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An Intrinsic Notion of Convexity for Minimax

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A proper extension of Fan's minimax theorem, sharp in a sense, is provided. In particular, a natural kind of convexity for minimax inequalities arises. In addition a vector-valued Lax–Milgram theorem is derived, which in turn implies a characterization of the solvability of systems with infinitely many variational equations.

Keywords: Minimax theorems, Lax-Milgram theorem, variational equations

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Contents

1. Introduction

A *minimax theorem* states that, under suitable assumptions, for a function $\Phi : A \times B \longrightarrow \mathbb{R}$ there holds

$$\inf_{y \in B} \sup_{x \in A} \Phi(x, y) \leq \sup_{x \in A} \inf_{y \in B} \Phi(x, y),$$

which yields

$$\inf_{y \in B} \sup_{x \in A} \Phi(x, y) = \sup_{x \in A} \inf_{y \in B} \Phi(x, y),$$

since the reverse inequality is always valid. For this reason, such a result is also known as a *minimax inequality*. Although this is not consistent with the

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use of "inf" and "sup", it is a standard abuse of language. Minimax theorems have their roots in optimization, more specifically in game theory, going back to the pioneering work of J. von Neumann [34], and have attracted increasing attention since then, due to their many applications not only in game or economic theory (see the classical reference [1]), but also in monotone multifunction theory ([30, 16]), as well as in duality in convex analysis ([9]) or variational principles in nonlinear analysis ([1, 4, 13]), to name a few. The survey paper [31] and the book [7] describe an overview of the subject, including a historical perspective and recent applications and developments. In addition, and despite the fact that minimax inequalities often have a nonlinear character, even nonconvex, in [11] it is shown that a wide variety of these results are equivalent to the finite-dimensional Hahn–Banach theorem.

Without a doubt, one of the most celebrated minimax theorems is that of K. Fan. With regard to it, let A and B be nonempty sets and let $\Phi : A \times B \longrightarrow \mathbb{R}$ be a function. K. Fan introduced in [10, p. 42] the following notion, less restrictive than convexity, which allows us to dispense with the linear structure: Φ is *convexlike* on B provided that for all $y_1, y_2 \in B$ and $0 \leq t \leq 1$ there exists $y_0 \in B$ such that

$$x \in A \;\Rightarrow\; \Phi(x, y_0) \leq t\Phi(x, y_1) + (1-t)\Phi(x, y_2).$$

As one can expect, Φ is said to be *concavelike* on B when $-\Phi$ is convexlike on B, and in an analogous way the same concepts are defined on A. Let us also mention that, as usual, we use the convention that $\Phi : A \times B \longrightarrow \mathbb{R}$ satisfies a one variable-depending condition on A (respectively on B) to express that for all $y \in B$ (respectively $x \in A$) such a condition holds for the function $x \mapsto \Phi(x, y)$ (respectively $y \mapsto \Phi(x, y)$). For instance, if A is a nonempty topological space and B is a nonempty set, a function $\Phi : A \times B \longrightarrow \mathbb{R}$ is upper semicontinuous on A when for all $y \in B$, the function

$$x \in A \mapsto \Phi(x, y)$$

is upper semicontinuous on A. The minimax theorem of Fan ([10, Theorem 2], see also [18, Minimax-Theorem], [32, Lemma 11 and Remark 6], [5, Theorem A] and [30, Theorem 3.1]) reads as follows:

Theorem 1.1 (Fan's minimax inequality). *Let A be a nonempty compact topological space and let B be a nonempty set. Let us also assume that $\Phi : A \times B \longrightarrow \mathbb{R}$ is a function which is*

(*i*) *upper semicontinuous and concavelike on A and*
(*ii*) *convexlike on B.*

Then

$$\inf_{y \in B} \sup_{x \in A} \Phi(x, y) = \sup_{x \in A} \inf_{y \in B} \Phi(x, y).$$

Indeed, since A is compact and Φ is upper semicontinuous on A, then one can replace "sup" with "max" in the above minimax identity.

In this paper we refine this earlier result, indeed, also another different and classical minimax inequality due to M. Sion, Theorem 2.21 below. In our main statement, Theorem 2.15, we obtain a minimax inequality which unifies and generalizes the two, and moreover, it is sharp for the topological assumptions under consideration, as shown in Theorem 2.20. For this reason, the extension of the concept of convexlikeness introduced in Definition 2.1, the *sup-convexlikeness*, seems to be an intrinsic type of convexity for dealing with minimax inequalities. In order to establish Theorem 2.15 we adapt the ideas in [30], and more precisely those in a nice and useful result of S. Simons on concave functions [32, Lemma 9] for a more general framework, in Proposition 2.9. These results form the core of Section 2.

Another motivation behind Theorem 2.15 is to state a vector-valued Lax–Milgram theorem for which the Fan or Sion theorems do not work as a tool for proving it, unlike the proper extension of them that we introduce here, dealt with in Section 3. The known vector-valued versions of the Lax–Milgram theorem are global (see for instance [19, Theorem 3] and [28, Theorem 9]); that is, they analyze when a vector-valued continuous bilinear mapping represents in one of its variables the family of all the corresponding continuous linear operators. For the sake of precision, we introduce some common notations: if E, F and G are vector spaces, for a bilinear mapping $a : E \times F \longrightarrow G$ and $x \in E$, $a(x, \cdot)$ stands for the linear operator from F into G

$$y \in F \longmapsto a(x, y) \in G.$$

Similarly, when $y \in F$ we define the corresponding linear operator $a(\cdot, y)$ from E into G. For real topological vector spaces E and F we write $\mathrm{L}(E, F)$ to denote the space of those continuous and linear operators from E into F, and when $F = \mathbb{R}$, E^* is $\mathrm{L}(E, \mathbb{R})$, i.e., the topological dual space of E. The Lax–Milgram type results [19, Theorem 3] and [28, Theorem 9] provide necessary and sufficient conditions for a continuous bilinear mapping $a : E \times F \longrightarrow G$ in order that

$$\textit{for all } T \in \mathrm{L}(E, F) \text{ there exists } x_0 \in E \text{ such that } T = a(x_0, \cdot),$$

by imposing additional conditions on G (it admits a Schauder basis). Instead, we adopt a more accurate approach, along the lines of [26, Corollary 1.3]: given a continuous bilinear mapping $a : E \times F \longrightarrow G$, we characterize in Theorem 3.2 when

$$\textit{for a fixed } T \in \mathrm{L}(E, F) \text{ there exists } x_0 \in E \text{ such that } T = a(x_0, \cdot).$$

Finally, as an application of that vectorial Lax–Milgram theorem, Section 4 is concerned with studying the solvability of systems with an infinite number of variational equations.

2. The minimax theorem

This section is focused on providing an extension of the minimax inequality of K. Fan, Theorem 1.1, in Theorem 2.15, as well as on showing that, under adequate topological assumptions, it cannot be improved. As mentioned above, we extend [32, Lemma 9] to a more general context. With this aim in mind, we previously need to generalize the notion of convexlikeness, and to establish some elementary facts.

Given $m \geq 1$, let us denote by Δ_m the *unit simplex* in $\mathbb{R}^m$, that is,

$$\Delta_m := \left\{ (t_1, \ldots, t_m) \in \mathbb{R}^m : t_1, \ldots, t_m \geq 0 \text{ and } \sum_{j=1}^m t_j = 1 \right\}.$$

As we shall prove in the characterizations given in Theorem 2.20 and Proposition 2.11, the following concept of convexity intrinsically arises when dealing with minimax inequalities:

Definition 2.1. If A and B are nonempty sets, a function $\Phi : A \times B \longrightarrow \mathbb{R}$ is *sup-convexlike* on B when

$$\left. \begin{array}{l} m \geq 1,\ y_1, \ldots, y_m \in B \\ (t_1, \ldots, t_m) \in \Delta_m \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \text{there exists } y_0 \in B \text{ such that} \\ \sup\limits_{x \in A} \Phi(x, y_0) \leq \sup\limits_{x \in A} \sum\limits_{j=1}^m t_j \Phi(x, y_j). \end{array} \right.$$

In a similar way, Φ is *inf-concavelike* on A as soon as

$$\left. \begin{array}{l} n \geq 1,\ x_1, \ldots, x_n \in A \\ (s_1, \ldots, s_n) \in \Delta_n \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \text{there exists } x_0 \in A \text{ such that} \\ \inf\limits_{y \in B} \sum\limits_{i=1}^n s_i \Phi(x_i, y) \leq \inf\limits_{y \in B} \Phi(x_0, y). \end{array} \right.$$

Let us first note that Φ is sup-convexlike on B provided it is convexlike on B, according to the following basic result, whose proof involves standard and simple arguments:

Lemma 2.2. *Assume that A and B are nonempty sets. Then a function $\Phi : A \times B \longrightarrow \mathbb{R}$ is convexlike on B if, and only if,*

$$\left. \begin{array}{l} m \geq 1,\ y_1, \ldots, y_m \in B \\ (t_1, \ldots, t_m) \in \Delta_m \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \textit{there exists } y_0 \in B \textit{ such that} \\ x \in A \ \Rightarrow\ \Phi(x, y_0) \leq \sum\limits_{j=1}^m t_j \Phi(x, y_j). \end{array} \right. \tag{1}$$

Proof. It is plain that condition (1) implies that Φ is convexlike on B, so we deal with the converse. To this end, let us proceed by induction on m. For $m = 1$ there is nothing to prove, thus we suppose that for some $m \geq 1$ there holds

$$\left. \begin{array}{l} y_1, \ldots, y_m \in B \\ (t_1, \ldots, t_m) \in \Delta_m \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \text{there exists } y_0 \in B \text{ such that} \\ x \in A \ \Rightarrow\ \Phi(x, y_0) \leq \sum\limits_{j=1}^m t_j \Phi(x, y_j) \end{array} \right.$$

and we are going to prove that

$$\left.\begin{array}{l} y_1,\dots,y_m,y_{m+1}\in B \\ (t_1,\dots,t_m,t_{m+1})\in\Delta_{m+1}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{there exists } y_0\in B \text{ with} \\ x\in A \;\Rightarrow\; \Phi(x,y_0)\le \displaystyle\sum_{j=1}^{m+1} t_j\Phi(x,y_j).\end{array}\right.$$

Therefore, let us choose $y_1,\dots,y_m,y_{m+1}\in B$ and $(t_1,\dots,t_m,t_{m+1})\in\Delta_{m+1}$, and suppose, without loss of generality, that $t_{m+1}\neq 0$. Then, for all $x\in A$,

$$\begin{aligned}
&\sum_{j=1}^{m+1} t_j\Phi(x,y_j)\\
&= \sum_{j=1}^{m-1} t_j\Phi(x,y_j) \qquad (2)\\
&\quad + \left(1-\sum_{j=1}^{m-1} t_j\right)\left[\frac{t_m}{\left(1-\sum_{j=1}^{m-1} t_j\right)}\Phi(x,y_m)+\frac{t_{m+1}}{\left(1-\sum_{j=1}^{m-1} t_j\right)}\Phi(x,y_{m+1})\right].
\end{aligned}$$

But, since Φ is convexlike on B, there exists $\widetilde{y}\in B$ satisfying

$$x\in A \;\Rightarrow\; \Phi(x,\widetilde{y})\le \frac{t_m}{\left(1-\sum_{j=1}^{m-1} t_j\right)}\Phi(x,y_m)+\frac{t_{m+1}}{\left(1-\sum_{j=1}^{m-1} t_j\right)}\Phi(x,y_{m+1}),$$

so, in view of (2) we arrive at

$$x\in A \;\Rightarrow\; \sum_{j=1}^{m-1} t_j\Phi(x,y_j)+\left(1-\sum_{j=1}^{m-1} t_j\right)\Phi(x,\widetilde{y})\le \sum_{j=1}^{m+1} t_j\Phi(x,y_j)$$

and it suffices to apply the induction hypothesis to conclude the proof. □

Corollary 2.3. *Let A and B be nonempty sets and let $\Phi : A\times B \longrightarrow \mathbb{R}$ be a convexlike function on B. Then Φ is sup-convexlike on B.*

The converse of Corollary 2.3 does not hold, as shown in the following example:

Example 2.4. Let E be a nontrivial real normed space, with closed unit ball B_E, and let $x_0^* : E \longrightarrow \mathbb{R}$ be a nonzero continuous linear functional. Consider the nonempty sets $A := B_E$ and the compact interval of $\mathbb{R}$ $B := [-1,1]$, and the real function $\Phi : A\times B\longrightarrow\mathbb{R}$ defined for each $(x,y)\in A\times B$ as

$$\Phi(x,y) := x_0^*(x)f(y),$$

where $f : [-1, 1] \longrightarrow \mathbb{R}$ is given for all $y \in B$ by

$$f(y) := \begin{cases} 1, & \text{if } -0.5 \leq y \leq 0.5 \\ 0, & \text{if } -1 \leq y < -0.5 \text{ or } 0.5 < y \leq 1. \end{cases}$$

Let us first check that Φ is not convexlike on B. Let $y_1 := 0.25,\ y_2 := 0.75 \in B$ and let $0 < t < 1$. We proceed by contradiction. Therefore, let us suppose that there exits $y_0 \in B$ in such a way that

$$x \in A \ \Rightarrow \ \Phi(x, y_0) \leq t\Phi(x, y_1) + (1 - t)\Phi(x, y_2),$$

i.e.,

$$x \in B_E \ \Rightarrow \ x_0^*(x) f(y_0) \leq x_0^*(x)(tf(y_1) + (1 - t)f(y_2)),$$

equivalently ($x_0^* \neq 0$),

$$f(y_0) = tf(y_1) + (1 - t)f(y_2).$$

But

$$tf(y_1) + (1 - t)f(y_2) = t$$

and thus

$$f(y_0) = t,$$

which is impossible, since $0 \neq t \neq 1$ and $f([0, 1]) = \{0, 1\}$.

Let us conclude by showing that, nevertheless, Φ is sup-convexlike on B. Indeed, given $m \geq 1$, $0 \leq y_1, \ldots, y_m \leq 1$, and $(t_1, \ldots, t_m) \in \Delta_m$, for $y_0 := 0.75 \in [0, 1]$ we have that

$$\begin{aligned} \max_{x \in B_E} \Phi(x, 0.75) &= 0 \\ &\leq \|x_0^*\| \left(\sum_{j=1}^m t_j f(y_j) \right) \\ &= \max_{x \in B_E} \sum_{j=1}^m t_j \Phi(x, y_j). \end{aligned}$$

□

It is obvious that as a consequence of Corollary 2.3 and Example 2.4, for a function $\Phi : A \times B \longrightarrow \mathbb{R}$

$$\Phi \text{ concavelike on } A \ \Rightarrow \ \Phi \text{ inf-concavelike on } A$$

but

$$\Phi \text{ concavelike on } A \ \not\Leftarrow \ \Phi \text{ inf-concavelike on } A.$$

In general, given $n \geq 1$, $x_1, \ldots, x_n \in A$ and $(s_1, \ldots, s_n) \in \Delta_n$ we have that

$$\inf_{y \in B} \sum_{i=1}^n s_i \Phi(x_i, y) \leq \inf_{y \in B} \sup_{x \in A} \Phi(x, y).$$

In the next easy result we obtain something better when Φ is inf-concavelike on A:

Lemma 2.5. *Suppose that A and B are nonempty sets and that $\Phi : A\times B \longrightarrow \mathbb{R}$ is inf-concavelike on A. If $n \geq 1$, $x_1,\ldots,x_n \in A$ and $(s_1,\ldots,s_n) \in \Delta_n$, then*

$$\inf_{y\in B} \sum_{i=1}^{n} s_i\Phi(x_i,y) \leq \sup_{x\in A}\inf_{y\in B} \Phi(x,y).$$

Proof. Given $n \geq 1$, $x_1,\ldots,x_n \in A$ and $(s_1,\ldots,s_n) \in \Delta_n$, let $x_0 \in A$ such that

$$\inf_{y\in B} \sum_{i=1}^{n} s_i\Phi(x_i,y) \leq \inf_{y\in B} \Phi(x_0,y),$$

which does exist by the inf-concavelikeness of Φ on A. Then,

$$\begin{aligned}\inf_{y\in B} \sum_{i=1}^{n} s_i\Phi(x_i,y) &\leq \inf_{y\in B} \Phi(x_0,y)\\ &\leq \sup_{x\in A}\inf_{y\in B} \Phi(x,y),\end{aligned}$$

and the proof is complete. □

Another basis fact regarding convex combinations is collected in the following result:

Lemma 2.6. *Assume that A and B are nonempty sets and consider a function $\Phi : A\times B \longrightarrow \mathbb{R}$. If moreover $n \geq 1$, $x_1,\ldots,x_n \in A$, $(s_1,\ldots,s_n) \in \Delta_n$ and $m \geq 1$, $y_1,\ldots,y_m \in B$, $(t_1,\ldots,t_m) \in \Delta_m$, then*

$$\min_{j=1,\ldots,m} \sum_{i=1}^{n} s_i\Phi(x_i,y_j) \leq \max_{i=1,\ldots,n} \sum_{j=1}^{m} t_j\Phi(x_i,y_j).$$

Proof. The announced inequality is obvious, since for all $n \geq 1$, $x_1,\ldots,x_n \in A$, $(s_1,\ldots,s_n) \in \Delta_n$ and for all $m \geq 1$, $y_1,\ldots,y_m \in B$, $(t_1,\ldots,t_m) \in \Delta_m$ we have that

$$\begin{aligned}\min_{j=1,\ldots,m} \sum_{i=1}^{n} s_i\Phi(x_i,y_j) &\leq \sum_{j=1}^{m} t_j\left(\sum_{i=1}^{n} s_i\Phi(x_i,y_j)\right)\\ &= \sum_{i=1}^{n} s_i\left(\sum_{j=1}^{m} t_j\Phi(x_i,y_j)\right)\\ &\leq \max_{i=1,\ldots,n} \sum_{j=1}^{m} t_j\Phi(x_i,y_j).\end{aligned}$$

□

The inequality can be strict:

Example 2.7. Let A and B be the open interval of $\mathbb{R}$ $(0,1)$. For the function $\Phi : (0,1) \times (0,1) \longrightarrow \mathbb{R}$ defined for each $0 < x, y < 1$ as

$$\Phi(x,y) := \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } y < x, \end{cases}$$

taking $n := 2 =: m$, $x_1 := 0.25 =: y_1$, $x_2 := 0.75 =: y_2$ and $(s_1, s_2) := (0.5, 0.5) =: (t_1, t_2)$ yields

$$\begin{aligned} \min_{j=1,2} \sum_{i=1}^{2} s_i \Phi(x_i, y_j) &= 0.5 \\ &< 1 \\ &= \max_{i=1,2} \sum_{j=1}^{2} t_j \Phi(x_i, y_j). \end{aligned}$$ □

Now we provide the announced extension of Fan's minimax theorem, Theorem 1.1, in Theorem 2.15. As mentioned above, we argue as in [30], but adapting the proof of [32, Lemma 9] to a more general context. In fact, we follow the proof given in [30, Lemma 2.1]. We shall proceed step by step, arriving at the proof from some previous results. The first of them is a well-known characterization of the minimax identity in terms of finite subsets of B (see the proofs of [10, Theorem 2], [5, Theorem A] or [30, Theorem 3.1]), although here we include a slight modification:

Lemma 2.8. *Let $\alpha \in \mathbb{R}$, let A be a nonempty compact topological space, let B be a nonempty set and let $\Phi : A \times B \longrightarrow \mathbb{R}$ be an upper semicontinuous function on A. Then*

$$\alpha \leq \max_{x \in A} \inf_{y \in B} \Phi(x,y)$$

if, and only if, there exists a finite subset B_1 of B such that

$$\left.\begin{array}{l} B_1 \subset B_0 \subset B \\ B_0 \textit{ finite} \end{array}\right\} \Rightarrow \alpha \leq \max_{x \in A} \min_{y \in B_0} \Phi(x,y).$$

Proof. A is compact and each subset of A of the form $\{a \in A : \alpha \leq \Phi(a,b)\}$ is closed, because Φ is upper semicontinuous on A, so thanks to the finite intersection property, it follows that

$$\bigcap_{b \in B} \{a \in A : \alpha \leq \Phi(a,b)\} \neq \emptyset$$

$$\Updownarrow$$

$$B_0 \text{ finite subset of } B \;\Rightarrow\; \bigcap_{b \in B_0} \{a \in A : \alpha \leq \Phi(a,b)\} \neq \emptyset,$$

which is clearly equivalent to the existence of a finite subset B_1 of B satisfying

$$\left.\begin{array}{l} B_1 \subset B_0 \subset B \\ B_0 \text{ finite} \end{array}\right\} \Rightarrow \bigcap_{b \in B_0} \{a \in A : \alpha \leq \Phi(a,b)\} \neq \emptyset,$$

and this is exactly what we needed to prove. □

In particular we deduce, under the assumptions of Lemma 2.8, that

$$\inf_{y \in B} \max_{x \in A} \Phi(x,y) = \max_{x \in A} \inf_{y \in B} \Phi(x,y)$$

whenever for some finite subset B_1 of B there holds

$$\left.\begin{array}{l} B_1 \subset B_0 \subset B \\ B_0 \text{ finite} \end{array}\right\} \Rightarrow \inf_{y \in B} \max_{x \in A} \Phi(x,y) \leq \max_{x \in A} \min_{y \in B_0} \Phi(x,y).$$

To handle $\max_{x\in A} \min_{y \in B_0} \Phi(x,y)$ in Lemma 2.8 we generalize [32, Lemma 9], but modifying the proof appearing in [30, Lemma 2.1] (see also [29, Lemma 3.1] for another proof). This will be essential in order to establish our minimax theorem and its generality.

Proposition 2.9. *Suppose that* A *is a nonempty set,* $B_0 = \{y_1, \ldots, y_m\}$ *is a finite set and that* $\Phi : A \times B_0 \longrightarrow \mathbb{R}$ *is a inf-concavelike function on* A. *Then there exists* $(t_1, \ldots, t_m) \in \Delta_m$ *such that*

$$\sup_{x \in A} \min_{j=1,\ldots,m} \Phi(x, y_j) = \sup_{x \in A} \sum_{j=1}^{m} t_j \Phi(x, y_j).$$

Proof. It suffices to consider the convex subset of $\mathbb{R}^m$

$$C := \mathrm{co}\{(-\Phi(x,y_1), \ldots, -\Phi(x,y_m)) : x \in A\}$$

("co" stands for "convex hull"), and the sublinear functional $S : \mathbb{R}^m \longrightarrow \mathbb{R}$ defined for each $(\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ as

$$S(\xi_1, \ldots, \xi_m) := \max_{j=1,\ldots,m} \xi_j.$$

Then the version of the Hahn–Banach theorem known as *Mazur–Orlicz's theorem* (see for instance [30, Theorem 1.1]) provides us with a linear functional $L : \mathbb{R}^m \longrightarrow \mathbb{R}$ in such a way that $L \leq S$ and

$$\inf_{(\xi_1,\ldots,\xi_m) \in C} L(\xi_1, \ldots, \xi_m) = \inf_{(\xi_1,\ldots,\xi_m) \in C} S(\xi_1, \ldots, \xi_m). \tag{3}$$

The linear functional L is of the form

$$L(\xi_1, \ldots, \xi_m) = \sum_{j=1}^{m} t_j \xi_j, \quad ((\xi_1, \ldots, \xi_m) \in \mathbb{R}^m),$$

for some $t_1, \ldots, t_m \in \mathbb{R}$, and since $L \leq S$, then

$$(t_1, \ldots, t_m) \in \Delta_m.$$

Finally

$$\begin{aligned}\inf_{(\xi_1,\ldots,\xi_m)\in C} L(\xi_1, \ldots, \xi_m) &= \inf_{(\xi_1,\ldots,\xi_m)\in C} \sum_{j=1}^m t_j \xi_j \\ &\leq \inf_{x\in A} \sum_{j=1}^m -t_j \Phi(x, y_j) \qquad (4) \\ &= -\sup_{x\in A} \sum_{j=1}^m t_j \Phi(x, y_j),\end{aligned}$$

while

$$\begin{aligned}&\inf_{(\xi_1,\ldots,\xi_m)\in C} S(\xi_1, \ldots, \xi_m) = \inf_{(\xi_1,\ldots,\xi_m)\in C} \max_{j=1,\ldots,m} \xi_j \\ &= \inf_{\substack{n\geq 1 \\ (s_1,\ldots,s_n)\in\Delta_n \\ x_1,\ldots,x_n\in A}} \max_{j=1,\ldots,m} \sum_{i=1}^n -s_i \Phi(x_i, y_j) \\ &= \inf_{\substack{n\geq 1 \\ (s_1,\ldots,s_n)\in\Delta_n \\ x_1,\ldots,x_n\in A}} \left(-\min_{j=1,\ldots,m} \sum_{i=1}^n s_i \Phi(x_i, y_j) \right) \qquad (5) \\ &\geq \inf_{\substack{n\geq 1 \\ (s_1,\ldots,s_n)\in\Delta_n \\ x_1,\ldots,x_n\in A}} \left(-\sup_{x\in A} \min_{j=1,\ldots,m} \Phi(x, y_j) \right) \quad \text{(by Lemma 2.5)} \\ &= -\sup_{x\in A} \min_{j=1,\ldots,m} \Phi(x, y_j),\end{aligned}$$

hence, according to (3), (4) and (5),

$$\sup_{x\in A} \sum_{j=1}^m t_j \Phi(x, y_j) \leq \sup_{x\in A} \min_{j=1,\ldots,m} \Phi(x, y_j).$$

To conclude, let us observe that the reverse inequality trivially holds, and therefore we have the equality. □

Remark 2.10. If we do not assume Φ is inf-concavelike and follow the proof of Proposition 2.9, then we derive the existence of $(t_1, \ldots, t_m) \in \Delta_m$ such that

$$\sup_{x\in A} \sum_{j=1}^m t_j \Phi(x, y_j) \leq \sup_{\substack{n\geq 1 \\ (s_1,\ldots,s_n)\in\Delta_n \\ x_1,\ldots,x_n\in A}} \min_{j=1,\ldots,m} \sum_{i=1}^n s_i \Phi(x_i, y_j),$$

so by Lemma 2.6 we have that

$$\min_{(t_1,\ldots,t_m)\in\Delta_m} \sup_{x\in A} \sum_{j=1}^{m} t_j\Phi(x,y_j) = \sup_{\substack{n\geq 1\\ (s_1,\ldots,s_n)\in\Delta_n\\ x_1,\ldots,x_n\in A}} \min_{j=1,\ldots,m} \sum_{i=1}^{n} s_i\Phi(x_i,y_j), \tag{6}$$

which is also a straight consequence of Wald's minimax theorem [35, Theorem 3.1]. Indeed, Proposition 2.9 is a sort of minimax result, since it can be equivalently reformulated in these terms: if A is a nonempty set, $B_0 = \{y_1,\ldots,y_m\}$ is a finite set and $\Phi : A\times B_0 \longrightarrow \mathbb{R}$ is inf-concavelike on A, then

$$\min_{(t_1,\ldots,t_m)\in\Delta_m} \sup_{x\in A} \sum_{j=1}^{m} t_j\Phi(x,y_j) = \sup_{x\in A} \min_{j=1,\ldots,m} \Phi(x,y_j),$$

and so, it follows the improvement of (6)

$$\begin{aligned}\min_{(t_1,\ldots,t_m)\in\Delta_m} \sup_{x\in A} \sum_{j=1}^{m} t_j\Phi(x,y_j) &= \sup_{\substack{n\geq 1\\ (s_1,\ldots,s_n)\in\Delta_n\\ x_1,\ldots,x_n\in A}} \min_{j=1,\ldots,m} \sum_{i=1}^{n} s_i\Phi(x_i,y_j)\\ &= \sup_{x\in A} \min_{j=1,\ldots,m} \Phi(x,y_j).\end{aligned} \tag{7}$$

The first time we prove the concept of sup-convexlikeness is an ad hoc hypothesis in a minimax context –in the presence of compactness and upper semicontinuity– is the next characterization, where it is shown that Proposition 2.9 is an optimal result:

Proposition 2.11. *Let A be a nonempty compact topological space, let $B_0 = \{y_1,\ldots,y_m\}$ be a finite set and let $\Phi : A\times B_0 \longrightarrow \mathbb{R}$ be an upper semicontinuous function on A. Then*

Φ is inf-concavelike on A

if, and only if,

$$\begin{aligned}&\textit{there exists } (t_1,\ldots,t_m)\in\Delta_m\\ &\textit{such that } \max_{x\in A} \min_{j=1,\ldots,m} \Phi(x,y_j) = \max_{x\in A} \sum_{j=1}^{m} t_j\Phi(x,y_j).\end{aligned} \tag{8}$$

Proof. According to Proposition 2.9, we only concentrate on the proof that the inf-concavelikeness of Φ follows from (8). Therefore, let $n\geq 1$, $(s_1,\ldots,s_n)\in\Delta_m$ and $x_1,\ldots,x_n\in A$. Since A is compact and Φ is upper semicontinuous on A, we fix $x_0\in A$ in such a way that

$$\min_{j=1,\ldots,m} \Phi(x_0,y_j) = \max_{x\in A} \min_{j=1,\ldots,m} \Phi(x,y_j). \tag{9}$$

Then, for $(t_1, \ldots, t_m) \in \Delta_m$ satisfying (8), we have that

$$\begin{aligned}
\min_{j=1,\ldots,m} \sum_{i=1}^{n} s_i \Phi(x_i, y_j) &\leq \max_{i=1,\ldots,n} \sum_{j=1}^{m} t_j \Phi(x_i, y_j) \quad \text{(by Lemma 2.6)} \\
&\leq \max_{x \in A} \sum_{j=1}^{m} t_j \Phi(x, y_j) \\
&= \max_{x \in A} \min_{j=1,\ldots,m} \Phi(x, y_j) \\
&= \min_{j=1,\ldots,m} \Phi(x_0, y_j) \quad \text{(by (9))},
\end{aligned}$$

and thus

$$\min_{j=1,\ldots,m} \sum_{i=1}^{n} s_i \Phi(x_i, y_j) \leq \min_{j=1,\ldots,m} \Phi(x_0, y_j)$$

which implies that Φ is inf-concavelike, because $n \geq 1$, $(s_1, \ldots, s_n) \in \Delta_m$ and $x_1, \ldots, x_n \in A$ are arbitrary. $\square$

Proposition 2.9 allows us to derive a first minimax result, which is a particular case of Theorem 2.15:

Corollary 2.12. *Suppose that A is a nonempty set, B_0 is a nonempty finite set and that $\Phi : A \times B_0 \longrightarrow \mathbb{R}$ is inf-concavelike on A and sup-convexlike on B_0. Then*

$$\min_{y \in B_0} \sup_{x \in A} \Phi(x, y) = \sup_{x \in A} \min_{y \in B_0} \Phi(x, y).$$

Proof. Since B_0 is finite, let us say $B_0 = \{y_1, \ldots, y_m\}$, and Φ is inf-concavelike on A, Proposition 2.9 provides us with $(t_1, \ldots, t_m) \in \Delta_m$ such that

$$\sup_{x \in A} \min_{y \in B_0} \Phi(x, y) = \sup_{x \in A} \sum_{j=1}^{m} t_j \Phi(x, y_j),$$

and the sup-convexlikeness of Φ on B_0 guarantees the existence of $y_{j_0} \in B_0$ with

$$\sup_{x \in A} \Phi(x, y_{j_0}) \leq \sup_{x \in A} \sum_{j=1}^{m} t_j \Phi(x, y_j),$$

therefore

$$\begin{aligned}
\min_{y \in B_0} \sup_{x \in A} \Phi(x, y) &\leq \sup_{x \in A} \Phi(x, y_{j_0}) \\
&\leq \sup_{x \in A} \min_{y \in B_0} \Phi(x, y).
\end{aligned}$$

$\square$

One can guess from Corollary 2.12 that the minimax identity for Φ follows from its inf-concavelikeness on A and its sup-convexlikeness on B. However, the function Φ in Example 2.7 is clearly inf-concavelike on A and sup-convexlike on B, but the minimax inequality is strict. Actually, we need a little more work for stating our minimax theorem. The next discretization of the inf-concavelikeness will be essential to this end. For a set X, a nonempty subset Y of X and a function $f : X \longrightarrow \mathbb{R}$, we write $f_{|Y}$ for the restriction of f to Y.

Lemma 2.13. *Assume that A is a nonempty compact topological space, B is a nonempty set and that $\Phi : A \times B \longrightarrow \mathbb{R}$ is an upper semicontinuous function on A for which there exists a finite subset B_1 of B in such a way that*

$$\left.\begin{array}{l} B_1 \subset B_0 \subset B \\ B_0 \textit{ finite} \end{array}\right\} \Rightarrow \ \Phi_{|A\times B_0} \textit{ is inf-concavelike on } A.$$

Then Φ is inf-concavelike on A.

Proof. Let $n \geq 1$, let $x_1, \ldots, x_n \in A$ and let $(s_1, \ldots, s_n) \in \Delta_n$. We can clearly suppose that $\inf_{y\in B} \sum_{i=1}^n s_i\Phi(x_i, y) > -\infty$. Given a finite subset B_0 of B with $B_1 \subset B_0$, the inf-concavelikeness of $\Phi_{|A\times B_0}$ provides us with an $x_0 \in A$ such that

$$\min_{y\in B_0} \sum_{i=1}^n s_i\Phi(x_i, y) \leq \min_{y\in B_0} \Phi(x_0, y),$$

so

$$\begin{aligned} \inf_{y\in B} \sum_{i=1}^n s_i\Phi(x_i, y) &\leq \min_{y\in B_0} \sum_{i=1}^n s_i\Phi(x_i, y) \\ &\leq \min_{y\in B_0} \Phi(x_0, y). \end{aligned}$$

Therefore

$$\left.\begin{array}{l} B_1 \subset B_0 \subset B \\ B_0 \text{ finite} \end{array}\right\} \Rightarrow \ \inf_{y\in B} \sum_{i=1}^n s_i\Phi(x_i, y) \leq \max_{x\in A} \min_{y\in B_0} \Phi(x, y),$$

and so Lemma 2.8 implies

$$\inf_{y\in B} \sum_{i=1}^n s_i\Phi(x_i, y) \leq \max_{x\in A} \inf_{y\in B} \Phi(x, y),$$

that is, Φ is inf-concavelike on A. □

Let us emphasize that the sufficient condition for the inf-concavelikeness given in Lemma 2.13 is not necessary, as shown by the next example:

Example 2.14. Let us consider the compact interval of $\mathbb{R}$ $A := [0,1] =: B$ and the function $\Phi : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ defined by

$$\Phi(x,y) := \begin{cases} 0, & \text{if } x \neq y \\ 1, & \text{if } x = y. \end{cases}$$

It is obvious that A is compact and that Φ is upper semicontinuous on A. The fact that Φ is inf-concavelike on its first variable is easily verified: if $n \geq 1$, $x_1, \ldots, x_n \in [0,1]$ and $(s_1, \ldots, s_n) \in \Delta_n$, then

$$\inf_{y \in [0,1]} \sum_{i=1}^{n} s_i \Phi(x_i, y) = 0,$$

because choosing any $\tilde{y} \in [0,1] \backslash \{x_1, \ldots, x_n\}$ yields

$$i = 1, \ldots, n \ \Rightarrow \ \Phi(x_i, \tilde{y}) = 0.$$

Therefore, since for all $x \in [0,1]$ we have that

$$\inf_{y \in [0,1]} \Phi(x,y) = 0,$$

then, for instance for $x_0 = 0$,

$$\inf_{y \in [0,1]} \sum_{i=1}^{n} s_i \Phi(x_i, y) \leq \inf_{y \in [0,1]} \Phi(x_0, y).$$

To conclude, we are going to prove not only that for any finite subset B_1 of $[0,1]$ there exists a finite subset B_0 of $[0,1]$ such that $B_1 \subset B_0$ and $\Phi_{|[0,1] \times B_0}$ is not inf-concavelike on $[0,1]$, but also that for each finite subset B_0 of $[0,1]$ with at least two different points, the function $\Phi_{|[0,1] \times B_0}$ is not inf-concavelike on $[0,1]$. Indeed, let $B_0 = \{y_1, \ldots, y_m\}$ be a finite subset of $[0,1]$ with $m \geq 2$ different elements, which implies

$$x \in [0,1] \ \Rightarrow \ \min_{j=1,\ldots,m} \Phi(x, y_j) = 0$$

and thus

$$\max_{x \in [0,1]} \min_{j=1,\ldots,m} \Phi(x, y_j) = 0.$$

Then, for $n := m$, $x_1 := y_1, \ldots, x_m := y_m$ and $(s_1, \ldots, s_m) := (1/m, \ldots, 1/m)$,

$$\begin{aligned} \min_{j=1,\ldots,m} \sum_{i=1}^{n} s_i \Phi(x_i, y_j) &= \frac{1}{m} \min_{j=1,\ldots,m} \sum_{i=1}^{m} \Phi(y_i, y_j) \\ &= \frac{1}{m}, \end{aligned}$$

so we arrive at

$$\begin{aligned}\max_{x\in[0,1]}\min_{j=1,\dots,m}\Phi(x,y_j) &= 0\\ &< \frac{1}{m}\\ &= \min_{j=1,\dots,m}\sum_{i=1}^{n} s_i\Phi(x_i,y_j)\end{aligned}$$

and $\Phi_{|[0,1]\times B_0}$ is not inf-concavelike on $[0,1]$. □

The main result in this work is the following minimax theorem:

Theorem 2.15. *Let A be a nonempty compact topological space, let B be a nonempty set and let $\Phi : A\times B \longrightarrow \mathbb{R}$ be an upper semicontinuous function on A. Suppose in addition that*

(i) *there exists a finite subset B_1 of B in such a way that*

$$\left.\begin{array}{l} B_1\subset B_0\subset B\\ B_0 \textit{ finite}\end{array}\right\} \Rightarrow \Phi_{|A\times B_0} \textit{ is inf-concavelike on } A$$

and

(ii) *Φ is sup-convexlike on B.*

Then

$$\inf_{y\in B}\max_{x\in A}\Phi(x,y) = \max_{x\in A}\inf_{y\in B}\Phi(x,y).$$

Proof. If $\inf_{y\in B}\max_{x\in A}\Phi(x,y) = -\infty$ there is nothing to prove. Otherwise, let B_1 be the finite subset of B whose existence we are assuming in (i). We are going to prove that

$$\left.\begin{array}{l} B_1\subset B_0\subset B\\ B_0 \text{ finite}\end{array}\right\} \Rightarrow \inf_{y\in B}\max_{x\in A}\Phi(x,y) \le \max_{x\in A}\min_{y\in B_0}\Phi(x,y), \tag{10}$$

which in view of Lemma 2.8 suffices to arrive at what we want to prove. Thus, let $B_0 = \{y_1,\dots,y_m\}$ be a finite subset of B with $B_1\subset B_0$. We deduce from the hypothesis (i) and Proposition 2.9 the existence of $(t_1,\dots,t_m)\in\Delta_m$ such that

$$\max_{x\in A}\min_{y\in B_0}\Phi(x,y) = \max_{x\in A}\sum_{j=1}^{m} t_j\Phi(x,y_j). \tag{11}$$

But since Φ is sup-convexlike on B, there exists $y_0\in B$ such that

$$\max_{x\in A}\Phi(x,y_0) \le \max_{x\in A}\sum_{j=1}^{m} t_j\Phi(x,y_j), \tag{12}$$

so, it follows from (11) and (12) that

$$\begin{aligned}\inf_{y\in B}\max_{x\in A}\Phi(x,y) &\le \max_{x\in A}\Phi(x,y_0)\\ &\le \max_{x\in A}\min_{y\in B_0}\Phi(x,y)\end{aligned}$$

and so (10) holds. □

Theorem 2.15 is an improved version of the Fan minimax theorem: it is enough to consider Corollary 2.3 and the function Φ in Example 2.4, with E a real reflexive Banach space which, as shown there, is sup-convexlike but not convexlike on B. But endowing A with its weak topology (inherited from E), it becomes compact and Φ is linear and continuous on A, so Theorem 2.15 applies, unlike Theorem 1.1, arriving at

$$\inf_{y\in[0,1]}\max_{x\in B_E}\Phi(x,y) = \max_{x\in B_E}\inf_{y\in[0,1]}\Phi(x,y)$$

(note that the saddle value is 0). Actually, Theorem 2.15 also generalizes the minimax inequality [15, Corollary 3.3] for $f = g$, which is a proper extension of Fan's minimax theorem. Its hypotheses are: A is a nonempty compact topological space, B is a nonempty set and $\Phi : A\times B \longrightarrow \mathbb{R}$ is an upper semicontinuous function on A which is convexlike on B and satisfies the so-called *weak concavelikeness* on A (for the definition see [15, p. 653]), which under those topological assumptions, is equivalent to the fact that for each finite subset B_0 of B, $\Phi_{|A\times B_0}$ is inf-concavelike on A. However, weak concavelikeness is stronger than (i) in Theorem 2.15, even when A is compact and Φ is upper semicontinuous on A:

Example 2.16. Let us consider the compact subsets of real numbers $A := \{0,1\}$ and $B := [0,1]$, and the function $\Phi : \{0,1\}\times[0,1] \longrightarrow \mathbb{R}$ defined as

$$\Phi(0,y) := \begin{cases} 1, & \text{when } 0\le y<0.5\\ 0, & \text{for } 0.5\le y\le 1\end{cases}$$

and

$$\Phi(1,y) := \begin{cases} 0, & \text{when } 0\le y\le 0.5\\ 1, & \text{for } 0.5< y\le 1.\end{cases}$$

A is compact, Φ is continuous on A and there exists a finite subset of B, $\tilde{B} := \{0,1\}$, such that $\Phi_{|A\times\tilde{B}}$ is not inf-concavelike on A, since for all $0<s<1$ there holds

$$\begin{aligned}\max_{x\in\{0,1\}}\min_{y\in\{0,1\}}\Phi(x,y) &= 0\\ &< \min\{s,1-s\}\\ &= \min_{y\in\{0,1\}}\left(s\Phi(0,y)+(1-s)\Phi(1,y)\right).\end{aligned}$$

However, it is easy to check that for each finite subset B_0 of B containing $B_1 := \{0, 0.5, 1\}$, $\Phi_{|A\times B_0}$ is inf-concavelike on A: given $0 \leq s \leq 1$,

$$\min_{y\in B_0} (s\Phi(0, y) + (1 - s)\Phi(1, y)) = 0$$
$$= \min_{y\in B_0} \Phi(0, y). \qquad \square$$

In addition, our minimax inequality is more general than [15, Corollary 3.3] with $f = g$: Theorem 2.15 applies to the function Φ in Example 2.16, unlike [15, Corollary 3.3] for $f = g$, since it is sup-convexlike on its second variable $[0, 1]$: given $m \geq 1$, $y_1, \ldots, y_m \in [0, 1]$ and $(t_1, \ldots, t_m) \in \Delta_m$,

$$\max_{x\in\{0,1\}} \Phi(x, 0.5) = 0$$
$$\leq \max_{x\in\{0,1\}} \sum_{j=1}^{m} t_j\Phi(x, y_j).$$

Moreover, and what is more remarkable, Theorem 2.15 is not far from being sharp, as shown in Theorem 2.20 below. It suffices to establish the next two easy lemmas:

Lemma 2.17. *Assume that A is a nonempty compact topological space, B is a nonempty set and that $\Phi : A \times B \longrightarrow \mathbb{R}$ is an upper semicontinuous function on A such that*

$$\inf_{y\in B} \max_{x\in A} \Phi(x, y) = \max_{x\in A} \inf_{y\in B} \Phi(x, y).$$

Then Φ is inf-concavelike on A.

Proof. Suppose that the minimax relation holds and let $n \geq 1$, $x_1, \ldots, x_n \in A$ and $(s_1, \ldots, s_n) \in \Delta_n$. Notice that the inequality

$$\inf_{y\in B} \sum_{i=1}^{n} s_i\Phi(x_i, y) \leq \inf_{y\in B} \max_{x\in A} \Phi(x, y)$$

is always valid, so our assumption yields

$$\inf_{y\in B} \sum_{i=1}^{n} s_i\Phi(x_i, y) \leq \max_{x\in A} \inf_{y\in B} \Phi(x, y).$$

Finally, we chose $x_0 \in A$ with

$$\inf_{y\in B} \Phi(x_0, y) = \max_{x\in A} \inf_{y\in B} \Phi(x, y)$$

to arrive at

$$\inf_{y\in B} \sum_{i=1}^{n} s_i\Phi(x_i, y) \leq \inf_{y\in B} \Phi(x_0, y),$$

and Φ is inf-concavelike on A. $\square$

Let us point out that Lemma 2.17 implies that the converse of Corollary 2.12 holds under some topological assumptions, leading to the next particular case of Theorem 2.20:

Corollary 2.18. *If A is a nonempty compact topological space, B_0 is a nonempty finite set and $\Phi : A \times B_0 \longrightarrow \mathbb{R}$ is an upper semicontinuous function on A, then*

$$\min_{y\in B_0}\max_{x\in A}\Phi(x,y)=\max_{x\in A}\min_{y\in B_0}\Phi(x,y)$$

if, and only if, Φ is inf-concavelike on A and sup-convexlike on B_0.

According to Lemma 2.17, the second announced result, Lemma 2.19 below, is a partial converse of Lemma 2.13:

Lemma 2.19. *Let A and B be nonempty compact topological spaces and let $\Phi : A \times B \longrightarrow \mathbb{R}$ be an upper semicontinuous function on A which is also lower semicontinuous on B. If in addition*

$$\min_{y\in B}\max_{x\in A}\Phi(x,y)=\max_{x\in A}\min_{y\in B}\Phi(x,y),$$

then there exists a finite subset B_1 of B in such a way that

$$\left.\begin{array}{l} B_1\subset B_0\subset B\\ B_0 \text{ finite}\end{array}\right\} \Rightarrow \Phi_{|A\times B_0} \text{ is inf-concavelike on } A.$$

Proof. Let us first choose, thanks to the compactness of B and the lower semicontinuity on B of Φ, $y_0 \in B$ such that

$$\max_{x\in A}\Phi(x,y_0)=\min_{y\in B}\max_{x\in A}\Phi(x,y).$$

Let us take $B_1 := \{y_0\}$ and a finite subset B_0 of B with $B_1 \subset B_0$. Then, taking into account the validity of the minimax equality, there hold

$$\begin{aligned}\min_{y\in B_0}\max_{x\in A}\Phi(x,y) &= \min_{y\in B}\max_{x\in A}\Phi(x,y)\\ &= \max_{x\in A}\min_{y\in B}\Phi(x,y)\\ &\le \max_{x\in A}\min_{y\in B_0}\Phi(x,y),\end{aligned}$$

hence the minimax inequality is valid for $\Phi_{|A\times B_0}$, and so finally Lemma 2.17 implies the inf-concavelikeness of $\Phi_{|A\times B_0}$ on A. □

The aforementioned fact that Theorem 2.15 cannot essentially be improved, as well as the fact that the sup-convexlikeness is an intrinsic concept of convexity for minimax inequalities, are stated in these terms:

Theorem 2.20. *Let A and B be nonempty compact topological spaces and let $\Phi : A \times B \longrightarrow \mathbb{R}$ be a function which is upper semicontinuous on A and lower semicontinuous on B. Then the following statements are equivalent:*

(*i*) *The minimax relation is valid:*

$$\min_{y\in B}\max_{x\in A}\Phi(x,y)=\max_{x\in A}\min_{y\in B}\Phi(x,y).$$

(*ii*) *There exists a finite subset B_1 of B satisfying*

$$\left.\begin{array}{l}B_1\subset B_0\subset B\\ B_0 \textit{ finite}\end{array}\right\}\Rightarrow\ \Phi_{|A\times B_0}\textit{ is inf-concavelike on } A$$

and for some finite subset A_1 of A there holds

$$\left.\begin{array}{l}A_1\subset A_0\subset A\\ A_0 \textit{ finite}\end{array}\right\}\Rightarrow\ \Phi_{|A_0\times B}\textit{ is sup-convexlike on } B.$$

(*iii*) *For some finite subset B_1 of B it is satisfied*

$$\left.\begin{array}{l}B_1\subset B_0\subset B\\ B_0 \textit{ finite}\end{array}\right\}\Rightarrow\ \Phi_{|A\times B_0}\textit{ is inf-concavelike on } A$$

and Φ is sup-convexlike on B.

(*iv*) *There exists a finite subset A_1 of A in such a way that*

$$\left.\begin{array}{l}A_1\subset A_0\subset A\\ A_0 \textit{ finite}\end{array}\right\}\Rightarrow\ \Phi_{|A_0\times B}\textit{ is sup-convexlike on } B$$

and Φ is inf-concavelike on A.

Proof. It follows from Theorem 2.15 and Lemmas 2.13, 2.17 and 2.19. □

Notice that [10, Theorem 1] states another equivalent condition for the minimax relation under the considered topological assumptions.

Let us emphasize that our Theorem 2.15 also generalizes the classical Sion minimax inequality, Theorem 2.21 below, in the sense that there is no function $\Phi : A\times B \longrightarrow \mathbb{R}$ for which Theorem 2.21 applies but Theorem 2.15 does not. To be more precise, let us recall that if A is a nonempty set and B is a nonempty convex subset of some vector space, a function $\Phi : A\times B \longrightarrow \mathbb{R}$ is said to be *quasi-convex* on B provided that for all $x \in A$ and for all $\beta \in \mathbb{R}$, the set $\{y \in B : \Phi(x,y) \leq \beta\}$ is convex. Analogously, we say that a function $\Phi : A \times B \longrightarrow \mathbb{R}$, defined in the cartesian product of a nonempty convex subset A of some linear space and a nonempty set B, is *quasi-concave* on A when for all $y \in B$ and for all $\alpha \in \mathbb{R}$, the set $\{x \in A : \Phi(x,y) \geq \alpha\}$ is convex. The result of M. Sion was stated as follows ([33, Theorem 3.4], see also [17, Theorem] and [22, Theorem 4]):

Theorem 2.21 (Sion's minimax inequality). *Assume that A and B are nonempty convex and compact subsets of some topological vector spaces and that $\Phi : A \times B \longrightarrow \mathbb{R}$ is a function which is*

(*i*) *upper semicontinuous and quasi-concave on* A *and*
(*ii*) *lower semicontinuous and quasi-convex on* B.

Then

$$\min_{y\in B}\max_{x\in A}\Phi(x,y)=\max_{x\in A}\min_{y\in B}\Phi(x,y).$$

Now we derive the next relationship between quasi-convexity and sup-convexlikeness from Theorem 2.21 and Lemma 2.17, and in this way, sup-convexlikeness comprises both kinds of convexity – convexlikeness and quasi-convexity – which are independent concepts (see [21, p. 70]).

Corollary 2.22. *If* A *and* B *are nonempty convex and compact subsets of some topological vector spaces and* $\Phi : A\times B \longrightarrow \mathbb{R}$ *is a function which is upper semicontinuous and quasi-concave on* A *and lower semicontinuous and quasi-convex on* B*, then* Φ *is inf-concavelike on* A *and sup-convexlike on* B.

Actually, Lemma 2.19 implies a stronger condition, i.e., the existence of finite subsets A_1 and B_1 of A and B, respectively, in such a way that

$$\left.\begin{array}{l} B_1\subset B_0\subset B\\ B_0 \text{ finite}\end{array}\right\} \Rightarrow\ \Phi_{|A\times B_0} \text{ is inf-concavelike on } A$$

and

$$\left.\begin{array}{l} A_1\subset A_0\subset A\\ A_0 \text{ finite}\end{array}\right\} \Rightarrow\ \Phi_{|A_0\times B} \text{ is sup-convexlike on } B.$$

According to Corollary 2.22 and Example 2.4 – also assuming that E is a real reflexive Banach space and taking in A its weak topology inherited from E, the hypotheses of Theorem 2.15 are satisfied, but not those of Theorem 2.21, because Φ is not quasi-convex on B – Theorem 2.15 improves Theorem 2.21.

In Section 3 we derive a consequence of Theorem 2.15, a vector-valued Lax–Milgram theorem, much subtler than Example 2.4, for which neither Theorem 1.1 nor Theorem 2.21 apply.

Let us finally mention that, although the classical minimax theory focuses on obtaining sufficient conditions for the minimax equality, the characterization in Theorem 2.20 provides clearly necessary and sufficient conditions for the opposite situation, that is,

$$\max_{x\in A}\min_{y\in B}\Phi(x,y)<\min_{y\in B}\max_{x\in A}\Phi(x,y).$$

Regarding to this failure of the minimax relation, known as a *strict minimax inequality*, let us say that recently an interest in studying it has arisen, as can be seen in [27, 24, 20, 23] and the references therein, and which is justified by its use in multiplicity-of-solution results for variational problems.

3. A vector-valued Lax–Milgram theorem

The results [19, Theorem 3] and [28, Theorem 9] quoted in the Introduction provide some vectorial versions of Lax–Milgram's theorem, and as also mentioned earlier, they have a global character. On the other hand, [26, Corollary 1.3] is a scalar Lax–Milgram type result, but involving only one fixed continuous linear functional:

Proposition 3.1. *If E is a real reflexive Banach space, F is a real normed space, $y_0^* \in F^*$ and $a : E \times F \longrightarrow \mathbb{R}$ is a bilinear form such that*

$$y \in F \;\Rightarrow\; a(\cdot, y) \in E^*,$$

then

$$\textit{there exists } x_0 \in E \textit{ such that } y_0^* = a(x_0, \cdot)$$

if, and only if,

$$\textit{there exists } \alpha \geq 0 \textit{ such that } y \in F \;\Rightarrow\; y_0^*(y) \leq \alpha \|a(\cdot, y)\|.$$

Moreover, if one of these conditions is satisfied and $a \neq 0$, then

$$\min\{\|x_0\| : x_0 \in E \textit{ and } y_0^*(y) = a(x_0, \cdot)\} = \sup_{y \in F,\ a(\cdot, y) \neq 0} \frac{y_0^*(y)}{\|a(\cdot, y)\|}.$$

Indeed, in [26, Corollary 1.3] the space E is not assumed to be reflexive, it is just a normed space, and so the representing element x_0 is not necessarily in E, but also maybe be in E^{**}.

The main result in this section is the following generalization of the Lax–Milgram theorem implying Proposition 3.1, which will be deduced from the minimax theorem, Theorem 2.15. As mentioned in the Introduction, neither the Fan minimax inequality nor Sion's minimax theorem apply, not even [15, Corollary 3.3] with $f = g$. It provides a characterization, by means of the existence of a certain nonnegative constant, of those continuous and linear operators between real topological vector spaces that are represented through a vector-valued bilinear mapping.

Theorem 3.2. *Let E be a real reflexive Banach space, F and G real topological vector spaces, $T_0 \in \mathrm{L}(F, G)$ and $a : E \times F \longrightarrow G$ a continuous bilinear mapping. Then*

$$\textit{there exists } x_0 \in E \textit{ such that } T_0 = a(x_0, \cdot) \tag{13}$$

if, and only if,

$$\begin{aligned} &\textit{there exists } \alpha \geq 0 \textit{ such that} \\ &\left.\begin{array}{l} m \geq 1,\ (y_1, z_1^*), \ldots, (y_m, z_m^*) \in F \times G^* \\ (t_1, \ldots, t_m) \in \Delta_m \end{array}\right\} \\ \Rightarrow\ &\sum_{j=1}^m t_j z_j^*(T_0 y_j) \leq \alpha \left\| \sum_{j=1}^m t_j z_j^* \circ a(\cdot, y_j) \right\|. \end{aligned} \tag{14}$$

Furthermore, assuming that these equivalent conditions hold and that a is nonzero, then the following estimate

$$\min\{\|x_0\| : x_0 \in E \text{ and } T_0 = a(x_0,\cdot)\}$$
$$= \sup_{\substack{m\geq 1,\ (y_1,z_1^*),\dots,(y_m,z_m^*)\in F\times G^* \\ (t_1,\dots,t_m)\in\Delta_m \\ \sum_{j=1}^m t_j z_j^*\circ a(\cdot,y_j)\neq 0}} \frac{\sum_{j=1}^m t_j z_j^*(T_0 y_j)}{\left\|\sum_{j=1}^m t_j z_j^*\circ a(\cdot,y_j)\right\|} \tag{15}$$

is valid.

Proof. Since the implication (13) $\Rightarrow$ (14) clearly holds (take $\alpha := \|x_0\|$), we may reduce our efforts and show only the converse (14) $\Rightarrow$ (13). Let us apply the minimax theorem, Theorem 2.15, with the nonempty sets

$$A := \alpha B_E, \qquad B := F \times G^*$$

(B_E denotes the closed unit ball of E) and the function $\Phi : A\times B \longrightarrow \mathbb{R}$ defined for each $(x,(y,z^*)) \in A\times B$ as

$$\Phi(x,(y,z^*)) := z^*(a(x,y)) - z^*(T_0 y).$$

A is weakly compact, because E is reflexive, and Φ is affine on A and weakly continuous. In addition Φ is sup-convexlike on B, since given $m\geq 1$, $(y_1,z_1^*),\dots,$ $(y_m,z_m^*)\in B$ and $(t_1,\dots,t_m)\in\Delta_m$, there exists $(y_0,z_0^*) := (0,0)\in B$ such that

$$\begin{aligned}\max_{x\in A}\Phi(x,(y_0,z_0^*)) &= 0\\ &\leq \alpha\left\|\sum_{j=1}^m t_j z_j^*\circ a(\cdot,y_j)\right\| - \sum_{j=1}^m t_j z_j^*(T_0 y_j) \quad \text{(by (14))}\\ &= \max_{x\in A}\sum_{j=1}^m t_j \Phi(x,(y_j,z_j^*)).\end{aligned}$$

Therefore,

$$\inf_{(y,z^*)\in B}\max_{x\in A}\Phi(x,(y,z^*)) = \max_{x\in A}\inf_{(y,z^*)\in B}\Phi(x,(y,z^*)).$$

But

$$\inf_{(y,z^*)\in B}\max_{x\in A}\Phi(x,(y,z^*)) = \inf_{(y,z^*)\in F\times G^*}(\alpha\|z^*\circ a(\cdot,y)\| - z^*(T_0 y))$$

which, in view of (14) for $m=1$, is nonnegative, hence

$$\max_{x\in A}\inf_{(y,z^*)\in B}\Phi(x,(y,z^*)) \geq 0,$$

that is,

$$\text{there exits } x_0 \in \alpha B_E \text{ such that } T_0 = a(x_0, \cdot)$$

and we have stated the equivalence (13) $\Leftrightarrow$ (14). Let us emphasize that in both directions some x_0 can be chosen in such a way that $\|x_0\| \leq \alpha$.

In order to conclude the proof we deduce the control of the norm given by (15). Thus, let us assume that (13), or equivalently (14), is satisfied. Let $x \in E$ such that

$$T_0 = a(x, \cdot).$$

In other words,

$$(y, z^*) \in F \times G^* \;\Rightarrow\; z^*(T_0 y) = z^*(a(x, y)),$$

so, if $m \geq 1$, $(y_1, z_1^*), \dots, (y_m, z_m^*)$, and $(t_1, \dots, t_m) \in \Delta_m$, then

$$\begin{aligned}\sum_{j=1}^m t_j z_j^*(T_0 y_j) &= \sum_{j=1}^m t_j z_j^*(a(x, y_j)) \\ &\leq \|x\| \left\| \sum_{j=1}^m t_j z_j^* \circ a(\cdot, y_j) \right\|.\end{aligned}$$

Then

$$\rho := \sup_{\substack{m \geq 1,\ (y_1, z_1^*), \dots, (y_m, z_m^*) \in F \times G^* \\ (t_1, \dots, t_m) \in \Delta_m \\ \sum_{j=1}^m t_j z_j^* \circ a(\cdot, y_j) \neq 0}} \frac{\sum_{j=1}^m t_j z_j^*(T_0 y_j)}{\left\| \sum_{j=1}^m t_j z_j^* \circ a(\cdot, y_j) \right\|} < \infty$$

and

$$\|x\| \geq \rho. \tag{16}$$

It is clear that for all $m \geq 1$, $(y_1, z_1^*), \dots, (y_m, z_m^*)$ and $(t_1, \dots, t_m) \in \Delta_m$,

$$\sum_{j=1}^m t_j z_j^*(T_0 y_j) \leq \rho \left\| \sum_{j=1}^m t_j z_j^* \circ a(\cdot, y_j) \right\|$$

(if $\left\| \sum_{j=1}^m t_j z_j^* \circ a(\cdot, y_j) \right\| \neq 0$ it is obvious, and otherwise, it follows from (14) that $\sum_{j=1}^m t_j z_j^*(T_0 y_j) = 0$ and it also holds), so, thanks to the equivalence just stated (13) $\Leftrightarrow$ (14), there exists $x_0 \in E$, and as we have proved, it can be chosen in ρB_E, such that

$$T_0 = a(x_0, \cdot)$$

and from this and (16) we arrive at (15). $\square$

Remark 3.3. It suffices, just like in Proposition 3.1, to suppose

$$y \in F \ \Rightarrow \ a(\cdot, y) \in \mathrm{L}(E, G),$$

and the proof has no changes: we have assumed that a is continuous as a matter of simplicity.

The reflexivity assumption cannot be dropped in this result, that is, when E is not reflexive x_0^{**} is not necessarily in E:

Example 3.4. Consider the real sequence Banach spaces

$$E := c_0 = \{x \in \mathbb{R}^{\mathbb{N}} : \lim_n x(n) = 0\}, \qquad F := \ell_1 = \left\{ y \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |y(n)| < \infty \right\}$$

and the scalar situation $G := \mathbb{R}$, with the continuous bilinear form $a : c_0 \times \ell_1 \longrightarrow \mathbb{R}$ given for each $(x, y) \in c_0 \times \ell_1$ by

$$a(x, y) := \sum_{n=1}^{\infty} x(n)y(n).$$

Then, for the sequence $y_0^* = (1, 1, 1, \dots) \in \mathrm{L}(F, G) = F^* = \ell_\infty = \{z \in \mathbb{R}^{\mathbb{N}} : z \text{ is bounded}\}$ there exists no $x_0 \in c_0$ with $y_0^* = a(x_0, \cdot)$, since otherwise, for all $y \in \ell_1$,

$$\sum_{n=1}^{\infty} y(n) = \sum_{n=1}^{\infty} x(n)y(n),$$

or in other words, $x_0 = (1, 1, 1, \dots)$, which contradicts the fact that $x_0 \in c_0$. $\square$

In a similar way, this example can be extended to any real nonreflexive Banach space: for such a Banach space E it suffices to take $F := E^*$, $G := \mathbb{R}$, $a : E \times E^* \longrightarrow \mathbb{R}$ as the corresponding duality pairing and $y_0^* \in F^* = E^{**} \backslash E$.

Let us also point out that the uniqueness of $x_0 \in E$ is ignored here, since it is a trivial issue: if E, F and G are real vector spaces, $a : E \times F \longrightarrow G$ is a bilinear mapping and $T_0 : F \longrightarrow G$ is a linear operator for which there exists $x_0 \in E$ with $T_0 = a(x_0, \cdot)$, then it is a very easy matter to check that such an x_0 is unique if, and only if,

$$x \in E \ \text{ and } \ a(x, \cdot) = 0 \ \Rightarrow \ x = 0.$$

Before we conclude this section, we emphasize that in spite of [19, Theorem 3], [28, Theorem 9] or Theorem 3.2, our knowledge of this vectorial subject is not satisfactory, including the hilbertian framework, as illustrated by the irritating question posed in [28, Problem 1]: Is it true that if $a : \ell^2 \times \ell^2 \longrightarrow \ell^2$ is a continuous and bilinear mapping, then $\{a(x, \cdot) : x \in \ell_2\} \neq \mathrm{L}(\ell_2, \ell_2)$?

4. Application to systems with an arbitrary number of variational equations

At this point the focus shift to systems of variational equations, systems of infinitely many equations. The most natural context for them is that of a continuous bilinear mapping $a : E \times F \longrightarrow G$, where $G = \mathbb{R}^\Lambda$ and with Λ being a nonempty set, and equipping this space with the topology of the pointwise convergence, i.e., the coarsest topology on $\mathbb{R}^\Lambda$ for the family $(P_\lambda)_{\lambda\in\Lambda}$ of the associated projections, where for $\lambda_0 \in \Lambda$, $P_{\lambda_0} : \mathbb{R}^\Lambda \longrightarrow \mathbb{R}$ is defined for each $\mathbf{t} = (t_\lambda)_{\lambda\in\Lambda}$ as

$$P_{\lambda_0}(\mathbf{t}) := t_{\lambda_0}.$$

As a direct consequence of the definition of this topology (see for instance [6, Proposition 3.2]) we have that for a real topological vector space F, any linear and continuous operator $T : F \longrightarrow \mathbb{R}^\Lambda$ is of the form

$$T = (y_\lambda^*)_{\lambda\in\Lambda},$$

for some $(y_\lambda^*)_{\lambda\in\Lambda} \in (F^*)^\Lambda$. Of course, for all $\lambda \in \Lambda$

$$y_\lambda^* = P_\lambda \circ T.$$

Similarly, for some real topological vector spaces E and F, each continuous and bilinear mapping $a : E \times F \longrightarrow \mathbb{R}^\Lambda$ is given by

$$a = (a_\lambda)_{\lambda\in\Lambda},$$

where for all $\lambda \in \Lambda$, a_λ is the continuous and bilinear form from $E \times F$ into $\mathbb{R}$ defined as

$$a_\lambda := P_\lambda \circ a.$$

Thus, a variational problem of the form: given real topological vector spaces E and F, a nonempty set Λ, a continuous and bilinear mapping $a : E \times F \longrightarrow \mathbb{R}^\Lambda$ and $T_0 \in \mathrm{T}(F, \mathbb{R}^\Lambda)$,

$$\text{find } x_0 \in E \text{ such that } T_0 = a(x_0, \cdot),$$

is nothing more than: if E and F are real topological vector spaces, Λ is a nonempty set, and for each $\lambda \in \Lambda$, $a_\lambda : E \times F \longrightarrow \mathbb{R}$ is a continuous bilinear form and $y_\lambda^* \in F^*$, then

$$\begin{gathered}\text{find } x_0 \in E \text{ solution of the system} \\ \text{(of possibly infinitely many) variational equations} \\ y_\lambda^* = a_\lambda(x_0, \cdot), \quad (\lambda \in \Lambda).\end{gathered}$$

Hence, Λ plays the role of the index set.

In order to apply Theorem 3.2, we first recall the following elementary and well-known result, whose proof follows from [8, Chapter §IV, Theorem 3.1]:

Lemma 4.1. *Assume that Λ is a nonempty set. Then $\Phi : \mathbb{R}^\Lambda \longrightarrow \mathbb{R}$ is a continuous and linear form on the real topological vector space $\mathbb{R}^\Lambda$ (pointwise convergence topology) if, and only if, for some $N \geq 1$, $\xi_1, \ldots, \xi_N \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_N \in \Lambda$ it is given for each $\mathbf{t} \in \mathbb{R}^\Lambda$ as*

$$\Phi(\mathbf{t}) = \sum_{j=1}^{N} \xi_j t_{\lambda_j}.$$

As a consequence of this description of the dual of $\mathbb{R}^\Lambda$ (topology of the pointwise convergence) and Theorem 3.2 we arrive at the following characterization of the solvability of a system of (not necessarily a finite number of) variational equations, in terms of the existence of a constant:

Theorem 4.2. *Let E be a real reflexive Banach space, F a real topological vector space and Λ a nonempty set. Assume in addition that for each $\lambda \in \Lambda$, $a_\lambda : E \times F \longrightarrow \mathbb{R}$ is a continuous bilinear form and $y_\lambda^* \in F^*$. Then*

$$\text{there exists } x_0 \in E \text{ such that } \lambda \in \Lambda \;\Rightarrow\; y_\lambda^* = a_\lambda(x_0, \cdot)$$

if, and only if,

$$\begin{aligned} &\text{there exists } \alpha \geq 0 \text{ such that} \\ &N \geq 1,\; y_1, \ldots, y_N \in F,\; \lambda_1, \ldots, \lambda_N \in \Lambda \\ \Rightarrow\; &\sum_{k=1}^{N} y_{\lambda_k}^*(y_k) \leq \alpha \left\| \sum_{k=1}^{N} a_{\lambda_k}(\cdot, y_k) \right\|. \end{aligned} \tag{17}$$

Moreover, if these equivalent conditions hold and a is nonzero, then

$$\begin{aligned} &\min\{\|x_0\| : x_0 \in E \text{ and } \lambda \in \Lambda \;\Rightarrow\; y_\lambda^* = a_\lambda(x_0, \cdot)\} \\ = &\sup_{\substack{N\geq 1,\; y_1,\ldots,y_N \in F \\ \lambda_1,\ldots,\lambda_N \in \Lambda,\; \sum_{k=1}^N a_{\lambda_k}(\cdot, y_k) \neq 0}} \frac{\sum_{k=1}^{N} y_{\lambda_k}^*(y_k)}{\left\| \sum_{k=1}^{N} a_{\lambda_k}(\cdot, y_k) \right\|}. \end{aligned} \tag{18}$$

Proof. To begin with, let $G := \mathbb{R}^\Lambda$, endowed with its pointwise convergence topology, and let us define $T_0 \in \mathrm{L}(F, G)$ for each $y \in F$ by

$$T_0(y) := (y_\lambda^*(y))_{\lambda \in \Lambda}$$

and the continuous bilinear form $a : E \times F \longrightarrow \mathbb{R}$

$$(x, y) \in E \times F \mapsto (a_\lambda(x, y))_{\lambda \in \Lambda}.$$

Then, Theorem 3.2 guarantees that the existence of $x_0 \in E$ in such a way that

$$\lambda \in \Lambda \;\Rightarrow\; y_\lambda^* = a_\lambda(x_0, \cdot)$$

is equivalent to that of $\alpha \geq 0$ such that

$$\underbrace{\begin{array}{c} m \geq 1,\ y_1, \ldots, y_m \in F,\ N_1, \ldots, N_m \geq 1 \\ \{\lambda_1^{(1)}, \ldots, \lambda_{N_1}^{(1)}\}, \ldots, \{\lambda_1^{(m)}, \ldots, \lambda_{N_m}^{(m)}\} \subset \Lambda \\ \xi_1^{(1)}, \ldots, \xi_{N_1}^{(1)}, \ldots, \xi_1^{(m)}, \ldots, \xi_{N_m}^{(m)} \in \mathbb{R} \\ (t_1, \ldots, t_m) \in \Delta_m \end{array}}$$

$$\Downarrow \tag{19}$$

$$\sum_{j=1}^{m} \sum_{k=1}^{N_j} t_j \xi_k^{(j)} y^*_{\lambda_k^{(j)}}(y_j) \leq \alpha \left\| \sum_{j=1}^{m} \sum_{k=1}^{N_j} t_j \xi_k^{(j)} a_{\lambda_k^{(j)}}(\cdot, y_j) \right\|.$$

So, let us prove that (19) is exactly (17). Let us first assume that (17) holds, for some $\alpha \geq 0$, and let us show that (19) is valid for the same α. Thus, let

$$m \geq 1,\ y_1, \ldots, y_m \in F,\ N_1, \ldots, N_m \geq 1,$$
$$\{\lambda_1^{(1)}, \ldots, \lambda_{N_1}^{(1)}\}, \ldots, \{\lambda_1^{(m)}, \ldots, \lambda_{N_m}^{(m)}\} \subset \Lambda,$$
$$\xi_1^{(1)}, \ldots, \xi_{N_1}^{(1)}, \ldots, \xi_1^{(m)}, \ldots, \xi_{N_m}^{(m)} \in \mathbb{R}$$

and

$$(t_1, \ldots, t_m) \in \Delta_m.$$

If we take

$$N := \sum_{j=1}^{m} N_j,$$

$$(\lambda_1, \ldots, \lambda_N) := (\lambda_1^{(1)}, \ldots, \lambda_{N_1}^{(1)}, \ldots, \lambda_1^{(m)}, \ldots, \lambda_{N_m}^{(m)}),$$
$$(\mu_1^{(1)}, \ldots, \mu_N^{(1)}) := (\xi_1^{(1)}, \ldots, \xi_{N_1}^{(1)}, \underbrace{0, \ldots, 0}_{N-N_1}),$$
$$(\mu_1^{(2)}, \ldots, \mu_N^{(2)}) := (\underbrace{0, \ldots, 0}_{N_1}, \xi_1^{(2)}, \ldots, \xi_{N_2}^{(2)}, \underbrace{0, \ldots, 0}_{N-N_1-N_2}),$$
$$\cdots \quad \cdots$$
$$(\mu_1^{(m)}, \ldots, \mu_N^{(m)}) := (\underbrace{0, \ldots, 0}_{N-N_m}, \xi_1^{(m)}, \ldots, \xi_{N_m}^{(m)}),$$

then

$$\sum_{j=1}^{m} \sum_{k=1}^{N_j} t_j \xi_k^{(j)} y^*_{\lambda_k^{(j)}}(y_j) = \sum_{k=1}^{N} y^*_{\lambda_k} \left(\sum_{j=1}^{m} t_j \mu_k^{(j)} y_j \right)$$

and

$$\sum_{j=1}^{m} \sum_{k=1}^{N_j} t_j \xi_k^{(j)} a_{\lambda_k^{(j)}}(\cdot, y_j) = \sum_{k=1}^{N} a_{\lambda_k} \left(\cdot, \sum_{j=1}^{m} t_j \mu_k^{(j)} y_j \right),$$

hence, defining for each $k = 1, \dots, N$

$$y_k := \sum_{j=1}^{m} t_j \mu_k^{(j)} y_j \in F$$

we conclude (19), according to (17). And conversely. Let $\alpha \geq 0$ satisfying (19), and let us prove (17) for that $\alpha \geq 0$. Thus, let $N \geq 1$, $y_1, \dots, y_N \in F$ and $\lambda_1, \dots, \lambda_N \in \Lambda$. We take

$$\begin{aligned} m &:= N, \\ N_1 := \cdots := N_N &= N, \\ t_1 := \cdots := t_m &= \frac{1}{N}, \end{aligned}$$

and if for $k = 1, \dots, N$, $j = 1, \dots, N$, δ_{kj} denotes the *Kronecker delta*, we define

$$\left(\xi_k^{(j)}\right)_{\substack{k=1,\dots,N \\ j=1,\dots,m}} := (N\delta_{kj})_{\substack{k=1,\dots,N \\ j=1,\dots,m}}$$

and

$$\left(\lambda_k^{(j)}\right)_{\substack{k=1,\dots,N \\ j=1,\dots,m}} := (\lambda_k \delta_{kj})_{\substack{k=1,\dots,N \\ j=1,\dots,m}}.$$

Then

$$\sum_{k=1}^{N} y_{\lambda_k}^*(y_k) = \sum_{j=1}^{m} \sum_{k=1}^{N_j} t_j \xi_k^{(j)} y_{\lambda_k^{(j)}}(y_j)$$

and

$$\sum_{k=1}^{N} a_{\lambda_k}^*(\cdot, y_k) = \sum_{j=1}^{m} \sum_{k=1}^{N_j} t_j \xi_k^{(j)} a_{\lambda_k^{(j)}}(\cdot, y_j),$$

which together with (19) implies (17).

For the estimate of the norm of the solutions (or the solution, if it is unique) of the system of variational equations, the previous argument yields

$$\sup_{\substack{m \geq 1,\ y_1,\dots,y_m \in F \\ \lambda_1,\dots,\lambda_N \in \Lambda \\ (\mu_1^{(1)},\dots,\mu_N^{(1)}),\dots,(\mu_1^{(m)},\dots,\mu_N^{(m)}) \in \mathbb{R}^N \\ (t_1,\dots,t_m) \in \Delta_m \\ \sum_{j=1}^m \sum_{k=1}^N t_j \mu_k^{(j)} a_{\lambda_k}(\cdot, y_j) \neq 0}} \frac{\displaystyle\sum_{j=1}^{m} \sum_{k=1}^{N} t_j \mu_k^{(j)} y_{\lambda_k}^*(y_j)}{\left\| \displaystyle\sum_{j=1}^{m} \sum_{k=1}^{N} t_j \mu_k^{(j)} a_{\lambda_k}(\cdot, y_j) \right\|}$$

$$= \sup_{\substack{N \geq 1,\ y_1,\dots,y_N \in F \\ \lambda_1,\dots,\lambda_N \in \Lambda,\ \sum_{k=1}^N a_{\lambda_k}(\cdot, y_k) \neq 0}} \frac{\displaystyle\sum_{k=1}^{N} y_{\lambda_k}^*(y_k)}{\left\| \displaystyle\sum_{k=1}^{N} a_{\lambda_k}(\cdot, y_k) \right\|},$$

so it suffices to apply Theorem 3.2, (15), to arrive at (18). □

Now we apply Theorem 4.2 to study the existence of solutions of systems with an arbitrary number of Fredholm integral equations of the second kind:

Example 4.3. Let $n \geq 1$, Ω a nonempty, bounded and open subset of $\mathbb{R}^n$, $1 < p < \infty$, Λ a nonempty set and, for each $\lambda \in \Lambda$, $f_\lambda \in L^p(\Omega)$ and $K_\lambda \in L^\infty(\Omega \times \Omega)$. The problem under consideration is: find $x_0 \in L^p(\Omega)$ in such a way that

$$f_\lambda(\omega) = x_0(\omega) - \int_\Omega K_\lambda(\omega, \xi) x_0(\xi) d\xi, \quad (\omega \text{ a.e. in } \Omega, \ \lambda \in \Lambda). \tag{20}$$

The simple fact that this problem does not always admit a solution reveals that we have to impose additional assumptions to guarantee its solvability. Specifically, we are going to show that the system of integral equations has one solution, provided that there exists $\rho > 0$ such that

$$\begin{aligned} & N \geq 1, \ \lambda_1, \ldots, \lambda_N \in \Lambda, \ f_{\lambda_1}, \ldots, f_{\lambda_N} \in L^p(\Omega) \\ \Rightarrow \ & \rho \sum_{k=1}^N \|f_{\lambda_k}\|_p \leq \left\| \sum_{k=1}^N f_{\lambda_k} \right\|_p \end{aligned} \tag{21}$$

and

$$\sup_{\lambda \in \Lambda} \|K_{\lambda_k}\|_\infty < \frac{\rho}{\operatorname{meas}(\Omega)}. \tag{22}$$

Indeed, system (20) is nothing more than

$$y_\lambda^* = a_\lambda(x_0, \cdot), \quad (\lambda \in \Lambda),$$

where $x_0 \in E := L^p(\Omega)$, $F := L^{p'}(\Omega)$, with p' being the conjugate exponent of p, and for all $\lambda \in \Lambda$, $y_\lambda^* : F \longrightarrow \mathbb{R}$ and $a_\lambda : E \times F \longrightarrow \mathbb{R}$ are given by

$$y_\lambda^*(y) := \int_\Omega f_\lambda(\omega) y(\omega) d\omega, \quad (y \in F)$$

and

$$a_\lambda(x, y) := \int_\Omega x(\omega) y(\omega) d\omega - \int_\Omega \int_\Omega K_\lambda(\omega, \xi) x(\xi) y(\omega) d\xi d\omega, \quad ((x, y) \in E \times F).$$

Therefore, Theorem 4.2 says that the above system of variational equations, in other words, the system (20), admits a solution as soon as there exists $\alpha \geq 0$ such that

$$\begin{aligned} & N \geq 1, \ y_1, \ldots, y_N \in L^{p'}(\Omega), \ \lambda_1, \ldots, \lambda_N \in \Lambda \\ \Rightarrow \ & \sum_{k=1}^N y_{\lambda_k}^*(y_k) \leq \alpha \left\| \sum_{k=1}^N a_{\lambda_k}(\cdot, y_k) \right\|. \end{aligned} \tag{23}$$

Notice that given $\omega \in \Omega$,

$$\sum_{k=1}^N a_{\lambda_k}(\cdot, y_k)(\omega) = \sum_{k=1}^N y_k(\omega) - \sum_{k=1}^N \int_\Omega K_{\lambda_k}(\omega, \xi) y_k(\xi) d\xi,$$

so, according to (21) and (22), we conclude that

$$\left\| \sum_{k=1}^{N} a_{\lambda_k}(\cdot, y_k) \right\| \geq \sum_{k=1}^{N} (\rho - \|K_{\lambda_k}\|_\infty \mathrm{meas}(\Omega)) \|y_k\|_{p'}$$

and thus

$$\sum_{k=1}^{N} y_k^*(y_k) \leq \frac{\sum_{k=1}^{N} \|f_{\lambda_k}\|_p \|y_k\|_{p'}}{\sum_{k=1}^{N} (\rho - \|K_{\lambda_k}\|_\infty \mathrm{meas}(\Omega)) \|y_k\|_{p'}} \left\| \sum_{k=1}^{N} a_{\lambda_k}(\cdot, y_k) \right\|$$
$$\leq \alpha \left\| \sum_{k=1}^{N} a_{\lambda_k}(\cdot, y_k) \right\|,$$

with

$$\alpha := \sup_{\lambda \in \Lambda} \frac{\|f_\lambda\|_p}{\rho - \|K_\lambda\|_\infty \mathrm{meas}(\Omega)} \geq 0,$$

so we have checked (23).

Finally, according to (23) and Theorem 4.2, exactly (18), for some solution x_0 of (20) –actually its unique solution– there holds $\|x_0\| \leq \alpha$. $\square$

Now let us analyze what Theorem 3.2 tells us when the final space G is finite-dimensional. In this case E is a real reflexive Banach space, F is a real topological vector space,

$$G = \mathbb{R}^N$$

for some positive integer N, $T_0 \in \mathrm{L}(F, \mathbb{R}^N)$ is of the form

$$T_0 y = (y_1^*(y), \ldots, y_n^*(y)), \quad (y \in F),$$

for adequate $y_1^*, \ldots, y_N^* \in F^*$, and the continuous bilinear mapping $a : E \times F \longrightarrow \mathbb{R}^N$ is given for each $(x, y) \in E \times F$ by

$$a(x, y) = (a_1(x, y), \ldots, a_N(x, y)),$$

where $a_1, \ldots, a_N : E \times F \longrightarrow \mathbb{R}$ are continuous bilinear forms.

Corollary 4.4. *Suppose that E is a real reflexive Banach space, F is a real topological vector space and that for some positive integer N, $y_1^*, \ldots, y_N^* \in F^*$ and that $a_1, \ldots, a_N : E \times F \longrightarrow \mathbb{R}$ are continuous bilinear forms. Then, the following statements are equivalent:*

(i) *There exists $x_0 \in E$ such that*

$$\begin{cases} y_1^* = & a_1(x_0, \cdot) \\ \cdots & \\ y_N^* = & a_N(x_0, \cdot). \end{cases} \tag{24}$$

(*ii*) *For some* $\alpha \geq 0$ *there holds*

$$y_1, \dots, y_N \in F \;\Rightarrow\; \sum_{k=1}^{N} y_k^*(y_k) \leq \alpha \left\| \sum_{k=1}^{N} a_k(\cdot, y_k) \right\|.$$

In addition, if these equivalent conditions hold and a is nonzero, then

$$\min\{\|x_0\| : x_0 \in E \text{ is a solution to system } (24)\} = \sup_{\substack{y_1,\dots,y_N \in F \\ \sum_{k=1}^{N} a_k(\cdot,y_k) \neq 0}} \frac{\displaystyle\sum_{k=1}^{N} y_k^*(y_k)}{\left\| \displaystyle\sum_{k=1}^{N} a_k(\cdot, y_k) \right\|}.$$

Some particular cases of this result have found applications in the study of variational equations with constraints and mixed variational formulations of linear elliptic boundary value problems (see [2, Theorem 2.1], [3, Theorem 5.2], [25, Theorem 3.3] or [12, Theorem 3.6]).

Let us observe, going on with the finite-dimensional case, that Corollary 4.4 admits an extension in which the space F is not necessarily fixed:

Corollary 4.5. *Let* E *be a real reflexive Banach space,* $N \geq 1$*,* $F_1, \dots, F_N$ *be real topological vector spaces,* $y_1^* \in F_1^*, \dots, y_N^* \in F_N^*$ *and let* $a_1 : E \times F_1 \longrightarrow \mathbb{R}, \dots, a_N : E \times F_N \longrightarrow \mathbb{R}$ *be continuous bilinear forms. Then, there exists* $x_0 \in E$ *such that*

$$\begin{cases} y_1^* = a_1(x_0, \cdot) \\ \cdots \\ y_N^* = a_N(x_0, \cdot) \end{cases} \tag{25}$$

if, and only if, for some $\alpha \geq 0$ *there holds*

$$(y_1, \dots, y_N) \in F_1 \times \cdots \times F_N \;\Rightarrow\; \sum_{k=1}^{N} y_k^*(y_k) \leq \alpha \left\| \sum_{k=1}^{N} a_k(\cdot, y_k) \right\|.$$

Moreover, when these equivalent conditions hold and at least one of the continuous bilinear forms is nonzero, then

$$\min\{\|x_0\| : x_0 \in E \text{ is a solution to system } (25)\} = \sup_{\substack{(y_1,\dots,y_N) \in F_1 \times \cdots \times F_N \\ \sum_{k=1}^{N} a_k(\cdot,y_k) \neq 0}} \frac{\displaystyle\sum_{k=1}^{N} y_k^*(y_k)}{\left\| \displaystyle\sum_{k=1}^{N} a_k(\cdot, y_k) \right\|}. \tag{26}$$

Proof. It suffices to apply Corollary 4.4 to the continuous linear functionals and to the continuous bilinear forms

$$v_k^* : \prod_{k=1}^{N} F_k \longrightarrow \mathbb{R}$$

and

$$b_k : E \times \left(\prod_{k=1}^{N} F_k \right) \longrightarrow \mathbb{R},$$

for $k = 1, \ldots, N$, and defined for each $(y_1, \ldots, y_N) \in \prod_{k=1}^{m} F_k$ and $(x, (y_1, \ldots, y_N)) \in E \times (\prod_{k=1}^{m} F_k)$ by

$$v_k^*(y_1, \ldots, y_N) := y_k^*(y_k)$$

and

$$b_k(x, (y_1, \ldots, y_N)) := a_k(x, y_k). \qquad \square$$

In Corollary 4.5 we characterize when N *concrete* continuous and linear functionals are represented by N continuous and bilinear forms. The following result, meanwhile, provides us with a characterization of those continuous bilinear forms that represent *any* continuous and linear functionals in the normed case:

Corollary 4.6. *Assume that E is a real reflexive Banach space, $N \geq 1$, $F_1, \ldots,$ F_N are Banach spaces and that $a_1 : E \times F_1 \longrightarrow \mathbb{R}, \ldots, a_N : E \times F_N \longrightarrow \mathbb{R}$ are continuous bilinear forms. Then, for all $y_1^* \in F_1^*, \ldots, y_N^* \in F_N^*$ there exists $x_0 \in E$ such that*

$$\begin{cases} y_1^* = a_1(x_0, \cdot) \\ \cdots \\ y_N^* = \ a_N(x_0, \cdot) \end{cases} \tag{27}$$

if, and only if, there exists $\rho > 0$ satisfying

$$(y_1, \ldots, y_N) \in F_1 \times \cdots \times F_N \ \Rightarrow \ \rho \sum_{k=1}^{N} \|y_k\| \leq \left\| \sum_{k=1}^{N} a_k(\cdot, y_k) \right\|. \tag{28}$$

Furthermore, if one of these equivalent conditions holds and at least one of the continuous bilinear forms is nonzero, then the estimate

$$\min\{\|x_0\| : x_0 \in E \textit{ is a solution to system } (27)\} \leq \frac{1}{\rho} \max_{k=1,\ldots,N} \|y_k^*\| \tag{29}$$

is valid.

Proof. Let us endow the vector space $F := \prod_{k=1}^{N} F_k$ with the Banach space's structure by means of the norm

$$\|(y_1, \ldots, y_N)\| := \sum_{k=1}^{N} \|y_k\|,$$

whenever $(y_1, \ldots, y_N) \in F$. It is clear that the "if part" straightforwardly follows from Corollary 4.5. So, let us prove the converse, therefore, suppose that for all $y_1^* \in F_1^*, \ldots, y_N^* \in F_N^*$ the variational system (27) admits a solution. Let us first notice that, under that assumption,

$$(y_1, \ldots, y_N) \in F, \ \sum_{k=1}^N a_k(\cdot, y_k) = 0 \ \Rightarrow \ y_1 = 0, \ldots, y_N = 0. \tag{30}$$

Indeed, let $(y_1, \ldots, y_N) \in F$ with $\sum_{k=1}^N a_k(\cdot, y_k) = 0$ and apply the Hahn–Banach theorem to deduce the existence of $(y_1^*, \ldots, y_N^*) \in \prod_{k=1}^N F_k^*$ such that

$$\max_{k=1,\ldots,N} \|y_k^*\| = 1$$

and

$$\sum_{k=1}^N y_k^*(y_k) = \sum_{k=1}^N \|y_k\|.$$

But in view of our hypothesis, we derive that there exists $x_0 \in E$ satisfying (27), hence in particular,

$$\begin{aligned} \sum_{k=1}^N \|y_k\| &= \sum_{k=1}^N y_k^*(y_k) \\ &= \sum_{k=1}^N a_k(x_0, y_k) \\ &= 0 \end{aligned}$$

and thus

$$y_1 = 0, \ldots, y_N = 0.$$

To establish the announced equivalence, let us consider the subset of F

$$A := \left\{ \frac{(y_1, \ldots, y_N)}{\left\| \sum_{k=1}^N a_k(\cdot, y_k) \right\|} : (y_1, \ldots, y_N) \in F \text{ with } \sum_{k=1}^N a_k(\cdot, y_k) \neq 0 \right\},$$

which, thanks to the nondegeneration condition (30), coincides with

$$A = \left\{ \frac{(y_1, \ldots, y_N)}{\left\| \sum_{k=1}^N a_k(\cdot, y_k) \right\|} : (y_1, \ldots, y_N) \in F \text{ and } y_1 \neq 0, \ldots, y_N \neq 0 \right\}.$$

But we are assuming that any continuous and linear functional on F is bounded on A (the easy part of Corollary 4.5), and finally, the uniform boundedness principle implies that, for some $\rho > 0$, (28) holds.

The stability condition (29) is a consequence of (26). $\square$

Let us add that this result is the version of the Lax–Milgram theorem in [14, Theorem 12] for systems with a finite number of variational equations.

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