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Characterization of Weakly Efficient Solutions for Nonlinear Multiobjective Programming Problems. Duality*

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Convexity and generalized convexity play a central role in mathematical programming for duality results and in order to characterize the solutions set. In this paper, taking in mind Craven's notion of K-invexity function (when K is a cone in $\mathbb{R}^n$) and Martin's notion of Karush-Kuhn-Tucker invexity (hereafter KKT-invexity), we define new notions of generalized convexity for a multiobjective problem with conic constraints. These new notions are both necessary and sufficient to ensure every Karush-Kuhn-Tucker point is a solution. The study of the solutions is also done through the solutions of an associated scalar problem. A Mond-Weir type dual problem is formulated and weak and strong duality results are provided. The notions and results that exist in the literature up to now are particular instances of the ones presented here.

Keywords: Generalized convexity, KKT-invexity, optimality conditions, duality

1. Introduction

Consider the following general optimization problem with conic constraints:

$$(CP) \quad \begin{cases} \text{Min} & \Phi(x) \\ \text{s.t.} & x \in D. \end{cases}$$

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Let X, Y and V be normed linear spaces, $D = \{x \in S \subset X / \ - F(x) \in K\}$. Let K and Q be closed convex cones in Y and V respectively, with $\operatorname{int} K \neq \emptyset$ and $\operatorname{int} Q \neq \emptyset$, S be an open subset of X, and $\Phi : S \subset X \longrightarrow V$, $F : S \subset X \longrightarrow Y$ be called the objective function and the constraint mapping, respectively. We denote $\tilde{K} = Q \times K \subset V \times Y$ and $f \equiv (\Phi, F)$. We observe that $\tilde{K}$ is a convex cone in $V \times Y$. Throughout this work, we assume that Φ and F are Fréchet differentiable vector functions.

Several important classes of optimization problems can be modeled in the above, rather abstract, framework: nonlinear, semi-definite and semi-infinite programming, variational problems in mechanics; and composite and min-max optimization problems (see [1]). The non-differentiable case has been studied by El Abdouni and Thibault [9] and Brandão et al. [2]. Under differentiability, Dos Santos et al. [8] and Suneja et al. [21] have studied the problem, but the solutions set has not been characterized using a generalized convexity notion.

Our notation is fairly standard. If Σ is a topological linear space, then Σ^* denotes its (topologically) dual space and $\langle \cdot, \cdot \rangle$ is the pairing of elements in Σ^* and Σ , i.e., $\langle \sigma^*, \sigma \rangle$ is the value of the linear functional $\sigma^* \in \Sigma^*$ on $\sigma \in \Sigma$. For a cone C in Σ, the positive dual cone (sometimes also referred to as the polar cone) of C is $C^* = \{\sigma^* \in \Sigma^* / \langle \sigma^*, \sigma \rangle \ \geq 0 \ \forall \sigma \in C\}$. If Υ and Σ are topological linear spaces and $\Lambda : \Upsilon \to \Sigma$ is a continuous linear operator, then $\Lambda^* : \Sigma^* \to \Upsilon^*$ denotes the adjoint operator of Λ. The interior of a set Ω (in appropriate topology) is denoted by $\operatorname{int} \Omega$.

Let us consider the following partial order in Y and V:

$$\begin{aligned}
u, \omega \in Y, &\quad u \preceq_Y \omega \Leftrightarrow \omega - u \in K, \\
u, \omega \in V, &\quad u \preceq_V \omega \Leftrightarrow \omega - u \in Q, \\
u, \omega \in Y, &\quad u \prec_Y \omega \Leftrightarrow \omega - u \in \operatorname{int} K, \\
u, \omega \in V, &\quad u \prec_V \omega \Leftrightarrow \omega - u \in \operatorname{int} Q.
\end{aligned}$$

$u \npreceq_Y \omega$ is the negation of $u \preceq_Y \omega$ and $u \npreceq_V \omega$ is the negation of $u \preceq_V \omega$. In the same form, $u \nprec_Y \omega$ and $u \nprec_V$ are the negation of $u \prec_Y$ and $u \prec_V$, respectively.

The weakly efficient solution is an important concept in mathematical modeling, economics, decision theory, optimal control and game theory.

Definition 1.1. $u \in D$ is said to be a weakly efficient solution of (CP) (hereafter WES), if there does not exist $\omega \in D$, such that $\Phi(\omega) \prec_V \Phi(u)$.

The study of the solutions of an optimization problem may be approached from two aspects: one, trying to relate them with the solutions of the scalar problems, whose resolution has been studied extensively and another, trying to locate conditions which are easier to deal with computationally and which guarantee efficiency. As much in one case as in the other, the convexity concept plays an important role as a fundamental condition in obtaining the desired results.

In the past few years, this kind of problems have been extensively studied, both in the differentiable and non-differentiable case. For instance, if $X = \mathbb{R}^n, V = \mathbb{R}^p$ and $Y = \mathbb{R}^m$ with $T = \mathbb{R}^p_+$ and $K = \mathbb{R}^m_+$ it was studied among others, by Clarke [3], Craven [6], Minami [14], Mishra and Mukherjee [15], Osuna-Gómez et al. [16]–[19]; and with cone constraints in $\mathbb{R}^m$, by Suneja et al. [22], [20], among others, in relation to optimality conditions and characterization of weakly efficient solutions by scalarization. The infinite dimensional case was considered by El Abdouni and Thibault [9], Dos Santos et al. [8], Brandão et al. [2], Suneja et al. [21] and references therein, among others.

Many authors have studied sufficient and necessary optimality conditions of the Karush-Kuhn-Tucker type (hereafter KKT conditions) involving weakly efficient solutions for a multiobjective programming problem. But, as it is well known, the KKT conditions are necessary for optimality and are sufficient if the problem is convex. In a pioneering work, Hanson [11] defined invex functions, which allow the use of the KKT conditions as sufficient conditions for optimality in constrained optimization problems. The term invex was coined by Craven in [5]. Later, Martin [13] considered a constrained optimization problem with inequality-type constraints and he established that it is not possible to infer (objective and constraints)-invexity from the equivalence of minimum points and KKT points. So he defined a weaker invexity notion, called KKT-invexity, which is both necessary and sufficient to establish the KKT type optimality conditions in scalar optimization.

Craven [5] gave the notion of K-invex function (when K is a cone in $\mathbb{R}^n$). Combining Martin and Craven's ideas we define a new concept of generalized convexity that is both necessary and sufficient to guarantee the characterization of weakly efficient solutions set basing on KKT-type optimality conditions. Then, with Osuna-Gómez's ideas [18] in mind we give a characterization of weakly efficient solutions by scalarization too. Moreover, we formulate a Mond-Weir type dual problem for (CP) and get weak and strong duality results.

2. Preliminaries

The notions of invex and pseudoinvex vector functions can be generalized to the case that the functions are defined between abstract spaces in the following form:

Definition 2.1. Let $\Phi : S \subset X \to V$ be a Fréchet differentiable function on $S \subset X$, open, convex and nonempty in X and, let $Q \subset V$ be a closed, convex cone with nonempty interior:

a) Φ is said to be invex at $u \in S$, if there exists $\eta : S \times S \to X$ such that for all $\omega \in S$ it is verified that

$$\Phi'(u)\eta(\omega, u) \preceq_V \Phi(\omega) - \Phi(u) \Leftrightarrow \Phi(\omega) - \Phi(u) - \Phi'(u)\eta(\omega, u) \in Q.$$

Φ is said to be invex on S, if it is invex at every $u \in S$.

b) Φ is said to be pseudoinvex at $u \in S$, if there exists $\eta : S \times S \to X$ such that for all $\omega \in S$ it is verified that

$$\begin{cases} \text{if } \ \Phi(\omega) - \Phi(u) \prec_V 0 \Rightarrow \Phi'(u)\eta(\omega, u) \prec_V 0, \\ \text{or} \\ \text{if } \ -\Phi'(u)\eta(\omega, u) \preceq_V 0 \Rightarrow \Phi(u) - \Phi(\omega) \preceq_V 0, \end{cases}$$

or in an equivalent form:

$$\begin{cases} \text{if } \Phi(u) - \Phi(\omega) \in \operatorname{int} Q \Rightarrow -\Phi'(u)\eta(\omega, u) \in \operatorname{int} Q, \\ \text{or} \\ \text{if } \Phi'(u)\eta(\omega, u) \in Q \Rightarrow \Phi(\omega) - \Phi(u) \in Q. \end{cases}$$

Φ is said to be pseudoinvex on S if it is pseudoinvex at every $u \in S$.

From the last definition it is immediate that every invex vector function is a pseudoinvex vector function with respect to the same η.

Further, for every $\alpha^* \in Q^*$, it is easily seen that $\alpha^* \circ \Phi$ is invex at $u \in S$ with the same η whenever Φ is invex.

Let us recall the following results from [7] and from [12] respectively, that we will use in the following sections:

Lemma 2.2. *Let $Q \subset V$ be a convex cone with $\operatorname{int} Q \neq \emptyset$, and let $p \in Q^* \setminus \{0\}$. Then $\langle p, s\rangle > 0$, for all $s \in \operatorname{int} Q$.*

Lemma 2.3. *Let X, Y and V be normed vector spaces and $\Lambda \in \mathcal{L}(X, V)$, $M \in \mathcal{L}(X, Y)$. Let $T \subset V$ and $Q \subset Y$ be convex cones with $\operatorname{int} T \neq \emptyset$, and $s \in -T$, $b \in -Q$. Let the cone $[M, b]^*(Q^*)$ be weak* closed. Then the system* $\begin{cases} Mx + b \in -Q \\ \Lambda x + s \in -\operatorname{int} T \end{cases}$

has no solution $\Leftrightarrow \exists\ \lambda \in T^* \setminus \{0\},\ \tau \in Q^*$ *solution of* $\begin{cases} \lambda\Lambda + \tau M = 0 \\ \langle \lambda, s\rangle = 0 \\ \langle \tau, b\rangle = 0. \end{cases}$

We consider that (CP) is a regular problem at $u \in D$, that is, its constraints satisfy a constraint qualification. We consider Slater-type constraint qualification.

Definition 2.4. The constraints of (CP) satisfy the Slater constraint qualification at $u \in D$ if $-F(u) \in \operatorname{int} K$.

Definition 2.5. Recalling the notation $\tilde{K} := Q \times K$ in the introduction, a feasible point u is said to be a Karush-Kuhn-Tucker point of (CP) (hereafter KKTP), if there exists $\lambda^* = (\alpha^*, \theta^*) \in \tilde{K}^*$, $\alpha^* \neq 0$ such that

$$\begin{cases} [\Phi'(u)]^*\alpha^* + [F'(u)]^*\theta^* = 0, \\ \langle \theta^*, F(u)\rangle = 0. \end{cases}$$

The first- order necessary optimality condition for (CP) is:

Theorem 2.6 ([4]). *If $u \in D$ is a WES of (CP) and the constraints of the problem satisfy the Slater constraint qualification at $u \in D$, then u is a KKTP of (CP).*

3. Sufficient optimality condition

The last necessary optimality condition is not sufficient. We need to consider the invexity of the vector function $f \equiv (\Phi, F)$ to get a sufficient optimality condition for (CP).

Definition 3.1. With $\tilde{K} := Q \times K$ as in the previous sections, we observe that $f \equiv (\Phi, F)$ is $\tilde{K}$-invex at $u \in S$, if there exists $\eta : S \times S \to X$ such that

$$f(\omega) - f(u) - f'(u)\eta(\omega, u) \in \tilde{K},$$

for all $\omega \in S$. In an equivalent form:

$$\begin{cases} \Phi(\omega) - \Phi(u) - \Phi'(u)\eta(\omega, u) \in Q, \\ F(\omega) - F(u) - F'(u)\eta(\omega, u) \in K. \end{cases}$$

This means that we have the same mapping $\eta(\cdot, \cdot)$ for defining the invexity of both Φ and F.

f is a $\tilde{K}$-invex function on $D \subseteq S$ if it is a $\tilde{K}$-invex function at every $u \in D \subseteq S$.

Observation 3.2. If $Q = \mathbb{R}_+ \subset V = \mathbb{R}$, that is, we have the scalar case, the definition of $\tilde{K}$-invex function given in [12] is a particular case of Definition 3.1 given here.

So the sufficient condition of optimality for (CP) is:

Proposition 3.3. *If f is a $\tilde{K}$-invex function on D, every KKTP of (CP) is a WES of (CP).*

Proof. Let $u \in D$ be a KKTP of (CP), then there exist $\alpha^* \in Q^* \setminus \{0\}$, and $\theta^* \in K^*$, such that

$$[\Phi'(u)]^*\alpha^* + [F'(u)]^*\theta^* = 0, \tag{1}$$

$$\langle \theta^*, F(u) \rangle = 0. \tag{2}$$

Since f is a $\tilde{K}$-invex at $u \in D$, there exists $\eta(\omega, u) \in X$ such that for all $\omega \in D$:

$$\begin{cases} \Phi(\omega) - \Phi(u) - \Phi'(u)\eta(\omega, u) \in Q, \\ F(\omega) - F(u) - F'(u)\eta(\omega, u) \in K. \end{cases}$$

Taking (1) and $\eta(\omega, u) \in X$, we have that

$$\langle \alpha^*, \Phi'(u)\eta(\omega, u) \rangle + \langle \theta^*, F'(u)\eta(\omega, u) \rangle = 0$$

$$\langle \theta^*, F'(u)\eta(\omega, u) \rangle = -\langle \alpha^*, \Phi'(u)\eta(\omega, u) \rangle \tag{3}$$

Let us argue by contradiction. Suppose that $u \in D$ is not a WES of (CP), then there exists $\omega \in D$, such that

$$\Phi(u) - \Phi(\omega) \in \operatorname{int} Q.$$

Using that Φ is an invex vector function at u (by hypothesis f is $\tilde{K}$-invex at u), it is a pseudoinvex vector function at u and then $-\Phi'(u)\eta(\omega, u) \in \operatorname{int} Q$.

By Lemma 2.2, as $\alpha^* \in Q^* \setminus \{0\}$,

$$-\langle \alpha^*, \Phi'(u)\eta(\omega, u)\rangle > 0,$$

and hence from (3) we obtain

$$\langle \theta^*, F'(u)\eta(\omega, u)\rangle > 0. \tag{4}$$

Since f is a $\tilde{K}$-invex function at $u \in D$, for all $\omega \in D$, in particular for the one that exists because u is not a WES for (CP), it follows that

$$F(\omega) - F(u) - F'(u)\eta(\omega, u) \in K.$$

As $\theta^* \in K^*$: $\langle \theta^*, F(\omega) - F(u) - F'(u)\eta(\omega, u)\rangle \geq 0$, that is, $\langle \theta^*, F(\omega)\rangle - \langle \theta^*, F(u)\rangle + \langle \theta^*, -F'(u)\eta(\omega, u)\rangle \geq 0$.

Using (2) we have that $\langle \theta^*, -F'(u)\eta(\omega, u)\rangle \geq \langle \theta^*, -F(\omega)\rangle \geq 0$, because $w \in D$. Hence $\langle \theta^*, F'(u)\eta(\omega, u)\rangle \leq 0$ holds contradicting (4).

So u is a WES of (CP). □

4. Scalarization Results

A classic technique to solve an optimization problem is to solve a scalar problem associated with it. Let us define it for our case:

Definition 4.1. The weighting scalar problem associated with (CP) is

$$(PC)_{\alpha^*} \quad \begin{cases} \text{Min} & \alpha^* \circ \Phi(x) \\ \text{s.t.} & x \in D \end{cases} \tag{5}$$

with $\alpha^* \in Q^* \setminus \{0\}$.

Observation 4.2. If $X = \mathbb{R}^n,\ Q = \mathbb{R}^p_+ \subset V = \mathbb{R}^p,\ K = \mathbb{R}^m_+ \subset Y = \mathbb{R}^m$, the definition of weighting scalar problem associated with a vector problem with a finite number of inequality-type constraints, given by Osuna-Gómez et al. [18], is a particular case of Definition 4.1.

Let us prove that $\tilde{K}$-invexity is a necessary condition for optimization problems, in order to every KKTP is an optimal solution of the associated weighting scalar problem.

Theorem 4.3. *If f is a $\tilde{K}$-invex function on D, every KKTP of (CP) solves an associated weighting scalar problem $(PC)_{\alpha^*}$, $\alpha^* \in Q^* \setminus \{0\}$.*

Proof. If $u \in D$ is a KKTP of (CP), there exist $\alpha^* \in Q^* \setminus \{0\}$ and $\theta^* \in K^*$, such that

$$\begin{cases} [\Phi'(u)]^*\alpha^* + [F'(u)]^*\theta^* = 0, \\ \langle \theta^*, F(u) \rangle = 0. \end{cases} \tag{6}$$

Let us prove that if $\lambda^* = (\alpha^*, \theta^*) \in \tilde{K}^*,\ \alpha^* \neq 0$, then:

$$\langle \lambda^*, f(\omega) \rangle - \langle \lambda^*, f(u) \rangle \leq \langle \alpha^*, \Phi(\omega) \rangle - \langle \alpha^*, \Phi(u) \rangle$$

Indeed,

$$\begin{aligned} & \langle \lambda^*, f(\omega) \rangle - \langle \lambda^*, f(u) \rangle \\ = {} & \langle \lambda^*, f(\omega) - f(u) \rangle \\ = {} & \langle (\alpha^*, \theta^*), (\Phi, F)(\omega) - (\Phi, F)(u) \rangle \\ = {} & \langle \alpha^*, \Phi(\omega) - \Phi(u) \rangle + \langle \theta^*, F(\omega) - F(u) \rangle \\ = {} & \langle \alpha^*, \Phi(\omega) \rangle - \langle \alpha^*, \Phi(u) \rangle + \langle \theta^*, F(\omega) \rangle - \langle \theta^*, F(u) \rangle \\ \leq {} & \langle \alpha^*, \Phi(\omega) \rangle - \langle \alpha^*, \Phi(u) \rangle, \end{aligned}$$

because $\langle \theta^*, F(u) \rangle = 0$ by the second inequality in (6) and $\langle \theta^*, F(\omega) \rangle \leq 0$ according to $\theta^* \in K^*$ and $-F(\omega) \in K$.

So

$$\begin{aligned} & \langle \alpha^*, \Phi(\omega) \rangle - \langle \alpha^*, \Phi(u) \rangle \\ \geq {} & \langle \lambda^*, f(\omega) \rangle - \langle \lambda^*, f(u) \rangle \\ \geq {} & \langle \lambda^* \circ f'(u), \eta(\omega, u) \rangle \\ = {} & \langle [\Phi'(u)]^*\alpha^* + [F'(u)]^*\theta^*, \eta(\omega, u) \rangle = 0, \end{aligned}$$

the second inequality being due to the invexity of f with respect to η and the last equality to the first equation of (6).

Therefore, u solves a weighting scalar problem associated with (CP). □

The next proposition is a direct consequence of the theorem above since a point $u \in D$ is a KKTP of (CP) whenever it is a WES of (CP) and satisfies the Salter constraint qualification (see Theorem 2.6). The proposition provides, for (CP), a connexion between the last definitions.

Proposition 4.4. *If f is a $\tilde{K}$-invex function at $u \in D$, u is a WES of (CP), satisfying the Slater constraint qualification at u, then u solves a weighting scalar problem $(PC)_{\alpha^*}$, $\alpha^* \in Q^* \setminus \{0\}$ associated with (CP).*

The following result is well-known and easy to prove. It will be used in the next section.

Theorem 4.5. *If $u \in D$ is a solution of $(PC)_{\alpha^*}$, $\alpha^* \in Q^* \setminus \{0\}$, then u is a WES of (CP).*

Remark 4.6. As Martin showed in [13] and we showed in [12], there is some obvious surplus fat in Definition 3.1. Some relaxations can be considered in that definition due to the complementary slackness property of a KKTP and the fact that proofs concern only feasible points u and w. These relaxations lead to the following weaker notion of generalized convexity which is both necessary and sufficient for optimality.

Definition 4.7. For $\tilde{K} = Q \times K$ (as in the previous sections) the vector function $f \equiv (\Phi, F)$ is said to be a $\tilde{K}$KKT-invex function at $u \in S$, if there exists $\eta : S \times S \to X$ such that, for all $\omega \in S$,

$$\begin{cases} \Phi(\omega) - \Phi(u) - \Phi'(u)\eta(\omega, u) \in Q, \\ -F'(u)\eta(\omega, u) \in K. \end{cases}$$

The vector function f is a $\tilde{K}KKT$-invex on $D \subseteq S$ if it is a $\tilde{K}KKT$-invex at every $u \in D \subseteq S$.

5. Characterization of weakly efficient solutions

Let us give a characterization of $\tilde{K}KKT$-invex function, where the KKTP, the WES and the solutions of the weighting scalar problems coincide.

Theorem 5.1. *Suppose that for each $u \in D$ that is a KKTP of (CP) the cone $[F'(u) \;\; 0, F(u)]^*(K^*)$ is weak* closed. Then every KKTP is a WES of (CP) and solves an associated weighting scalar problem $(PC)_{\alpha^*}$, $\alpha^* \in Q^* \setminus \{0\}$ if and only if f is a $\tilde{K}KKT$-invex function on D.*

Proof. Suppose that every $u \in D$ that is a KKTP is a WES of (CP) and solves a weighting scalar problem $(PC)_{\alpha^*}$, $\alpha^* \in Q^* \setminus \{0\}$. Then for any $\theta^* \in K^*$ and $u, \omega \in D$, the following system has no solution $\delta \in \mathbb{R}_+ \setminus \{0\}$:

$$\begin{cases} [\Phi'(u)]^*\alpha^* + [F'(u)]^*\theta^* = 0, \\ \langle \theta^*, F(u) \rangle = 0, \\ \langle \alpha^*, \Phi(\omega) - \Phi(u) \rangle + \delta = 0, \end{cases}$$

with $\theta^* \in K^*$, $\forall\, u,\ \omega \in D$.

Equivalently the system

$$\begin{cases} \begin{bmatrix} \delta & \alpha^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \Phi'(u) & \Phi(\omega) - \Phi(u) \end{bmatrix} + \theta^* \begin{bmatrix} F'(u) & 0 \end{bmatrix} = 0, \\ \langle \theta^*, F(u) \rangle = 0, \end{cases}$$

has no solution $\delta \in \mathbb{R}_+ \setminus \{0\}$.

Since the cone $[F'(u) \quad 0, F(u)]^*(K^*)$ is weak* closed by hypothesis, applying Lemma 2.3 and taking $M \equiv [F'(u) \ \ 0] \in \mathcal{L}(X\times\mathbb{R}, Y)$, $\Lambda \equiv \begin{bmatrix} 0 & 1 \\ \Phi'(u) & \Phi(\omega)-\Phi(u)\end{bmatrix} \in \mathcal{L}(\mathbb{R}_+ \times X, \mathbb{R}_+ \times \mathbb{R})$, $Q \subset V,\ K \subset Y,\ \tau = \theta^* \in K^*,\ \lambda = (\delta, \alpha^*) \in (\mathbb{R}_+ \setminus \{0\}) \times (Q^* \setminus \{0\}) = \tilde{Q} \setminus \{0\} \subset \mathbb{R}_+ \times \mathbb{R},\ b = F(u) \in -K,\ s = (0,0) \in (\mathbb{R}_+ \times \mathbb{R})$, there exists $[\eta(u), \xi(u)] \in X \times \mathbb{R}$ such that

$$\begin{cases} -[\eta(u), \xi(u)] \begin{bmatrix} F'(u) \\ 0 \end{bmatrix} - F(u) \in K, \\ -[\eta(u), \xi(u)] \begin{bmatrix} 0 & \Phi'(u) \\ 1 & \Phi(\omega)-\Phi(u) \end{bmatrix} \in \operatorname{int}(\mathbb{R}_+ \times Q), \end{cases}$$

that is,

$$\begin{cases} -F'(u)\eta(u) - F(u) \in K, \\ -\xi(u) \in (0, +\infty),\ \text{ i.e. } \xi(u) < 0, \\ -\Phi'(u)\eta(u) - \xi(u)(\Phi(\omega) - \Phi(u)) \in \operatorname{int} Q. \end{cases}$$

If $-F'(u)\eta(u) - F(u) \in K \Rightarrow \langle \theta^*, -F'(u)\eta(u) - F(u)\rangle \geq 0$

$$\begin{aligned} &\langle \theta^*, -F'(u)\eta(u) - F(u)\rangle \\ &= \langle \theta^*, -F'(u)\eta(u)\rangle - \langle \theta^*, F(u)\rangle = \langle \theta^*, -F'(u)\eta(u)\rangle \geq 0 \end{aligned}$$

since u is a KKTP. Then $-F'(u)\eta(u) \in K$.

Therefore there exists $[\eta(u), \xi(u)] \in X \times \mathbb{R}$ such that

$$\begin{cases} -F'(u)\eta(u) \in K, \\ \xi(u) < 0, \\ -\Phi'(u)\eta(u) - \xi(u)(\Phi(\omega) - \Phi(u)) \in \operatorname{int} Q. \end{cases}$$

Dividing by $-\xi(u) > 0$:

$$\begin{cases} -F'(u)\left(\dfrac{-\eta(u)}{\xi(u)}\right) \in K, \\ -\Phi'(u)\left(\dfrac{-\eta(u)}{\xi(u)}\right) + (\Phi(\omega) - \Phi(u)) \in \operatorname{int} Q, \end{cases}$$

and taking $\eta(\omega, u) := -\frac{\eta(u)}{\xi(u)} \in X,\ \forall\, \omega \in D$ we have that:

$$\begin{cases} \Phi(\omega) - \Phi(u) - \Phi'(u)\eta(\omega, u) \in \operatorname{int} Q \subset Q, \\ -F'(u)\eta(\omega, u) \in K, \end{cases}$$

So f is a $\tilde{K}KKT$-invex function on D, because the latter properties hold true for all $u \in D$.

Conversely, suppose that f is a $\tilde{K}$KKT-invex function on D and let us show that every KKTP is a WES for (CVP) and solves an associated weighting scalar problem $(PC)_{\alpha^*}$, $\alpha^* \in Q^* \setminus \{0\}$.

Let u be a KKTP of (CP). There exist $\alpha^* \in Q^* \setminus \{0\}$ and $\theta^* \in K^*$, such that

$$[\Phi'(u)]^* \alpha^* + [F'(u)]^* \theta^* = 0, \tag{7}$$

$$\langle \theta^*, F(u) \rangle = 0. \tag{8}$$

Since f is a $\tilde{K}$KKT-invex function at $u \in D$, there exists $\eta(\omega, u) \in X$ such that for all $\omega \in D$:

$$\begin{cases} \Phi(\omega) - \Phi(u) - \Phi'(u)\eta(\omega, u) \in Q, \\ -F'(u)\eta(\omega, u) \in K. \end{cases}$$

Then

$$\langle \alpha^*, \Phi(\omega) - \Phi(u) - \Phi'(u)\eta(\omega, u) \rangle \geq 0,$$

and

$$\langle \theta^*, -F'(u)\eta(\omega, u) \rangle \geq 0.$$

Summing them, it follows

$$\langle \alpha^*, \Phi(\omega) - \Phi(u) \rangle + \langle \alpha^*, -\Phi'(u)\eta(\omega, u) \rangle + \langle \theta^*, -F'(u)\eta(\omega, u) \rangle \geq 0,$$

or equivalently

$$\langle \alpha^*, \Phi(\omega) - \Phi(u) \rangle - \langle [\Phi'(u)]^* \alpha^* + [F'(u)]^* \theta^*, \eta(\omega, u) \rangle \geq 0.$$

The latter inequality and (7) yield for all $u \in D$,

$$\langle \alpha^*, \Phi(\omega) \rangle \geq \langle \alpha^*, \Phi(u) \rangle.$$

Therefore u solves a weighting scalar problem $(PC)_{\alpha^*}$, $\alpha^* \in Q^* \setminus \{0\}$.

It remains to show that u is a WES of (CP). Suppose, contrary to the result, that $u \in D$ is not a WES of (CP), then there exists $\omega_0 \in D$, such that

$$\Phi(u) - \Phi(\omega_0) \in \operatorname{int} Q.$$

Since Φ is an invex vector function at u (by hypothesis f is a $\tilde{K}$KKT-invex at u), it is pseudoinvex at u and then $-\Phi'(u)\eta(\omega_0, u) \in \operatorname{int} Q$.

By Lemma 2.2, with $\alpha^* \in Q^* \setminus \{0\}$, we obtain $-\langle \alpha^*, \Phi'(u)\eta(\omega, u) \rangle > 0$. Combining this and (7) we obtain

$$\langle \theta^*, F'(u)\eta(\omega, u) \rangle > 0. \tag{9}$$

On the other hand, since f is $\tilde{K}$KKT-invex at $u \in D$, $-F'(u)\eta(\omega, u) \in K$. From this and the inclusion $\theta^* \in K^*$ we deduce $\langle \theta^*, F'(u)\eta(\omega, u) \rangle \leq 0$, contradicting (9).

Therefore u is a WES of (CP). $\square$

Some relations proved in Theorem 5.1 remain valid even under weak hypotheses on f. For that, we introduce the definition of $\tilde{K}$KKT-pseudoinvex function.

Definition 5.2. For $\tilde{K} = Q \times K$, the vector function $f \equiv (\Phi, F)$ is said to be a $\tilde{K}$KKT-pseudoinvex function at $u \in S$, if there exists $\eta : S \times S \longrightarrow X$ such that for all $\omega \in S$,

$$\begin{cases} \begin{cases} \text{if } \Phi(u) - \Phi(\omega) \in \operatorname{int} Q \Rightarrow -\Phi'(u)\eta(\omega, u) \in \operatorname{int} Q \\ \text{or} \\ \text{if } \Phi'(u)\eta(\omega, u) \in Q \Rightarrow \Phi(\omega) - \Phi(u) \in Q. \end{cases} \\ -F'(u)\eta(\omega, u) \in K. \end{cases}$$

If f is a $\tilde{K}$KKT-pseudoinvex function at every $u \in D \subseteq S$, it is a $\tilde{K}$KKT-pseudoinvex function on $D \subseteq S$.

Obviously, any invex function being pseudoinvex, we have:

Observation 5.3. If f is a $\tilde{K}$KKT-invex function at $u \in D$, then f is a $\tilde{K}$KKT-pseudoinvex function at the same point.

Theorem 5.4. *Suppose that for each $u \in D$ that is a KKTP of (CP) the cone $[F'(u), F(u)]^*(K^*)$ is weak* closed, then every KKTP of (CP) is a WES if and only if f is a $\tilde{K}$KKT-pseudoinvex function on D.*

Proof. Let us prove that if every KKTP of (CP) is a WES of (CP), then f is a $\tilde{K}$KKT-pseudoinvex function on D.

Consider $u, \omega \in D$ solution of

$$\begin{cases} \Phi(\omega) - \Phi(u) \in -\operatorname{int} Q, \\ -F(\omega) \in K. \end{cases}$$

Then it follows that u is not a WES of (CP), and by hypothesis, it is not be a KKTP either. Therefore the system

$$\begin{cases} [\Phi'(u)]^*\alpha^* + [F'(u)]^*\theta^* = 0, \\ \langle \theta^*, F(u) \rangle = 0, \end{cases}$$

has no solution $\alpha^* \in Q^* \setminus \{0\},\ \theta^* \in K^*$.

Since the cone $[F'(u), F(u)]^*(K^*)$ is weak* closed, taking $M \equiv F'(u) \in \mathcal{L}(X, Y)$, $\Lambda \equiv \Phi'(u) \in \mathcal{L}(X, V),\ s = 0 \in -Q,\ b = F(u) \in -K$ and applying Lemma 2.3, it follows that there exists $x_0 \in X$ solution of the system in x:

$$\begin{cases} F'(u)x + F(u) \in -K, \\ \Phi'(u)x \in -\operatorname{int} Q. \end{cases} \tag{10}$$

Taking $\eta(\omega, u) := x_0$, it is enough to prove that f is a $\tilde{K}$KKT-pseudoinvex function on D with respect to $\eta(\cdot, \cdot)$. Indeed:

- if $\Phi(\omega) - \Phi(u) \in -\operatorname{int} Q$, we have to prove that $\Phi'(u)\eta(\omega, u) \in -\operatorname{int} Q$, but this holds true because $\eta(\omega, u) = x_0$, and x_0 is a solution of system (10).
- We also have $-F'(u)\eta(\omega, u) \in K$: arguing by contradiction, there exists some $\theta^* \in K^*$ such that $\langle \theta^*, -F'(u)\eta(\omega, u)\rangle < 0$. Further, since u is a KKTP by hypothesis, $\langle \theta^*, -F(u)\rangle = 0$. So $\langle \theta^*, -F(u) - F'(u)\eta(\omega, u)\rangle < 0$ which stands in contradiction to the first equation of system (10). Therefore, $-F'(u)\eta(\omega, u) \in K$.

Then f is a $\tilde{K}$KKT-pseudoinvex function on D.

Conversely, suppose that $f \equiv (\Phi, F)$ is a $\tilde{K}$KKT-pseudoinvex function at u. Fix any KKTP u of (CP) and let us prove that u is a WES of (CP).

By hypothesis there exists $\alpha^* \in Q^* \setminus \{0\}$ and $\theta^* \in K^*$, such that

$$\begin{cases} [\Phi'(u)]^*\alpha^* + [F'(u)]^*\theta^* = 0, \\ \langle \theta^*, F(u)\rangle = 0. \end{cases} \tag{11}$$

Arguing by contradiction, suppose that u is not a WES of (CP), then there exists $\omega_0 \in D$ such that

$$\Phi(u) - \Phi(\omega_0) \in \operatorname{int} Q,$$

and since Φ is a pseudoinvex function at u, then $-\Phi'(u)\eta(\omega_0, u) \in \operatorname{int} Q$.

Since $\alpha^* \in Q^* \setminus \{0\}$, by Lemma 2.2, it follows that $-\langle \alpha^*, \Phi'(u)\eta(\omega_0, u)\rangle > 0$. Combining the first equation in (11) we obtain

$$\langle \theta^*, F'(u)\eta(\omega_0, u)\rangle > 0. \tag{12}$$

On the other hand, since f is a $\tilde{K}$-pseudoinvex function at $u \in D$, we have $-F'(u)\eta(\omega_0, u) \in K$. Then $\langle \theta^*, F'(u)\eta(\omega_0, u)\rangle \le 0$ which stands in contradiction to (12).

So u is a WES of (CP). □

6. Duality

Let us define the Mond-Weir type duality problem for (CP):

$$(CD_{M-W}) \quad \begin{cases} \text{Max} \;\; \Phi(\bar{x}) \quad \text{s.t.} & [\Phi'(\bar{x})]^*\alpha^* + [F'(\bar{x})]^*\theta^* = 0, \\ & \langle \theta^*, F(\bar{x})\rangle \ge 0. \\ & \theta^* \in K^*, \;\; \alpha^* \in Q^* \setminus \{0\}. \\ & \bar{x} \in S. \end{cases}$$

Let us denote by UD_{M-W} the feasible set of (CD_{M-W}). We establish results on weak and strong duality with the appropriate generalized convexity notion on the vector function $f \equiv (\Phi, F)$.

Theorem 6.1 (Weak duality). *Let u be feasible for (CP) and $(\bar{x}, \bar{\alpha}^*, \bar{\theta}^*)$ feasible for (CD_{M-W}). Let $f \equiv (\Phi, F)$ be a $\tilde{K}$-invex function at $\bar{x} \in S$. Then*

$$\Phi(u) \not\prec_V \Phi(\bar{x}).$$

Proof. Let us suppose that there exists $u \in D$ and $(\bar{x}, \bar{\alpha}^*, \bar{\theta}^*) \in UD_{M-W}$ such that $\Phi(u) \prec_V \Phi(\bar{x})$, i.e., $\Phi(\bar{x}) - \Phi(u) \in \operatorname{int} Q$.

Since $\bar{\alpha}^* \in Q^* \setminus \{0\}$, by Lemma 2.2, we have

$$\langle \bar{\alpha}^*, \Phi(\bar{x}) - \Phi(u) \rangle > 0. \tag{13}$$

Since f is a $\tilde{K}$-invex function at $\bar{x} \in S$, there exists $\eta(u, \bar{x}) \in X$, such that

$$\begin{cases} \Phi(u) - \Phi(\bar{x}) - \Phi'(\bar{x})\eta(u, \bar{x}) \in Q, \\ F(u) - F(\bar{x}) - F'(\bar{x})\eta(u, \bar{x}) \in K. \end{cases}$$

Since $\bar{\alpha}^* \in Q^* \setminus \{0\}$ and $\bar{\theta}^* \in K^*$, we obtain:

$$\begin{aligned} &\langle \bar{\alpha}^*, \Phi(u) - \Phi(\bar{x}) - \Phi'(\bar{x})\eta(u, \bar{x}) \rangle \geq 0, \\ &\langle \bar{\theta}^*, F(u) - F(\bar{x}) - F'(\bar{x})\eta(u, \bar{x}) \rangle \geq 0. \end{aligned}$$

Adding them, it follows

$$\begin{aligned} &\langle \bar{\alpha}^*, \Phi(u) - \Phi(\bar{x}) \rangle + \langle \bar{\alpha}^*, -\Phi'(\bar{x})\eta(u, \bar{x}) \rangle \\ &\quad + \langle \bar{\theta}^*, F(u) \rangle + \langle \bar{\theta}^*, -F(\bar{x}) \rangle + \langle \bar{\theta}^*, -F'(\bar{x})\eta(u, \bar{x}) \rangle \geq 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} &\langle \bar{\alpha}^*, \Phi(u) - \Phi(\bar{x}) \rangle - \langle [\Phi'(\bar{x})]^* \bar{\alpha}^* + [F'(\bar{x})]^* \bar{\theta}^*, \eta(u, \bar{x}) \rangle \\ &\quad + \langle \bar{\theta}^*, -F(\bar{x}) \rangle + \langle \bar{\theta}^*, F(u) \rangle \geq 0. \end{aligned}$$

Due to $(\bar{x}, \bar{\alpha}^*, \bar{\theta}^*) \in UD_{M-W}$ and $u \in D$:

$$\begin{cases} [\Phi'(\bar{x})]^* \alpha^* + [F'(\bar{x})]^* \theta^* = 0, \\ \langle \theta^*, F(\bar{x}) \rangle \geq 0, \\ -F(u) \in K \Rightarrow \langle \bar{\theta}^*, -F(u) \rangle \geq 0. \end{cases}$$

Then, $\langle \bar{\alpha}^*, \Phi(u) - \Phi(\bar{x}) \rangle \geq \langle \bar{\theta}^*, -F(u) \rangle + \langle \bar{\theta}^*, F(\bar{x}) \rangle \geq 0$.

Hence, $\langle \bar{\alpha}^*, \Phi(\bar{x}) - \Phi(u) \rangle \leq 0$ holds contradicting (13). □

Observation 6.2. The Weak Duality Theorem is also true if one changes the hypothesis on f by $\tilde{K}$KKT-invexity.

Proof. Arguing in the same way that in Theorem 6.1, you get

$$\langle \bar{\alpha}^*, \Phi(\bar{x}) - \Phi(u) \rangle > 0. \tag{14}$$

If now f is $\tilde{K}$KKT-invex at $\bar{x} \in S$, then there exists $\eta(\bar{x}, u) \in X$ such that

$$\begin{cases} \Phi(u) - \Phi(\bar{x}) - \Phi'(\bar{x})\eta(u, \bar{x}) \in Q, \\ -F'(\bar{x})\eta(u, \bar{x}) \in K. \end{cases}$$

As $\bar{\alpha}^* \in Q^* \setminus \{0\}$ and $\bar{\theta}^* \in K^*$, we get:

$$\langle \bar{\alpha}^*, \Phi(u) - \Phi(\bar{x}) - \Phi'(\bar{x})\eta(u, \bar{x}) \rangle \geq 0,$$

and

$$\langle \bar{\theta}^*, -F'(\bar{x})\eta(u, \bar{x}) \rangle \geq 0.$$

Summing them, it yields

$$\langle \bar{\alpha}^*, \Phi(u) - \Phi(\bar{x}) \rangle + \langle \bar{\alpha}^*, -\Phi'(\bar{x})\eta(u, \bar{x}) \rangle + \langle \bar{\theta}^*, -F'(\bar{x})\eta(u, \bar{x}) \rangle \geq 0,$$

$$\langle \bar{\alpha}^*, \Phi(u) - \Phi(\bar{x}) \rangle - \langle [\Phi'(\bar{x})]^*\bar{\alpha}^* + [F'(\bar{x})]^*\bar{\theta}^*, \eta(u, \bar{x}) \rangle \geq 0.$$

Since $(\bar{x}, \bar{\alpha}^*, \bar{\theta}^*) \in UD_{M-W}$, then $[\Phi'(\bar{x})]^*\bar{\alpha}^* + [F'(\bar{x})]^*\bar{\theta}^* = 0$.

Hence $\langle \bar{\alpha}^*, \Phi(u) - \Phi(\bar{x}) \rangle \geq 0$ and $\langle \bar{\alpha}^*, \Phi(\bar{x}) - \Phi(u) \rangle \leq 0$ holds, which stands in contradiction to (14). □

Theorem 6.3 (Strong duality). *Assume that $u \in D$ is a WES of (CP) and the Slater constraint qualification is satisfied at u. If $f \equiv (\Phi, F)$ is a $\tilde{K}$-invex function at u, then there exists $(\alpha^*, \theta^*) \in \tilde{K}^*, \alpha^* \neq 0$ such that $\langle \theta^*, F(u) \rangle = 0$ and $(u, \alpha^*, \theta^*) \in UD_{M-W}$ is a WES of (CD_{M-W}) and the values of the objective functions are equal.*

Proof. If $u \in D$ is a WES of (CP) and the Slater constraint qualification is satisfied at u, using Theorem 2.6, u is a KKTP of (CP). Then there exists $\lambda^* = (\alpha^*, \theta^*) \in \tilde{K}^*, \alpha^* \neq 0$ such that

$$\begin{cases} [\Phi'(u)]^*\alpha^* + [F'(u)]^*\theta^* = 0, \\ \langle \theta^*, F(u) \rangle = 0. \end{cases}$$

Hence $(u, \bar{\alpha}^*, \bar{\theta}^*) \in UD_{M-W}$ and the values of the two objectives functions are equal.

Suppose that (u, α^*, θ^*) is not a WES of (CD_{M-W}), then there exists $(\bar{\omega}, \bar{\alpha}^*, \bar{\theta}^*) \in UD_{M-W}$ such that

$$\Phi(u) \prec_V \Phi(\bar{\omega}). \tag{15}$$

Since f is a $\tilde{K}$-invex function at $u \in D \subset S$, using Theorem 6.1, we have that $\Phi(u) \not\prec_V \Phi(\bar{\omega})$ contradicting (15). So $(u, \bar{\alpha}^*, \bar{\theta}^*)$ is a WES of (VCD_{M-W}). □

Observation 6.4. The strong duality result is also true if f is a $\tilde{K}$KKT-invex function at $u \in D$.

Observation 6.5. In the particular case one has a scalar problem with conic constraints, we have established weak and strong duality results for it.

7. Conclusions

If $X = \mathbb{R}^n,\ Q = \mathbb{R}^p_+ \subset V = \mathbb{R}^p,\ K = \mathbb{R}^m_+ \subset Y = \mathbb{R}^m$ we have a vector problem with a finite number of inequality-type constraints, that was studied by Osuna-Gómez et al. in [18]. In this case, the hypothesis on the cones $[F'(u), F(u)]^*(K^*)$ and $[F'(u)\ \ 0, F(u)]^*(K^*)$ are certain by Minkowski-Farkas' theorem [10].

- Definition of vector KKTP point, KT-invex problem and KT-pseudoinvex problem in [18] are particular instances of Definition 2.5 (because $\alpha^* \in Q^* \setminus \{0\}$ is the nonnegative condition for the multiplier of the objective function, and $\langle \theta^*, F(u) \rangle = 0$ is the complementary slackness condition), Definition 4.7 and Definition 5.2, respectively.
- All theorems about characterization of weakly efficient solutions and scalarization given in [18] are particular cases of Propositions 3.3, 4.4 and Theorems 4.3, 4.5, 5.1 and 5.4 presented here.

If $Q \subset V = \mathbb{R}^p,\ K \subset Y = \mathbb{R}^m$ we have a vector problem with conic constraints in finite dimensional spaces, studied by some authors. In particular, our paper characterizes the weakly efficient solutions, relates the solutions of the associated scalar problem and provides weak and strong duality results for the Mond-Weir dual problem. All the results are proved using generalized convexity notions that in the case of the characterization of the weakly efficient solutions are the weakest ones.

Up to now there were no generalized convexity notions and results to characterize the weakly efficient solutions for (CP). In this paper we present new generalized convexity notions and results that allow us to characterize the weakly efficient solutions in the sense that the necessary KKT-optimality condition is sufficient too and solve the problem with the solutions of a scalar problem associated. So we extend the definitions and results given in [12], [18] and in other papers in finite dimensional spaces to the general problem with conic constraints (CP), establishing weak and strong duality results.

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