m(k)}) \quad \text{for all } k \ge 0,$$

by using the nondecreasing character of ϕ, we have

$$0 < \phi(\epsilon) \le \phi\left(d(x_{n(k)}, x_{m(k)})\right) \quad \text{for all } k \ge 0,$$

and, from this fact, we infer that

$$0 < \phi(\epsilon) \le \lim_{k\to\infty} \phi\left(d(x_{n(k)}, x_{m(k)})\right),$$

which contradicts (16).

Therefore, $\{x_n\}$ is a Cauchy sequence.

Since $\{x_n\} \subset A$ and A is a closed subset of the complete metric space (X, d), we can find an $x^* \in A$ such that $x_n \to x^*$.

Now, since T is continuous, $Tx_n \to Tx^*$.

Therefore, we have that $d(x_{n+1}, Tx_n) \to d(x^*, Tx^*)$. Taking into account that the sequence $d(x_{n+1}, Tx_n)$ is a constant sequence with value $d(A, B)$, we deduce that

$$d(x^*, Tx^*) = d(A, B).$$

This means that x^* is a best proximity point of T.

This proves the existence part of our theorem.

To prove uniqueness, suppose that x_1 and x_2 are two best proximity points of T and $x_1 \neq x_2$. This means that

$$d(x_i, Tx_i) = d(A, B) \quad \text{for } i = 1, 2.$$

By using the P-property of the pair (A, B), we have

$$d(x_1, x_2) = d(Tx_1, Tx_2).$$

Now, taking into account the contractive condition, we get

$$\psi\left(d(x_1, x_2)\right) = \psi\left(d(Tx_1, Tx_2)\right) \leq \psi\left(M(x_1, x_2) - d(A, B)\right) - \phi\left(d(x_1, x_2)\right). \quad (17)$$

There are four cases.

Case 1: Suppose that $M(x_1, x_2) = d(x_1, x_2)$. In this case inequality (17) takes the form

$$\psi\left(d(x_1, x_2)\right) \leq \psi\left(d(x_1, x_2) - d(A, B)\right) - \phi\left(d(x_1, x_2)\right),$$

and, since $x_1 \neq x_2$ and $\phi(t) = 0$ if and only if $t = 0$, we have that $\phi\left(d(x_1, x_2)\right) > 0$. Consequently, the last inequality gives us

$$\begin{aligned} \psi\left(d(x_1, x_2)\right) &\leq \psi\left(d(x_1, x_2) - d(A, B)\right) - \phi\left(d(x_1, x_2)\right) \\ &< \psi\left(d(x_1, x_2) - d(A, B)\right) \\ &\leq \psi\left(d(x_1, x_2)\right), \end{aligned}$$

and this is a contradiction.

Case 2: Suppose that $M(x_1, x_2) = d(x_1, Tx_1)$. Since $d(x_1, Tx_1) = d(A, B)$, the inequality (17) takes the form

$$\begin{aligned} \psi\left(d(x_1, x_2)\right) &\leq \psi\left(d(x_1, Tx_1) - d(A, B)\right) - \phi\left(d(x_1, x_2)\right) \\ &= \psi\left(d(A, B) - d(A, B)\right) - \phi\left(d(x_1, x_2)\right) \\ &= \psi(0) - \phi\left(d(x_1, x_2)\right). \end{aligned}$$

As $\psi(0) = 0$ and $x_1 \neq x_2$ and, consequently, $\phi\left(d(x_1, x_2)\right) > 0$, we have

$$\psi\left(d(x_1, x_2)\right) \leq -\phi\left(d(x_1, x_2)\right) < 0,$$

which is a contradiction.

Case 3: Suppose that $M(x_1, x_2) = d(x_2, Tx_2)$. Using the same argument as in Case 2, we obtain

$$\psi\left(d(x_1, x_2)\right) \leq -\phi\left(d(x_1, x_2)\right) < 0,$$

which is a contradiction.

Case 4: Suppose that $M(x_1, x_2) = \frac{1}{2}\left[d(x_1, Tx_2) + d(x_2, Tx_1)\right]$. Since

$$\begin{aligned}
&M(x_1, x_2) - d(A, B) \\
&= \frac{1}{2}\left[d(x_1, Tx_2) + d(x_2, Tx_1)\right] - d(A, B) \\
&\le \frac{1}{2}\left[d(x_1, Tx_1) + d(Tx_1, Tx_2) + d(x_2, Tx_2 + d(Tx_1, Tx_2))\right] - d(A, B) \\
&= \frac{1}{2}\left[d(A, B) + 2d(Tx_1, Tx_2) + d(A, B)\right] - d(A, B) \\
&= d(Tx_1, Tx_2),
\end{aligned}$$

from (17), we get

$$\begin{aligned}
&\psi\left(d(x_1, x_2)\right) \\
&= \psi\left(d(Tx_1, Tx_2)\right) \le \psi\left(d(Tx_1, Tx_2)\right) - \phi\left(d(x_1, x_2)\right) \le \psi\left(d(Tx_1, Tx_2)\right).
\end{aligned}$$

Consequently, $\phi\left(d(x_1, x_2)\right) = 0$.
This is a contradiction since $x_1 \neq x_2$.

Therefore $x_1 = x_2$. □

In what follows we will present some particular cases and their consequences.

When the mapping ψ in Theorem 3.1 is the identity, we obtain the following corollary.

Corollary 3.2. *Let (A, B) a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$. Let $T\colon A \to B$ be a continuous mapping satisfying that $T(A_0) \subset B_0$ and such that*

$$d(Tx, Ty) \le M(x, y) - d(A, B) - \phi\left(d(x, y)\right) \quad \textit{for any } x, y \in A.$$

Suppose that the pair (A, B) has the P-property. Then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$.

When $d(A, B) = 0$, Corollary 3.2 takes the following form.

Corollary 3.3. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and $d(A, B) = 0$. Let $T\colon A \to B$ be a continuous mapping such that $T(A_0) \subset B_0$ and satisfying*

$$d(Tx, Ty) \le M(x, y) - \phi\left(d(x, y)\right) \quad \textit{for any } x, y \in A.$$

Then there exists a unique $x^* \in A$ such that $Tx^* = x^*$.

Proof. Notice that, if $d(A, B) = 0$, then (A, B) has the P-property. □

The condition

$$d(Tx, Ty) \le d(x, y) - \phi\left(d(x, y)\right), \text{ for any } x, y \in A$$

implies that

$$d(Tx,Ty) \leq d(x,y) - \phi\left(d(x,y)\right) \leq d(x,y) \text{ for any } x,y \in A.$$

Thus T is continuous and satisfies

$$d(Tx,Ty) \leq d(x,y) - \phi\left(d(x,y)\right) \leq M(x,y) - \phi\left(d(x,y)\right) \text{ for any } x,y \in A.$$

By Theorem 3.1 we obtain the desired result.
As a consequence of Corollary 3.3 we have the following result.

Corollary 3.4. *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq$ and $d(A,B)=0$. Let $T\colon A \to B$ be a mapping such that $T(A_0) \subset B_0$ and satisfies*

$$d(Tx,Ty) \leq d(x,y) - \phi\left(d(x,y)\right), \text{ for any } x,y \in A.$$

Then there exists a unique $x^* \in A$ such that $Tx^* = x^*$.

Corollary 3.4 is Theorem 3.1 of [12] for the particular case $d(A,B)=0$.

The following result presents a best proximity point theorem for a mapping satisfying a Kannan type contractive condition.

Corollary 3.5. *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$. Let $T\colon A \to B$ be a continuous mapping satisfying $T(A_0) \subset B_0$ and such that*

$$\begin{aligned}&\psi\left(d(Tx,Ty)\right)\\ &\leq \psi\left(\frac{1}{2}[d(x,Tx)+d(y,Ty)] - d(A,B)\right) - \phi\left(d(x,y)\right), \text{ for any } x,y \in A,\end{aligned}$$

where $\psi\colon [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t)=0$ if and only if $t=0$ and $\phi\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function such that $\phi(t)=0$ if and only if $t=0$.

Suppose that the pair (A,B) has the P-property. Then there exists a unique $x^* \in A$ such that $d(x^*,Tx^*) = d(A,B)$.

Proof. First, notice that

$$\frac{1}{2}[d(x,Tx)+d(y,Ty)] \geq \frac{1}{2}[d(A,B)+d(A,B)] = d(A,B),$$

and, consequently, the contractive condition is well defined.

On the other hand, since

$$\frac{1}{2}[d(x,Tx)+d(y,Ty)] \leq \frac{1}{2}[M(x,y)+M(x,y)] = M(x,y)$$

we have, by virtue of the nondecreasing character of ψ that, for any $x, y \in A$

$$\begin{aligned}\psi\left(d(Tx,Ty)\right) &\leq \psi\left(\frac{1}{2}[d(x,Tx)+d(y,Ty)] - d(A,B)\right) - \phi\left(d(x,y)\right)\\ &\leq \psi\left(M(x,y) - d(A,B)\right) - \phi\left(d(x,y)\right),\end{aligned}$$

and the desired result is obtained by applying Theorem 3.1. □

The following consequence is a best proximity point theorem for mappings satisfying a condition of integral type in the case $d(A,B) = 0$.

Corollary 3.6. *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $d(A,B) = 0$ and $A_0 \neq \emptyset$. Let $T\colon A \to B$ be a continuous mapping such that $T(A_0) \subset B_0$ and suppose that there exists a $k \in [0,1)$ such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \leq k \int_0^{M(x,y)} \varphi(t)\,dt \quad \text{for any } x, y \in A,$$

where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping such that $\int_0^\epsilon \varphi(t)\,dt > 0$ for $\epsilon > 0$.

Then there exists a unique $x^* \in A$ such that $Tx^* = x^*$.

Proof. First notice that, if $d(A,B) = 0$, then it is easily seen that (A,B) has the P-property. On the other hand, it is easily seen that the function $\psi\colon [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = \int_0^t \varphi(s)\,ds$$

is continuous and nondecreasing and, since $\int_0^\epsilon \varphi(t)\,dt > 0$ for each $\epsilon > 0$, $\psi(t) = 0$ if and only if $t = 0$.

Let ϕ be the function defined by $\phi(t) = (1-k)\psi(t)$ on $[0,\infty)$. Therefore our contractive mapping can be rewritten of the following way

$$\begin{aligned}\psi\left(d(Tx,Ty)\right) &\leq k\psi\left(M(x,y)\right)\\ &= \psi\left(M(x,y)\right) - (1-k)\psi\left(M(x,y)\right)\\ &= \psi\left(M(x,y)\right) - \phi\left(M(x,y)\right), \quad \text{for any } x, y \in A.\end{aligned}$$

By using the nondecreasing character of ϕ, the last inequality implies that

$$\psi\left(d(Tx,Ty)\right) \leq \psi\left(M(x,y)\right) - \phi\left(d(x,y)\right), \quad \text{for any } x, y \in A.$$

By applying Theorem 3.1, we obtain the desired result. □

When $A = B$ Theorem 3.1 has, as consequences, the following fixed point theorems.

Corollary 3.7. *Let (X,d) be a complete metric space and let $T\colon X \to X$ be a mapping satisfying*

$$\psi\left(d(Tx,Ty)\right) \leq \psi\left(M(x,y)\right) - \phi\left(d(x,y)\right), \quad \textit{for any } x,y \in X,$$

where $\psi\colon [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$ and $\phi\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Corollary 3.7 has as a consequence, the following result.

Corollary 3.8. *Let (X,d) be a complete metric space and let $T\colon X \to X$ be a mapping satisfying*

$$\psi\left(d(Tx,Ty)\right) \leq \psi\left(d(x,y)\right) - \phi\left(d(x,y)\right), \quad \textit{for any } x,y \in X,$$

where ψ and ϕ satisfy the same conditions as those in Corollary 3.7. Then T has a unique fixed point.

Proof. Notice that, since ψ is nondecreasing, for any $x,y \in X$

$$\psi\left(d(Tx,Ty)\right) \leq \psi\left(d(x,y)\right) - \phi\left(d(x,y)\right) \leq \psi\left(M(x,y)\right) - \phi\left(d(x,y)\right).$$

Moreover, since $\psi\left(d(Tx,Ty)\right) \leq \psi\left(d(x,y)\right)$ for any $x,y \in X$ and ψ is nondecreasing, $d(Tx,Ty) \leq d(x,y)$ for any $x,y \in X$ and thus T is continuous. Finally, Corollary 3.7 give us the desired result. □

Corollary 3.8 improves Theorem 2.1 of [4] since we do not assume the continuity of ϕ.

If we put $\psi = 1_X$(identity mapping) in Corollary 3.8 then we obtain the following result.

Corollary 3.9. *Let (X,d) be a complete metric space and $T\colon X \to X$ a mapping satisfying*

$$d(Tx,Ty) \leq d(x,y) - \phi\left(d(x,y)\right), \quad \textit{for any } x,y \in X$$

where ϕ satisfies the same conditions as in Corollary 3.7. Then T has a unique fixed point.

Corollary 3.9 improves Theorem 1 of [10] because we do not impose the continuity of ϕ.

A consequence of Corollary 3.6 is the following fixed point theorem with a condition of integral type.

Corollary 3.10. *Let (X,d) be a complete metric space and $k \in [0,1)$ and let $T\colon X \to X$ be a mapping such that for each $x,y \in X$*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \leq k \int_0^{M(x,y)} \varphi(t)\,dt,$$

where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a Lebesgue-integrable mapping such that* $\int_0^\epsilon \varphi(t)\,dt > 0$ *for* $\epsilon > 0$. *Then* T *has a unique fixed point.*

Since the contractive condition

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le k \int_0^{d(x,y)} \varphi(t)\,dt \quad \text{for any } x, y \in X$$

implies the one appearing in Corollary 3.10, we have the following result.

Corollary 3.11. *Let* (X, d) *be a complete metric space and* $k \in [0, 1)$ *and* $T\colon X \to X$ *a continuous mapping satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le k \int_0^{d(x,y)} \varphi(t)\,dt \quad \textit{for any } x, y \in X$$

where φ *satisfies the same conditions that in Corollary* 3.10. *Then* T *has a unique fixed point.*

Corollary 3.11 is the main result of [3] when T is continuous.

The following example shows that the P-property condition cannot be relaxed from Theorem 3.1.

Example 3.12. Consider $X = \mathbb{R}$ with the usual metric and $A = \{8, -8\}$ and $B = \{1, -1\}$.

Obviously, A and B are closed subsets of X with $A_0 = A$, $B_0 = B$ and $d(A, B) = 7$. Let $T\colon A \to B$ be the mapping defined by $T(8) = -1$ and $T(-8) = 1$. It is easily seen that

$$d(Tx, Ty) \le \frac{1}{2}\,(d(x, y) - d(A, B)) \quad \text{for } x, y \in A,$$

and, thus, T satisfies the contractive condition appearing in Theorem 3.1. On the other hand, since $d(x, Tx) = 9 > d(A, B) = 7$ for any $x, y \in A$, T does not have a best proximity point.

Notice that the pair (A, B) does not have the P-property

4. Examples.

In order to illustrate our results, we present some examples.

Example 4.1. Consider $X = \mathbb{R}^2$ with the usual metric and the closed subsets

$$A = [0, \infty) \times \{0\} \quad \text{and} \quad B = (-\infty, 0] \times \{0\}.$$

Obviously, $d(A, B) = 0$ and, consequently, the pair (A, B) has the P-property. Notice that $A_0 = \{(0, 0)\}$ and $B_0 = \{(0, 0)\}$.

Let $T\colon A \to B$ be the mapping defined by

$$T(x, 0) = (-\log(1 + x), 0).$$

Obviously $T(A_0) \subset B_0$.

In the sequel, we will prove that T satisfies the contractive condition of Theorem 3.1.

In fact, for $(x,0),(y,0) \in A$, we have

$$\begin{aligned} d(T(x,0),T(y,0)) &= d\left((-\log(1+x),0),(-\log(1+y),0)\right) \\ &= |\log(1+x)-\log(1+y)| \\ &= \left|\log\left(\frac{1+x}{1+y}\right)\right|. \end{aligned}$$

We claim that $\left|\log\left(\frac{1+x}{1+y}\right)\right| \leq \log\left(1+|x-y|\right)$.

Suppose that $x>y$(the same argument works for $y>x$).

Since $\phi(t)=\log(1+t)$ is strictly increasing in $[0,\infty)$, we have

$$\begin{aligned} \log\left(\frac{1+x}{1+y}\right) &= \log\left(\frac{1+y+x-y}{1+y}\right) = \log\left(1+\frac{x-y}{1+y}\right) \\ &\leq \log\left(1+x-y\right) = \log\left(1+|x-y|\right). \end{aligned}$$

This prove our claim.

Therefore, for $(x,0),(y,0) \in A$, we have

$$\begin{aligned} & d\left(T(x,0),T(y,0)\right) \\ = & \left|\log\left(\frac{1+x}{1+y}\right)\right| \\ \leq & \log(1+|x-y|) \\ = & \log\left(1+d\left((x,0),(y,0)\right)\right) \\ = & d\left((x,0),(y,0)\right)-\left(d\left((x,0),(y,0)\right)-\log\left(1+d\left((x,0),(y,0)\right)\right)\right). \end{aligned}$$

In our case, $\psi = 1_{[0,\infty)}$(identity mapping) and ϕ is given by $\phi(t)=t-\log(1+t)$ which is nondecreasing and satisfies that $\phi(t)=0$ if and only if $t=0$. T is obviously continuous and, therefore, the assumptions of Theorem 3.1 are satisfied. Consequently, by Theorem 3.1 there exists a unique $(x^*,0) \in A$ such that

$$(d\left((x^*,0),T(x^*,0)\right) = 0 = d(A,B).$$

Notice that $(x^*,0) \in A$ is the point $(0,0)$.

The following example shows that the condition A and B are closed subsets of (X,d) is not a necessary condition for the existence of a unique best proximity point.

Example 4.2. Consider $X=\mathbb{R}^2$ with the usual metric and the subsets

$$A=[0,\infty)\times\{0\} \quad \text{and} \quad B=\left(-\frac{\pi}{2},0\right]\times\{0\}.$$

Notice that B is not a closed subset of X.

It is easily seen that $d(A,B)=0$, $A_0=\{(0,0)\}$ and $B_0=\{(0,0)\}$.

Since $d(A,B)=0$ then the pair (A,B) has the P-property.

Let $T\colon A\to B$ the mapping defined by

$$T(x,0)=(-\arctan x,0).$$

Obviously, $T(A_0)\subset B_0$.

Now, we check that T satisfies the contractive condition of Theorem 3.1.

In fact, for $(x,0),(y,0)\in A$, we have

$$d\left(T(x,0),T(y,0)\right)=d\left((-\arctan x,0),(-\arctan y,0)\right)=\left|\arctan x-\arctan y\right|.$$

We claim that $|\arctan x-\arctan y|\leq\arctan|x-y|$.

In fact, suppose that $x>y$(the same reasoning works for $y>x$).

Put $\arctan x=\alpha$ and $\arctan y=\beta$ (notice that $\alpha>\beta$ since the function $\arctan t$ for $t\geq 0$ is strictly increasing).

Now, since

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta},$$

and, since $\alpha,\beta\in\left[0,\frac{\pi}{2}\right)$, we have that $\tan\alpha,\tan\beta\in[0,\infty)$. Thus, from the last expression it follows that

$$\tan(\alpha-\beta)\leq\tan\alpha-\tan\beta.$$

Applying ϕ to the last inequality (where $\phi(t)=\arctan t$) and, by using the nondecreasing character of ϕ, we infer that

$$\alpha-\beta\leq\arctan(\tan\alpha-\tan\beta).$$

This means that

$$\arctan x-\arctan y=\alpha-\beta\leq\arctan(x-y).$$

This proves our claim.

Therefore, we get, for $(x,0),(y,0)\in A$

$$\begin{aligned}
& d\left(T(x,0),T(y,0)\right)\\
=\ & |\arctan x-\arctan y|\\
\leq\ & \arctan|x-y|\\
=\ & \arctan\left(d\left((x,0),(y,0)\right)\right)\\
=\ & d\left((x,0),(y,0)\right)-\left(d\left((x,0),(y,0)\right)-\arctan\left(d\left((x,0),(y,0)\right)\right)\right).
\end{aligned}$$

In our case, $\psi = 1_{[0,\infty)}$(identity mapping) and ϕ is given by $\phi(t) = t - \arctan t$. It is easily seen that ϕ is nondecreasing and $\phi(t) = 0$ if and only if $t = 0$. Obviously, T is continuous, and satisfies assumptions of Theorem 3.1.

Notice that $(0,0) \in A$ is the unique best proximity point of T, since

$$d\left((0,0), T(0,0)\right) = 0 = d(A,B).$$

Therefore, the conclusion of Theorem 3.1 is satisfied, but B is not a closed subset.

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