

Multiple Ordered Positive Solutions of an Elliptic Problem Involving the p&q-Laplacian

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In this paper we will be concerned with questions of existence and multiplicity of positive solutions of the elliptic problem

$$(P_\lambda) \quad \begin{cases} -\operatorname{div} (\mathcal{K}(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $1 < p < \infty$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, λ is a positive real parameter, $\mathcal{K} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a C^1 -function and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous functions which changes sign. We will use variational methods.

Keywords: p&q-Laplacian, variational methods, multiplicity of positive solutions

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1. Introduction

In this paper we shall focus our attention on questions of existence, multiplicity and positivity of solutions for the following problem

$$(P_\lambda) \quad \begin{cases} -\operatorname{div} (\mathcal{K}(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $1 < p < \infty$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, λ is a positive real parameter, $\mathcal{K} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a C^1 -function and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous

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functions which changes sign. More precisely, we will suppose that the functions f and $\mathcal{K}$ enjoys the following assumptions:

(f_1) $f(0) \geq 0$ and there are $0 < a_1 < b_1 < a_2 < \dots < b_{m-1} < a_m$, zeroes of f , such that

$$\begin{cases} f \leq 0 & \text{in } (a_k, b_k) \\ f \geq 0 & \text{in } (b_k, a_{k+1}); \end{cases}$$

(f_2) $\int_{a_k}^{a_{k+1}} f(s)ds > 0$, $\forall k \in \{1, \dots, m-1\}$.

($\mathcal{K}_1$) There are real constants $0 < \alpha_0 \leq \alpha_2$, $0 < \alpha_1$, $0 \leq \alpha_3$ and $1 < p < q < \infty$ such that

$$\alpha_0 + H(\alpha_3)\alpha_1 t^{\frac{q-p}{p}} \leq \mathcal{K}(t) \leq \alpha_2 + \alpha_3 t^{\frac{q-p}{p}}, \quad \forall t \geq 0$$

where $H : [0, +\infty) \rightarrow \{0, 1\}$ is the function given by

$$H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

($\mathcal{K}_2$) If $p \geq 2$, the mapping $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by $g(t) = \mathcal{K}(t^p)t^{p-2}$ is non-decreasing. If $1 < p < 2$, the mapping $\mathcal{K} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is non-decreasing.

Setting

$$\gamma = (1 - H(\alpha_3))p + H(\alpha_3)q$$

we define $X(\Omega) = W_0^{1,\gamma}(\Omega)$ equipped with the norm

$$\|u\| = \|u\|_{1,p} + H(\alpha_3)\|u\|_{1,q}$$

where $\|u\|_{1,r}^r = \int_{\Omega} |\nabla u|^r dx$, for $r \geq 1$.

We say that $u \in X(\Omega)$ is a weak solution of problem (P_{λ}) if

$$\int_{\Omega} \mathcal{K}(|\nabla u|^p) |\nabla u|^{p-2} \nabla u \nabla v = \lambda \int_{\Omega} f(u)v, \quad \forall v \in X(\Omega).$$

We are able to enunciate our existence result:

Theorem 1.1. *Assume that conditions (f_1) – (f_2) and ($\mathcal{K}_1$) – ($\mathcal{K}_2$) hold. Then, there exists $\lambda^* > 0$, such that for each $\lambda \in (\lambda^*, +\infty)$ problem (P_{λ}) possesses at least $(m-1)$ nonnegative weak solutions u_i with*

$$u_i \in X(\Omega) \cap L^{\infty}(\Omega) \quad \text{and} \quad a_i \leq \|u_i\|_{\infty} \leq a_{i+1}, \quad \forall i \in \{1, \dots, m-1\}.$$

In the second result we prove that the condition (f_2) is still necessary for the existence of multiple solutions. More precisely, we have

Theorem 1.2. *Assume that conditions (f_1) and $(\mathcal{K}_1) - (\mathcal{K}_2)$ hold. If (P_λ) possesses a nonnegative weak solution $u \in L^\infty(\Omega)$ such that $\|u\|_\infty \in (a_i, a_{i+1}]$, then*

$$\int_{a_i}^{a_{i+1}} f(s)ds > 0,$$

for $i \in \{1, \dots, m-1\}$.

We recall that f is a nonlinear function that changes sign in $[0, +\infty)$ and the condition (f_2) says that the area between the graph of f and the x-axis, corresponding to the interval $[a_i, b_i]$, is strictly less than that corresponding to the interval $[b_i, a_{i+1}]$. This kind of area condition, at least to our knowledge, was first used by Brown-Budin [3] (1979) who proved a multiplicity result in case

$$\begin{cases} Lu = \lambda f(x, u) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega \end{cases} \quad (1)$$

where L is the operator

$$(Lu)(x) = - \sum_{i,j=1}^N \partial_i(a_{ij}(x)\partial_j u(x)) + c(x)u(x).$$

For $N \geq 1$ the authors use a combination of a variational approach and monotone iteration method in order to show the existence and multiplicity of ordered positive solutions for the problem (1). More detailed results are obtained for autonomous ordinary differential equations by using quadrature arguments. Later, in 1981, Hess [16] studied this problem, with $L = -\Delta$, by using variational methods and degree theory. In 1984, de Figueiredo [11] considered question on uniqueness of positive solution for the problem $-\Delta u = \lambda \sin u$ in bounded domains of $\mathbb{R}^N$ under Dirichlet boundary condition. In this work the author shows that this problem has no multiplicity of solution if Ω satisfies a sort of symmetry. Note that the function $\sin u$ does not enjoy the area condition (f_2) .

Still with respect to the problem (P_λ) , by considering the usual Laplacian, the authors Sweers [25], Dancer-Schmitt [8], Clément-Sweers [4], Dancer-Yan [9], Wang-Kazarinoff [27] and Liu-Wang-Shi [23] have considered questions of existence and multiplicity of positive solutions as well the necessity of condition (f_2) . For the p-Laplacian we cite Loc-Schmitt [24].

For nonlocal problems we refer Corrêa-Delgado-Suárez [5] and Dai [7]. See also Corrêa-Figueiredo [6] and Klaasen-Mitidieri [19]. In de Figueiredo [12] the author treats, besides the multiplicity of solutions, questions on boundary layer formations of the solutions u_λ as $\lambda \rightarrow +\infty$.

Moreover, problems involving operators of the type p&q-Laplacian has received special attention in the last years, not only for its mathematical interest, but also because it models situations in physics, biophysics and chemical. In fact,

this class of problems has applications in the study of a general reaction-diffusion system:

$$u_t = \operatorname{div}[D(u)\nabla u] + c(x, u), \quad (2)$$

where $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system has a wide range of application in physics and related sciences such as biophysics, plasma physics and chemical reaction design. In such applications, the function u describes a concentration, the first term on the right hand side of (2) corresponds to the diffusion with a diffusion coefficient $D(u)$, whereas the second one is the reaction and relates to source and loss process. Typically, in chemical and biological applications, the reaction term $c(x, u)$ is a polynomial of u with variable coefficients, see He-Li [15], Li-Liang [21] and Wu-Yang [28].

Just to illustrate the degree of generality of the problem (P_λ) let us consider some special cases, depending on the function $\mathcal{K}$, that are covered in this article.

Example 1.3. If $\mathcal{K} \equiv 1$ our operator is the p-Laplacian and so problem (P_λ) becomes

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

which was studied by several authors previously cited for the case $p = 2$ and by Loc-Schmitt [24] for $1 < p < N$.

Example 1.4. If $\mathcal{K}(t) = 1 + t^{\frac{q-p}{p}}$ we obtain

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

which has been studied by several authors by considering nonlinear terms distinct of ours. In particular, we cite Figueiredo [14], Alves-Figueiredo [1] and the references therein.

Example 1.5. Taking $\mathcal{K}(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ we get

$$\begin{cases} -\operatorname{div} \left(|\nabla u|^{p-2} \nabla u + \frac{|\nabla u|^{p-2} \nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}} \right) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Example 1.6. We now consider $\mathcal{K}(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ to obtain

$$\begin{cases} -\Delta_p u - \Delta_q u - \operatorname{div} \left(\frac{|\nabla u|^{p-2} \nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}} \right) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For more problems involving p&q-Laplacian type operators, see do Ó [10] and Hirano [17] and their references.

This paper is organized as follows. In Section 2 we establish some preliminary results. In Section 3, we prove the existence and multiplicity result of solution for (P_λ) using Ekeland variational principle, the Brézis-Lieb Lemma and a version of the Simon's inequality. Finally, in Section 4, we prove that the condition (f_2) is also necessary for the existence of weak solution of the problem (P_λ) .

2. Preliminary Results

In this section we establish two preliminary results that will be used in the proof of the existence theorem.

Lemma 2.1. *Let $g \in C(\mathbb{R})$ and $s_0 > 0$ be such that*

$$g(s) \geq 0 \quad \text{if } s \in (-\infty, 0),$$

$$g(s) \leq 0 \quad \text{if } s \in [s_0, \infty).$$

If $u \in X(\Omega)$ is a weak solution of

$$(P) \quad \begin{cases} -\operatorname{div} (\mathcal{K}(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \geq 0$ a.e. in Ω , $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leq s_0$.

Proof. If $u \in X(\Omega)$ we set $u^- = \max\{-u, 0\}$. It is well known that $u^- \in X(\Omega)$ and

$$\frac{\partial u^-}{\partial x_i} = \begin{cases} -\frac{\partial u}{\partial x_i} & \text{if } u < 0, \\ 0 & \text{if } u \geq 0. \end{cases}$$

If u is a weak solution of (P) we have

$$\int_{\Omega} \mathcal{K}(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla v = \int_{\Omega} g(u)v, \quad \forall v \in X(\Omega).$$

In particular, setting $v = u^-$ we obtain

$$\int_{\Omega} \mathcal{K}(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla u^- = \int_{\Omega} g(u)u^-,$$

that is,

$$\int_{u < 0} \mathcal{K}(|\nabla u|^p)|\nabla u|^p = \int_{u < 0} g(u)u$$

and so

$$\int_{\Omega} \mathcal{K}(|\nabla u|^p)|\nabla u^-|^p \leq 0.$$

As $\mathcal{K} \geq \alpha_0 > 0$ we get $\int_{\Omega} |\nabla u^-|^p = \|u^-\|_{1,p}^p \equiv 0$ and so $u^- = 0$. Thus $u = u^+ \geq 0$.

We now take $v = (u - s_0)^+ \in X(\Omega)$ which implies that

$$\frac{\partial(u - s_0)^+}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i} & \text{if } u > s_0, \\ 0 & \text{if } u \leq s_0. \end{cases}$$

So,

$$\int_{\Omega} \mathcal{K}(|\nabla u|^p) |\nabla u|^{p-2} \nabla u \nabla (u - s_0)^+ = \int_{\Omega} g(u) (u - s_0)^+$$

which yields

$$\int_{u > s_0} \mathcal{K}(|\nabla u|^p) |\nabla u|^p = \int_{u > s_0} g(u) (u - s_0) \leq 0$$

and we get

$$\int_{\Omega} \mathcal{K}(|\nabla u|^p) |\nabla (u - s_0)^+|^p \leq 0$$

and because $\mathcal{K} \geq \alpha_0 > 0$ we have

$$\int_{\Omega} |\nabla (u - s_0)^+|^p = \|(u - s_0)^+\|_{1,p}^p = 0.$$

Hence $(u - s_0)^+ \equiv 0$ and we conclude that $u \leq s_0$ a.e. in Ω . Consequently, $0 \leq \|u\|_{\infty} \leq s_0$ and the proof of the lemma is over. $\square$

We now consider, for each $k \in \{2, \dots, m\}$, the truncation of the function f given by

$$f_k(s) = \begin{cases} f(0) & \text{if } s \leq 0, \\ f(s) & \text{if } 0 \leq s \leq a_k, \\ 0 & \text{if } s > a_k. \end{cases}$$

For each $\lambda > 0$, let us consider the functional $\Phi_{k,\lambda} : X(\Omega) \rightarrow \mathbb{R}$ defined by

$$\Phi_{k,\lambda}(u) = \frac{1}{p} \int_{\Omega} \widehat{\mathcal{K}}(|\nabla u|^p) - \lambda \int_{\Omega} F_k(u)$$

where $\widehat{\mathcal{K}}(t) = \int_0^t \mathcal{K}(s) ds$ and $F_k(t) = \int_0^t f_k(s) ds$. Let us denote by $\mathbb{K}_{k,\lambda}$ the set of the critical points of $\Phi_{k,\lambda}$.

We point out that $\Phi_{k,\lambda}$ is the energy functional of the problem

$$(P_{\lambda})_k \quad \begin{cases} -\operatorname{div} (\mathcal{K}(|\nabla u|^p) |\nabla u|^{p-2} \nabla u) = \lambda f_k(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and its weak solutions are critical points of the functional $\Phi_{k,\lambda}$.

Lemma 2.2. *$u \in \mathbb{K}_{k,\lambda}$ if, and only if, u is a nonnegative weak solution of the problem $(P_\lambda)_k$ and $u \in L^\infty(\Omega)$ with $\|u\|_\infty \leq a_k$. Consequently, u is a nonnegative weak solution of the problem (P_λ) .*

Proof. It is enough to see that the definition of f_k leads us to the assumptions of the Lemma 2.1. $\square$

3. Existence Result

This section will be devoted to the proof of the existence result.

Lemma 3.1. *If $(\mathcal{K}_1)$ and $(\mathcal{K}_2)$ hold, then there exists a positive constant c_p , such that*

$$c_p|x - y|^p \leq \langle \mathcal{K}(|x|^p)|x|^{p-2}x - \mathcal{K}(|y|^p)|y|^{p-2}y, x - y \rangle, \quad \forall x, y \in \mathbb{R}^N, \quad (3)$$

when $p \geq 2$, and

$$c_p \frac{|x - y|^2}{(|x| + |y|)^{2-p}} \leq \langle \mathcal{K}(|x|^p)|x|^{p-2}x - \mathcal{K}(|y|^p)|y|^{p-2}y, x - y \rangle, \quad \forall x, y \in \mathbb{R}^N, \quad (4)$$

when $1 < p < 2$.

Proof. The case $p \geq 2$ follows of arguments used in Figueiredo [14]. Now, if $1 < p < 2$ we observe that, as in Figueiredo [14], we have

$$\begin{aligned} & \sum_{i,j=1}^N \frac{\partial}{\partial z_i} (\mathcal{K}(|z|^p)|z|^{p-2}z_j) \xi_i \xi_j \\ &= \left(\sum_{i,j=1}^N z_j \xi_j \right)^2 |z|^{p-4} [(p-2)\mathcal{K}(|z|^p) + p\mathcal{K}'(|z|^p)|z|^p] + \mathcal{K}(|z|^p)|z|^{p-2}|\xi|^2, \end{aligned}$$

for all $z, \xi \in \mathbb{R}^N$. From $(\mathcal{K}_2)$ and Schwarz's inequality, we obtain

$$\begin{aligned} & \sum_{i,j=1}^N \frac{\partial}{\partial z_i} (\mathcal{K}(|z|^p)|z|^{p-2}z_j) \xi_i \xi_j \\ & \geq - (2-p)\mathcal{K}(|z|^p)|z|^{p-2}|\xi|^2 + \mathcal{K}(|z|^p)|z|^{p-2}|\xi|^2. \end{aligned}$$

So,

$$\sum_{i,j=1}^N \frac{\partial}{\partial z_i} (\mathcal{K}(|z|^p)|z|^{p-2}z_j) \xi_i \xi_j \geq (p-1)\mathcal{K}(|z|^p)|z|^{p-2}|\xi|^2,$$

for all $z, \xi \in \mathbb{R}^N$. Choosing $\xi = x - y$ and $z = tx + (1-t)y = y + (1-t)x$ with $t \in (0, 1)$ and using $(\mathcal{K}_1)$, it follows that

$$\sum_{i,j=1}^N \frac{\partial}{\partial z_i} (\mathcal{K}(|z|^p)|z|^{p-2}z_j) \xi_i \xi_j \geq c_p \frac{|x - y|^2}{(|x| + |y|)^{2-p}}, \quad (5)$$

for all $x, y \in \mathbb{R}^N$, where $c_p = \alpha_0(p-1)$.

On the other hand,

$$\begin{aligned} & \sum_{i,j=1}^N (\mathcal{K}(|x|^p)|x|^{p-2}x_j - \mathcal{K}(|y|^p)|y|^{p-2}y_j)(x_j - y_j) \\ &= \int_0^1 \sum_{i,j=1}^N \frac{\partial}{\partial z_i} (\mathcal{K}(|z|^p)|z|^{p-2}z_j) \xi_i \xi_j, \end{aligned} \quad (6)$$

for all $x, y \in \mathbb{R}^N$. From (5) and (6), we conclude the proof. $\square$

Proposition 3.2. *Problem (P_λ) possesses a nonnegative solution for all $\lambda > 0$.*

Proof. Since f_k bounded we get

$$\begin{aligned} \Phi_{k,\lambda}(u) &= \frac{1}{p} \int_{\Omega} \widehat{\mathcal{K}}(|\nabla u|^p) - \lambda \int_{\Omega} F_k(u) \\ &\geq \frac{1}{p} \int_{\Omega} \widehat{\mathcal{K}}(|\nabla u|^p) - \lambda M_k \int_{\Omega} |u|. \end{aligned}$$

In view of the assumption $(\mathcal{K}_1)$ we have

$$\Phi_{k,\lambda}(u) \geq \frac{\alpha_0}{p} \int_{\Omega} |\nabla u|^p + \frac{H(\alpha_3)\alpha_1}{q} \int_{\Omega} |\nabla u|^q - \lambda M_k \int_{\Omega} |u|.$$

The boundedness of Ω yields

$$\Phi_{k,\lambda}(u) \geq C_1 (\|u\|_{1,p}^p + H(\alpha_3)\|u\|_{1,q}^q) - \lambda M_k \|u\|, \quad (7)$$

where $C_1 = \min \left\{ \frac{\alpha_0}{p}, \frac{\alpha_1}{q} \right\}$. To conclude the coerciveness of $\Phi_{k,\lambda}$ we should note that if $\|u\| \rightarrow +\infty$ it is enough to consider the cases:

- (i) $\|u\|_{1,p} \rightarrow +\infty$ and $\|u\|_{1,q} \rightarrow +\infty$;
- (ii) $\|u\|_{1,p}$ is bounded and $\|u\|_{1,q} \rightarrow +\infty$;

From the analysis of these two cases we conclude that $\Phi_{k,\lambda}$ is coercive. Moreover, the inequality in (7) implies that $\Phi_{k,\lambda}$ is bounded from below in $X(\Omega)$ and so we set

$$\Phi_{\infty}^k := \inf_{X(\Omega)} \Phi_{k,\lambda}.$$

Since $\Phi_{k,\lambda}$ is continuous (indeed $\Phi_{k,\lambda}$ is of C^1 -class), we have that $\Phi_{k,\lambda}$ is lower semi-continuous. These facts, combined with the boundedness from below of the functional $\Phi_{k,\lambda}$, permits us to apply the Ekeland Variational Principle (see de Figueiredo [13]) in the complete metric space $(X(\Omega), d)$ with $d(u, v) = \|u - v\|$ to conclude that there exists $(u_n) \subset X(\Omega)$ such that

$$\Phi_{k,\lambda}(u_n) \rightarrow \Phi_{\infty}^k$$

and

$$\Phi'_{k,\lambda}(u_n) \rightarrow 0 \text{ in } X(\Omega)^*.$$

Hence (u_n) is a $(PS)_{\Phi_{\infty}^k}$ sequence and the coerciveness of $\Phi_{k,\lambda}$ yields the boundedness of (u_n) in $X(\Omega)$. Since $X(\Omega)$ is a reflexive Banach space, up to a subsequence, we have

$$u_n \rightharpoonup u_{k,\lambda} \text{ in } X(\Omega). \quad (8)$$

From the compact Sobolev immersion

$$u_n \rightarrow u_{k,\lambda} \text{ in } L^s(\Omega), 1 \leq s < \gamma^*. \quad (9)$$

By a Vainberg result (see [26])

$$u_n(x) \rightarrow u_{k,\lambda}(x) \text{ a.e. in } \Omega. \quad (10)$$

Claim 3.3. $\frac{\partial u_n}{\partial x_i}(x) \rightarrow \frac{\partial u_{k,\lambda}}{\partial x_i}(x)$ a.e. in Ω for all $i \in \{1, \dots, N\}$.

Proof of the Claim 3.3. Since that $\{u_n\}$ is bounded and in view of the Lemma 3.1, there is $\xi_0 > 0$ such that

$$\begin{aligned} & \int_{\Omega} \langle \mathcal{K}(|\nabla u_n|^p) |\nabla u_n|^{p-2} \nabla u_n - \mathcal{K}(|\nabla u_{k,\lambda}|^p) |\nabla u_{k,\lambda}|^{p-2} \nabla u_{k,\lambda}, \nabla u_n - \nabla u_{k,\lambda} \rangle \\ & \geq \xi_0 \int_{\Omega} |\nabla u_n - \nabla u_{k,\lambda}|^p. \end{aligned} \quad (11)$$

From the weak convergence in (8), we get

$$\begin{aligned} & \int_{\Omega} \mathcal{K}(|\nabla u_n|^p) |\nabla u_n|^p - \int_{\Omega} \mathcal{K}(|\nabla u_n|^p) |\nabla u_n|^{p-2} \nabla u_n \nabla u_{k,\lambda} + o_n(1) \\ & \geq \xi_0 \int_{\Omega} |\nabla u_n - \nabla u_{k,\lambda}|^p \end{aligned} \quad (12)$$

The convergence in (9) yields

$$o_n(1) = -\lambda \int_{\Omega} f_k(u_n) u_n + \lambda \int_{\Omega} f_k(u_n) u_{k,\lambda} \quad (13)$$

and so, from (11) and (13),

$$0 \leq \frac{\xi_0}{4} \int_{\Omega} |\nabla u_n - \nabla u_{k,\lambda}|^p \leq \Phi'_{k,\lambda}(u_n) u_n - \Phi'_{k,\lambda}(u_n) u_{k,\lambda} = o_n(1) \quad (14)$$

because (u_n) is bounded and is a $(PS)_{\Phi_{\infty}^k}$ sequence. Invoking, again, a Vainberg result (see [26]) we conclude that $\frac{\partial u_n}{\partial x_i}(x) \rightarrow \frac{\partial u_{k,\lambda}}{\partial x_i}(x)$ a.e. in Ω for all $i \in \{1, \dots, N\}$ and the proof of the Claim 3.3 is over.

Claim 3.4. $u_{k,\lambda}$ is a global minimum point of the functional $\Phi_{k,\lambda}$ on $X(\Omega)$.

Proof of the Claim 3.4. By virtue of the continuity of $\mathcal{K}$ and the Claim 3.3, we have

$$\mathcal{K}(|\nabla u_n(x)|^p)|\nabla u_n(x)|^{p-2}\frac{\partial u_n}{\partial x_i}(x) \rightarrow \mathcal{K}(|\nabla u_{k,\lambda}(x)|^p)|\nabla u_{k,\lambda}(x)|^{p-2}\frac{\partial u_{k,\lambda}}{\partial x_i}(x) \quad (15)$$

a.e. in Ω . On the other hand, by $(\mathcal{K}_1)$

$$\begin{aligned} \left| \mathcal{K}(|\nabla u_n|^p)|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \right| &\leq \alpha_2 |\nabla u_n|^{p-2} \left| \frac{\partial u_n}{\partial x_i} \right| + \alpha_3 |\nabla u_n|^{q-2} \left| \frac{\partial u_n}{\partial x_i} \right| \\ &\leq \alpha_2 |\nabla u_n|^{p-1} + \alpha_3 |\nabla u_n|^{q-1}. \end{aligned}$$

If $\alpha_3 = 0$, then $\gamma = p$ and so

$$\int_{\Omega} \left| \mathcal{K}(|\nabla u_n|^p)|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \right|^{\frac{p}{p-1}} \leq \alpha_2 \int_{\Omega} |\nabla u_n|^p \leq K,$$

because, in this case, $\|u_n\|^p = \int_{\Omega} |\nabla u_n|^p$ and (u_n) is bounded in $X(\Omega)$.

If $\alpha_3 > 0$, then $\gamma = q$ and so

$$\begin{aligned} \int_{\Omega} \left| \mathcal{K}(|\nabla u_n|^p)|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \right|^{\frac{q}{q-1}} &\leq \int_{\Omega} [\alpha_2 |\nabla u_n|^{p-1} + \alpha_3 |\nabla u_n|^{q-1}]^{\frac{q}{q-1}} \\ &\leq K_1 \int_{\Omega} |\nabla u_n|^{\frac{q(p-1)}{q-1}} + K_2 \int_{\Omega} |\nabla u_n|^q \\ &\leq K_1 |\Omega|^{\frac{q-p}{q-1}} \|u_n\|_{1,q}^{\frac{q(p-1)}{q-1}} + K_2 \|u_n\|_{1,q}^q \\ &\leq K_1 |\Omega|^{\frac{q-p}{q-1}} K^{\frac{q(p-1)}{q-1}} + K_2 K^q, \end{aligned}$$

because (u_n) is bounded in $X(\Omega)$ and $\|u_n\| = \|u_n\|_{1,p} + \|u_n\|_{1,q}$ when $\gamma = q$. This shows that

$$\int_{\Omega} \left| \mathcal{K}(|\nabla u_n|^p)|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \right|^{\frac{\gamma}{\gamma-1}} \leq C$$

for all $n \in \mathbb{N}$ and $i \in \{1, \dots, N\}$. Consequently, by using the Brézis-Lieb Lemma [2] (see also Kavian [18, Lemma 4.6]), we obtain

$$\int_{\Omega} \mathcal{K}(|\nabla u_n|^p)|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} h \rightarrow \int_{\Omega} \mathcal{K}(|\nabla u_{k,\lambda}|^p)|\nabla u_{k,\lambda}|^{p-2}\frac{\partial u_{k,\lambda}}{\partial x_i} h, \quad \forall h \in L^{\gamma}(\Omega)$$

because $\frac{\gamma-1}{\gamma} + \frac{1}{\gamma} = 1$ and so

$$\begin{aligned} &\int_{\Omega} \mathcal{K}(|\nabla u_n|^p)|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \frac{\partial h}{\partial x_i} \\ &\rightarrow \int_{\Omega} \mathcal{K}(|\nabla u_{k,\lambda}|^p)|\nabla u_{k,\lambda}|^{p-2}\frac{\partial u_{k,\lambda}}{\partial x_i} \frac{\partial h}{\partial x_i}, \quad \forall h \in X(\Omega) \end{aligned}$$

and summing up these terms for all $i \in \{1, \dots, N\}$ we get

$$\begin{aligned} & \int_{\Omega} \mathcal{K}(|\nabla u_n|^p) |\nabla u_n|^{p-2} \nabla u_n \nabla h \\ & \rightarrow \int_{\Omega} \mathcal{K}(|\nabla u_{k,\lambda}|^p) |\nabla u_{k,\lambda}|^{p-2} \nabla u_{k,\lambda} \nabla h, \quad \forall h \in X(\Omega). \end{aligned} \quad (16)$$

In view of the compact immersions

$$\lambda \int_{\Omega} f(u_n) h \rightarrow \lambda \int_{\Omega} f(u_{k,\lambda}) h, \quad \forall h \in X(\Omega) \quad (17)$$

and by virtue of

$$o_n(1) = \Phi'_{k,\lambda}(u_n)h, \quad \forall h \in X(\Omega),$$

it follows, from (16) and (17), that

$$o_n(1) = \Phi'_{k,\lambda}(u_{k,\lambda})h + o_n(1), \quad \forall h \in X(\Omega)$$

which shows that $\Phi_{k,\lambda}(u_{k,\lambda})h = 0$ for all $h \in X(\Omega)$. Consequently $u_{k,\lambda} \in X(\Omega)$ is a weak solution of our problem. $\square$

Theorem 3.5. *For each $k \in \{2, \dots, m\}$ there is $\lambda_k > 0$ such that for all $\lambda > \lambda_k$ we have*

$$u_{k,\lambda} \notin \mathbb{K}_{k-1,\lambda} \quad (18)$$

where

$$\Phi_{k,\lambda}(u_{k,\lambda}) = \min_{v \in X(\Omega)} \Phi_{k,\lambda}(v). \quad (19)$$

Proof. We will show that there exists $\lambda_k > 0$ and $\omega \in X(\Omega)$, with $\omega \geq 0$ and $\|\omega\|_{\infty} \leq a_k$ such that

$$\Phi_{k,\lambda}(\omega) < \Phi_{k-1,\lambda}(u_{k-1,\lambda}) \quad \forall \lambda > \lambda_k \quad (20)$$

and so

$$\Phi_{k,\lambda}(u_{k,\lambda}) < \Phi_{k-1,\lambda}(u_{k-1,\lambda}) \quad \forall \lambda > \lambda_k \quad (21)$$

Invoking Lemma 2.2, the inequality (21) leads us to $u_{k,\lambda} \neq u_{k-1,\lambda}$, that is, $u_{k,\lambda}$ and $u_{k-1,\lambda}$ are two distinct solutions of the problem (P_{λ}) . From (21) we obtain

$$a_{k-1} < \|u_{k,\lambda}\|_{\infty} \leq a_k. \quad (22)$$

Indeed, if

$$0 \leq u_{k,\lambda} \leq a_{k-1} \quad (23)$$

we would have

$$\Phi_{k-1,\lambda}(u_{k-1,\lambda}) \leq \Phi_{k-1,\lambda}(u_{k,\lambda}) = \Phi_{k,\lambda}(u_{k,\lambda}) \quad (24)$$

where this last inequality follows from (23) and from the definition of F_k . Hence

$$\Phi_{k-1,\lambda}(u_{k-1,\lambda}) \leq \Phi_{k,\lambda}(u_{k,\lambda}) \quad (25)$$

which contradicts the inequality (21). Hence (22) holds true. We are going to show the existence of the function ω satisfying (20). First we note that, from assumption (f_2) we get

$$\begin{aligned} 0 < \alpha &:= F(a_k) - \max_{0 \leq s \leq a_{k-1}} F(s) \\ &= F(a_k) - F(a_{k-1}) \\ &= \int_{a_{k-1}}^{a_k} f(s) ds \end{aligned} \quad (26)$$

where $F(s) = \int_0^s f(r) dr$. Consequently, for all $0 \leq u(x) \leq a_{k-1}$ a.e. in Ω we have

$$\begin{aligned} \int_{\Omega} F(u) &= \int_{\Omega} F(a_k) - \left[\int_{\Omega} F(a_k) - \int_{\Omega} F(u) \right] \\ &\leq \int_{\Omega} F(a_k) - \int_{\Omega} \alpha \end{aligned} \quad (27)$$

$$= \int_{\Omega} F(a_k) - \alpha |\Omega|. \quad (28)$$

Given a $\delta > 0$ let us consider the open set

$$\Omega_{\delta} := \{x \in \Omega; \text{dist}(x, \partial\Omega) < \delta\}. \quad (29)$$

We now take $\omega_{\delta} \in C_c^{\infty}(\Omega)$ with $0 \leq \omega_{\delta} \leq a_k$ and $\omega_{\delta} \equiv a_k$ on $\Omega \setminus \Omega_{\delta}$. Hence,

$$\begin{aligned} \int_{\Omega} F(\omega_{\delta}) &= \int_{\Omega \setminus \Omega_{\delta}} F(a_k) - \int_{\Omega_{\delta}} F(\omega_{\delta}) \\ &= \int_{\Omega} F(a_k) - \int_{\Omega_{\delta}} [F(a_k) + F(\omega_{\delta})] \\ &\geq \int_{\Omega} F(a_k) - 2C|\Omega_{\delta}| \end{aligned} \quad (30)$$

where $C = \max_{0 \leq s \leq a_k} |F(s)|$. Consequently, for all $u \in X(\Omega)$ with $0 \leq u \leq a_{k-1}$, we have

$$\int_{\Omega} F(\omega_{\delta}) \geq \int_{\Omega} F(u) + \alpha |\Omega| - 2C|\Omega_{\delta}|. \quad (31)$$

Choosing $\bar{\delta} > 0$ such that

$$\eta = \alpha |\Omega| - 2C|\Omega_{\bar{\delta}}| > 0$$

we have

$$\begin{aligned} \Phi_{k,\lambda}(\omega_{\bar{\delta}}) - \Phi_{k-1,\lambda}(u) &= \frac{1}{p} \int_{\Omega} [\widehat{\mathcal{K}}(|\nabla \omega_{\bar{\delta}}|^p) - \widehat{\mathcal{K}}(|\nabla u|^p)] - \lambda \int_{\Omega} [F(\omega_{\bar{\delta}}) - F(u)] \\ &\leq \frac{1}{p} \int_{\Omega} \widehat{\mathcal{K}}(|\nabla \omega_{\bar{\delta}}|^p) - \lambda \eta \end{aligned} \quad (32)$$

where in the last inequality we used (31). This shows that for $\omega = \omega_{\bar{\delta}}$ and $\lambda > 0$ large enough we have

$$\Phi_{k,\lambda}(\omega) < \Phi_{k-1,\lambda}(u_{k-1,\lambda})$$

which concludes the proof of the theorem. $\square$

4. Condition (f_2) is necessary

In this section we will show that the condition (f_2) is necessary for the existence of weak solution $u \in X(\Omega) \cap L^\infty(\Omega)$ with $\|u\|_\infty \in [a_k, a_{k+1})$, we argue in the same spirit of [24]. We will consider only the case $\lambda = k = 1$ because the other cases, with slight adaptations, are proved in a similar way. Furthermore, we assume, at first, that $f(0) > 0$.

Remark 4.1. Since the application $\mathcal{K}$ satisfies the assumption $(\mathcal{K}_1)$ and f satisfies the assumption (f_1) , we can invoke Theorem 1 in Lieberman [22] and assume that weak solutions in $L^\infty(\Omega)$ are in $C^1(\Omega)$.

Lemma 4.2. *Let $u \in C^1(\Omega)$ be a weak nonnegative solution of the problem (P_λ) . If $f(0) > 0$, then u is positive in Ω .*

Proof. Let us suppose that there is $x_0 \in \Omega$ such that

$$u(x_0) = 0.$$

We now consider $y_0 \in \Omega, B_r(y_0) \subset \Omega$ with $x_0 \in \partial B_r(y_0)$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ a continuous function, strictly decreasing on $[0, \infty)$ such that $g(0) = f(0) > 0$ and

$$\gamma := g\left(\frac{a_1}{2}\right) = \inf_{0 \leq s \leq \frac{a_1}{2}} g(s) > 0.$$

We now define a function $b : B_r(y_0) \rightarrow \mathbb{R}$ by

$$b(x) := \epsilon \left[e^{-|\frac{x-y_0}{r}|^2} - e^{-1} \right] \quad (33)$$

where $\epsilon > 0$ is sufficiently small such that

$$\sup_{B_r(y_0)} |\operatorname{div}(\mathcal{K}(|\nabla b|^p)|\nabla b|^{p-2}\nabla b)| \leq \gamma. \quad (34)$$

Consequently, the function b is a subsolution of the problem

$$\begin{cases} -\operatorname{div}(\mathcal{K}(|\nabla v|^p)|\nabla v|^{p-2}\nabla v) = g(v) & \text{in } B_r(y_0), \\ v = 0 & \text{on } \partial B_r(y_0) \end{cases} \quad (35)$$

and so, for all $\varphi \geq 0$ in $X(\Omega)$, we obtain

$$\begin{aligned} & \int_{B_r(y_0)} (\mathcal{K}(|\nabla b|^p)|\nabla b|^{p-2}\nabla b - \mathcal{K}(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) \nabla \varphi \\ & \leq \int_{B_r(y_0)} [g(b) - g(u)]\varphi \end{aligned} \quad (36)$$

In view of condition $(\mathcal{K}_2)$ the function

$$t \rightarrow \mathcal{K}(t^p)t^{p-2}, \quad t \geq 0$$

is nondecreasing and by an argument used in [14], we take $\varphi = (b - u)^+$ to obtain

$$\begin{aligned} 0 &\leq \int_{B_r(y_0)^+} (\mathcal{K}(|\nabla b|^p)|\nabla b|^{p-2}\nabla b - \mathcal{K}(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) (\nabla b - \nabla u) \\ &\leq \int_{B_r(y_0)^+} [g(b) - g(u)](b - u) \\ &\leq 0 \end{aligned}$$

where in the last inequality we used the fact that g is a decreasing function and

$$B_r(y_0)^+ = \{x \in B_r(y_0); b(x) > u(x)\}.$$

Hence $B_r(y_0)^+ = \emptyset$. Because $u(x_0) = b(x_0) = 0$ and $b > 0$ in $B_r(y_0)$, a straightforward computations leads us to

$$\frac{\partial u}{\partial \eta}(x_0) \leq \frac{\partial b}{\partial \eta}(x_0) < 0$$

which yields $|\nabla u(x_0)| \neq 0$ and this contradicts the fact that x_0 is a minimum point of u in Ω . $\square$

We now consider an open ball B with center $0 \in \mathbb{R}^N$ and $\Omega \subset B$. Let us define the function

$$\alpha(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega}, \\ 0 & \text{if } x \in \overline{B} \setminus \Omega. \end{cases}$$

Since Ω is a bounded smooth domain we have

$$\alpha \in X(B) = W_0^{1,\gamma}(B).$$

Lemma 4.3. α is a subsolution of the problem

$$\begin{cases} -\operatorname{div}(\mathcal{K}(|\nabla v|^p)|\nabla v|^{p-2}\nabla v) = f(v) & \text{in } B, \\ v = 0 & \text{on } \partial B. \end{cases} \quad (37)$$

Proof. For each $n \in \mathbb{N}$ we define

$$v_n(x) = n \min \left\{ u(x), \frac{1}{n} \right\}, \quad x \in \Omega$$

where u is as in the previous lemma. Note that

$$v_n(x) = \begin{cases} nu(x), & u(x) < \frac{1}{n}, \\ 1, & u(x) \geq \frac{1}{n}. \end{cases}$$

and so $\nabla u \nabla v_n \geq 0$. It follows from the previous lemma that $v_n(x) \rightarrow 1$ a.e. in Ω . Let us consider a function $\omega \in C_c^\infty(\Omega)$, $\omega \geq 0$. Thus $\omega v_n \in X(\Omega)$ and because u is a weak solution of (P_λ)

$$\int_{\Omega} \mathcal{K}(|\nabla u|^p) |\nabla u|^{p-2} \nabla u \nabla (\omega v_n) = \int_{\Omega} f(u) \omega v_n$$

or

$$\int_{\Omega} \omega \mathcal{K}(|\nabla u|^p) |\nabla u|^{p-2} \nabla u \nabla v_n + \int_{\Omega} v_n \mathcal{K}(|\nabla u|^p) |\nabla u|^{p-2} \nabla u \nabla \omega = \int_{\Omega} f(u) \omega v_n.$$

Since $0 \leq v_n \leq 1$ and $v_n \rightarrow 1$ a.e. in Ω , it follows from the Lebesgue Dominated Convergence Theorem that

$$\begin{aligned} \int_B \mathcal{K}(|\nabla \alpha|^p) |\nabla \alpha|^{p-2} \nabla \alpha \nabla \omega &= \int_{\Omega} \mathcal{K}(|\nabla u|^p) |\nabla u|^{p-2} \nabla u \nabla \omega \\ &= \lim_{n \rightarrow +\infty} \int_{\Omega} v_n \mathcal{K}(|\nabla u|^p) |\nabla u|^{p-2} \nabla u \nabla \omega \\ &= \lim_{n \rightarrow +\infty} \left(\int_{\Omega} f(u) \omega v_n - \int_{\Omega} \omega \mathcal{K}(|\nabla u|^p) |\nabla u|^{p-2} \nabla u \nabla v_n \right) \\ &\leq \lim_{n \rightarrow +\infty} \int_{\Omega} f(u) \omega v_n \\ &= \int_{\Omega} f(u) \omega \leq \int_B f(\alpha) \omega \end{aligned}$$

where the last inequality follows from the fact that $f(0) > 0$. This proves the lemma. $\square$

Theorem 4.4. *Assume that $f(0) > 0$. If (P_λ) possesses a weak nonnegative solution $u \in L^\infty(\Omega)$ such that $\|u\|_\infty \in (a_1, a_2]$, then*

$$\int_{a_1}^{a_2} f(s) ds > 0. \quad (38)$$

Proof. We set $\beta(x) = a_2$, $\forall x \in B$. Hence,

$$\begin{cases} -\operatorname{div}(\mathcal{K}(|\nabla \beta|^p) |\nabla \beta|^{p-2} \nabla \beta) = f(\beta) & \text{in } B, \\ \beta > 0 & \text{on } \partial B. \end{cases} \quad (39)$$

Whence β is a supersolution of (37). Since α is a subsolution of (37), with $\alpha \leq \beta$ (because $\|u\|_\infty \in (a_1, a_2]$), we get a maximal solution $\bar{u}$ in $[\alpha, \beta]$, namely, for each solution v of (37) with

$$\alpha(x) \leq v(x) \leq \beta(x),$$

we have $v \leq \bar{u}$. This allows us to show that $\bar{u}$ is radially symmetric. In fact, suppose by contradiction that exists $x_1, x_2 \in B$ with $|x_1| = |x_2|$ and $\bar{u}(x_1) <$

$\bar{u}(x_2)$. Let A a $N \times N$ orthogonal matrix such that $x_2 = Ax_1$. Set $\tilde{u}(x) = \bar{u}(Ax)$. So,

$$\nabla \tilde{u}(x) = A^T \nabla \bar{u}(Ax), \quad \forall x \in B.$$

Once $x \mapsto Ax$ is a isometry, we get

$$|\nabla \tilde{u}(x)| = |A^T \nabla \bar{u}(Ax)| = |\nabla \bar{u}(Ax)|.$$

Let's see that $\tilde{u}$ is a weak solution of (37), in other words, we will show that

$$\int_B \mathcal{K}(|\nabla \tilde{u}|^p) |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \nabla h = \int_B f(\tilde{u}) h, \quad \forall h \in X(B).$$

For this, note that putting $\psi(x) = h(A^T x) \in X(B)$, follows the change of variable theorem that

$$\begin{aligned} & \int_B \mathcal{K}(|\nabla \tilde{u}|^p) |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \nabla h dx \\ &= \int_B \mathcal{K}(|\nabla \bar{u}(Ax)|^p) |\nabla \bar{u}(Ax)|^{p-2} A^T \nabla \tilde{u}(Ax) (A^T \nabla \psi(Ax)) dx \\ &= \int_B \mathcal{K}(|\nabla \bar{u}(y)|^p) |\nabla \bar{u}(y)|^{p-2} \nabla \tilde{u}(y) \nabla \psi(y) |\det A| dy \\ &= \int_B f(\bar{u}(y)) \psi(y) dy \\ &= \int_B f(\bar{u}(AA^T y)) \psi(AA^T y) dy \\ &= \int_B f(\bar{u}(Ax)) \psi(Ax) |\det A^T| dx \\ &= \int_B f(\tilde{u}(x)) h(x) dx, \end{aligned}$$

as we had promised. Since α and $\tilde{u}$ are two solutions it results that the problem (37) admits another solution $\hat{u}$ such that

$$\max\{\alpha, \tilde{u}\} \leq \hat{u} \leq \beta. \quad (40)$$

Once $\bar{u}$ the maximal solution in (α, β) , we conclude of (40) that

$$\bar{u}(x_1) \geq \hat{u}(x_1) \geq \tilde{u}(x_1) = \bar{u}(x_2) > \bar{u}(x_1),$$

which is a contradiction. Thus, $\bar{u}$ is radially symmetric. Now, let us set a C^1 -function $u : [0, R) \rightarrow \mathbb{R}_+$ by $u(r) = u(|x|) = \bar{u}(x)$. Thus, we have

$$\frac{\partial \bar{u}}{\partial x_i} = u' \frac{x_i}{r} \quad \text{and} \quad |\nabla \bar{u}| = |u'|.$$

For each $v \in C_0^\infty(0, R)$ we set

$$w(r) = \frac{v(r)}{r^{N-1}}, \quad r \in (0, R) \quad \text{and} \quad w(0) = 0,$$

yet let us set $\bar{v}(x) = v(|x|)$ and $\bar{w}(x) = w(|x|)$. Since $\bar{u}$ is a weak solution of (37), we have

$$\int_B \mathcal{K}(|\nabla \bar{u}|^p) |\nabla \bar{u}|^{p-2} \nabla \bar{u} \nabla \bar{w} dx = \int_B f(\bar{u}) \bar{w} dx.$$

As

$$\frac{\partial \bar{w}}{\partial x_i} = w' \frac{x_i}{r} \quad \text{and} \quad |\nabla \bar{w}| = |w'|,$$

it follows that,

$$\int_0^R \mathcal{K}(|u'|^p) |u'|^{p-2} u' w' r^{N-1} dr = \int_0^R f(u) w r^{N-1} dr,$$

replacing

$$w = \frac{1}{r^{N-1}} v \quad \text{and} \quad w' = \frac{1}{r^{N-1}} v' - \frac{(N-1)}{r^N} v$$

in the previous equality, we get

$$\int_0^R \mathcal{K}(|u'|^p) |u'|^{p-2} u' \left(\frac{1}{r^{N-1}} v' - \frac{(N-1)}{r^N} v \right) r^{N-1} dr = \int_0^R f(u) \frac{1}{r^{N-1}} v r^{N-1} dr,$$

whence

$$\int_0^R \mathcal{K}(|u'|^p) |u'|^{p-2} u' v' dr - \int_0^R \frac{(N-1)}{r} \mathcal{K}(|u'|^p) |u'|^{p-2} u' v dr = \int_0^R f(u) v dr,$$

for all $v \in C_0^\infty(0, R)$ which tell us that u is a weak solution of the class $C^1(0, R)$ of the equation

$$-\partial (\mathcal{K}(|u'|^p) |u'|^{p-2} u') = \frac{N-1}{r} \mathcal{K}(|u'|^p) |u'|^{p-2} u' + f(u). \quad (41)$$

Since the right hand side of the previous equality is continuous, we conclude that the distributional derivative which we denote for ∂ is a classical derivative and u is a classical solution of (41). Since $\bar{u}$ radially symmetric we have $u'(0) = 0$ and then u is a classical solution of (41) and verifies $u'(0) = 0 = u(R)$. Now, let $r_0 \in [0, R)$ such that

$$u(r_0) = \max\{u(r) : r \in [0, R)\}.$$

Multiplying both sides of (41) for u' and integrating it, we obtain

$$\begin{aligned} & - \left((p-1) \int_{r_0}^r \mathcal{K}(|u'|^p) |u'|^{p-2} u' u'' dt + p \int_{r_0}^r \mathcal{K}'(|u'|^p) |u'|^{2p-2} u' u'' dt \right) \\ & - (N-1) \int_{r_0}^r \mathcal{K}(|u'|^p) \frac{|u'|^p}{t} dt \\ & = \int_{r_0}^r f(u) u' dt, \end{aligned}$$

for all $r \in (0, R)$. Since $u(r_0) > a_1$ we can choose $r \in (0, R)$ such that $u(r) = a_1$. Hence,

$$\begin{aligned} & \int_{u(r_0)}^{a_1} f(t) dt \\ &= - \int_0^{u'(r)} [(p-1)\mathcal{K}(|t|^p) + p\mathcal{K}'(|t|^p)|t|^p] |t|^{p-2} t dt - (N-1) \int_{r_0}^r \mathcal{K}(|u'|^p) \frac{|u'|^p}{t} dt. \end{aligned}$$

So, since $\mathcal{K}$ a C^1 -function, follow from $(\mathcal{K}_2)$ that

$$\int_{a_1}^{u(r_0)} f(t) dt > 0$$

and since $f \leq 0$ in $(a_1, b_1]$ then $u(r_0) \in (a_1, b_1]$ and f is nonnegative in $[u(r_0), a_2]$. Therefore,

$$\int_{a_1}^{a_2} f(s) ds \geq \int_{a_1}^{u(r_0)} f(s) ds > 0. \quad \square$$

Remark 4.5. Now let us show that it's possible remove the condition $f(0) > 0$ in the previous Theorem. In fact, suppose that $f(0) \leq 0$ and $u \in X(\Omega) \cap L^\infty(\Omega)$ is a nonnegative solution of (P_λ) such that $\|u\|_\infty \in (a_1, a_2]$. The idea is to set a continuous function $\hat{f}$ such that $\hat{f}(0) > 0$, $\hat{f} \geq f$ in $[0, a_1]$ and $\hat{f} = f$ in $[a_1, \infty)$. Thus u is a subsolution of

$$\begin{cases} -\operatorname{div} (\mathcal{K}(|\nabla u|^p) |\nabla u|^{p-2} \nabla u) = \hat{f}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

using again $\beta(x) = a_2$ as supersolution we get a solution $\tilde{u}$ verifying $u \leq \tilde{u} \leq a_2$.

We proceed as in the first part of the proof with $\hat{f}$ in place of f and obtain

$$\int_{a_1}^{a_2} f(s) ds = \int_{a_1}^{a_2} \hat{f}(s) ds > 0.$$

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