Journal of Convex Analysis
Volume 21 (2014), No. 4, 1043–1064

Primal Attainment in Convex Infinite Optimization Duality*

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Received: April 17, 2013
Revised manuscript received: October 1, 2013
Accepted: October 3, 2013

This article provides results guarateeing that the optimal value of a given convex infinite optimization problem and its corresponding surrogate Lagrangian dual coincide and the primal optimal value is attainable. The conditions ensuring converse strong Lagrangian (in short, minsup) duality involve the weakly-inf-(locally) compactness of suitable functions and the linearity or relative closedness of some sets depending on the data. Applications are given to different areas of convex optimization, including an extension of the Clark-Duffin Theorem for ordinary convex programs.

Keywords: Convex infinite programming, converse strong duality, minsup duality

1991 Mathematics Subject Classification: 90C25, 90C48, 49N15

1. Introduction

This paper deals with some duality principles for the convex infinite programming problem

$$\begin{array}{lll} (P) & \text{Minimize} & f(x) \\ & \text{subject to} & f_t(x) \le 0,\ t \in T, \\ & & x \in C, \end{array} \tag{1}$$

*This research was partially supported by MINECO of Spain, Grant MTM2011-29064-C03-02, and Generalitat Valenciana, Grant ACOMP/2013/062.

ISSN 0944-6532 / $ 2.50

where T is an arbitrary (possibly infinite) index set, C is a non-empty convex subset of a locally convex Hausdorff topological vector space X, whose zero is represented by 0_X, and $f,\ f_t : X \to \mathbb{R} \cup \{+\infty\}$ are proper convex functions on X. The feasible set of (P) is $F \cap C$, where

$$F := \{x \in X : f_t(x) \leq 0,\ t \in T\}.$$

Throughout the paper we assume that

$$E := C \cap \operatorname{dom} f \cap F \neq \emptyset, \tag{2}$$

which amounts to saying that the optimal value of (P), $\inf(P)$, is not equal to $+\infty$. The *optimal set* of (P), say

$$S(P) = \{x \in E : f(x) = \inf(P)\},$$

is possibly empty.

We represent by $\mathbb{R}_+^{(T)}$ the non-negative cone in $\mathbb{R}^{(T)}$, the so-called *space of generalized finite sequences* $\lambda = (\lambda_t)_{t\in T}$ such that $\lambda_t \in \mathbb{R}$, for each $t \in T$, and with only finitely many λ_t different from zero. We denote by 0_T the element $(\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ such that $\lambda_t = 0$ for all $t \in T$. Given $\lambda \in \mathbb{R}_+^{(T)}$, we define for every $x \in X$

$$\sum_{t\in T} \lambda_t f_t(x) = \begin{cases} 0, & \text{if } \lambda = 0_T, \\ \sum_{t\in T,\ \lambda_t>0} \lambda_t f_t(x), & \text{if } \lambda \neq 0_T. \end{cases}$$

The *Lagrangian dual problem* of (P) is, by definition ([8], [13]–[17]),

$$\begin{array}{lll} (D) & \text{Maximize} & \inf_{x\in C} \left(f(x) + \sum_{t\in T} \lambda_t f_t(x)\right) \\ & \text{subject to} & \lambda \in \mathbb{R}_+^{(T)}. \end{array} \tag{3}$$

The optimal value of (D), $\sup(D)$, is always less or equal to $\inf(P)$, i.e. the *weak dual inequality* holds:

$$\sup(D) \leq \inf(P).$$

Several closedness criteria have been provided (see [8], [13]–[17]) ensuring that the *strong Lagrangian duality* holds between (P) and (D). This means that there is *no duality gap* (i.e., $\sup(D) = \inf(P)$) and the dual problem (D) has an optimal solution:

$$\exists \overline{\lambda} \in \mathbb{R}_+^{(T)} : \inf(P) = \inf_{x\in C} \left(f(x) + \sum_{t\in T} \overline{\lambda}_t f_t(x) \right) \in \mathbb{R} \cup \{-\infty\},$$

a property that we can summarize by

$$\inf(P) = \max(D) \in \mathbb{R} \cup \{-\infty\}.$$

The aim of this paper is to give conditions under which the so-called *converse strong Lagrangian duality* holds, i.e.

$$\min(P) = \sup(D) \in \mathbb{R}.$$

To this end we introduce the *surrogate Lagrangian dual* (Δ) of (P) by setting

$$\begin{array}{lll} (\Delta) & \text{Maximize} & \inf_{x\in C}\left(f(x) + \sum_{t\in T}\lambda_t f_t(x)\right) \\ & \text{subject to} & \lambda \in \mathbb{P}(T), \end{array}$$

where $\mathbb{P}(T) = \mathbb{R}_+^{(T)} \setminus \{0_T\}$. Since $\sum_{t\in T}\lambda_t f_t(x) = 0$ for $\lambda = 0_T$, it holds that

$$+\infty > \inf(P) \geq \sup(D) = \max\left(\sup(\Delta), \inf_C f\right) \geq \sup(\Delta) \geq -\infty. \tag{4}$$

It may happen that $\sup(\Delta) < \sup(D)$, even if the familiar *Slater condition* holds. This is, for instance, the case if $X = C = \mathbb{R}$, $f(x) = \exp(x)$, $T = \{1\}$, and $f_1(x) = x$. One has

$$\sup(\Delta) = \sup_{\lambda>0}\inf_{x\in\mathbb{R}}\{\exp(x) + \lambda x\} = -\infty < 0 = \inf_C f = \sup(D).$$

Let us now consider an important case in which $\sup(\Delta) = \sup(D)$.

Proposition 1.1. *Assume that there exists* $\overline{\lambda} \in \mathbb{P}(T)$ *such that* $\sum_{t\in T}\overline{\lambda}_t f_t$ *is bounded below on* $C \cap \operatorname{dom} f$*. Then we have* $\sup(\Delta) = \sup(D)$.

Proof. By hypothesis, there exist $\overline{\lambda} \in \mathbb{P}(T)$ and $K \in \mathbb{R}$ such that $\sum_{t\in T}\overline{\lambda}_t f_t \geq K$ on $C \cap \operatorname{dom} f$. Since $E \neq \emptyset$ one has $K \leq 0$ and

$$\begin{aligned} \sup(\Delta) &\geq \sup_{0<\tau\leq 1}\inf_C\left\{f + \tau\sum_{t\in T}\overline{\lambda}_t f_t\right\} \geq \sup_{0<\tau\leq 1}\inf_C\{f + \tau K\} \\ &= \inf_C f + \sup_{0<\tau\leq 1}(\tau K) = \inf_C f, \end{aligned}$$

and according to (4), $\sup(D) = \max(\sup(\Delta), \inf_C f) = \sup(\Delta)$. □

We intend to give conditions ensuring that

$$\min(P) = \sup(\Delta) \in \mathbb{R}.$$

This amounts to the existence of $\overline{x} \in E$ such that

$$f(\overline{x}) = \sup(\Delta).$$

In such a case we have also, by (4),

$$\min(P) = \sup(D) \in \mathbb{R}.$$

To the best of our knowledge, this topic has not been tackled in full generality in the frame of convex infinite optimization (see however [28, Theorem 2], [22, Proposition 7.9.8], [31, Proposition 3.3], and [16, Theorem 8.8], where special cases are investigated).

The paper is organized as follows. Section 2 introduces the necessary preliminaries. Section 3 and 4 provide results guaranteeing the minsup duality by assuming the weakly-inf-(locally) compactness and the linearity of suitable functions and sets, respectively. Finally, Section 5 provides a characterization of the minsup duality for a modified surrogate Lagrange dual of (P) under a closedness assumption. The strategy used in the latter section, consisting of modifying one of the problems in duality in order to get minsup duality, traces back to R. J. Duffin ([9]), who replaced feasible points of the primal problem by feasible sequences in linear infinite programming. The results are illustrated with applications to different instances of the optimization convex problem (P), including linear and non-linear, ordinary, semi-infinite, and infinite programs.

2. Notation and convex analysis requirements

A set $K \subset X$ is a cone if $\lambda > 0$ and $x \in K$ implies $\lambda x \in K$. Given $B \subset X$, we denote by $\operatorname{co} B$, $\overset{\vee}{\operatorname{cone}} B$, $\operatorname{cone} B$, and $\operatorname{aff} B$ the convex hull of B, the smallest convex cone containing B, the smallest convex cone containing $B \cup \{0_X\}$, and the smallest linear manifold containing B, respectively. Obviously,

$$\overset{\vee}{\operatorname{cone}} B = \{\lambda x : \lambda > 0 \text{ and } x \in \operatorname{co} B\}$$

and

$$\operatorname{cone} B = \overset{\vee}{\operatorname{cone}} (B \cup \{0_X\}) = \{\lambda x : \lambda \geq 0 \text{ and } x \in \operatorname{co} B\}.$$

If B is non-empty, then

$$\operatorname{cone} B = \left(\overset{\vee}{\operatorname{cone}} B\right) \cup \{0_X\}$$

and

$$\operatorname{cl} \operatorname{cone} B = \operatorname{cl} \overset{\vee}{\operatorname{cone}} B.$$

Given a family $(B_t)_{t \in T}$ of non-empty convex subsets of X, it holds

$$\overset{\vee}{\operatorname{cone}} \left(\bigcup_{t \in T} B_t \right) = \bigcup_{\lambda \in \mathbb{P}(T)} \sum_{t \in T} \lambda_t B_t$$

and

$$\operatorname{cone} \left(\bigcup_{t \in T} B_t \right) = \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \sum_{t \in T} \lambda_t B_t.$$

We begin with an easy lemma involving i_F, the *indicator function* of F, namely, $i_F(x) = 0$ if $x \in F$, $i_F(x) = +\infty$ if $x \in X \backslash F$.

Lemma 2.1. $\sup_{\lambda\in\mathbb{P}(T)}\sum_{t\in T}\lambda_t f_t = \sup_{\lambda\in\mathbb{R}_+^{(T)}}\sum_{t\in T}\lambda_t f_t = i_F$.

Proof. Let $x\in F$. For any $\lambda\in\mathbb{R}_+^{(T)}$ one has $\sum_{t\in T}\lambda_t f_t(x)\leq 0$ and

$$\sup_{\lambda\in\mathbb{P}(T)}\sum_{t\in T}\lambda_t f_t(x)\leq \sup_{\lambda\in\mathbb{R}_+^{(T)}}\sum_{t\in T}\lambda_t f_t(x)\leq 0 = i_F(x).$$

If $(f_t(x))_{t\in T}=0_T$ we are done. If not, there exists $\bar{t}\in T$ such that $f_{\bar{t}}(x)<0$. Defining $(\lambda^r)_{r\geq 1}\subset\mathbb{P}(T)$ by $\lambda_{\bar{t}}^r=\frac{1}{r}$, $\lambda_t^r=0$ if $t\neq\bar{t}$, we get

$$\sup_{\lambda\in\mathbb{P}(T)}\sum_{t\in T}\lambda_t f_t(x)\geq \sup_{r\geq 1}\frac{1}{r}f_{\bar{t}}(x)=0$$

and we are done.

Assume now that $x\notin F$. Then, there exists $\bar{t}\in T$ such that $f_{\bar{t}}(x)>0$. Defining $(\lambda^r)_{r\geq 1}\subset\mathbb{P}(T)$ by $\lambda_{\bar{t}}^r=r$, $\lambda_t^r=0$ if $t\neq\bar{t}$, we get

$$\sup_{\lambda\in\mathbb{R}_+^{(T)}}\sum_{t\in T}\lambda_t f_t(x)\geq \sup_{\lambda\in\mathbb{P}(T)}\sum_{t\in T}\lambda_t f_t(x)\geq \sup_{r\geq 1} r f_{\bar{t}}(x)=+\infty,$$

which completes the proof. □

The duality product of $x^*\in X^*$ (the topological dual of X) and $x\in X$ is represented by $\langle x^*,x\rangle$ or by $x^*(x)$. The *positive* and *negative dual cones* of a non-empty set $C\subset X$ are

$$C^+=\{x^*\in X^*: x^*(x)\geq 0\ \forall x\in C\}$$

and

$$C^-=\{x^*\in X^*: x^*(x)\leq 0\ \forall x\in C\}$$

while the *recession cone* of a non-empty convex set $C\subset X$ is

$$C_\infty=\{v\in X: c+v\in C\ \forall c\in C\}.$$

Given $f:X\to\overline{\mathbb{R}}:=\mathbb{R}\cup\{\pm\infty\}$ and $r\in\mathbb{R}$, $[f\leq r]$ denotes the lower-level set $\{x\in X: f(x)\leq r\}$. Such a function is said to be *inf-compact* (*inf-locally-compact*) when $[f\leq r]$ is a compact set (a locally compact set, respectively) for every $r\in\mathbb{R}$. The *domain,* the *epigraph*, and the *strict epigraph* of f are $\operatorname{dom} f=\{x\in X: f(x)<+\infty\}$, $\operatorname{epi} f=\{(x,\mu)\in X\times\mathbb{R}: f(x)\leq\mu\}$, and $\operatorname{epi}_s f=\{(x,\mu)\in X\times\mathbb{R}: f(x)<\mu\}$, respectively.

Recall that the *Legendre-Fenchel conjugate* of $f:X\to\overline{\mathbb{R}}$ (respectively, $h:X^*\to\overline{\mathbb{R}}$) is the function $f^*:X^*\to\overline{\mathbb{R}}$ (respectively, $h^*:X\to\overline{\mathbb{R}}$) defined by

$$f^*(x^*)=\sup_{x\in X}(x^*(x)-f(x))$$

(respectively, $h^*(x) = \sup_{x^* \in X^*} (x^*(x) - h(x^*))$). If $f \in \Gamma(X)$, the class of proper lower semicontinuous (lsc, in brief) convex functions on X, then $f^{**} = f$.

The *support function* of $C \subset X$ is the conjugate of its indicator, i.e., i_C^*. The *recession function* of $f \in \Gamma(X)$ is $f_\infty = i_{\operatorname{dom} f^*}^*$.

For any family $(h_i)_{i \in I} \subset \overline{\mathbb{R}}^{X^*}$ one has

$$\left(\inf_{i \in I} h_i\right)^* = \sup_{i \in I} h_i^*. \tag{5}$$

A function $p : X^* \to \overline{\mathbb{R}}$ is said to be *subdifferentiable* at a point $\overline{x}^* \in p^{-1}(\mathbb{R})$ if there exists $\overline{x} \in X$ such that

$$p(x^*) \geq p(\overline{x}^*) + (x^* - \overline{x}^*)(\overline{x}), \quad \forall x^* \in X^*.$$

We then write $\overline{x} \in \partial p(\overline{x}^*)$. For example, a convex function $p : X^* \to \overline{\mathbb{R}}$ which is finite and continuous with respect to the the Mackey topology $\tau(X^*, X)$ at a point $\overline{x}^* \in X^*$ is subdifferentiable at $\overline{x}^*$ ([11, Proposition 5.2, p. 22]). In the case when p belongs to $\Gamma(X^*)$, the set of proper convex functions that are $\tau(X^*, X)$-lsc or, equivalently, w^*-lsc, then one has ([22, Corollary in p. 348])

$$\begin{array}{c} p \text{ is finite and } \tau(X^*, X)\text{-continuous at } \overline{x}^* \\ \Updownarrow \\ p^* - \overline{x}^* \text{ is } w\text{-inf-compact,} \end{array} \tag{6}$$

i.e., the sublevel sets of $p^* - \overline{x}^*$ are $\sigma(X, X^*)$-compact (compact for the weak topology on X).

According to [30, Theorem 2.4], if X is a Banach space, and $p \in \Gamma(X^*)$ is such that $p^* - x^*$ attains its minimum on X for any $x^* \in X^*$, then $p^* - x^*$ is w-inf-compact for any $x^* \in X^*$. [30, Theorem 2.4] gave a positive answer to a conjecture raised by J. Orihuela and M. Ruiz-Galán in [25], and it furnishes a sufficient condition for our main assumptions in Section 3.

We shall also use the following subdifferentiability criterion (see, for instance, Proposition 2.107 in [2]): Let $p, q : X^* \to \overline{\mathbb{R}}$ be convex functions which are both finite at a point $\overline{x}^* \in X^*$. Then,

$$\begin{array}{l} q \text{ is } \tau(X^*, X)\text{-continuous at } \overline{x}^* \text{ and } p \leq q \\ \qquad \Rightarrow p \text{ is } \tau(X^*, X)\text{-continuous at } \overline{x}^*. \\ \text{In that case, } \partial p(\overline{x}^*) \neq \emptyset. \end{array} \tag{7}$$

The *infimal convolution* of two proper functions $f, g : X \to \overline{\mathbb{R}}$ is the function $f \square g$ defined on X by

$$(f \square g)(x) = \inf_{u \in X} \{f(x - u) + g(u)\}, \quad \forall x \in X,$$

with the convention $(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty$ when necessary.

If $p, q : X^* \to \mathbb{R} \cup \{+\infty\}$ are convex and p is finite and $\tau(X^*, X)$-continuous at a point where q is finite, then ([33, Theorem 2.8.7])

$$(p+q)^* = p^* \square q^*;$$

and for each $x \in X$ there exists $u \in X$ such that

$$(p+q)^*(x) = p^*(x-u) + q^*(u). \tag{8}$$

For any $f, g : X \to \overline{\mathbb{R}}$ the next classical relation always holds:

$$(f+g)^* \leq f^* \square g^*. \tag{9}$$

3. Minsup duality under inf-compactness

In this section we consider the problem (P) in (1), assuming that C is closed and $f, f_t \in \Gamma(X)$. We introduce the functions $f_C := f + i_C$, and $h : X^* \to \overline{\mathbb{R}}$ defined by

$$h := \inf_{\lambda \in \mathbb{P}(T)} \left(f_C + \sum_{t \in T} \lambda_t f_t \right)^* . \tag{10}$$

Lemma 3.1. *The function h defined in* (10) *is proper and convex, and its conjugate is $h^* = f_C + i_F$.*

Proof. Let us first calculate h^*. By (5) one has

$$h^* = \sup_{\lambda \in \mathbb{P}(T)} \left(f_C + \sum_{t \in T} \lambda_t f_t \right)^{**} .$$

Now, for each $\lambda \in \mathbb{P}(T)$, the function $f_C + \sum_{t \in T} \lambda_t f_t$ is proper (as the set E defined in (2) is non-empty), convex, and lsc. Consequently, it coincides with its biconjugate. We thus have

$$h^* = \sup_{\lambda \in \mathbb{P}(T)} \left(f_C + \sum_{t \in T} \lambda_t f_t \right) = f_C + \sup_{\lambda \in \mathbb{P}(T)} \left(\sum_{t \in T} \lambda_t f_t \right)$$

and, by Lemma 2.1, $h^* = f_C + i_F$. Let us observe that $\operatorname{dom} h^* = E \neq \emptyset$, and this entails that h does not take the value $-\infty$. On the other hand, fixing $\lambda \in \mathbb{P}(T)$, one has $h \leq \left(f_C + \sum_{t \in T} \lambda_t f_t\right)^*$ which is proper since $f_C + \sum_{t \in T} \lambda_t f_t \in \Gamma(X)$, so h is proper. It remains to prove that h is convex. Take $(u^*, r), (v^*, s) \in \operatorname{epi}_s h$ and $\gamma \in]0,1[$. Then there exist $\lambda, \mu \in \mathbb{P}(T)$ such that $\mathbb{R} \ni r' := \left(f + \sum_{t \in T} \lambda_t f_t\right)^*(u^*) < r$ and $\mathbb{R} \ni s' := \left(f + \sum_{t \in T} \mu_t f_t\right)^*(v^*) < s$, that is

$$\langle u^*, x \rangle - r' \leq f(x) + \sum_{t \in T} \lambda_t f_t(x),$$

$$\langle v^*, x \rangle - s' \leq f(x) + \sum_{t \in T} \mu_t f_t(x), \quad \forall x \in X.$$

Multiplying the terms of the first inequality by γ and those of the second one by $(1-\gamma)$, then adding term by term, we get

$$\langle \gamma u^* + (1-\gamma) v^*, x \rangle - (\gamma r' + (1-\gamma) s') \leq f(x) + \sum_{t \in T} \upsilon_t f_t(x), \quad \forall x \in X,$$

where $\upsilon := \gamma\lambda + (1-\gamma)\mu \in \mathbb{P}(T)$. It follows that

$$\begin{aligned} h(\gamma u^* + (1-\gamma) v^*) &\leq \left(f + \sum_{t \in T} \upsilon_t f_t \right)^* (\gamma u^* + (1-\gamma) v^*) \\ &\leq \gamma r' + (1-\gamma) s' < \gamma r + (1-\gamma) s. \end{aligned}$$

Hence $\gamma(u^*, r) + (1-\gamma)(v^*, s) \in \operatorname{epi}_s$, and so h is convex. $\square$

Lemma 3.2. *Let h be as in* (10) *and $\overline{x} \in X$. The following statements are equivalent:*

(i) $\overline{x} \in \partial h(0_{X^*})$.
(ii) $\overline{x} \in E$ *and* $f(\overline{x}) = \sup(\Delta)$.
(iii) $\overline{x} \in S(P)$ *and* $\min(P) = \sup(\Delta)$.

Proof. We first observe that

$$\begin{aligned} -h(0_{X^*}) &= \sup_{\lambda \in \mathbb{P}(T)} - \left(f_C + \sum_{t \in T} \lambda_t f_t \right)^* (0_{X^*}) \\ &= \sup_{\lambda \in \mathbb{P}(T)} \inf_C \left(f_C + \sum_{t \in T} \lambda_t f_t \right) = \sup(\Delta). \end{aligned}$$

(i) $\Rightarrow$ *(ii)*. Since $\overline{x} \in \partial h(0_{X^*})$ one has $h^*(\overline{x}) = -h(0_{X^*}) \in \mathbb{R}$. By Lemma 3.1 we thus have $h^*(\overline{x}) = f(\overline{x})$ with $\overline{x} \in E$.

(ii) $\Rightarrow$ *(i)*. Since $\infty > \sup(\Delta) = f(\overline{x})$ one has $f(\overline{x}) \in \mathbb{R}$ and, by Lemma 3.1, $h^*(\overline{x}) = \sup(\Delta) = -h(0_{X^*}) \in \mathbb{R}$, and this means that $\overline{x} \in \partial h(0_{X^*})$.

(ii) $\Leftrightarrow$ *(iii)*. It follows from (4). $\square$

So, if h is subdifferentiable at 0_{X^*}, then $\partial h(0_{X^*}) = S(P)$. In particular, if h is finite and $\tau(X^*, X)$-continuous at 0_{X^*}, then $S(P)$ is weakly compact (see, for instance, [22, Théorème 6.4.6]). This is the case in Theorems 3.3 and 3.7 below, but not necessarily in Theorems 4.7, 4.8, 5.2, and 5.5. We are now in position to state our first result.

Theorem 3.3. *Assume there exists $\overline{\lambda} \in \mathbb{P}(T)$ such that the function $f_C + \sum_{t \in T} \overline{\lambda}_t f_t$ is w-inf-compact. Then,*

$$\min(P) = \sup(D) = \sup(\Delta) \in \mathbb{R}.$$

In other words,

$$\exists \overline{x} \in E \text{ such that } f(\overline{x}) = \inf(P) = \sup(D) = \sup(\Delta). \tag{11}$$

Moreover, $S(P)$ is weakly compact.

Proof. We have noticed in the proof of Lemma 3.2 that the proper convex function h defined by (10) satisfies $-h(0_{X^*}) = \sup(\Delta) \le \inf(P) < +\infty$.

Now $f_C + \sum_{t\in T} \overline{\lambda}_t f_t$ belongs to $\Gamma(X)$ and is w-inf-compact, by assumption. By (6), $\left(f_C + \sum_{t\in T} \overline{\lambda}_t f_t\right)^*$ is finite and $\tau(X^*, X)$-continuous at 0_{X^*}. On the other hand, $h \le \left(f_C + \sum_{t\in T} \overline{\lambda}_t f_t\right)^*$, and so

$$-h(0_{X^*}) \ge \min_X \left(f_C + \sum_{t\in T} \overline{\lambda}_t f_t\right) \in \mathbb{R}.$$

It follows from (7) that h is subdifferentiable at 0_{X^*}. Since $f_C + i_F \ge f_C + \sum_{t\in T} \overline{\lambda}_t f_t$, $f_C + i_F$ is lsc, and $f_C + \sum_{t\in T} \overline{\lambda}_t f_t$ is w-inf-compact, $f_C + i_F$ is w-inf-compact too. Then (11) holds and $S(P) = \operatorname{argmin}(f_C + i_F)$ is weakly compact by Lemma 3.2. □

The next result shows that one can take $\overline{\lambda} \in \mathbb{R}_+^{(T)}$ instead of $\overline{\lambda} \in \mathbb{P}(T)$ in the assumption of Theorem 3.3.

Corollary 3.4. *Assume that f_C is w-inf-compact. Then, the conclusions of Theorem* 3.3 *still hold.*

Proof. By (6), f_C^* is finite and $\tau(X^*, X)$-continuous at 0_{X^*}, and there will exist an open neighborhood V of 0_{X^*} on which f_C^* is bounded from above and $\tau(X^*, X)$-continuous ([2, Proposition 2.107]). Pick $\lambda \in \mathbb{P}(T)$ and, for each $t \in T$ such that $\lambda_t > 0$, take a continuous affine minorant $x_t^* - r_t$ of f_t. Since V is absorbing, there will exist $s > 0$ such that

$$u^* := -s \sum_{t\in T} \lambda_t x_t^* \in V,$$

and so, f_C^* is finite and $\tau(X^*, X)$-continuous at u^*. Again by (6), $f_C - u^*$ is w-inf-compact. Now, since

$$f_C + s \sum_{t\in T} \lambda_t f_t \ge f_C + s \sum_{t\in T} \lambda_t (x_t^* - r_t) = f_C - u^* - s \sum_{t\in T} \lambda_t r_t,$$

we have that $f_C + s \sum_{t\in T} \lambda_t f_t$ is w-inf-compact too, and we apply Theorem 3.3 with $\overline{\lambda} = s\lambda$. □

Remark 3.5. Statement (11) holds in particular whenever the closed convex set C is w-compact, since f_C is then w-inf-compact.

We now state our second result. To this end next lemma will be useful.

Lemma 3.6. *Let $p, q \in \Gamma(X)$ and assume that there exists $x^* \in \operatorname{dom} q^*$ such that $p + x^*$ is w-inf-compact. Then $p + q$ is w-inf-compact.*

Proof. Take $s := q^*(x^*) \in \mathbb{R}$. Then $p + q \geq p + x^* - s$, and so $[p + q \leq r] \subset [p + x^* \leq r + s]$ for every $r \in \mathbb{R}$. Since $p + q$ is lsc, it follows that $p + q$ is w-inf-compact. □

Theorem 3.7. *Assume that there exists*

$$\overline{x}^* \in \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \operatorname{dom}\left(\sum_{t \in T} \lambda_t f_t\right)^*$$

such that $f_C + \overline{x}^*$ is w-inf-compact. Then (11) *holds and $S(P)$ is w-compact.*

Proof. Let $\lambda \in \mathbb{R}_+^{(T)}$ be such that $\overline{x}^* \in \operatorname{dom}\left(\sum_{t \in T} \lambda_t f_t\right)^*$. Since $f_C + \overline{x}^*$ is w-inf-compact, Lemma 3.6 says that $f_C + \sum_{t \in T} \lambda_t f_t$ is w-inf-compact too. We conclude the proof by applying Theorem 3.3 or Corollary 3.4, depending on whether $\lambda \in \mathbb{P}(T)$ or $\lambda = 0_T$, respectively. □

Remark 3.8. Since $(\alpha f_t)^*(x^*) = \alpha (f_t)^*\left(\frac{x^*}{\alpha}\right)$ for $\alpha > 0$ and $x^* \in X^*$, and according to (9), it is easy to observe that for $\lambda \in \mathbb{P}(T)$ one has

$$\sum_{t \in T} \lambda_t \operatorname{dom} f_t^* \subset \operatorname{dom}\left(\sum_{t \in T} \lambda_t f_t\right)^*,$$

and, as a consequence of that,

$$\overset{\vee}{\operatorname{cone}}\left(\bigcup_{t \in T} \operatorname{dom} f_t^*\right) = \bigcup_{\lambda \in \mathbb{P}(T)} \left(\sum_{t \in T} \lambda_t \operatorname{dom} f_t^*\right) \subset \bigcup_{\lambda \in \mathbb{P}(T)} \operatorname{dom}\left(\sum_{t \in T} \lambda_t f_t\right)^*. \tag{12}$$

According to Theorem 3.7 and (12) we can state:

Corollary 3.9. *Assume there exists $\overline{x}^* \in \operatorname{cone}\left(\bigcup_{t \in T} \operatorname{dom} f_t^*\right)$ such that $f_C + \overline{x}^*$ is w-inf-compact. Then* (11) *holds and $S(P)$ is w-compact.*

Remark 3.10. Consider the linear semi-infinite programming (LSIP) problem

$$\begin{array}{lll} (P_1) & \text{Minimize} & \langle c^*, x\rangle \\ & \text{subject to} & \langle x_t^*, x\rangle \leq r_t, \ t \in T, \\ & & x \in C, \end{array} \tag{13}$$

with $c^*, x_t^* \in X^*$, $r_t \in \mathbb{R}$, and $X = \mathbb{R}^n$. Theorem 3.7 (or Corollary 3.9) asserts that, if there exists $\overline{x}^* \in \operatorname{cone}\{x_t^* : t \in T\}$ such that $C_\infty \cap [c^* + \overline{x}^* \leq 0] = \{0_{\mathbb{R}^n}\}$, then (11) holds and $S(P_1)$ is compact. When $C = \mathbb{R}^n$, $C_\infty \cap [c^* + \overline{x}^* \leq 0] = [c^* + \overline{x}^* \leq 0]$ is $\mathbb{R}^n$, if $\overline{x}^* = -c^*$, or a half-space, otherwise, so that Theorem 3.7 does not apply. This is one reason to avoid the inf-compactness assumption in the next section.

4. Minsup duality without inf-compactness

Let us recall that a convex function $h : Y \to \overline{\mathbb{R}}$, Y being a locally convex Hausdorff topological vector space, is said to be *quasicontinuous* ([19], [20]) if:

a) $\operatorname{aff}\operatorname{dom} h$ is closed and of finite codimension;
b) the relative interior of $\operatorname{dom} h$, represented by $\operatorname{ri}\operatorname{dom} h$, is non-empty and the restriction of h to $\operatorname{aff}\operatorname{dom} h$ is continuous on $\operatorname{ri}\operatorname{dom} h$.

We will use the following fundamental results which are more or less explicitly contained in [19] and [20].

Lemma 4.1 ([22, Theorem 7.7.6]). *For any $g \in \Gamma(X)$, the following statements are equivalent:*

(i) *g is w-inf-locally-compact.*
(ii) *g^* is $\tau(X^*, X)$-quasicontinuous.*

Lemma 4.2 (see, e.g., [23, Proposition 6], [24, Theorem 2.4]). *Let $p, q : Y \to \overline{\mathbb{R}}$ be convex. Then,*

$$p \leq q \text{ and } q \text{ quasicontinuous} \Rightarrow p \text{ quasicontinuous}.$$

Lemma 4.3 ([24, Theorem 3.3]). *Let $h : Y \to \overline{\mathbb{R}}$ be convex, quasicontinuous, and let $y_0 \in Y$ be such that $h(y_0) > -\infty$ and $\operatorname{cl}\operatorname{cone}(\operatorname{dom} h - y_0)$ is a linear subspace. Then, h is subdifferentiable at y_0 and $\partial h(y_0)$ is the sum of a w^*-compact convex set and a finite dimensional linear subspace.*

Remark 4.4. Under the assumptions of Lemma 4.3 one has $y_0 \in \operatorname{ri}\operatorname{dom} h$ ([24, Proposition II.1, Theorem 3.2]) and so the restriction of h to $\operatorname{aff}\operatorname{dom} h$ is continuous at y_0. If, moreover, $\operatorname{cl}\operatorname{cone}(\operatorname{dom} h - y_0) = Y$, then $\operatorname{aff}\operatorname{dom} h = Y$ (recall that $\operatorname{aff}\operatorname{dom} h$ is closed) and h is continuous at y_0. It follows that $\partial h(y_0)$ is non-empty and w^*-compact.

Throughout this section we consider again the problem (P) in (1), assuming that C is closed and $f, f_t \in \Gamma(X)$. We intend to apply Lemma 4.3 to the convex proper function h defined in (10):

$$h := \inf_{\lambda \in \mathbb{P}(T)} \left(f_C + \sum_{t \in T} \lambda_t f_t \right)^* .$$

To this end we assume that

$$\exists \overline{\lambda} \in \mathbb{P}(T) : f_C + \sum_{t \in T} \overline{\lambda}_t f_t \text{ is } w-\text{inf-locally-compact.} \tag{14}$$

By Lemma 4.1, $\left(f_C + \sum_{t \in T} \overline{\lambda}_t f_t\right)^*$ is $\tau(X^*, X)$-quasicontinuous and, by Lemma 4.2, h is $\tau(X^*, X)$-quasicontinuous too.

We now use Lemma 4.3 to provide a subdifferentiability criterion for the function h at 0_{X^*}. To this end the following condition must be satisfied (remember that

$h(0_{X^*}) = -\sup(\Delta) \geq -\inf(P) > -\infty$):

$$\mathrm{cl}^{w^*} \operatorname{cone} \operatorname{dom} h \text{ is a linear subspace.} \tag{15}$$

Condition (15) amounts to saying that 0_{X^*} belongs to the quasi relative interior of $\operatorname{dom} h$ in the sense of [3]. In order to guarantee this condition we shall apply the following lemma.

Lemma 4.5. *For any $\lambda \in \mathbb{P}(T)$ one has*

$$\mathrm{cl}^{w^*} \operatorname{dom} \left(f_C + \sum_{t\in T} \lambda_t f_t \right)^* = \mathrm{cl}^{w^*} \left(\operatorname{dom} f^* + b\,(C) + \sum_{t\in T} \lambda_t \operatorname{dom} f_t^* \right), \tag{16}$$

where $b\,(C) = \operatorname{dom} i_C^*$ denotes the barrier cone of C.

Proof. Let $\lambda \in \mathbb{P}(T)$. If $k := f^* \square i_C^* \square \left(\square_{t\in T} (\lambda_t f_t)^* \right)$, one has $k^* = f_C + \sum_{t\in T} \lambda_t f_t$ with k convex and $\operatorname{dom} k^* \neq \emptyset$ by (2). It follows that

$$\begin{aligned} \mathrm{cl}^{w^*} \operatorname{dom} \left(f_C + \sum_{t\in T} \lambda_t f_t \right)^* &= \mathrm{cl}^{w^*} \operatorname{dom} k^{**} \\ &= \mathrm{cl}^{w^*} \operatorname{dom} k \\ &= \mathrm{cl}^{w^*} \left(\operatorname{dom} f^* + b\,(C) + \sum_{t\in T} \lambda_t \operatorname{dom} f_t^* \right). \qquad \square \end{aligned}$$

Now we can provide an explicit formula for $\mathrm{cl}^{w^*} \operatorname{cone} \operatorname{dom} h$.

Lemma 4.6. *The following equation holds:*

$$\mathrm{cl}^{w^*} \operatorname{cone} \operatorname{dom} h = \mathrm{cl}^{w^*} \left(\operatorname{cone} \operatorname{dom} f^* + b\,(C) + \operatorname{cone} \left(\bigcup\nolimits_{t\in T} \operatorname{dom} f_t^* \right) \right).$$

Proof. By Lemma 4.5, and the observations made in Section 2,

$$\begin{aligned} \mathrm{cl}^{w^*} \operatorname{cone} \operatorname{dom} h &= \mathrm{cl}^{w^*} \operatorname{cone} \bigcup_{\lambda\in\mathbb{P}(T)} \mathrm{cl}^{w^*} \left(\operatorname{dom} f^* + b\,(C) + \sum_{t\in T} \lambda_t \operatorname{dom} f_t^* \right) \\ &= \mathrm{cl}^{w^*} \operatorname{cone} \bigcup_{\lambda\in\mathbb{P}(T)} \left(\operatorname{dom} f^* + b\,(C) + \sum_{t\in T} \lambda_t \operatorname{dom} f_t^* \right) \\ &= \mathrm{cl}^{w^*} \operatorname{cone} \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \left(\operatorname{dom} f^* + b\,(C) + \sum_{t\in T} \lambda_t \operatorname{dom} f_t^* \right) \\ &= \mathrm{cl}^{w^*} \left(\operatorname{cone} \operatorname{dom} f^* + b\,(C) + \operatorname{cone} \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \lambda_t \operatorname{dom} f_t^* \right) \\ &= \mathrm{cl}^{w^*} \left(\operatorname{cone} \operatorname{dom} f^* + b\,(C) + \operatorname{cone} \left(\bigcup_{t\in T} \operatorname{dom} f_t^* \right) \right). \qquad \square \end{aligned}$$

Now we provide the min-sup duality theorem in convex infinite programming without inf-compactness.

Theorem 4.7. *Assume that* (14) *holds and that*

$$\mathrm{cl}^{w^*}\left(\operatorname{cone}\operatorname{dom} f^* + b\,(C) + \operatorname{cone}\left(\bigcup\nolimits_{t\in T} \operatorname{dom} f_t^*\right)\right) \text{ is a linear subspace.} \tag{17}$$

Then (11) *holds. Moreover, $S\,(P)$ is the sum of a non-empty weakly compact convex set and a finite dimensional linear subspace.*

Proof. According to (14), Lemma 4.1 and Lemma 4.2, we have that h is $\tau\,(X^*, X)$ -quasicontinuous at 0_{X^*}. According to (17), Lemma 4.6, and Lemma 4.3, we then obtain that $\partial h\,(0_{X^*})$ is the sum of a non-empty weakly compact convex set and a finite dimensional linear subspace. We complete the proof of Theorem 4.7 by using Lemma 3.2. □

The next result states that if one replaces the condition $\overline{\lambda} \in \mathbb{P}(T)$ by $\overline{\lambda} \in \mathbb{R}_+^{(T)}$ in (14) then the conclusions of Theorem 4.7 still holds.

Theorem 4.8. *Assume that f_C is w-inf-locally-compact and that* (17) *holds. Then the conclusions of Theorem* 4.7 *are also true.*

Proof. Pick $\overline{\lambda} \in \mathbb{P}(T)$ and, for each $t \in T$ such that $\overline{\lambda}_t > 0$, take a continuous affine minorant $x_t^* - r_t$ of f_t. We thus have

$$\varphi := f_C + \sum_{t\in T} \overline{\lambda}_t f_t \geq f_C + \sum_{t\in T} \overline{\lambda}_t x_t^* - r =: \psi,$$

with $r := \sum_{t\in T} \overline{\lambda}_t r_t$. Since f_C is w-inf-locally-compact, ψ enjoys the same property ([6, Lemma I-13]). Now $\varphi \in \Gamma(X)$ is minorized by ψ which is w-inf-locally-compact. Consequently, φ is w-inf-locally-compact too, entailing (14). Then, Theorem 4.8 follows from Theorem 4.7. □

Remark 4.9. Observe that f_C is w-inf-locally-compact whenever C is locally compact.

Remark 4.10. If either f_C is w-inf-compact or there exists $\lambda \in \mathbb{P}(T)$ such that $f_C + \sum_{t\in T} \lambda_t f_t$ is w-inf-compact one has $\mathrm{cl}^{w^*} \operatorname{cone}\operatorname{dom} h = X^*$. According to Remark 4.4, h is then continuous at 0_{X^*}. In this way one can derive Theorem 3.3 (resp. Corollary 3.4) from Theorem 4.7 (resp. Theorem 4.8).

Remark 4.11. When X is finite dimensional, we deal with convex semi-infinite programming. In such a frame, (14) always holds and condition (17) can be reformulated as

$$\operatorname{cone}\operatorname{dom} f^* + b\,(C) + \operatorname{cone}\left(\bigcup\nolimits_{t\in T} \operatorname{dom} f_t^*\right) \quad \text{is a linear subspace,}$$

or, equivalently,

$$0_{\mathbb{R}^n} \in \operatorname{ri}\left(\operatorname{cone}\operatorname{dom} f^* + b\,(C) + \operatorname{cone}\left(\bigcup\nolimits_{t\in T} \operatorname{dom} f_t^*\right)\right),$$

or, equivalently,

$$0_{\mathbb{R}^n} \in \operatorname{ri}\operatorname{cone}\operatorname{dom} f^* + \operatorname{ri} b\left(C\right) + \operatorname{ri}\operatorname{cone}\left(\bigcup\nolimits_{t\in T} \operatorname{dom} f_t^*\right). \tag{18}$$

The set in (18) only depends on f, C, and F. Indeed, we can represent F by means of the linear system $\{u(x) \leq f_t^*(u),\ u \in \operatorname{dom} f_t^*,\ t \in T\}$ (see, e.g., [7, Section 3] or [16, (8.14)]; then, as a consequence of [16, Theorem 8.1(iv)],

$$\operatorname{ri}\operatorname{cone}\left(\bigcup\nolimits_{t\in T} \operatorname{dom} f_t^*\right) = \operatorname{ri}\operatorname{cone}\{x_s^* : s \in S\}$$

for any linear system $\{\langle x_s^*, x\rangle \leq r_s,\ \ s \in S\}$ whose solution set is F. Consequently, $\operatorname{ri}\operatorname{cone}\left(\bigcup_{t\in T} \operatorname{dom} f_t^*\right)$ is the same for all convex representations of F.

We now point out the dual form of condition (17).

Lemma 4.12. *Condition* (17) *holds if and only if*

$$[f_\infty \leq 0] \cap C_\infty \cap \left(\bigcap\nolimits_{t\in T} [(f_t)_\infty \leq 0]\right) \quad \textit{is a linear subspace.} \tag{19}$$

Proof. Since $C_\infty = (b\left(C\right))^-$, one has

$$\begin{aligned}
&\operatorname{cl}^{w^*}\left(\operatorname{cone}\operatorname{dom} f^* + b\left(C\right) + \operatorname{cone}\left(\bigcup\nolimits_{t\in T} \operatorname{dom} f_t^*\right)\right)\\
=\ &\operatorname{cl}^{w^*}\left(\operatorname{cl}^{w^*}\operatorname{cone}\operatorname{dom} f^* + \operatorname{cl}^{w^*} b\left(C\right) + \operatorname{cl}^{w^*}\operatorname{cone}\left(\bigcup\nolimits_{t\in T} \operatorname{dom} f_t^*\right)\right)\\
=\ &\operatorname{cl}^{w^*}\left([f_\infty \leq 0]^- + C_\infty^- + \left(\bigcap\nolimits_{t\in T} [(f_t)_\infty \leq 0]^-\right)\right)\\
=\ &\left([f_\infty \leq 0] \cap C_\infty \cap \left(\bigcap\nolimits_{t\in T} [(f_t)_\infty \leq 0]\right)\right)^-.
\end{aligned}$$

Now the closed convex cone $[f_\infty \leq 0] \cap C_\infty \cap \left(\bigcap_{t\in T} [(f_t)_\infty \leq 0]\right)$ is a linear subspace if and only if its negative dual is a linear subspace too, that is, if and only if (19) holds. □

Remark 4.13. The set in (19) depends on f, C, and $F = \bigcap_{t\in T} [f_t \leq 0]$, but not on the specific convex representation of F, $\{f_t\left(x\right) \leq 0, t \in T\}$, as we have $\bigcap_{t\in T} [(f_t)_\infty \leq 0] = F_\infty$. Thus, its negative dual, the set in (17), depends on f, C, and F too.

Let us now consider the ordinary convex program (P_2) we obtain from (P) in (1) by taking $X = \mathbb{R}^n$ and $T = \{1, ..., m\}$:

$$\begin{array}{lll}
(P_2) & \text{Minimize} & f(x)\\
 & \text{subject to} & f_i(x) \leq 0,\ i = 1, ..., m,\\
 & & x \in C.
\end{array}$$

The corresponding dual problem amounts to

$$(D_2) \quad \text{Maximize} \quad \inf_{x\in C}\left(f(x)+\sum_{i=1}^{m}\lambda_i f_i(x)\right)$$
$$\text{subject to} \quad \lambda=(\lambda_1,...,\lambda_m)\in\mathbb{R}_+^m.$$

According to the general dual problem (D) in (3) one has to use the rule $0\times(+\infty)=0$ in the formulation of (D_2).

Applying Theorem 5.2 and Lemma 4.12 we get a very general form of Clark-Duffin Theorem ([10, Theorem 2]) to be compared with [29, Corollary 27.3.3], [18, Theorem 5.1], [1, Proposition 5.3.3], [4, Theorem 4.3.8] and, very recently, [12, Theorem 3.1].

Corollary 4.14 (Generalized Clark-Duffin Theorem). *Let $f, f_1, ..., f_m \in \Gamma(\mathbb{R}^n)$ and C convex closed in $\mathbb{R}^n$ be such that*

$$[f_\infty \le 0]\cap C_\infty \cap \left(\bigcap_{i=1,\dots,m}[(f_i)_\infty \le 0]\right)$$

is a linear subspace. Then $\min(P_2)=\sup(D_2)=\sup(\Delta_2)\in\mathbb{R}$. Moreover, $S(P_2)$ is the sum of a non-empty compact convex set and a linear subspace.

Let us apply Theorem 4.8 to the linear infinite programming (LIP) problem (P_1) in (13), with X infinite dimensional, where C is a closed convex cone.

Corollary 4.15. *Assume that the cone C is w-locally-compact and that*

$$[c^*\le 0]\cap C\cap\left(\bigcap\nolimits_{t\in T}[x_t^*\le 0]\right) \quad \textit{is a linear subspace.} \tag{20}$$

Then,

$$\begin{aligned}\min(P_1) &= \sup_{\lambda\in\mathbb{R}_+^{(T)}}\left(-\sum_{t\in T}\lambda_t r_t : \sum_{t\in T}\lambda_t x_t^*\in C^+ - c^*\right)\\ &= \sup_{\lambda\in\mathbb{P}(T)}\left(-\sum_{t\in T}\lambda_t r_t : \sum_{t\in T}\lambda_t x_t^*\in C^+ - c^*\right).\end{aligned}$$

Moreover, $S(P_1)$ is the sum of a non-empty w-compact convex set and a finite dimensional linear subspace.

Proof. Let us apply Theorem 4.8 with $f=c^*$, $f_t=x_t^*-r_t$, $t\in T$. Since $f_\infty=c^*$, $C_\infty=C$, and $(f_t)_\infty=x_t^*$, $t\in T$, condition (20) coincides with (17). Since C is w-locally-compact, f_C will be w-inf-locally-compact, and Theorem 4.8 applies. Now, for each $\lambda\in\mathbb{R}_+^{(T)}$,

$$\inf_{x\in C}\left(\langle c^*,x\rangle+\sum_{t\in T}\lambda_t(\langle x_t^*,x\rangle-r_t)\right)=-\sum_{t\in T}\lambda_t r_t+\inf_{x\in C}\left\langle c^*+\sum_{t\in T}\lambda_t x_t^*,x\right\rangle,$$

but

$$\inf_{x\in C}\left\langle c^* + \sum_{t\in T}\lambda_t x_t^*, x\right\rangle = \begin{cases} 0, & \text{if } c^* + \sum\limits_{t\in T}\lambda_t x_t^* \in C^+, \\ -\infty, & \text{otherwise,} \end{cases}$$

and we are done. □

Remark 4.16. When we are dealing with the LSIP problem (P_1) in (13) with C closed convex cone, condition (20), and its reformulation (18), are equivalent to

$$-c^* \in \operatorname{ri} C^- + \operatorname{ri}\operatorname{cone}\{x_t^* : t \in T\}. \tag{21}$$

If moreover, $C = \mathbb{R}^n$, (20) is equivalent to

$$-c^* \in \operatorname{ri}\operatorname{cone}\{x_t^* : t \in T\}, \tag{22}$$

condition guaranteeing that $\min(P_1) = \sup(D_1) \in \mathbb{R}$ and $S(P_1)$ is the sum of a compact convex set with a linear subspace by Corollary 4.15. If, additionally, $-c^* \in \operatorname{int}\operatorname{cone}\{x_t^* : t \in T\}$, one has

$$\begin{aligned} 0_{\mathbb{R}^n} \in\ & \operatorname{int}\left(c^* + \{0_{\mathbb{R}^n}\} + \operatorname{cone}\{x_t^* : t \in T\}\right) \\ =\ & \operatorname{int}\left(\operatorname{cone}\operatorname{dom} f^* + b(C) + \operatorname{cone}\left(\bigcup\nolimits_{t\in T}\operatorname{dom} f_t^*\right)\right), \end{aligned}$$

so that, by Lemma 4.6, $\operatorname{cl}\operatorname{cone}\operatorname{dom} h = \mathbb{R}^n$ and $S(P_1)$ is compact by Remark 4.4. Now, let us denote by v the optimal value function of the parametric problem which one obtains replacing the gradient of the objective function in (13) with a parameter ranging on $\mathbb{R}^n$. Then, according to [16, Theorem 8.1] and [15, Lemma 4.3], (22) implies that $\min(P_1) = \sup(D_1) \in \mathbb{R}$ and $S(P_1) = \partial(-v)(c^*)$, where $-v$ is convex and

$$c^* \in -\operatorname{ri}\operatorname{cone}\{x_t^* : t \in T\} = \operatorname{ri}\operatorname{dom}(-v),$$

so that $S(P_1)$ is the sum of a compact convex set with a linear subspace. If additionally, $\operatorname{cone}\{x_t^* : t \in T\}$ is full-dimensional, $S(P_1)$ is compact by [16, Theorem 8.1.6] (note that boundedness can be extended to continuous LIP, see [2, Theorem 5.99]). Thus, Corollary 4.15 allows to recover all known results on minsup duality in LSIP.

5. A necessary and sufficient condition for minsup duality

In this section we consider the problem (P) in (1), assuming that the functions $f, f_t : X \to \mathbb{R}\cup\{+\infty\}$, $t \in T$, are convex (not necessarily lsc) and C is a convex subset (not necessarily closed) of X. Let us introduce the set M defined as

$$M := \bigcap\nolimits_{t\in T}\operatorname{dom} f_t \supset F = \bigcap\nolimits_{t\in T}[f_t \le 0].$$

Let us now consider the problem

$$\begin{array}{lll} (D_0) & \text{Maximize} & \inf_{x\in C\cap M}\left(f(x) + \sum_{t\in T}\lambda_t f_t(x)\right) \\ & \text{subject to} & \lambda \in \mathbb{R}_+^{(T)}. \end{array} \tag{23}$$

It holds that

$$+\infty > \inf(P) \geq \sup(D_0) \geq \sup(D) \geq \sup(\Delta) \geq -\infty,$$

and (D_0) can be seen as another surrogate dual of (P). If $C \subset M = \bigcap_{t\in T} \operatorname{dom} f_t$, then the problems (D) and (D_0) coincide. This is in particular true whenever the functions f_t, $t \in T$, are finite-valued, e.g., in LIP and LSIP. In fact, (D_0) is a *perturbational dual* associated with the mapping $H : M \to \mathbb{R}^T$ defined by $H(x) = (f_t(x))_{t\in T}$ (see, e.g., [5, p. 74]). The *epigraph* of H is the convex set

$$\begin{aligned}\operatorname{epi} H &= \left\{(x,\mu) \in M \times \mathbb{R}^T : f_t(x) \leq \mu_t,\ t \in T\right\} \\ &= \left\{(x,\mu) \in X \times \mathbb{R}^T : f_t(x) \leq \mu_t,\ t \in T\right\}.\end{aligned}$$

Now the function $\Phi : X \times \mathbb{R}^T \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\Phi(x,\mu) = f_C(x) + i_{\operatorname{epi} H}(x,-\mu)$$

is convex and, since $C \cap M \cap \operatorname{dom} f \neq \emptyset$ by (2), we easily get an explicit expression for its conjugate Φ^* : for any $(x^*,\lambda) \in X^* \times \mathbb{R}^{(T)}$, it holds

$$\Phi^*(x^*,\lambda) = \begin{cases} \left(f_{C\cap M} + \sum_{t\in T} \lambda_t f_t\right)^*(x^*), & \text{if } (x^*,\lambda) \in X^* \times \mathbb{R}_+^{(T)}, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $f_{C\cap M} := f_C + i_M = f + i_{C\cap M}$.

Considering the optimal value function $m : \mathbb{R}^T \to \overline{\mathbb{R}}$ given by

$$m(\mu) := \inf_{x\in X} \Phi(x,\mu),$$

one has, classically, $m(0_T) = \inf(P)$, $m^* = \Phi^*(0_{X^*},\cdot)$ and $m^{**}(0_T) = \sup(D_0)$. Let us assume that

$$\exists \overline{\lambda} \in \mathbb{R}_+^{(T)} : \inf_{C\cap M} \left(f + \sum_{t\in T} \overline{\lambda}_t f_t \right) > -\infty, \tag{24}$$

or, equivalently, that $\operatorname{dom} m^* \neq \emptyset$, or, equivalently, that $\sup(D_0) > -\infty$. Since $\sup(D_0) \leq \inf(P) < +\infty$, we then have $r_0 := \sup(D_0) \in \mathbb{R}$ and so $(0_T, r_0) \in \operatorname{epi} m^{**}$.

Now, since m is convex (because Φ is so [33, Theorem 2.1.3(v)]) and $m^{**}(0_T) \in \mathbb{R}$, one has (according to [33, Theorem 2.3.4(i)]) $m^{**} = \overline{m}$ (the lsc hull of m) and $(0_T, r_0) \in \operatorname{epi} \overline{m} = \operatorname{cl} \operatorname{epi} m$. Denoting by $P_{\mathbb{R}^T\times\mathbb{R}}$ the projection of $X \times \mathbb{R}^T \times \mathbb{R}$ onto $\mathbb{R}^T \times \mathbb{R}$, by [33, (2.8)] and the fact that epigraphs and strict epigraphs have the same closure, one has that $\operatorname{cl} \operatorname{epi} m = \operatorname{cl} P_{\mathbb{R}^T\times\mathbb{R}} \operatorname{epi} \Phi$ and, consequently, $(0_T, r_0)$ belongs to $\operatorname{cl} P_{\mathbb{R}^T\times\mathbb{R}} \operatorname{epi} \Phi$:

$$(0_T, \sup(D_0)) \in \operatorname{cl} \left(\bigcup_{x\in C} \left\{(\mu,r) \in \mathbb{R}^T \times \mathbb{R} : f(x) \leq r, f_t(x) \leq -\mu_t, t \in T\right\} \right). \tag{25}$$

Remark 5.1. In the present setting, the minsup duality property coincides with the so-called d-stability at point 0_T, a notion introduced in [26]. The d-stability on the whole of the perturbational space has been studied in several papers ([32, Theorem 2.1], [21, Theorem 3.2], [27, Theorem 2.1], etc.).

We now characterize the minsup duality property in convex infinite programming. This characterization is based on the notion of closedness regarding to a set (see [5, Section 9]) that we recall: having R, a topological space, and two subsets A, B of R, one says that A is closed regarding B if $B \cap \operatorname{cl} A = B \cap A$.

Theorem 5.2. *Assume that* (24) *holds. Then the following sentences are equivalent:*

(i) $\min(P) = \sup(D_0) \in \mathbb{R}$, *that is, there exists* $\overline{x} \in E = C \cap F \cap \operatorname{dom} f$ *such that* $f(\overline{x}) = \sup(D_0)$.

(ii) *The set*

$$\mathfrak{C} := \bigcup_{x \in C} \left\{ (\mu, r) \in \mathbb{R}^T \times \mathbb{R} : f(x) \leq r, f_t(x) \leq -\mu_t, t \in T \right\} \tag{26}$$

is closed regarding to $\{0_T\} \times \{\sup(D_0)\}$.

Proof. By (25), condition (ii) is equivalent to the existence of $\overline{x} \in C$ such that $f(\overline{x}) \leq \sup(D_0)$, $f_t(x) \leq 0$, $t \in T$, that means that there exists $\overline{x} \in E = C \cap F \cap \operatorname{dom} f$ such that $\inf(P) \leq f(\overline{x}) \leq \sup(D_0) \leq \inf(P)$ which is equivalent to (i). $\square$

Remark 5.3. In the condition (ii) of Theorem 5.2, one can replace $\{0_T\} \times \{\sup(D_0)\}$ by $\{0_T\} \times [\sup(D_0), +\infty[$ or by $\{0_T\} \times \mathbb{R}$. In fact, (ii) means that the intersection of the set $\mathfrak{C}$ in (26) with the line $\{0_T\} \times \mathbb{R}$ is equal to $\{0_T\} \times [\sup(D_0), +\infty[$ (recall that $\sup(D_0) \in \mathbb{R}$).

As an immediate corollary of Theorem 5.2, and according to Remark 5.3, we have the following non-standard Farkas lemma:

Corollary 5.4. *Assume that* (24) *holds. Then the following sentences are equivalent:*

(i) *For any* $\alpha \in \mathbb{R}$,

$$\left(\inf_{C \cap M} \left(f + \sum_{t \in T} \lambda_t f_t \right) \leq \alpha \ \forall \lambda \in \mathbb{R}_+^{(T)} \right) \Leftrightarrow \left(\exists \overline{x} \in C \cap F : f(\overline{x}) \leq \alpha \ \right).$$

(ii) *For any* $\alpha \in \mathbb{R}$,

$$\left(x \in C \cap F \Rightarrow f(x) > \alpha \right) \Leftrightarrow \left(\exists \lambda \in \mathbb{R}_+^{(T)} : \inf_{C \cap M} \left(f + \sum_{t \in T} \lambda_t f_t \right) > \alpha \right).$$

(*iii*) *The set* $\mathfrak{C}$ *in* (26) *satisfies*

$$\mathfrak{C} \cap \{0_T\} \times \mathbb{R} = \{0_T\} \times [\sup(D_0), +\infty[\,.$$

(*iv*) *The set* $\mathfrak{C}$ *in* (26) *is closed regarding* $\{0_T\} \times \mathbb{R}$.

Let us return to the LIP problem (P_1) defined in (13), with C being an arbitrary convex cone. We assume that

$$\exists \overline{\lambda} \in \mathbb{R}_+^{(T)} : c^* + \sum_{t \in T} \lambda_t x_t^* \in C^+. \tag{27}$$

Condition (27) can clearly be reformulated as

$$c^* \in C^+ - \operatorname{cone}\{x_t^* : t \in T\}. \tag{28}$$

We are now in position to characterize the primal attainment in LIP duality in full generality.

Theorem 5.5. *Assume that* (28) *holds. Then, the following statements are equivalent:*

(*i*) $\exists \overline{x} \in C$, $\langle x_t^*, \overline{x} \rangle \leq r_t$, $t \in T$, *such that*

$$\langle c^*, \overline{x} \rangle = \min(P_1) = \sup_{\lambda \in \mathbb{R}_+^{(T)}} \left(-\sum_{t \in T} \lambda_t r_t : \sum_{t \in T} \lambda_t x_t^* \in C^+ - c^* \right).$$

(*ii*) *The set*

$$\left\{ \left((x_t^*(x))_{t \in T}, c^*(x) \right) : x \in C \right\} + \mathbb{R}_+^T \times \mathbb{R}_+$$

is closed regarding $\left\{ (r_t)_{t \in T} \right\} \times \mathbb{R}$.

Proof. Let us explicit the set $\mathfrak{C}$ in Theorem 5.2:

$$\begin{aligned} \mathfrak{C} &= \bigcup_{x \in C} \left(\left\{ \left((r_t - x_t^*(x))_{t \in T}, c^*(x) \right) - \mathbb{R}_+^T \times \mathbb{R}_+ \right\} \right) \\ &= \left(\left\{ (r_t)_{t \in T} \right\} \times \{0\} \right) + \bigcup_{x \in C} \left(\left\{ \left((-x_t^*(x))_{t \in T}, c^*(x) \right) \right\} \right) - \mathbb{R}_+^T \times \mathbb{R}_+. \end{aligned}$$

Now $\mathfrak{C}$ is closed regarding $\{0_T\} \times \mathbb{R}$ if and only if the set

$$\bigcup_{x \in C} \left(\left\{ \left((-x_t^*(x))_{t \in T}, c^*(x) \right) \right\} \right) - \mathbb{R}_+^T \times \mathbb{R}_+$$

is closed regarding $-\left\{ (r_t)_{t \in T} \right\} \times \mathbb{R}$, that holds if and only if (via the homeomorphism $(\mu, r) \mapsto (-\mu, r)$) the set

$$\bigcup_{x \in C} \left(\left\{ \left((x_t^*(x))_{t \in T}, c^*(x) \right) \right\} \right) + \mathbb{R}_+^T \times \mathbb{R}_+$$

is closed regarding $\left\{ (r_t)_{t \in T} \right\} \times \mathbb{R}$.

As in the proof of Corollary 4.15 we quote that the dual of (P_1), (D_1), is nothing else than the Haar's dual of (P_1), i.e.,

$$\begin{array}{ll} \text{Maximize} & \sum_{t\in T} \lambda_t r_t \\ \text{subject to} & \lambda \in \mathbb{R}_+^{(T)}, \sum_{t\in T} \lambda_t x_t^* \in C^+ - c^*, \end{array}$$

and the proof is achieved, applying Theorem 5.2. □

Remark 5.6. Let us introduce the linear mapping $L : X \longrightarrow \mathbb{R}^T \times \mathbb{R}$ such that $L(x) = \left((x_t^*(x))_{t\in T}, c^*(x)\right)$, which is continuous (equipping $\mathbb{R}^T$ with the weak topology). Then, condition (ii) in Theorem 5.5 amounts to

(ii') $L(C) + \mathbb{R}_+^T \times \mathbb{R}_+$ is closed regarding $\left\{(r_t)_{t\in T}\right\} \times \mathbb{R}$.

Condition (ii') is obviously satisfied whenever

$$L(C) + \mathbb{R}_+^T \times \mathbb{R}_+ \text{ is closed.} \tag{29}$$

Since L is continuous we have at hand closedness criteria for $L(C)$, and then closedness criteria for the sum $L(C) + \mathbb{R}_+^T \times \mathbb{R}_+$.

Remark 5.7. In order to make the link between the approach followed in Section 5 and the ones in Sections 3 and 4, let us assume, as in the latter sections, that the convex functions f, f_t $(t \in T)$ are lsc (which is not required in Section 5). We thus have $\Phi \in \Gamma\left(X \times \mathbb{R}^T\right)$. Let us introduce the problem

$$\begin{array}{lll} (Q) & \text{Minimize} & \Phi^*(0_{X^*}, \lambda) \\ & \text{subject to} & \lambda \in \mathbb{R}^{(T)}, \end{array}$$

and the corresponding (convex) value function

$$k(x^*) := \inf_{\lambda\in\mathbb{R}^{(T)}} \Phi^*(x^*, \lambda), \quad x^* \in X^*.$$

It holds that $h \leq k$ and $k^* = h^* (= f_C + i_F)$. So, if $\partial h(0_{X^*}) \neq \emptyset$, then $\partial h(0_{X^*}) = \partial k(0_{X^*})$ is the optimal solution set of the perturbational dual problem (Q') associated with (Q), namely (since $\Phi^{**} = \Phi$),

$$\begin{array}{lll} (Q) & \text{Maximize} & -\Phi(x, 0_T) \\ & \text{subject to} & x \in X, \end{array}$$

which is nothing else but the opposite of the problem (P).

Acknowledgements. The authors are very grateful to an anonymous referee for his judicious observations, which motivated Remarks 4.10, 5.1, 5.7, and also for a notable simplification of the proof of Theorem 3.7 via Lemma 3.6.

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