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Expectations of Random Sets in Banach Spaces

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The Aumann and Herer expectations represent two different approaches to defining the expectation of a random set (analytical versus geometrical). This paper investigates their relationships in the setting of Banach spaces, which are shown to be related to the geometry of the dual unit ball, to the bornological differentiability properties of the norm, and to the kind of sets the random set takes on as values. We also show that both expectations are identical for almost all (in the Baire sense) norms equivalent to any given norm provided the dual is strongly separable, and that in general the Herer expectation is the ball hull of the Aumann expectation for almost all norms. The Aumann expectation can be regarded as a special case of the Herer expectation after an embedding into a suitably larger space. This suggests that the Herer expectation is a suitable extension of Aumann's to non-linear metric spaces.

Keywords: Aumann expectation, Herer expectation, intersection of balls, Mazur Intersection Property, random set, weak* denting point

1. Introduction

The definitions of the Aumann and Herer expectations of a random set are very different. The well-known Aumann expectation, either in his original definition or in Debreu's (equivalent in non-atomic probability spaces), relies on integration in Banach spaces, be it by integrating the selections of the random set or by using an embedding into a larger Banach space where integrals are available. Thus it can be called an *analytic* approach to the expectation of a random set. On the other hand, Herer's relies not on the linear structure but on the metric structure of the space. Explicitly defined as the locus of the points whose expected distances to the random set satisfy a certain condition, it is a *geometric* approach.

Since the Herer expectation can be defined in a general metric space, its scope of application goes beyond linear spaces. In order to establish whether it is a good substitute for Aumann's in such spaces, one has to inquire first how they are related in Banach spaces. The Herer expectation is always convex, and is a proper subset of the whole space for integrably bounded random sets. In its turn,

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the Aumann expectation is convex too under mild conditions but can be empty in non-separable spaces. Therefore the natural setting for a comparison is that of integrably bounded random closed sets in a separable Banach space.

From a result of Bru, Heinich and Lootgieter [2], the Herer expectation of a random singleton $\{\xi\}$ equals the singleton of its Bochner integral $\{E\xi\}$. It follows then that the Aumann expectation is a subset of the Herer expectation. The study of conditions for the reverse inclusion was initiated by Hess [15] (see also [1, Theorem 3.10.3]) and highlighted in [21, Open Problem 3.11, p. 190].

Hess showed that both expectations are identical in Hilbert spaces, provided the random set satisfies a square integrability condition. Then Terán [26] characterized the Banach spaces where the equality holds for random compact sets as those with the K-Mazur Intersection Property, and for Hausdorff approximable random closed bounded sets[1] as those with the Mazur Intersection Property (or MIP: every closed bounded set is the intersection of a family of balls).

This paper is concerned with the more general situation of spaces that may fail the Mazur Intersection Property. That is a natural question since spaces as simple as $\mathbb{R}^2$ with the maximum norm do not have the MIP. To that end, the results in [27] will prove useful.

The Herer expectation covers Aumann's and is, by definition, the intersection of a family of balls. Hence the simplest possible relationship, when both expectations are not identical, happens when the Herer expectation is the ball hull of the Aumann expectation (i.e. the intersection of all closed balls covering it). The main effort goes along three lines:

1. Sufficient conditions on the norm for the equality between the Herer expectation and the ball hull of the Aumann expectation.
2. Sufficient conditions on the kind of sets the random set takes on as values.
3. General inclusion relationships valid without restricting either the norm or the possible values of the random set.

The structure of the paper is as follows. After the preliminaries, in Section 3 we apply the bornological differentiability properties of the norm to give answers to 1. and 2. above. In essence, bornological differentiability on a dense set yields the sought equality for random closed sets whose values are elements of the bornology.

We also show that, for all but a meager set of renormings of any given norm, it holds that both expectations are identical (if the dual is separable) or at least the Herer expectation is the ball hull of the Aumann expectation (in the general case). Moreover, the Aumann expectation can always be regarded as a special case of the Herer expectation, by way of an embedding into a larger space. If one is interested only in the expectations of a sequence of random sets, then the larger space can still be taken separable.

[1]The result was claimed without the Hausdorff approximability assumption, which is nonetheless necessary to avoid a gap in the proof. Theorem 3.3(b.3) in this paper will show with a different proof that the result as stated was correct.

Due to their very different definitions, the fact that such strong relationships exist is positive and supports using the Herer expectation in non-linear metric spaces.

Section 4 is about goal 3. above. A key tool is a generalization of the weak* denting points of the dual unit ball, which we call Mazur functionals. We show that, for an arbitrary norm, the Herer expectation is bounded between the ball hull and a suitable H-convex hull of the Aumann expectation. That result is then applied to goal 1. in polyhedral and Mazur spaces, and to goal 2. for random Mazur sets.

Readers unfamiliar with random sets or the MIP and related properties are referred to [21] (especially Chapter 2) and the survey [10], as well the recent papers [11, 12, 26, 27].

2. Preliminaries

Unless otherwise stated, $\mathbb{E}$ denotes a separable Banach space with closed unit ball B, unit sphere S, and dual $\mathbb{E}^*$ with unit ball B^* and unit sphere S^*. The norm of $\mathbb{E}$ is denoted by $\|\cdot\|$.

A Banach space where convex continuous real functions defined on an open subset are Fréchet differentiable in a dense G_δ subset of their domain is called an *Asplund space*. If $\mathbb{E}$ is separable, then $\mathbb{E}$ is Asplund if and only if $\mathbb{E}^*$ is separable.

A biorthogonal system $\{x_i, f_i\}_i \subset \mathbb{E} \times \mathbb{E}^*$ is called *fundamental* if the linear span of $\{x_i\}_i$ is dense in $\mathbb{E}$. A subset $\mathcal{H} \subset S^*$ is called *norming* (or 1-norming) if $\|x\| = \sup_{f\in\mathcal{H}} |f(x)|$ for each $x \in \mathbb{E}$.

Spaces of closed sets. We adopt the following notation:

$\mathcal{F}_b$: The family of all non-empty closed bounded subsets of $\mathbb{E}$.

$\mathcal{F}_{bc}$: The family of all non-empty closed bounded convex subsets of $\mathbb{E}$.

$\mathcal{F}_c$: The family of all non-empty closed convex subsets of $\mathbb{E}$.

$\mathcal{K}$: The family of all non-empty compact subsets of $\mathbb{E}$.

$\mathcal{K}_c$: The family of all non-empty compact convex subsets of $\mathbb{E}$.

$\mathcal{WK}$: The family of all non-empty weakly compact subsets of $\mathbb{E}$.

$\mathcal{WK}_c$: The family of all non-empty weakly compact convex subsets of $\mathbb{E}$.

$\mathcal{B}$: The family of all closed balls $B(a, \varepsilon)$ for $a \in \mathbb{E}$, $\varepsilon > 0$.

For $A, C \in \mathcal{F}_b$, we denote by $A + C$ the closure of $\{x + y \mid x \in A, y \in C\}$.

The closure of A will be denoted by $\operatorname{cl} A$, its closed convex hull by $\overline{\operatorname{co}}\, A$, and the quantity $\|A\| = \sup_{x\in A} \|x\|$ will be called the *norm* of A. The *support function* of A is the mapping $s(\cdot, A) : S^* \to \mathbb{R}$ defined by $s(f, A) = \sup f(A)$.

We denote by d_H the Hausdorff metric in $\mathcal{F}_b$, given by

$$d_H(A, C) = \max\{\sup_{x\in A} \inf_{y\in C} \|x - y\|, \sup_{x\in C} \inf_{y\in A} \|x - y\|\}.$$

Random sets. A *random closed set* is a mapping X on a measurable space $(\Omega, \mathcal{A})$ whose values are closed subsets and which is *Effros measurable*: the events

$$\{\omega \in \Omega \mid X(\omega) \cap U \neq \emptyset\}$$

are measurable for each open $U \subset \mathbb{E}$. Borel functions from $(\Omega, \mathcal{A})$ to $(\mathcal{F}_b, d_H)$ are random closed sets.

The space of all random closed bounded sets X with $E\|X\| < \infty$ will be denoted by $L^1(\mathcal{F}_b)$; the underlying probability space $(\Omega, \mathcal{A}, P)$ is fixed and not indicated explicitly, while the codomain is indicated between parentheses. A random closed set X in $L^1(\mathcal{F}_b)$ is called *integrably bounded*. To avoid confusion, the usual Lebesgue spaces of real functions on $(\Omega, \mathcal{A}, P)$ will be denoted by just L^p.

If X is additionally assumed to be *Hausdorff approximable* (i.e. approximable by simple functions in the Hausdorff metric), we write $X \in L^1_H(\mathcal{F}_b)$. Under the set-theoretical assumption that the cardinality of the continuum is measure-free (see e.g. [8, Proposition 438D] or [6, Appendix C, Theorem 3.2]), every Borel random closed bounded set becomes Hausdorff approximable.

If $\mathcal{C} \subset \mathcal{F}_b$, then $L^1(\mathcal{C})$ will denote the family of all $X \in L^1(\mathcal{F}_b)$ which are almost surely $\mathcal{C}$-valued, and analogously for $L^1_H(\mathcal{C})$.

A *selection* of X is a random element ξ such that $\xi \in X$ almost surely. If X is integrably bounded, its *Aumann expectation* is the closure of the set of all Bochner expectations of integrable selections of X. It is denoted $E_A X$, and it is a bounded set.

The Herer expectation of an integrably bounded X [13, 14] is given by

$$E_H X = \{x \in \mathbb{E} \mid \forall a \in \mathbb{E},\ \|x - a\| \leq E d_H(X, \{a\})\}.$$

When need arises to specify the norm, we will write $E_H[X; \|\cdot\|]$.

Intersections of balls. The family of all sets that can be written as intersections of closed balls is denoted by $\mathcal{M}$. The space $\mathcal{M}$ is called *sum stable* if $\mathcal{M} + \mathcal{M} = \mathcal{M}$. If every closed bounded convex set is in $\mathcal{M}$, then $\mathbb{E}$ is said to have the *Mazur Intersection Property (MIP)*. We denote by β the function mapping each element of $\mathcal{F}_b$ to the intersection of all balls covering it (its *ball hull*). Observe that $E_H X$ is always in $\mathcal{M}$, and in fact $E_H A = \beta(A)$ for any deterministic $A \in \mathcal{F}_b$. If the Herer expectation equals the ball hull of the Aumann expectation, we will naturally write $E_H = \beta \circ E_A$.

A set $A \subset \mathbb{E}$ is called a *Mazur set* if, for every hyperplane H missing A, there is a ball covering A which is still missed by H [11, 12]. The family of all Mazur sets in $\mathbb{E}$ is denoted by $\mathcal{P}$. The Hahn-Banach theorem implies $\mathcal{P} \subset \mathcal{M}$. If $\mathcal{P} = \mathcal{M}$, then $\mathbb{E}$ is called a *Mazur space*. As shown in [11], Mazur spaces include all reflexive spaces with Fréchet differentiable norm, all 2-dimensional spaces, $C(K)$ for K extremally disconnected Hausdorff compact, and c_0.

Generalizations of extreme points. We will be using some subsets of S^* with special geometric significance: those of all weak* denting, semidenting, weak* strongly exposed and extreme points of the dual unit ball, denoted $\mathcal{W}^*$, $\mathcal{SD}$, $\mathcal{W}^*\mathcal{S}$ and $\mathcal{E}^*$ respectively. Their definitions are as follows.

A *weak* slice* of a subset $A \subset \mathbb{E}^*$ is a set

$$S(A, x, \delta) = \{f \in A \mid f(x) > \sup_{g \in A} g(x) - \delta\},$$

where $x \in \mathbb{E}, \delta > 0$.

A *weak* denting* point of A is a point $f \in A$ such that for all $\varepsilon > 0$ there exists a weak* slice $S(A, x, \delta)$, having diameter at most ε, that contains f.

$f \in A$ is a *semidenting* point [3] of A if for all $\varepsilon > 0$ there exists a weak* slice $S(A, x, \delta)$ such that the diameter of $S(A, x, \delta) \cup \{f\}$ is at most ε.

$f \in A$ is a *weak* strongly exposed* point of A if there exists some $x \in \mathbb{E}$ such that $f(x) > g(x)$ for all $g \in A \backslash \{f\}$, and $f_n(x) \to f(x)$ implies $f_n \to f$ in the dual norm whenever $\{f_n\}_n \subset A$.

$f \in A$ is an *extreme* point of A if $f = 2^{-1}(g+h)$ with $g, h \in A$ implies $f = g = h$.

The following inclusions are known: $\mathcal{W}^*\mathcal{S} \subset \mathcal{W}^* \subset \mathcal{E}^*$ and $\operatorname{cl} \mathcal{W}^* \subset \mathcal{SD}$. Besides, some of those sets admit the following characterizations adapted from [3, 4]. For any $f \in B^*$,

(i) $f \in \mathcal{W}^*$ if and only if, for all $A \in \mathcal{F}_b$ and $\lambda \in \mathbb{R}$ such that $s(f, A) < \lambda$, there exists $C \in \mathcal{B}$ covering A such that $s(f, C) < \lambda$.

(ii) $f \in \mathcal{SD}$ if and only if, for all $A \in \mathcal{F}_b$ and $x \in \mathbb{E}$ such that $s(f, A) < f(x)$, there exists $C \in \mathcal{B}$ covering A such that $x \notin C$.

(iii) $f \in \mathcal{E}^*$ if and only if, for all $A \in \mathcal{K}$ and $\lambda \in \mathbb{R}$ such that $s(f, A) < \lambda$, there exists $C \in \mathcal{B}$ covering A such that $s(f, C) < \lambda$.

Whenever $\mathcal{H} \subset \mathcal{S}^*$, the $\mathcal{H}$-*convex hull* of $A \in \mathcal{F}_b$ is defined to be

$$\operatorname{co}_{\mathcal{H}} A = \bigcap_{f \in \mathcal{H}} \{x \in \mathbb{E} \mid f(x) \leq s(f, A)\}.$$

An important property which will be used several times is

$$f \in \mathcal{H} \Rightarrow s(f, \operatorname{co}_{\mathcal{H}} A) = s(f, A).$$

3. Bornological differentiability and the equality $E_H = \beta \circ E_A$

Let $\mathfrak{B}$ be a bornology; actually, for our purposes we could take any covering of $\mathbb{E}$. A real function f on $\mathbb{E}$ is called $\mathfrak{B}$-*differentiable* at $x \in \mathbb{E}$ if there exists $D_x f \in \mathbb{E}^*$ such that, for every $V \in \mathfrak{B}$,

$$\frac{f(x+tu) - f(x)}{t} \to D_x f(u)$$

uniformly with respect to $u \in V$ as $t \to 0$. We will say that a random set $X : \Omega \to \mathcal{F}_{bc}$ is $\mathfrak{B}$-*covered* if almost all $X(\omega)$ are contained in some element of $\mathfrak{B}$ and also $E_A X$ is contained in some element of $\mathfrak{B}$.

Lemma 3.1. *Let $\mathfrak{B}$ be a bornology, $x \neq 0$ a point of $\mathfrak{B}$-differentiability of the norm of $\mathbb{E}$, and $X \in L^1(\mathcal{F}_{bc})$ a $\mathfrak{B}$-covered random set. Then,*

$$Ed_H(X, \{nx\}) - d_H(E_A X, \{nx\}) \to 0.$$

Proof. For any $\omega \in \Omega$ such that $X(\omega) \subset V$ for some $V \in \mathfrak{B}$,

$$\begin{aligned} d_H(X(\omega), \{nx\}) - \|nx\| &= \frac{d_H(n^{-1}X(\omega), \{x\}) - \|x\|}{n^{-1}} \\ &= \sup_{y \in X(\omega)} \frac{\|x - n^{-1}y\| - \|x\|}{n^{-1}}. \end{aligned}$$

By the hypotheses,

$$\sup_{y \in V} \left| \frac{\|x - ty\| - \|x\|}{t} - D_x\| \cdot \|(y) \right| \to 0.$$

From this we deduce routinely that

$$\sup_{y \in X(\omega)} \frac{\|x - n^{-1}y\| - \|x\|}{n^{-1}} \to \sup_{y \in X(\omega)} D_x\| \cdot \|(y) = s(D_x\| \cdot \|, X(\omega)).$$

Since $\|nx\| = d_H(\{0\}, \{nx\})$, the triangle inequality for d_H yields

$$d_H(X, \{nx\}) - \|nx\| \leq \|X\| \in L^1$$

so, by the dominated convergence theorem,

$$Ed_H(X, \{nx\}) - \|nx\| \to Es(D_x\| \cdot \|, X).$$

Following similar steps, we prove

$$d_H(E_A X, \{nx\}) - \|nx\| \to s(D_x\| \cdot \|, E_A X).$$

Therefore,

$$Ed_H(X, \{nx\}) - d_H(E_A X, \{nx\}) \to Es(D_x\| \cdot \|, X) - s(D_x\| \cdot \|, E_A X) = 0,$$

where the equality comes from e.g. [21, Theorem 1.22, p. 159]. □

Bornological differentiability properties tell us how to restrict the values taken by X in order to ensure $E_H = \beta \circ E_A$. Note that, for any given norm in a separable Banach space, there exist bornologies (including the Hadamard bornology) to which the following proposition applies.

Proposition 3.2. *Let $\mathfrak{B}$ be a bornology. If the norm of $\mathbb{E}$ has a dense set of $\mathfrak{B}$-differentiability points, then $E_H X = \beta(E_A X)$ for every $\mathfrak{B}$-covered $X \in L^1(\mathcal{F}_{bc})$.*

Proof. The inclusion '$\supset$' always holds. In order to prove the converse, we need to show that $a \notin \beta(E_A X) \Rightarrow a \notin E_H X$. We claim that, for each $a \in \mathbb{E}$ and $Y \in L^1(\mathcal{F}_{bc})$, we have $E_H(Y+\{a\}) = E_H Y+\{a\}$ and $\beta(E_A(Y+\{a\})) = \beta(E_A Y)+\{a\}$. Indeed, for the first equality,

$$\begin{aligned}
& E_H Y + \{a\} \\
=\ & \{y + a \in \mathbb{E} \mid \forall b \in \mathbb{E}, \|y - b\| \leq Ed_H(Y, \{b\})\} \\
=\ & \{y + a \in \mathbb{E} \mid \forall b \in \mathbb{E}, \|(y + a) - (b + a)\| \leq Ed_H(Y + \{a\}, \{b + a\})\} \\
=\ & \{y \in \mathbb{E} \mid \forall b \in \mathbb{E}, \|y - b\| \leq Ed_H(Y + \{a\}, \{b\})\} = E_H(Y + \{a\}).
\end{aligned}$$

The second equality comes from [27, Proposition 3.2.(ii)]. That reduces the proof to the clearer case $a = 0$.

Assume $0 \notin \beta(E_A X)$. There exists a ball $B(y, r)$ such that $E_A X \subset B(y, r)$ and $\|y\| > r$. Since the $\mathfrak{B}$-differentiability set is dense, we can assume without loss of generality that the norm is $\mathfrak{B}$-differentiable at y (otherwise, there is a slightly larger ball whose center is a $\mathfrak{B}$-differentiability point close enough to y so that the ball still separates $E_A X$ from 0).

Note that

$$\|y\| > r \geq d_H(E_A X, \{y\}).$$

Now take a point $x \neq 0$ in the segment $[0, y]$ such that $\|y - x\| > d_H(E_A X, \{y\})$, and set $y_n = y + n(y - x)$. Then,

$$\begin{aligned}
d_H(E_A X, \{y_n\}) &= d_H(E_A X, \{y + n(y - x)\}) \leq d_H(E_A X, \{y\}) + n\|y - x\| \\
&< (n + 1)\|y - x\| = \|y_n - x\|.
\end{aligned}$$

From the collinearity of $0, x, y, y_n$ we obtain

$$\|y_n\| = \|y_n - x\| + \|x\| > d_H(E_A X, \{y_n\}) + \|x\|.$$

Notice that $y - x$ is a $\mathfrak{B}$-differentiability point of the norm, since it is a multiple of y. Applying Lemma 3.1 to $X + \{-y\}$,

$$\begin{aligned}
& d_H(E_A X, \{y_n\}) - Ed_H(X, \{y_n\}) \\
=\ & d_H(E_A(X + \{-y\}), \{n(y - x)\}) - Ed_H(X + \{-y\}, \{n(y - x)\}) \to 0;
\end{aligned}$$

thence, for n large enough,

$$\|y_n\| \geq Ed_H(X, \{y_n\}) + \|x\| > Ed_H(X, \{y_n\})$$

and so $0 \notin E_H X$, as wished. $\square$

We will present now some consequences of this general result in the cases $\mathfrak{B} = \mathcal{K}$, $\mathfrak{B} = \mathcal{WK}$ and $\mathfrak{B} = \mathcal{F}_b$ (Hadamard, weak Hadamard and Fréchet bornologies, and their corresponding differentiability notions).

Two variants of the MIP are the K-MIP and the W-MIP (see e.g. [24, 28]). A space is said to have the *K-MIP* if compact convex sets are intersections of balls.

The *W-MIP* is defined similarly but with weakly compact convex sets. Also, denote by $\mathfrak{N}$ the set of all norms equivalent to that of $\mathbb{E}$; it is a metric space with the distance between two norms being defined as the Hausdorff distance between their unit balls.

Theorem 3.3. *Let $\mathbb{E}$ be a separable Banach space. The following hold:*

$(a.1)$ [26, Theorem 3] $E_H = \beta \circ E_A$ *on* $L^1(\mathcal{K})$*;*

$(a.2)$ *If the norm has a dense set of weak Hadamard differentiability points, then* $E_H = \beta \circ E_A$ *on* $L^1(\mathcal{WK})$*;*

$(a.3)$ *If the norm has a dense set of Fréchet differentiability points (in particular, if $\mathbb{E}^*$ is separable), then* $E_H = \beta \circ E_A$ *on* $L^1(\mathcal{F}_b)$*;*

$(b.1)$ [26, Corollary 7] $E_H = E_A$ *on* $L^1(\mathcal{K}_c)$ *if and only if $\mathbb{E}$ has the K-MIP;*

$(b.2)$ *If the norm has a dense set of weak Hadamard differentiability points, then* $E_H = E_A$ *on* $L^1(\mathcal{WK}_c)$ *if and only if $\mathbb{E}$ has the W-MIP;*

$(b.3)$ $E_H = E_A$ *on* $L^1(\mathcal{F}_{bc})$ *if and only if $\mathbb{E}$ has the MIP;*

$(c.1)$ *The set of equivalent norms for which* $E_H = \beta \circ E_A$ *on* $L^1(\mathcal{F}_b)$ *is residual in* $\mathfrak{N}$*;*

$(c.2)$ *If $\mathbb{E}^*$ is separable, then the set of equivalent norms for which* $E_H = E_A$ *on* $L^1(\mathcal{F}_{bc})$ *is residual in* $\mathfrak{N}$*;*

$(d.1)$ *There exist a Banach space $\mathbb{F}$ and an embedding $j : \mathbb{E} \to \mathbb{F}$, such that $\mathbb{E}$ is isomorphic to a subspace of $\mathbb{F}$ and* $E_A X = j^{-1}(E_H j(X))$ *for all* $X \in L^1(\mathcal{F}_{bc})$.

$(d.2)$ *For any fixed sequence* $\{Y_i\}_i \subset L^1(\mathcal{F}_{bc})$*, there exist a separable Banach space $\mathbb{F}$ and an embedding $j : \mathbb{E} \to \mathbb{F}$, such that $\mathbb{E}$ is isomorphic to a subspace of $\mathbb{F}$ and* $E_A Y_i = j^{-1}(E_H j(Y_i))$ *for each* $i \in \mathbb{N}$.

Proof. Part (a) follows from a direct application of Proposition 3.2. For $(a.1)$, notice that every separable norm has a dense set of Hadamard (equivalently, Gâteaux) differentiability points. The fact, for $\mathfrak{B} \in \{\mathcal{K}, \mathcal{WK}, \mathcal{F}_b\}$, that $E_A X \in \mathfrak{B}$ when X is integrably bounded and is in $\mathfrak{B}$ almost surely, needed to check that X is $\mathfrak{B}$-covered, follows from the fact that support functions of sets in each of those bornologies are continuous with respect to the topology of uniform convergence on elements of the bornology [18].

(b): The Herer expectation of a deterministic set is its ball hull, hence the implications '$\Rightarrow$'. The converse implications follow from (a) and the definitions of the K-MIP, W-MIP and MIP. For $(b.3)$, notice that a separable space with the MIP is Asplund [25], in particular its norm is Fréchet differentiable in a dense subset.

$(c.1)$: Every separable Banach space has a Markuševič basis [20, Theorem 1.f.4], in particular a fundamental biorthogonal system. Then, by [10, p. 69–71], the set of equivalent norms with densely many points of Fréchet differentiability is residual in $\mathfrak{N}$. The result follows then from $(a.3)$.

$(c.2)$: By $(a.3)$, all equivalent norms satisfy $E_H = \beta \circ E_A$. In particular, $E_H = E_A$ on $L^1(\mathcal{F}_{bc})$ for every equivalent norm with the MIP. Now, by the results of Georgiev [9], the set of equivalent norms with the MIP is residual provided at least one such norm exists. But the latter is the case: from the separability of $\mathbb{E}^*$, the space $\mathbb{E}$ is Asplund, so it admits a Fréchet differentiable renorm (e.g. [5]); and, by Mazur's classical result (e.g. [10, Theorem 1.1]), every Banach space with Fréchet differentiable norm has the MIP.

$(d.1)$: Let T be the separability character (the smallest cardinality of a dense subset) of $\mathbb{E}^*$, and consider $\mathbb{E}\oplus\ell_2(T)$ with the sum norm. It admits a fundamental biorthogonal system since $\mathbb{E}$ is separable and $\ell_2(T)$ is a Hilbert space, so the set of equivalent norms in $\mathbb{E}\oplus\ell_2(T)$ which have a dense set of Fréchet differentiability points is residual in $\mathfrak{N}$.

On the other hand, by [19, Theorem 2.5], $\mathbb{E}\oplus\ell_2(T)$ admits an equivalent norm with the MIP. Like in the proof of $(c.2)$, the set of equivalent norms with the MIP is residual. Thus, the intersection of both sets is residual; in particular, $\mathbb{E}\times\ell_2(T)$ admits a MIP norm $|||\cdot|||$ with a dense set of Fréchet differentiability points.

We take $\mathbb{F}$ to be the space $(\mathbb{E}\times\ell_2(T), |||\cdot|||)$, with the natural embedding $j : x\mapsto(x,0)$. The proof of Proposition 3.2 does not rely on the separability of $\mathbb{E}$, so it will apply to random sets taking on values in a separable subset of $\mathbb{F}$, as is $j(X)=X\times\{0\}$, provided we show that $E_H j(X)$ is well-defined. Denote by $d_H^{|||\cdot|||}$ the Hausdorff metric associated to the norm $|||\cdot|||$. We need to show that $d_H^{|||\cdot|||}(j(X),\{a\})$ is an integrable random variable for each $a\in\mathbb{E}\times\ell_2(T)$.

As regards measurability, by taking a countable dense subset $D\subset\mathbb{E}$ we write

$$\begin{aligned} d_H^{|||\cdot|||}(j(X),\{a\}) &= \sup_{x\in X}|||j(x)-a||| = \inf_{m\in\mathbb{N}}\sup_{x\in(X+m^{-1}B)\cap D}|||j(x)-a||| \\ &= \inf_{m\in\mathbb{N}}\sup_{x\in D}[|||j(x)-a|||\cdot I_{\{d(x,X+m^{-1}B)=0\}}] \end{aligned}$$

Since X is Effros measurable, for any $x\in D$ the distance function $d(x,X)$ is a random variable (e.g. [21]) and

$$d(x,X+m^{-1}B)=\max\{d(x,X)-m^{-1},0\}$$

is so as well. Then, the indicator function of the event $\{d(x,X+m^{-1}B)=0\}$ is a random variable too. We conclude that $d_H^{|||\cdot|||}(j(X),\{a\})$ is a random variable, as wished.

As regards integrability, for some $k>0$

$$\begin{aligned} Ed_H^{|||\cdot|||}(j(X),a) &\le E|||j(X)|||+|||a||| \\ &\le k\cdot E||j(X)||_{\mathbb{E}\oplus\ell_2(T)}+|||a|||=k\cdot E\|X\|+|||a|||<\infty. \end{aligned}$$

Finally, $(a.3)$ and $(b.3)$ imply that $E_A j(X)$ equals the Herer expectation of $j(X)$ in the space $\mathbb{F}$. Since j is an homeomorphism onto its image and $\mathbb{E}$ is complete, using the definitions of Aumann and Bochner expectations one can check that $E_A j(X)=j(E_A X)$. Thus, $E_A X=j^{-1}(E_H j(X))$.

$(d.2)$: Let $(\mathbb{E}\times\ell_2(T),|||\cdot|||)$, j, $d_H^{|||\cdot|||}$ and D be as above. By $(d.1)$, for each $i,m\in\mathbb{N}$ and $x\in j(D)\backslash E_H[j(Y_i+m^{-1}B_{|||\cdot|||});|||\cdot|||]$, there exists $a_{x,i,m}\in\mathbb{E}\times\ell_2(T)$ such

that
$$|||x - a_{x,i,m}||| > Ed_H^{|||\cdot|||}(j(Y_i) + m^{-1}B_{|||\cdot|||}, \{a_{x,i,m}\}).$$
Let A be the set of all the $a_{x,i,m}$, and take $\mathbb{F} = \overline{\mathrm{span}}(j(\mathbb{E}) \cup A)$ with the trace norm $\|y\|_{\mathbb{F}} = |||y|||$. Note that j embeds isomorphically $\mathbb{E}$ into $\mathbb{F}$, which is separable by the countability of A. For the remainder of this proof, we write d_H for the Hausdorff metric associated to $\|\cdot\|_{\mathbb{F}}$.

The statement to be proved is equivalent to
$$j(E_A Y_i) = E_H[j(Y_i); \|\cdot\|_{\mathbb{F}}] \cap j(\mathbb{E}), \quad \forall\, i \in \mathbb{N}.$$
Fix $i \in \mathbb{N}$. Since, by $(d.1)$,
$$E_H[j(Y_i); |||\cdot|||] = j(E_A Y_i) = E_A j(Y_i) \subset E_H[j(Y_i); \|\cdot\|_{\mathbb{F}}],$$
it suffices to prove
$$E_H[j(Y_i); \|\cdot\|_{\mathbb{F}}] \cap j(\mathbb{E}) \subset E_H[j(Y_i); |||\cdot|||].$$

Let $y \in j(\mathbb{E})$ be such that $y \notin E_H[j(Y_i); |||\cdot|||]$. Denote the unit balls of the norms $\|\cdot\|_{\mathbb{F}}$ and $|||\cdot|||$ by $B_{\mathbb{F}}$ and $B_{|||\cdot|||}$. Since
$$\begin{aligned}
&\bigcap_{\delta>0} E_H[j(Y_i) + \delta B_{|||\cdot|||}; |||\cdot|||] \\
&= \bigcap_{\delta>0} \{x \in \mathbb{F} \mid |||x - a||| \le Ed_H^{|||\cdot|||}(j(Y_i) + \delta B_{|||\cdot|||}, \{a\})\} \\
&\subset \bigcap_{\delta>0} \{x \in \mathbb{F} \mid |||x - a||| \le Ed_H^{|||\cdot|||}(j(Y_i), \{a\}) + \delta\} = E_H[j(Y_i); |||\cdot|||]
\end{aligned}$$
and the right-hand side is a closed set, we find $m \in \mathbb{N}$ and $\varepsilon > 0$ for which the ball $y + \varepsilon B_{|||\cdot|||}$ misses $E_H[j(Y_i) + m^{-1}B_{|||\cdot|||}; |||\cdot|||]$, in particular
$$(y + \varepsilon B_{\mathbb{F}}) \cap E_H[j(Y_i) + m^{-1}B_{|||\cdot|||}; |||\cdot|||] = \emptyset.$$

Accordingly, by the density of $j(D)$ in $j(\mathbb{E})$, there is a sequence $\{x_n\}_n \subset j(D)$ converging to y with $\|x_n - y\|_{\mathbb{F}} < \varepsilon$ for all n. Then,
$$x_n \notin E_H[j(Y_i) + m^{-1}B_{|||\cdot|||}; |||\cdot|||]$$
so, by construction, there exist $a_n \in A \subset \mathbb{F}$ such that
$$||x_n - a_n||_{\mathbb{F}} = |||x_n - a_n||| > Ed_H^{|||\cdot|||}(j(Y_i) + m^{-1}B_{|||\cdot|||}, \{a_n\}).$$
But
$$\begin{aligned}
Ed_H^{|||\cdot|||}(j(Y_i) + m^{-1}B_{|||\cdot|||}, \{a_n\}) &= E \sup_{z \in j(Y_i) + m^{-1}B_{|||\cdot|||}} |||z - a_n||| \\
\ge E \sup_{z \in j(Y_i) + m^{-1}B_{\mathbb{F}}} \|z - a_n\|_{\mathbb{F}} &= Ed_H(j(Y_i) + m^{-1}B_{\mathbb{F}}, \{a_n\}),
\end{aligned}$$

therefore $x_n \notin E_H[j(Y_i) + m^{-1}B_{\mathbb{F}}; \|\cdot\|_{\mathbb{F}}]$.

Since $x_n \to y$, we deduce

$$y \notin \operatorname{int} E_H[j(Y_i) + m^{-1}B_{\mathbb{F}}; \|\cdot\|_{\mathbb{F}}].$$

We claim that the right-hand side contains $E_H[j(Y_i); \|\cdot\|_{\mathbb{F}}]$. Indeed, let $z \in E_H[j(Y_i); \|\cdot\|_{\mathbb{F}}]$. For any point z' with $\|z' - z\| < m^{-1}$, and any $b \in \mathbb{F}$,

$$d(z', b) \leq d(z, b) + m^{-1} \leq E d_H(j(Y_i), \{b\}) + m^{-1} = E d_H(j(Y_i) + m^{-1}B_{\mathbb{F}}, \{b\}).$$

Thus, $E_H[j(Y_i) + m^{-1}B_{\mathbb{F}}; \|\cdot\|_{\mathbb{F}}]$ contains an open ball centered at each point of $E_H[j(Y_i); \|\cdot\|_{\mathbb{F}}]$, so the claim is proven and $y \notin E_H[j(Y_i); \|\cdot\|_{\mathbb{F}}]$. The proof is complete. □

Remark 3.4. Part $(b.3)$ of Theorem 3.3 fixes the gap in [26] mentioned in the Introduction. □

4. Results for arbitrary norms

This section is devoted to proving bounds of the type

$$\beta(E_A X) \subset E_H X \subset \gamma(E_A X)$$

for some function γ, valid without restrictions on the norm or the possible values of the random set. Then they will be applied to specific cases in which $\gamma = \beta$.

We start by introducing the notion of a *Mazur functional*. The name is justified by a certain duality with the notion of a Mazur set. We say that $f \in S^*$ is a Mazur functional if

$$s(f, \beta(A)) = \inf\{s(f, C) \mid A \subset C \in \mathcal{B}\}$$

for all $A \in \mathcal{F}_{bc}$. The set of all Mazur functionals will be denoted by $\mathcal{P}^*$.

Let us remark that the functionals $s(f, \cdot)$ on $\mathcal{F}_{bc}$ are complete sup-semilattice homomorphisms in the sense that $s(f, \bigcup_i A_i) = \sup_i s(f, A_i)$, but they do not preserve infima. The notion of a Mazur functional just demands infima preservation over families of balls.

Mazur sets and Mazur functionals are defined by a similar condition, as shown if we restate the definition as follows. For every $f \in S^*$, the following conditions are equivalent:

(i) $f \in \mathcal{P}^*$,
(ii) For all $A \in \mathcal{F}_b$ and $\lambda \in \mathbb{R}$ such that $s(f, \beta(A)) < \lambda$, there exists $C \in \mathcal{B}$ covering A such that $s(f, C) < \lambda$,
(iii) For all $A \in \mathcal{M}$ and $\lambda \in \mathbb{R}$ such that $s(f, A) < \lambda$, there exists $C \in \mathcal{B}$ covering A such that $s(f, C) < \lambda$.

Given $f \in S^*$ and $A \in \mathcal{M}$, let $\mathbf{P}(f, A)$ be the property that, whenever $s(f, A) < \lambda \in \mathbb{R}$, there exists a ball C covering A such that also $s(f, C) < \lambda$. Then, f is Mazur if $\mathbf{P}(f, A)$ holds for every A, whereas A is Mazur if $\mathbf{P}(f, A)$ holds for every f. In particular, $\mathbb{E}$ is a Mazur space if and only if $\mathcal{P}^* = S^*$.

Remark 4.1. Since condition (*iii*) above is weaker than the condition characterizing $\mathcal{W}^*$, the notion of a Mazur functional is more general than that of a weak* denting point of B^*. If fact, using the characterizations of $\mathcal{W}^*$ and $\mathcal{SD}$, we deduce $\mathcal{W}^* = \mathcal{P}^* \cap \mathcal{SD}$. □

Example 4.2. Let $\mathbb{E} = \mathbb{R}^2$ with the maximum norm. Then, B^* has as weak* denting points its four corners, but every point in S^* is a Mazur functional. The latter can be checked directly or obtained from [12, Theorem 6.5], which states that every 2-dimensional Banach space is a Mazur space.

Two important tools are given in the following lemmas. The first is [22, Proposition 3.2], the second is a fragment of the proof of [26, Theorem 1].

Lemma 4.3 (Moreno–Schneider). *The mapping $\beta : (\mathcal{F}_{bc}, d_H) \to (\mathcal{M}, d_H)$ is continuous at every element of $\mathcal{F}_{bc}$ having non-empty interior.*

Lemma 4.4. *Let $X \in L^1(\mathcal{B})$. Then, $E_H X = E_A X$.*

Another property which will often be used is the following. Observe that, in general, $\beta(A) + \beta(C) \neq \beta(A + C)$ [11].

Lemma 4.5 ([27, Proposition 6.1]). *Let $A, C \in \mathcal{F}_b$. Then, $\beta(\beta(A)+\beta(C)) = \beta(A + C)$.*

Since we will consider the ball hull of a random set, we need to check that it is also a random set.

Lemma 4.6. *If $X \in L^1_H(\mathcal{F}_b)$, then $\beta(X) \in L^1(\mathcal{F}_{bc})$.*

Proof. Since X is d_H-approximable by simple functions, $\overline{\text{co}}\, X$ is so also and, except in a null set, it takes on values in a separable subspace $\mathcal{V} \subset \mathcal{F}_{bc}$. We can take $\mathcal{V}$ to be closed, and so d_H-Borel.

We define the inradius mapping $\text{inr} : \mathcal{F}_{bc} \to [0, \infty)$, and set $\mathcal{V}_0 = \{A \in \mathcal{V} \mid \text{inr}(A) > 0\}$. Also define the mapping $\phi_n : \mathcal{V} \to \mathcal{V}_0$ given by $\phi_n(A) = \overline{\text{co}}\, A + n^{-1}B$.

Since

$$d_H(\phi_n(A), \phi_n(C)) = d_H(\overline{\text{co}}\, A, \overline{\text{co}}\, C) \leq d_H(A, C),$$

the mapping ϕ_n is d_H-continuous. The inradius mapping defined on convex sets is d_H-continuous as well, so $\{A \in \mathcal{F}_{bc} \mid \text{inr}(A) > 0\}$ is d_H-open. Accordingly, $\mathcal{V}_0$ is a d_H-Borel set containing all the values of $\phi_n(X)$ modulo a null set.

Then,

$$\beta(X) = \bigcap_{n\in\mathbb{N}} \beta(X + n^{-1}B) = \bigcap_{n\in\mathbb{N}} \beta(\overline{\mathrm{co}}\, X + n^{-1}B) = \bigcap_{n\in\mathbb{N}} \beta|_{\mathcal{V}_0} \circ\phi_n(X)$$

almost surely (for the first equality, see [27, Lemma 3.3]). By Lemma 4.3, $\beta|_{\mathcal{V}_0}$ is continuous, implying that each mapping $\beta|_{\mathcal{V}_0} \circ\phi_n(X)$ is d_H-Borel (hence a random closed bounded set) and so is their intersection $\beta(X)$.

Finally, since $X \in L^1(\mathcal{F}_b)$, taking into account [27, Lemma 3.1] we have

$$E\|\beta(X)\| = Ed_H(\beta(X), \{0\}) = Ed_H(X, \{0\}) = E\|X\| < \infty. \qquad \square$$

We are finally ready for the main theorem of this section.

Theorem 4.7. *Let $\mathbb{E}$ be a separable Banach space. Then, on $L^1_H(\mathcal{F}_b)$,*

$(a.1)$ *E_H is bounded between $\beta \circ E_A$ and $\mathrm{co}_{\mathcal{P}^*} \circ\beta \circ E_A$.*
$(a.2)$ *Moreover, $\mathrm{co}_{\mathcal{P}^*} \circ\beta \circ E_A \subset \mathrm{co}_{\mathcal{W}^*} \circ E_A$.*
(b) *If $\mathcal{P}^*$ is norming, then $E_H = \beta \circ E_A$.*

Proof. $(a.1)$: Let $X \in L^1_H(\mathcal{F}_b)$. The inner bound $\beta(E_AX) \subset E_HX$ holds even if X is not Hausdorff approximable. Indeed, as mentioned in the Introduction, $E_AX \subset E_HX$ always holds. The bound follows from the fact that $E_HX \in \mathcal{M}$.

The proof of the outer bound $E_HX \subset \mathrm{co}_{\mathcal{P}^*}\, \beta(E_AX)$ is split into several steps.

Step 1. We begin by proving $E_HY \subset \mathrm{co}_{\mathcal{P}^*}\, E_A\beta(Y)$ for any simple random convex set $Y = \sum_{i=1}^r I_{\Omega_i}A_i \in L^1(\mathcal{F}_{bc})$.

Let $x \notin \mathrm{co}_{\mathcal{P}^*}\, E_A\beta(Y)$, then there exists $f \in \mathcal{P}^*$ with $f(x) > s(f, E_A\beta(Y))$. Let $\varepsilon = f(x) - s(f, E_A\beta(Y))$. Since $f \in \mathcal{P}^*$, there exist balls C_i such that $A_i \subset C_i$ and $s(f, C_i) < s(f, \beta(A_i)) + \varepsilon$.

Setting $Y' = \sum_{i=1}^r I_{\Omega_i}C_i$, we have

$$\begin{aligned} f(x) = s(f, E_A\beta(Y)) + \varepsilon &= \sum_{i=1}^r P(\Omega_i)s(f, A_i) + \varepsilon \\ &> \sum_{i=1}^r P(\Omega_i)s(f, C_i) = s(f, E_AY'). \end{aligned}$$

Hence $x \notin E_AY'$, but $E_HY \subset E_HY' = E_AY'$ by Lemma 4.4 because Y' is a random ball. We deduce $x \notin E_HY$.

Step 2. Now, take an arbitrary $\eta > 0$ and let $X \in L^1_H(\mathcal{F}_{bc})$. However, instead of taking $x \notin \mathrm{co}_{\mathcal{P}^*}\, \beta(E_AX)$ and proving $x \notin E_HX$, we do take $x \notin \mathrm{co}_{\mathcal{P}^*}\, \beta(E_A(X+2\eta B))$ for the moment. Similar to Step 1, we find some $f \in \mathcal{P}^*$ such that

$$\varepsilon := \frac{f(x) - s(f, \beta(E_A(X + 2\eta B)))}{2} > 0.$$

Let $\{Y_n\}_n$ be a sequence of simple $\mathcal{F}_b$-valued random sets such that $Y_n \to X$ almost surely, $Ed_H(Y_n, X) \to 0$ and $\|Y_n\| \leq \|X\| + 1$.

By Lemma 4.3,

$$d_H(\beta(Y_n + \eta B), \beta(X + \eta B)) = d_H(\beta(\overline{\text{co}}(Y_n + \eta B)), \beta(\overline{\text{co}}(X + \eta B))) \to 0$$

almost surely and also

$$\begin{aligned} d_H(\beta(Y_n + \eta B), \beta(X + \eta B)) &\leq \|\beta(Y_n + \eta B)\| + \|\beta(X + \eta B)\| \\ &= \|Y_n + \eta B\| + \|X + \eta B\| \leq 2\|X\| + 1 + 2\eta. \end{aligned}$$

Using the inequality in [17, Theorem 4.1] and the dominated convergence theorem,

$$d_H(E_A\beta(Y_n + \eta B), E_A\beta(X + \eta B)) \leq Ed_H(\beta(Y_n + \eta B), \beta(X + \eta B)) \to 0.$$

Therefore, for n large enough,

$$d_H(E_A\beta(Y_n + \eta B), E_A\beta(X + \eta B)) < \eta$$

and so

$$E_A\beta(Y_n + \eta B) \subset E_A\beta(X + \eta B) + \eta B.$$

Since the random set $\beta(Y_n + \eta B) + \eta B$ is simple, we have the chain of inclusions

$$\begin{aligned} &E_H X \subset E_H\beta(X + \eta B) \\ &\subset \{x \in \mathbb{E} \mid \forall a \in \mathbb{E}, \|x - a\| \leq Ed_H(\beta(Y_n + \eta B), \{a\}) \\ &\qquad + Ed_H(\beta(X + \eta B), \beta(Y_n + \eta B))\} \\ &= \{x \in \mathbb{E} \mid \forall a \in \mathbb{E}, \|x - a\| \leq Ed_H(\beta(Y_n + \eta B) \\ &\qquad + Ed_H(\beta(X + \eta B), \beta(Y_n + \eta B))B, \{a\})\} \\ &= E_H(\beta(Y_n + \eta B) + d_H(E_A\beta(Y_n + \eta B), E_A\beta(X + \eta B))B) \\ &\subset E_H(\beta(Y_n + \eta B) + \eta B) \subset \text{co}_{\mathcal{P}^*} E_A\beta(\beta(Y_n + \eta B) + \eta B). \end{aligned}$$

Lemma 4.5 finally simplifies the last expression into $\text{co}_{\mathcal{P}^*} E_A\beta(Y_n + 2\eta B)$.

Moreover, since $\beta(Y_n + 2\eta B)$ is a simple random set, a repeated application of Lemma 4.5 yields

$$\beta(E_A\beta(Y_n + 2\eta B)) = \beta(E_A(Y_n + 2\eta B)) = \beta(E_A Y_n + 2\eta B),$$

therefore

$$E_A\beta(Y_n + 2\eta B) \subset \beta(E_A Y_n + 2\eta B). \tag{1}$$

On the other hand,

$$\begin{aligned} &d_H(\overline{\text{co}}(E_A Y_n + 2\eta B), \overline{\text{co}}(E_A X + 2\eta B)) \\ &\leq d_H(E_A Y_n + 2\eta B, E_A X + 2\eta B) \\ &= d_H(E_A Y_n, E_A X) \leq Ed_H(Y_n, X) \to 0, \end{aligned}$$

whence, by Lemma 4.3,

$$d_H(\beta(E_A Y_n + 2\eta B), \beta(E_A X + 2\eta B)) \to 0.$$

Accordingly, eventually

$$\beta(E_A Y_n + 2\eta B) \subset \beta(E_A X + 2\eta B) + \varepsilon B. \tag{2}$$

Taking (1) and (2) into account, the facts that $E_H X \subset \mathrm{co}_{\mathcal{P}^*} E_A \beta(Y_n + 2\eta B)$ and $f \in \mathcal{P}^*$ imply

$$\begin{aligned} s(f, E_H X) &\le s(f, \mathrm{co}_{\mathcal{P}^*} E_A \beta(Y_n + 2\eta B)) = s(f, E_A \beta(Y_n + 2\eta B)) \\ &\le s(f, \beta(E_A X + 2\eta B) + \varepsilon B) = s(f, \beta(E_A X + 2\eta B)) + \varepsilon < f(x) \end{aligned}$$

and we conclude $x \notin E_H X$. Thus we have

$$E_H X \subset \mathrm{co}_{\mathcal{P}^*} \beta(E_A X + 2\eta B).$$

By the arbitrariness of η,

$$E_H X \subset \bigcap_{\eta > 0} \mathrm{co}_{\mathcal{P}^*} \beta(E_A X + \eta B).$$

Step 3. For the third step, let $x \notin \mathrm{co}_{\mathcal{P}^*} \beta(E_A X)$. Then, $f(x) > s(f, \beta(E_A X))$ for some $f \in \mathcal{P}^*$. But, by the definition of a Mazur functional,

$$s(f, \beta(E_A X)) = \inf\{s(f, C) \mid E_A X \subset C \in \mathcal{B}\}.$$

Thus we find a ball C covering $E_A X$ and a small $\eta > 0$ such that

$$\begin{aligned} f(x) &> s(f, C) + \eta = s(f, C + \eta B) \ge \inf\{s(f, D) \mid E_A X + \eta B \subset D \in \mathcal{B}\} \\ &= s(f, \beta(E_A X + \eta B)) = s(f, \mathrm{co}_{\mathcal{P}^*} \beta(E_A X + \eta B)), \end{aligned}$$

so that $x \notin \mathrm{co}_{\mathcal{P}^*} \beta(E_A X + \eta B)$ and, *a fortiori*, $x \notin E_H X$. That proves $E_H X \subset \mathrm{co}_{\mathcal{P}^*} \beta(E_A X)$, as wished.

Step 4. Finally, there only remains to prove the nonconvex case. But, if $X \in L^1_H(\mathcal{F}_b)$, then

$$E_H X = E_X \overline{\mathrm{co}}\, X \subset \mathrm{co}_{\mathcal{P}^*} \beta(E_A \overline{\mathrm{co}}\, X) = \mathrm{co}_{\mathcal{P}^*} \beta(\overline{\mathrm{co}}\, E_A X) = \mathrm{co}_{\mathcal{P}^*} \beta(E_A X).$$

$(a.2)$: As previously mentioned, every weak* denting point of B^* is both a Mazur functional and a semidenting point. The former yields $\mathrm{co}_{\mathcal{P}^*} \subset \mathrm{co}_{\mathcal{W}^*}$, the latter $\mathrm{co}_{\mathcal{W}^*} \circ \beta = \mathrm{co}_{\mathcal{W}^*}$ since $\mathcal{H}$-convex hulls with $\mathcal{H} \subset \mathcal{SD}$ absorb β [27, Proposition 7.3, $(ii) \Leftrightarrow (iv)$].

(b): By part $(i) \Leftrightarrow (v)$ in [27, Proposition 7.4], $\mathcal{P}^*$ is norming if and only if $\mathrm{co}_{\mathcal{P}^*} \subset \beta$. Then both bounds in $(a.1)$ are identical. $\square$

Remark 4.8. The upper bound $(a.2)$ implies that the Herer expectation is always outer bounded by an $\mathcal{H}$-convex hull of the Aumann expectation, and is tight in the following sense: if $\mathrm{co}_{\mathcal{P}^*} \circ \beta = \mathrm{co}_{\mathcal{P}^*}$ (i.e. the left-hand side of $(a.2)$ is already an $\mathcal{H}$-hull), then necessarily $\mathcal{P}^* = \mathcal{W}^*$. Indeed, by [27, Proposition 7.3, $(ii) \Leftrightarrow (iv)$] we would have $\mathcal{P}^* \subset \mathcal{SD}$; but then $\mathcal{P}^* = \mathcal{P}^* \cap \mathcal{SD} = \mathcal{W}^*$ by Remark 4.1. □

In view of Theorem 4.7(b), the question is what spaces have $\mathcal{P}^*$ norming. These include Asplund spaces, spaces with the Mazur Intersection Property, Mazur spaces (where $\mathcal{P}^* = S^*$), and polyhedral spaces in the sense that every finite-dimensional section of their unit ball is a polytope. Polyhedral spaces have a boundary $\mathcal{J} \subset S^*$ such that $f^{-1}(1) \cap S$ has non-empty interior in $f^{-1}(1)$ for all $f \in \Omega$ [7] (recall that $\mathcal{J}$ is a *boundary* if every $x \in S$ has some $f \in \mathcal{J}$ for which $f(x) = \|x\|$).

Corollary 4.9. *Let $\mathbb{E}$ be a separable Banach space which is Asplund or polyhedral. Then, $\mathcal{P}^*$ is norming, whence $E_H = \beta \circ E_A$ on $L^1_H(\mathcal{F}_b)$.*

Proof. In those spaces, $\mathcal{W}^*\mathcal{S}$ is norming. Indeed, if $\mathbb{E}$ is Asplund then B^* is the weak* closure of the convex hull of $\mathcal{W}^*\mathcal{S}$ (e.g. [23]), whence it follows readily that $\mathcal{W}^*\mathcal{S}$ is norming. Similarly, if $\mathbb{E}$ is polyhedral then $\mathcal{J}$ is, by definition, norming. By the property that $f^{-1}(1) \cap S$ has non-empty interior in $f^{-1}(1)$, we have $\mathcal{J} \subset \mathcal{W}^*\mathcal{S}$.

Since $\mathcal{W}^*\mathcal{S} \subset \mathcal{W}^* \subset \mathcal{P}^*$, the statement follows from Theorem 4.7(b). □

The case of Asplund spaces was already covered in the previous section (Theorem 3.3). As regards Mazur spaces, they enjoy a special situation. Note that, in general, $\beta \circ E_A \neq E_A \circ \beta$ [26, Example 1].

Proposition 4.10. *If $\mathbb{E}$ is Mazur, then $E_H = \beta \circ E_A = E_A \circ \beta$ on $L^1_H(\mathcal{F}_b)$.*

Proof. Since $\mathbb{E}$ is Mazur, $\mathcal{P}^* = S^*$ so Theorem 4.7(b) yields the first equality.

The proof of the second one reduces to the convex case, since $\beta(A) = \beta(\overline{\mathrm{co}}\, A)$ for every $A \in \mathcal{F}_b$, and similarly $E_H X = E_H \overline{\mathrm{co}}\, X$ for any $X \in L^1_H(\mathcal{F}_b)$. Note that $\mathcal{M}$ is sum stable because $\mathcal{M} = \mathcal{P}$ and, by [11, Proposition 5.1], $\mathcal{P} + \mathcal{P} = \mathcal{P}$. Moreover, $\mathcal{M}$ is sum stable if and only if β is linear [27, Corollary 6.2], and then β is non-expansive in the Hausdorff metric [27, Proposition 6.3].

Clearly, it follows from the linearity of β that $E_A\beta(Y) = \beta(E_A Y)$ for any $\mathcal{F}_b$-valued simple random set Y. To extend that equality to the general case, fix $\varepsilon > 0$ and take a simple random set Y such that $Ed_H(X, Y) \leq \varepsilon/2$.

From [17, Theorem 4.1] and the non-expansiveness of β,

$$d_H(E_A\beta(X), E_A\beta(Y)) \leq Ed_H(\beta(X), \beta(Y)) \leq \varepsilon$$

and

$$d_H(\beta(E_A X), \beta(E_A Y)) \leq d_H(E_A X, E_A Y) \leq Ed_H(X, Y) \leq \varepsilon.$$

That yields $d_H(E_A\beta(X), \beta(E_AX)) \le 2\varepsilon$, and the arbitrariness of ε proves $E_A\beta(X) = \beta(E_AX)$. □

As a complement to the equivalence between the equality $E_H = E_A$ and the Mazur Intersection Property (Theorem 3.3(b.3)), it is interesting to give sufficient conditions on X implying $E_AX = E_HX$ even in spaces without the MIP.

Contrary to what one might expect, the condition '$X \in \mathcal{M}$ almost surely' does *not* serve that purpose [26, Example 1]. It follows from Proposition 4.10 that, if $\mathbb{E}$ is a Mazur space and X is a random Mazur set, then $E_HX = E_AX$; let us show that the second assumption is enough.

Proposition 4.11. *Let $\mathbb{E}$ be a separable Banach space. Then, $E_H = E_A$ on $L^1_H(\mathcal{P})$.*

Proof. We begin by showing that $E_AX \in \mathcal{P}$ whenever $X \in L^1_H(\mathcal{P})$. Indeed, let $\{X_n\}_n \subset L^1_H(\mathcal{P})$ be independent and identically distributed as X, then by Hess's strong law of large numbers for random closed bounded convex sets (see [16, Theorem 9.1.1]),

$$n^{-1}\sum_{i=1}^{n} X_i(\omega) \to E_AX$$

in the sense of d_H for almost every ω.

The sum stability of $\mathcal{P}$ yields $n^{-1}\sum_{i=1}^{n} X_i(\omega) \in \mathcal{P}$. Now let $f \in S^*$ and $\lambda \in \mathbb{R}$ be such that $s(f, E_AX) < \lambda$. Take $\varepsilon \in (0, \lambda - s(f, E_AX))$. Since $s(f, n^{-1}\sum_{i=1}^{n} X_i(\omega)) \to s(f, A)$ eventually $s(f, n^{-1}\sum_{i=1}^{n} X_i(\omega)) < \lambda - \varepsilon$. But $n^{-1}\sum_{i=1}^{n} X_i(\omega) \in \mathcal{P}$, hence there exist balls B_n, each covering a term of the sequence, such that $s(f, B_n) < \lambda - \varepsilon$.

Eventually $E_AX \subset n^{-1}\sum_{i=1}^{n} X_i(\omega) + \varepsilon B$, whence $E_AX \subset B_n + \varepsilon B \in \mathcal{B}$ and $s(f, B_n + \varepsilon B) = s(f, B_n) + \varepsilon < \lambda$. Thus $E_AX \in \mathcal{P}$.

The remainder of the proof is similar in structure to the proof of the upper bound in Theorem 4.7(a.1). One just has to note that, under the assumption that X takes on values in $\mathcal{P}$, it works for arbitrary $f \in S^*$ (instead of $f \in \mathcal{P}^*$) because Mazur sets are precisely those on which every functional behaves like a Mazur functional. Hence, one can prove $E_HX \subset \beta(E_AX)$ rather than $E_HX \subset \mathrm{co}_{\mathcal{P}^*}\, \beta(E_AX)$; we leave the details to the reader. That, together with the lower bound in Theorem 4.7(a.1), yields $E_HX = \beta(E_AX)$. Finally, notice that $E_AX \in \mathcal{P} \subset \mathcal{M}$, so $\beta(E_AX) = E_AX$. □

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