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Choquet Theory for Vector-Valued Functions on a Locally Compact Space

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We consider spaces of continuous vector-valued functions on a locally compact Hausdorff space X, endowed with suitable locally convex topologies. Using a family of sets of such functions an order on the dual of the function space is defined. This order yields minimal elements which as in classical Choquet theory can be characterized in terms of a subset (Choquet boundary) of X, thus providing information about the support of their respective representation measures.

Keywords: Vector-valued functions, integral representation

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1. Introduction

This paper is based on earlier publications by the author [9] and a generalization of the techniques therein by Batty [2]. These address Choquet theory for complex- and Banach space-valued functions on a compact Hausdorff space, respectively. A more recent paper [12] contains a classification of spaces of vector-valued functions on locally compact spaces and establishes an integral representation for linear functionals. We shall use a special case of this to transfer Choquet theory to function spaces of this type and use the notion of minimal (rather than maximal, as in the classical theory) measures to demonstrate the existence of linear functionals with desired properties. The parameters of the theory can be adjusted to suit a wide range of situations. We shall use the following notations:

Let $(E, \mathcal{V}, \leq)$ be a locally convex ordered topological vector space, that is a real vector space E endowed with an order relation $\leq$ and a system $\mathcal{V}$ of balanced convex neighborhoods of the origin. For our purposes we assume that $E \in \mathcal{V}$. The order is supposed to be reflexive, transitive and compatible with the algebraic operations, that is $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in E$ and $\alpha \geq 0$. Since equality in E is obviously such an order, this concept will a apply to locally convex vector spaces in general. E_+ denotes the subcone of all positive elements of E. A non-empty subset A of E is called *decreasing* or *increasing*, respectively, if $b \in A$ whenever $b \leq a$, or $a \leq b$, for some $a \in A$. The *decreasing*

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hull $A{\downarrow}$ and the *increasing hull* $A{\uparrow}$ of A are defined as

$$A{\downarrow} = \{b \in E \mid b \leq a \text{ for some } a \in A\} \quad \text{and}$$
$$A{\uparrow} = \{b \in E \mid a \leq b \text{ for some } a \in A\},$$

respectively. Note that $(-A){\downarrow} = -(A{\uparrow})$. The decreasing, or increasing, hull of a convex or balanced set is again convex or balanced and the sum of two decreasing, increasing or convex sets is again decreasing, increasing or convex. A subset A of E is called *order convex* if $A = A{\downarrow} \cap A{\uparrow}$, that is $c \in A$ whenever $a \leq c \leq b$ for some $a, b \in A$. As usual, we assume that the neighborhoods in $\mathcal{V}$ are also order convex. The topological closure of a subset A of E is denoted by $\overline{A}$. By E^* we denote the topological dual of E, that is the space of all continuous real-valued linear functionals on E. A functional $\mu \in E^*$ is called *monotone* (or *positive*) if $a \leq b$ implies $\mu(a) \leq \mu(b)$, or equivalently if $\mu(a) \geq 0$ for all $a \geq 0$. E^*_+ will denote the subcone of all positive functionals in E^*. Every functional $\mu \in E^*$ can be expressed as a difference of two positive ones, that is $\mu = \mu_1 - \mu_2$ for some $\mu_1, \mu_2 \in E^*_+$ (see Section V.3 in [13]). The *polar* A° of a subset A of E consists of all $\mu \in E^*$ such that $\mu(a) \leq 1$ for all $a \in A$.

Let X be a locally compact Hausdorff space. $\mathfrak{R}$ represents the family of all relatively compact Borel subsets of X and $\mathfrak{O}$ the subfamily of all open sets in $\mathfrak{R}$. As usual, for a subset Y of a topological space, the sets $\overline{Y}$ and Y° denote its topological closure, and interior, respectively. Let $\mathcal{F}(X, E)$ be the space of all E-valued functions on X, endowed with the pointwise algebraic operations and order, and let $\mathcal{C}(X, E)$ denote the subspace of all continuous functions in $\mathcal{F}(X, E)$. The *support* of a function $f \in \mathcal{F}(X, E)$ is the closure of the set $\{x \in X \mid f(x) \neq 0\}$ and referred to by $\operatorname{supp}(f)$. The subspace of all functions in $\mathcal{C}(X, E)$ with compact support is denoted by $\mathcal{C}_\mathcal{K}(X, E)$. In case that $E = \mathbb{R}$ we write $\mathcal{F}(X)$, $\mathcal{C}(X)$ and $\mathcal{C}_\mathcal{K}(X)$ for short. The subscript $_+$ is used to denote the subcone of positive elements in any of these function spaces. For $\varphi \in \mathcal{F}(X)$ and $f \in \mathcal{F}(X, E)$ the function $\varphi{\otimes}f \in \mathcal{F}(X, E)$ is the mapping $x \mapsto \varphi(x)f(x) : X \to E$. If f is the constant function $x \mapsto a$ for some $a \in E$, we write $\varphi{\otimes}a$ for the function $x \mapsto \varphi(x)a$. If $\varphi \in \mathcal{C}(X)$ and $f \in \mathcal{C}(X, E)$, then $\varphi{\otimes}f \in \mathcal{C}(X, E)$. The *characteristic function* $\chi_Y \in \mathcal{F}(X)$ of a subset Y of X takes the value 1 on Y and 0 else. $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ denotes the extended real number system with the usual order and algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. The multiplication of $+\infty$ with negative numbers is not defined. We shall similarly use $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$

Let $\operatorname{Conv}(E)$ be the cone of all non-empty convex subsets of E, endowed with its canonical addition and multiplication by non-negative scalars and ordered by set inclusion. Its positive cone $\operatorname{Conv}(E)_+$ consists of all sets $C \in \operatorname{Conv}(E)$ that contain $0 \in E$. A set-valued function $F : X \to \operatorname{Conv}(E)_+$ is said to be *lower semicontinuous* at a point $x \in X$ if for every $\varepsilon > 0$ there is a neighborhood U of x in X such that $F(x) \subset (1 + \varepsilon)F(y)$ for all $y \in U$. For non-negative $\overline{\mathbb{R}}$-valued functions on X this formulation agrees with the usual notion of lower semicontinuity. Clearly, sums or multiples by non-negative reals of lower semicontinuous $\mathrm{Conv}(E)_+$-valued functions are again lower semicontinuous. For a non-negative $\overline{\mathbb{R}}$-valued function φ on X and $C \in \mathrm{Conv}(E)_+$, we denote the mapping $x \mapsto \varphi(x)C : X \to \mathrm{Conv}(E)_+$ by $\varphi{\otimes}C$, where $(+\infty)\,C$ is meant to be the full space E. If φ is lower semicontinuous, then the $\mathrm{Conv}(E)_+$-valued function $\varphi{\otimes}C$ is lower semicontinuous in the above sense.

In Section 2 we quote and summarize the relevant terminology and results and provide some examples for spaces of vector-valued functions. Due to technical requirements, our notation of function space neighborhoods will be narrower than that in [12] which also includes inductive limit type topologies. Section 3 contains our main results, including a Choquet-type representation theorem for continuous linear functionals. Section 4 is concerned with special cases and examples.

2. Spaces of functions that vanish at infinity

A *function space neighborhood* $\mathfrak{v}$ in $\mathcal{F}(X,E)$ is defined by a lower semicontinuous mapping

$$V_{\mathfrak{v}}(x) : X \to \mathcal{V}$$

such that for every $A \in \mathfrak{R}$ there is $W \in \mathcal{V}$ such that $W \subset V_{\mathfrak{v}}(x)$ for all $x \in A$. The neighborhood $\mathfrak{v}$ then consists of all functions $f \in \mathcal{F}(X,E)$ such that $f(x) \in V_v(x)$ for all $x \in X$. This is a special (reduced) case of the concept of function space neighborhoods defined in [12] which fits ours purpose. A simple argument shows that r-lower continuity implies that for every $x \in X$ the neighborhood $V_{\mathfrak{v}}(x)$ is contained in the topological closure of the set $\{f(x) \mid f \in \mathfrak{v}\}$. A family $\mathfrak{V}$ of function space neighborhoods is called a *function space neighborhood system* for $\mathcal{F}(X,E)$ if it is closed for multiplication by positive scalars and downwards directed with respect to set inclusion.

A function $f \in \mathcal{F}(X,E)$ *vanishes at infinity* (relative to the function space neighborhood system $\mathfrak{V}$) if for every $\mathfrak{v} \in \mathfrak{V}$ there is $A \in \mathfrak{R}$ such that $\chi_{(X\setminus A)}{\otimes}f \in \mathfrak{v}$. Let $\mathcal{C}_{\mathfrak{V}}(X,E)$ denote the space of all functions in $\mathcal{C}(X,E)$ that vanish at infinity, and abbreviate $\mathcal{C}_{\mathfrak{V}}(X)$ if $E = \mathbb{R}$. All neighborhoods $\mathfrak{v} \in \mathfrak{V}$ are absorbing in $\mathcal{C}_{\mathfrak{V}}(X,E)$, hence for every choice of $\mathfrak{V}$, the intersection of $\mathcal{C}_{\mathfrak{V}}(X,E)$ with the sets in $\mathfrak{V}$ forms a basis for a locally convex topology on $\mathcal{C}_{\mathfrak{V}}(X,E)$. Endowed with the pointwise order, $(\mathcal{C}_{\mathfrak{V}}(X,E),\mathfrak{V})$ is a locally convex ordered topological vector space. We shall mention a few examples, the details for which can be found in [12], Examples 2.2.

Examples 2.1. (a) The topology of uniform convergence on X is generated by the function space neighborhoods

$$\mathfrak{v}_V = \{f \in \mathcal{F}(X,E) \mid f(x) \in V \text{ for all } x \in X\},$$

rendered by the neighborhood function $\chi_K{\otimes}V$, corresponding to all $V \in \mathcal{V}$. A function $f \in \mathcal{F}(X,E)$ vanishes at infinity if for every $V \in \mathcal{V}$ there is a compact subset K of X such that $f(x) \in V$ for all $x \in X \setminus K$.

(b) The topology of compact convergence is generated by the lower semicontinuous function space neighborhoods

$$\mathfrak{v}_{(K,V)} = \{f \in \mathcal{F}(X,E) \mid f(x) \in V \text{ for all } x \in K\},$$

rendered by the neighborhood function $\chi_K \otimes V$, corresponding to all $V \in \mathcal{V}$ and compact subsets K of X. We have $\mathcal{C}_{\mathfrak{V}}(X,E) = \mathcal{C}(X,E)$ in this case.

(c) The topology of pointwise convergence is generated by the the lower semicontinuous function space neighborhoods

$$\mathfrak{v}_{(Y,V)} = \{f \in \mathcal{F}(X,E) \mid f(x) \in V \text{ for all } x \in Y\},$$

rendered by the neighborhood function $\chi_Y \otimes V$, corresponding to all $V \in \mathcal{V}$ and finite subsets Y of X. We have $\mathcal{C}_{\mathfrak{V}}(X,E) = \mathcal{C}(X,E)$.

(d) Parts (a), (b) and (c) are special cases for *weighted space topologies* which were introduced for real-valued functions by Nachbin and Prolla (see [7] and [8]). A family Ω of non-negative real-valued upper semicontinuous functions on X is called a *family of weights* if for all $\omega_1, \omega_2 \in \Omega$ there are $\omega_3 \in \Omega$ and $\rho > 0$ such that $\omega_1 \leq \rho\,\omega_3$ and $\omega_2 \leq \rho\,\omega_3$. For $V \in \mathcal{V}$ and $\omega \in \Omega$ the

$$\mathfrak{v}_{(\omega,V)} = \{f \in \mathcal{F}(X,E) \mid \omega(x)f(x) \in V \text{ for all } x \in X\}$$

is rendered by the neighborhood function $(1/\omega) \otimes V$. (Recall that $(+\infty)\,V = E$.) Sums of neighborhoods of this type establish a function space neighborhood system $\mathfrak{V}$ such that $\mathcal{C}_{\mathcal{K}}(X,E) \subset \mathcal{C}_{\mathfrak{V}}(X,E) \subset \mathcal{C}(X,E)$.

(e) The case $X = \mathbb{N}$ covers a variety of sequence spaces.

If E is either an algebra or a vector lattice, then the algebra or lattice operations transfer pointwise to the functions in $\mathcal{F}(X,E)$. Proposition 2.6 in [12] states:

Proposition 2.2. *Let $(\mathcal{C}_{\mathfrak{V}}(X,E), \mathfrak{V})$ be a function space.*

(*a*) *If E is a topological algebra and if for every $\mathfrak{v} \in \mathfrak{V}$ there is $\mathfrak{u} \in \mathfrak{V}$ such that $fg \in \mathfrak{v}$ whenever $f, g \in \mathfrak{u}$, then $\mathcal{C}_{\mathfrak{V}}(X,E)$ is also a topological algebra.*

(*b*) *If E is a topological vector lattice and if for every $\mathfrak{v} \in \mathfrak{V}$ there is $\mathfrak{u} \in \mathfrak{V}$ such that $f \in \mathfrak{v}$ whenever $|f| \leq |u|$ for some $u \in \mathfrak{u}$, then $\mathcal{C}_{\mathfrak{V}}(X,E)$ is also a topological vector lattice.*

The main result in [12] states that every continuous linear functional on a function space $\mathcal{C}_{\mathfrak{V}}(X,E)$ can be represented as an integral with respect to an E^*-valued measure θ. This measure is defined on the sets in $\mathfrak{R}$, which forms a weak σ-ring, but does generally not contain the space X itself. θ is countably additive within $\mathfrak{R}$. For $A \in \mathfrak{R}$ and for $a \in E$ we write $\theta_A(a)$ for the real number $(\theta(A))\,(a)$. The measure θ is also required to be *bounded*, that is for every $A \in \mathfrak{R}$ there is a neighborhood $V \in \mathcal{V}$ such that its *modulus*, that is

$$|\theta|_A(V) = \sup\left\{ \sum_{i=1}^{n} \theta_{A_i}(a_i) \;\middle|\; a_i \in V,\ A_i \in \mathfrak{R} \text{ are disjoint subsets of } A \right\} \leq 1.$$

Note that $|\theta|_A(V) = 0$ for some $V \in \mathcal{V}$ if and only if $|\theta|_A(V) = 0$ for all $V \in \mathcal{V}$. In this case we denote $|\theta|_A = 0$. It is straightforward to verify that $|\theta|_{(\bigcup_{n=1}^{\infty} A_n)}(V) = \sum_{n=1}^{\infty} |\theta|_{A_n}(V)$ whenever the sets $A_n \in \mathfrak{R}$ are disjoint and $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{R}$ (see Lemma II.3.4 in [10]). For a fixed $V \in \mathcal{V}$ the set-function $A \mapsto |\theta|_A(V)$ is therefore itself a positive real-valued measure.

The measure θ integrates *measurable* E-valued functions (see Section 3 in [12]) over sets of the σ-field

$$\mathfrak{A} = \{B \subset X \mid A \cap B \in \mathfrak{R} \text{ for all } A \in \mathfrak{R}\}$$

which contains all Borel subsets of X and coincides with the latter if X is countably compact. The values of the integrals are in $\overline{\mathbb{R}}$. The integral of f is first defined over the sets in $\mathfrak{R}$ and then over a set $B \in \mathfrak{A}$ by

$$\int_B f\, d\theta = \lim_{A \in \mathfrak{R}} \int_{A \cap B} f\, d\theta,$$

where the limit on the right-hand side is taken with respect to the upward directed family of all sets in $\mathfrak{R}$ (see [12]). One obtains the usual properties for integrals including convergence theorems. We have $\int_A f\, d\theta \leq |\theta|_A(V)$ for $A \in \mathfrak{R}$, whenever $f(x) \in V$ for all $x \in A$ (see Proposition 3.8 in [12]). For a suitable function space neighborhood system $\mathfrak{V}$ such a measure is seen to represent a monotone continuous linear functional on $\mathcal{C}_{\mathfrak{V}}(X, E)$. The measure θ is called *regular* if

$$\lim_{K \subset A} |\theta|_{(A \setminus K)}(V) = 0 \qquad \text{and} \qquad \lim_{O \supset A} |\theta|_{(O \setminus A)}(V) = 0$$

holds for all $A \in \mathfrak{R}$ and $V \in \mathcal{V}$ such that $|\theta|_O(V) < +\infty$ for some $A \subset O \in \mathfrak{O}$. These limits are taken with respect to the upwards directed family of all compact subsets of K of A in the first case, and the downwards directed family of all supersets $O \in \mathfrak{O}$ of A in the second case. Regularity implies that

$$\theta_O(a) = \lim_{\varphi \prec O} \int_X \varphi \otimes a\, d\theta \qquad \text{and} \qquad \theta_O(a) = \lim_{K \prec \varphi} \int_X \varphi \otimes a\, d\theta$$

holds for all $a \in E$, where $O \in \mathfrak{R}$ is an open and K is a compact set and φ ranges over the upward directed set of all $\varphi \in \mathcal{C}(X)$ such that $\varphi \leq \chi_O$ in the first case and the downward directed set of all $\varphi \in \mathcal{C}(X)$ such that $\varphi \geq \chi_K$ in the second case. A set $Z \in \mathfrak{A}$ is *of measure zero (with respect to* θ*)* if $\theta_A = 0 \in E^*$, that is $|\theta|_A = 0$, for all subset $A \in \mathfrak{R}$ of Z. In this case, θ is said to be *supported* by its complement $Y = X \setminus Z$, since $\theta_A = \theta_{(A \cap Y)}$ holds for all $A \in \mathfrak{R}$, and consequently, $\int_B f\, d\theta = \int_{B \cap Y} f\, d\theta$ for every integrable function $f \in \mathcal{F}(X, E)$; that is θ coincides with its *restriction to* Y. The usual notion of properties holding θ*-almost everywhere* is based on this concept. For a regular measure θ to be of measure zero it suffices to verify that $|\theta|_K = 0$ for every compact subset K of Z. In this case the complement in X of the union of all open sets of measure zero is called the *support of* θ and denoted by $\operatorname{supp}(\theta)$. A simple argument involving regularity and compactness shows that $\operatorname{supp}(\theta)$ is indeed the smallest closed set which supports θ, that is its complement in X is the largest open set of measure zero. A *point evaluation* is a measure with singleton support, that is a measure δ_x^η, for $x \in X$ and $0 \neq \eta \in E^*$ such that for every $A \in \mathfrak{R}$ it evaluates $\delta_{xA}^\eta = \eta$ if $x \in A$, and $\delta_{xA}^\eta = 0$, else. Consequently, $\int_X f \, d\delta_x^\eta = \eta\,(f(x))$ for every integrable E-valued function. We quote Theorem 4.4 in [12]:

Theorem 2.3. *Let $\mathfrak{V}$ be a function space neighborhood system. For every linear functional $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$ there exists a unique regular E^*-valued measure θ on $\mathfrak{R}$ such that all functions in $\mathcal{C}_{\mathfrak{V}}(X,E)$ are integrable with respect to θ and $\mu(f) = \int_X f \, d\theta$ holds for all $f \in \mathcal{C}_{\mathfrak{V}}(X,E)$. The functional μ is positive if and only if θ is E_+^*-valued.*

An E_+^*-valued measure is therefore called *positive.* Theorem 2.3 can be immediately applied to the Examples 2.1. It allows to consider linear functionals on $\mathcal{C}_{\mathfrak{V}}(X,E)$ both as elements of its topological dual and as vector-valued measures, thus permitting to employ techniques from either theory. This interchangeability will be used to a great extend in the following section.

3. Order and minimality for linear functionals

The concepts and techniques in this section rely on ideas developed by the author in [9] for complex-valued Choquet theory and their extension to the vector-valued case by Batty in [2]. While maximality of measures with respect to an order relation is used in the classical case, reversing the order and using minimality instead appears more suitable to our approach.

Let $\mathcal{C}_{\mathfrak{V}}(X,E)$ be a function space. Its topological dual $\mathcal{C}_{\mathfrak{V}}(X,E)^*$ is endowed with the weak* topology induced by $\mathcal{C}_{\mathfrak{V}}(X,E)$. We consider the pointwise order for $\overline{\mathbb{R}}$- or $\underline{\mathbb{R}}$-valued functions on $\mathcal{C}_{\mathfrak{V}}(X,E)^*$. The cone of all lower semicontinuous $\overline{\mathbb{R}}$-valued sublinear functionals on $\mathcal{C}_{\mathfrak{V}}(X,E)^*$ is denoted by $\mathfrak{P}$, and the cone of all upper semicontinuous $\underline{\mathbb{R}}$-valued superlinear functionals by $\mathfrak{Q}$, that is $\mathfrak{Q} = -\mathfrak{P}$. There is a canonical correspondence between non-empty convex subsets $\mathfrak{u}$ of $\mathcal{C}_{\mathfrak{V}}(X,E)$ and functionals $\mathfrak{p}_\mathfrak{u} \in \mathfrak{P}$ (or $\mathfrak{q}_\mathfrak{u} \in \mathfrak{Q}$), defined by

$$\mathfrak{p}_\mathfrak{u}(\mu) = \sup\{\mu(f) \mid f \in \mathfrak{u}\} \quad (\text{or} \quad \mathfrak{q}_\mathfrak{u}(\mu) = \inf\{\mu(f) \mid f \in \mathfrak{u}\}) \quad \text{for } \mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*.$$

Since $\mathfrak{p}_\mathfrak{u} = \mathfrak{p}_{\bar{\mathfrak{u}}}$ and $\mathfrak{q}_\mathfrak{u} = \mathfrak{q}_{\bar{\mathfrak{u}}}$, where $\bar{\mathfrak{u}}$ denotes the topological closure of $\mathfrak{u}$ in $\mathcal{C}_{\mathfrak{V}}(X,E)$, this correspondence is indeed one-to-one and onto if one regards only closed convex subsets of $\mathcal{C}_{\mathfrak{V}}(X,E)$. This follows from the Hahn-Banach theorem. We have $f \in \bar{\mathfrak{u}}$ for $f \in \mathcal{C}_{\mathfrak{V}}(X,E)$ if and only if $\mu(f) \leq \mathfrak{p}_\mathfrak{u}(\mu)$ (or $\mu(f) \geq \mathfrak{q}_\mathfrak{u}(\mu)$) holds for all $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$. The duality of both approaches by either sub- or superlinear functionals follows from the observation that $-\mathfrak{p}_\mathfrak{u} = \mathfrak{q}_{(-\mathfrak{u})}$ holds for every non-empty convex set $\mathfrak{u} \in \mathcal{C}_{\mathfrak{V}}(X,E)$. If $\mathfrak{u}$ is bounded, then both $\mathfrak{p}_\mathfrak{u}$ and $\mathfrak{q}_\mathfrak{u}$ are indeed real-valued on $\mathcal{C}_{\mathfrak{V}}(X,E)^*$. Note that $\mathfrak{p}_{(\mathfrak{u}+\mathfrak{w})} = \mathfrak{p}_\mathfrak{u} + \mathfrak{p}_\mathfrak{w}$ and $\mathfrak{p}_{(\alpha\mathfrak{u})} = \alpha\mathfrak{p}_u$ for convex subsets $\mathfrak{u}, \mathfrak{w}$ of $\mathcal{C}_{\mathfrak{V}}(X,E)$ and $\alpha \geq 0$. If $\mathfrak{u} = \operatorname{conv}\{\mathfrak{u}_i \mid i \in \mathcal{I}\}$ denotes the convex hull of the union of the sets $\mathfrak{u}_i$, then $\mathfrak{p}_\mathfrak{u} = \vee_{i \in \mathcal{I}} \, \mathfrak{p}_{\mathfrak{u}_i}$, that is the pointwise supremum of the functionals $\mathfrak{p}_{\mathfrak{u}_i}$. Dual statements apply to the superlinear functionals $\mathfrak{q}_{\mathfrak{u}}$. Every vector-valued function $f \in \mathcal{C}_{\mathfrak{V}}(X,E)$ can be considered as a continuous linear functional $\mu \mapsto f(\mu) = \mu(f)$ on $\mathcal{C}_{\mathfrak{V}}(X,E)^*$, and every continuous linear functional on $\mathcal{C}_{\mathfrak{V}}(X,E)^*$ is of this type. Hence $\mathcal{C}_{\mathfrak{V}}(X,E) = \mathfrak{P} \cap \mathfrak{Q}$ in this sense.

Batty [2] proved part of the following proposition for a compact space X and bounded sets $\mathfrak{u} \subset C(X)$. His argument can be adapted to suit our more general setting. We shall use the following notations:

A subset $\mathfrak{u}$ of $\mathcal{C}_{\mathfrak{V}}(X,E)$ is called $\mathcal{C}(X)$-*convex* (see [2]) if $\varphi \otimes f + (1-\varphi) \otimes g \in \mathfrak{u}$ whenever $f, g \in \mathfrak{u}$ and $0 \leq \varphi \leq 1$ for $\varphi \in \mathcal{C}(X)$. A simple argument shows that sums, multiples, topological closures, decreasing and increasing hulls of $\mathcal{C}(X)$-convex sets are again $\mathcal{C}(X)$-convex. Note that all function space neighborhoods $\mathfrak{v} \in \mathfrak{V}$ are $\mathcal{C}(X)$-convex.

With a non-empty convex subset $\mathfrak{u}$ of $\mathcal{C}_{\mathfrak{V}}(X,E)$ we associate the set-valued function $F_{\mathfrak{u}} : X \to \mathrm{Conv}(E)$ such that

$$F_{\mathfrak{u}}(x) = \{f(x) \mid f \in \mathfrak{u}\},$$

and similarly, with a set-valued function $F : \to \mathrm{Conv}(E)$ we associate the convex subset of $\mathcal{C}_{\mathfrak{V}}(X,E)$

$$\mathfrak{u}_F = \{f \in \mathcal{C}_{\mathfrak{V}}(X,E) \mid f(x) \in F(x) \text{ for all } x \in X\}.$$

Two functionals $\mu, \nu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$ are called *mutually singular* if their respective unique regular representation measures θ and ϑ are mutually singular, that is if they are supported by disjoint sets in $\mathfrak{A}$.

Proposition 3.1. *For a non-empty closed convex subset $\mathfrak{u}$ of $\mathcal{C}_{\mathfrak{V}}(X,E)$ the following are equivalent.*

(*i*) $\mathfrak{u}$ *is $\mathcal{C}(X)$-convex.*

(*ii*) $\varphi_1 f_1 + \ldots + \varphi_n f_n \in \mathfrak{u}$ *whenever* $f_1, \ldots, f_n \in \mathfrak{u}$ *and* $0 \leq \varphi_i \in \mathcal{C}(X)$ *such that* $\varphi_1 + \ldots + \varphi_n = 1$.

(*iii*) $\mathfrak{p}_{\mathfrak{u}}(\mu+\nu) = \mathfrak{p}_{\mathfrak{u}}(\mu) + \mathfrak{p}_{\mathfrak{u}}(\nu)$ (*or* $\mathfrak{q}_{\mathfrak{u}}(\mu+\nu) = \mathfrak{q}_{\mathfrak{u}}(\mu) + \mathfrak{q}_{\mathfrak{u}}(\nu)$) *holds for mutually singular linear functionals* $\mu, \nu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$.

(*iv*) $\mathfrak{u} = \mathfrak{u}_{F_{\mathfrak{u}}}$, *that is* $\mathfrak{u} = \{f \in \mathcal{C}_{\mathfrak{V}}(X,E) \mid f(x) \in F_{\mathfrak{u}}(x)$ *for all* $x \in X\}$.

Proof. (*i*) $\Rightarrow$ (*iii*): Let $\mathfrak{u} \subset \mathcal{C}_{\mathfrak{V}}(X,E)$ be $\mathcal{C}(X)$-convex. Let $\mu, \nu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$ be mutually singular, θ, ϑ their respective representation measures and let $C, D \in \mathfrak{A}$ be disjoint sets supporting θ and ϑ, respectively. We may assume that $C \cup D = X$. Then $\theta + \vartheta$ is the representing measure for the functional $\mu + \nu$. We have $\int_X h\, d\theta = \int_C h\, d\theta$ and $\int_X h\, d\vartheta = \int_D h\, d\vartheta$ for every $h \in \mathcal{C}_{\mathfrak{V}}(X,E)$. Let $f, g \in \mathfrak{u}$ be any two functions, let $A \in \mathfrak{R}$ and $\varepsilon > 0$. Choose $O \in \mathfrak{O}$ such that $A \subset O$. There is a neighborhood $V \in \mathcal{V}$ such that both $|\theta|_O(V) < +\infty$ and $|\vartheta|_O(V) < +\infty$. This follows from the boundedness of both measures. Because both sets $f(O)$ and $g(O)$ are relatively compact, hence bounded in E, there is $\lambda > 0$ such that both $f(O), g(O) \subset \lambda V$. Next we use the regularity of the measures θ and ϑ. Let $\varepsilon > 0$. There is $O' \in \mathfrak{O}$ such that $A \subset O' \subset O$ and $|\theta|_{(O'\setminus A)}(V), |\vartheta|_{(O'\setminus A)}(V) \leq \varepsilon/\lambda$, and there are compact subsets $K_C \subset A \cap C$ and $K_D \subset A \cap D$ such that $|\theta|_{((A\cap C)\setminus K_C)}(V) \leq \varepsilon/\lambda$ and $|\vartheta|_{((A\cap D)\setminus K_D)}(V) \leq \varepsilon/\lambda$. There is $\varphi \in \mathcal{C}(X)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for all $x \in K_C$ and $\varphi(x) = 0$ for all $x \in K_D \cup (X \setminus O')$. Then $h = \varphi \otimes f + (1-\varphi) \otimes g \in \mathfrak{u}$ by (i) and $h(O') \subset \lambda V$. We observe that h coincides with f on K_C and with g on $K_D \cup (X \setminus O')$. Furthermore

$$\left| \int_{(A\cap C)} h \, d\theta - \int_{(A\cap C)} f \, d\theta \right| = \left| \int_{((A\cap C)\setminus K_C)} (h - f) \, d\theta \right| \leq 2\varepsilon,$$

since $h(x) - f(x) \in 2\lambda V$ for all $x \in (A \cap C) \setminus K_C$. Likewise,

$$\left| \int_{(A\cap D)} h \, d\vartheta - \int_{(A\cap D)} g \, d\vartheta \right| \leq 2\varepsilon,$$

as well as

$$\left| \int_{(X\setminus A)} h \, d(\theta + \vartheta) - \int_{(X\setminus A)} g \, d(\theta + \vartheta) \right| = \left| \int_{(O'\setminus A)} (h - g) \, d(\theta + \vartheta) \right| \leq 4\varepsilon.$$

Combining all of the above yields

$$\begin{aligned} \int_X h \, d(\theta + \vartheta) &= \int_C h \, d\theta + \int_D h \, d\vartheta \\ &= \int_{(A\cap C)} h \, d\theta + \int_{(A\cap D)} h \, d\vartheta + \int_{(X\setminus A)} h \, d(\theta + \vartheta) \\ &\geq \int_{A\cap C} f \, d\theta + \int_{A\cap D} g \, d\vartheta + \int_{X\setminus A} g \, d(\theta + \vartheta) - 8\varepsilon. \end{aligned}$$

Since $\mathfrak{p}_{\mathfrak{u}}(\mu + \nu) \geq \int_X h \, d(\theta + \vartheta)$ and since the above holds for all $\varepsilon > 0$, we have

$$\mathfrak{p}_{\mathfrak{u}}(\mu + \nu) \geq \int_{A\cap C} f \, d\theta + \int_{A\cap D} g \, d\vartheta + \int_{X\setminus A} g \, d(\theta + \vartheta).$$

Because

$$\lim_{A\in\mathfrak{R}} \int_{A\cap C} f \, d\theta = \int_C f \, d\theta \quad \text{as well as} \quad \lim_{A\in\mathfrak{R}} \int_{A\cap D} g \, d\vartheta = \int_D g \, d\vartheta,$$

and

$$\lim_{A\in\mathfrak{R}} \int_{X\setminus A} g \, d(\theta + \vartheta) = \int_X g \, d(\theta + \vartheta) - \lim_{A\in\mathfrak{R}} \int_A g \, d(\theta + \vartheta) = 0$$

by the definition of the integral, passing to the limit over all $A \in \mathfrak{R}$ in the above inequality leads to

$$\mathfrak{p}_{\mathfrak{u}}(\mu + \nu) \geq \int_C f \, d\theta + \int_D g \, d\vartheta = \int_X f \, d\theta + \int_X g \, d\vartheta.$$

Finally, since $f, g \in \mathfrak{u}$ were independently chosen, taking the supremum over these functions in $\mathfrak{u}$ on the right-hand side yields $\mathfrak{p}_{\mathfrak{u}}(\mu+\nu) \geq \mathfrak{p}_{\mathfrak{u}}(\mu) + \mathfrak{p}_{\mathfrak{u}}(\nu)$, as claimed. The case for the functionals $\mathfrak{q}_{\mathfrak{u}}$ follows, since $-\mathfrak{u}$ is $\mathcal{C}(X)$-convex whenever $\mathfrak{u}$ is.

$(iii) \Rightarrow (iv)$: Suppose that $\mathfrak{u} \subset \mathcal{C}_{\mathfrak{V}}(X,E)$ is closed and convex and that (iii) holds. Clearly,

$$\mathfrak{u} \subset \{f \in \mathcal{C}_{\mathfrak{V}}(X,E) \mid f(x) \in F_{\mathfrak{u}}(x) \text{ for all } x \in X\}.$$

Conversely, let $f \in \mathcal{C}_{\mathfrak{V}}(X,E)$ such that $f(x) \in F_{\mathfrak{u}}(x)$ for all $x \in X$. We shall verify that $\mu(f) \leq \mathfrak{p}_{\mathfrak{u}}(\mu)$ holds for all $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$, hence $f \in \mathfrak{u}$, since $\mathfrak{u}$ is closed in $\mathcal{C}_{\mathfrak{V}}(X,E)$. For this let $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$, let θ be its representation measure, and let $\varepsilon > 0$. Let f_* be any function in $\mathfrak{u}$. According to our definition of the integral in Section 2 there is a compact subset K of X such that both $|\int_{(X\setminus K)} f\,d\theta| \leq \varepsilon/3$ and $|\int_{(X\setminus K)} f_*\,d\theta| \leq \varepsilon/3$. In turn, there is $V \in \mathcal{V}$ such that $|\theta|_K(V) \leq 1$, that is $|\int_K g\,d\theta| \leq 1$ for every integrable function $g \in \mathcal{F}(X,E)$ satisfying $g(x) \in V$ for all $x \in K$. We use a compactness argument to find functions $f_1 \ldots, f_n \in \mathfrak{u}$ and a corresponding disjoint partition of K consisting of measurable sets $A_1, \ldots, A_n$ such that

$$f(x) - f_i(x) \in \frac{\varepsilon}{3} V \quad \text{for all } x \in A_i,$$

for $i = 1, \ldots, n$. Let θ_i denote the restriction of the measure θ to A_i, and let $\mu_i \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$ be the linear functional represented by θ_i. Then

$$\mu_i(f - f_i) = \int_{A_i} (f - f_i)\,d\theta \leq \frac{\varepsilon}{3}\,|\theta|_{A_i}(V)$$

for all $i = 1, \ldots, n$. Hence

$$\sum_{i=1}^{n} \mu_i(f) \leq \sum_{i=1}^{n} \mu_i(f_i) + \frac{\varepsilon}{3} \sum_{i=1}^{n} |\theta|_{A_i}(V) \leq \sum_{i=1}^{n} \mathfrak{p}_{\mathfrak{u}}(\mu_i) + \frac{\varepsilon}{3},$$

since $\mu_i(f_i) \leq \mathfrak{p}_{\mathfrak{u}}(\mu_i)$ and $\sum_{i=1}^{n} |\theta|_{A_i}(V) = |\theta|_K(V) \leq 1$. Let μ_0 be the linear functional represented by $\theta_{(X\setminus K)}$, the restriction of θ to $X \setminus K$. Then $\mu = \sum_{i=0}^{n} \mu_i$, and since the functionals μ_i are mutually singular, we infer that $\mathfrak{p}_{\mathfrak{u}}(\mu) = \sum_{i=0}^{n} \mathfrak{p}_{\mathfrak{u}}(\mu_i)$ using (ii). Finally, $f_* \in \mathfrak{u}$ and $|\mu_0(f_*)| \leq \varepsilon/3$ yields that $\mathfrak{p}_{\mathfrak{u}}(\mu_0) \geq -\varepsilon/3$. Now combining all of the above, we obtain

$$\begin{aligned} \mu(f) = \mu_0(f) + \sum_{i=1}^{n} \mu_i(f) &\leq \frac{\varepsilon}{3} + \left(\sum_{i=1}^{n} \mathfrak{p}_{\mathfrak{u}}(\mu_i) + \frac{\varepsilon}{3} \right) \\ &= \mathfrak{p}_{\mathfrak{u}}(\mu) - \mathfrak{p}_0(\mu) + \frac{2\varepsilon}{3} \leq \mathfrak{p}_{\mathfrak{u}}(\mu) + \varepsilon. \end{aligned}$$

Our claim follows, since $\varepsilon > 0$ was arbitrarily chosen.

The implications $(iv) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are both obvious. □

For a non-closed convex set $\mathfrak{u}$ Proposition 3.1 can be applied to its closure $\bar{\mathfrak{u}}$. Since $\mathfrak{p}_{\mathfrak{u}} = \mathfrak{p}_{\bar{\mathfrak{u}}}$, statement 3.1(*iii*) holds provided that $\bar{\mathfrak{u}}$ is $\mathcal{C}(X)$-convex. On the subcone $\mathcal{C}_{\mathfrak{V}}(X,E)^*_+$ of $\mathcal{C}_{\mathfrak{V}}(X,E)^*$ the functionals $p_{\mathfrak{u}}$ and $p_{\mathfrak{u}\downarrow}$ coincide. Thus 3.1(*iii*) holds for $\mu,\nu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*_+$ provided that $\overline{\mathfrak{u}\downarrow}$ is $\mathcal{C}(X)$-convex. Note that $\mathcal{C}(X)$-convexity of $\mathfrak{u}$ implies that the sets $\bar{\mathfrak{u}}$, $\mathfrak{u}\downarrow$ and $\overline{\mathfrak{u}\downarrow}$ are also $\mathcal{C}(X)$-convex.

A neighborhood in $\mathcal{C}_{\mathfrak{V}}(X,E)$ is a (not necessarily balanced) convex subset $\mathfrak{u}$ of $\mathcal{C}_{\mathfrak{V}}(X,E)$ such that $\mathfrak{v} \subset \mathfrak{u}$ for some $\mathfrak{v} \in \mathfrak{V}$. For the following note that if $\mathfrak{u} \subset \mathcal{C}_{\mathfrak{V}}(X,E)$ is a neighborhood, then for every $x \in X$ the set $F_{\mathfrak{u}}(x) = \{f(x) \mid f \in \mathfrak{u}\}$ is a convex neighborhood (of the origin) in E.

Corollary 3.2. *If $\mathfrak{u} \subset \mathcal{C}_{\mathfrak{V}}(X,E)$ is a $\mathcal{C}(X)$-convex neighborhood, then every non-zero lineal functional $\mu \in \mathfrak{u}^\circ$ which is not a point evaluation can be expressed as a convex combination $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$, where $0 < \lambda < 1$ and μ_1, μ_2 are non-zero mutually singular functionals in $\mathfrak{u}^\circ$. Consequently, every non-zero extreme point of $\mathfrak{u}^\circ$ is a point evaluation δ_x^η, where $x \in X$ and η is an extreme point of the polar of $F_{\mathfrak{u}}(x)$ in E^*.*

Proof. If $\mathfrak{u}$ is as stated, then $\mathfrak{p}_{\mathfrak{u}}(\mu) = 0$ for $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$ implies that $\mu = 0$. Let $0 \neq \mu \in \mathfrak{u}^\circ$ and assume that the support C of its representing measure θ contains two distinct points x and y. Let O be an open neighborhood of x not containing y. Then $D = C \setminus O$ is a closed set not supporting θ, the restrictions θ_1 and θ_2 to D and $C \setminus D$ are both non-zero. Moreover, $\theta = \theta_1 + \theta_2$ implies that $0 < \mathfrak{p}_{\mathfrak{u}}(\theta) = \mathfrak{p}_{\mathfrak{u}}(\theta_1) + \mathfrak{p}_{\mathfrak{u}}(\theta_2)$ by 3.1(*iii*). Set $\alpha_1 = \mathfrak{p}_{\mathfrak{u}}(\theta_1)/\mathfrak{p}_{\mathfrak{u}}(\theta)$ and $\alpha_2 = \mathfrak{p}_{\mathfrak{u}}(\theta_2)/\mathfrak{p}_{\mathfrak{u}}(\theta)$. Then $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$. The functionals μ_1 and μ_2 in E^*, represented by the measures $(1/\alpha_1)\theta_1$ and $(1/\alpha_2)\theta_2$ respectively, do not coincide, are contained in $\mathfrak{u}^\circ$, and $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$. Consequently, μ is not an extreme point of $\mathfrak{u}^\circ$, hence every non-zero extreme point is some point evaluation δ_x^η. Finally, if $\eta \in E^*$ is not an extreme point of $F_{\mathfrak{u}}(x)^\circ$, then $\eta = \alpha\eta_1 + (1-\alpha)\eta_2$ for distinct $\eta_1, \eta_2 \in F_{\mathfrak{u}}^\circ$ and $0 < \alpha < 1$. Therefore $\delta_x^\eta = \alpha\delta_x^{\eta_1} + (1-\alpha)\delta_x^{\eta_2}$, hence δ_x^η is not an extreme point of $\mathfrak{u}^\circ$. □

Let $\mathfrak{u} \subset \mathcal{C}_{\mathfrak{V}}(X,E)$ be a neighborhood. A subset $\mathfrak{w}$ of $\mathcal{C}_{\mathfrak{V}}(X,E)$ is called $\mathfrak{u}$-*bounded* if $\mathfrak{w} \subset \lambda\mathfrak{u}$ for some $\lambda \geq 0$. We shall denote $\mathfrak{V}\!\downarrow = \{\mathfrak{v}\!\downarrow \mid \mathfrak{v} \in \mathfrak{V}\}$. A set is *bounded* (or *bounded above*), if it is $\mathfrak{v}$-bounded for every $\mathfrak{v} \in \mathfrak{V}$ (or every $\mathfrak{v} \in \mathfrak{V}\!\downarrow$). Note that a linear functional $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$ is positive if and only if it is contained in the polar of some element of $\mathfrak{V}\!\downarrow$.

A family $\mathfrak{C}$ of non-empty subsets of $\mathcal{C}_{\mathfrak{V}}(X,E)$ is called a *Choquet cone,* if

(C1) All sets in $\mathfrak{C}$ are $\mathcal{C}(X)$-convex.
(C2) $\mathfrak{C}$ forms a cone, that is $\mathfrak{u}+\mathfrak{w} \in \mathfrak{C}$ and $\alpha\mathfrak{u} \in \mathfrak{C}$ whenever $\mathfrak{u}, \mathfrak{w} \in \mathfrak{C}$ and $\alpha \geq 0$.
(C3) $\mathfrak{C}$ contains $\mathfrak{V}\!\downarrow$.
(C4) For every $\mathfrak{u} \in \mathfrak{C}$, every $f \in \mathfrak{u}$ and $\mathfrak{v} \in \mathfrak{V}\!\downarrow$ there is a $\mathfrak{v}$-bounded set $\mathfrak{u}_0 \in \mathfrak{C}$ such that $f \in \mathfrak{u}_0 \subset \mathfrak{u} + \mathfrak{v}$.

The last condition is of course guaranteed if the set $\mathfrak{u} \in \mathfrak{C}$ itself is bounded above in $\mathcal{C}_{\mathfrak{V}}(X,E)$. If the system $\mathfrak{V}\!\downarrow$ is closed for intersections of finitely many of its elements, then it satisfies (C4) and therefore $\mathfrak{V}{\downarrow} \cup \{\{0\}\}$ forms the smallest Choquet cone for $\mathcal{C}_{\mathfrak{V}}(X,E)$.

Given a Choquet cone $\mathfrak{C}$ we define an order relation $\prec_{\mathfrak{C}}$ on $\mathcal{C}_{\mathfrak{V}}(X,E)^*_+$ setting

$$\nu \prec_{\mathfrak{C}} \mu \ \text{ if } \mathfrak{p}_{\mathfrak{u}}(\nu) \leq \mathfrak{p}_{\mathfrak{u}}(\mu) \text{ for all } \mathfrak{u} \in \mathfrak{C} \quad (\text{or } \mathfrak{q}_{\mathfrak{u}}(\mu) \leq \mathfrak{q}_{\mathfrak{u}}(\nu) \text{ for all } \mathfrak{u} \in -\mathfrak{C})\,.$$

This order is compatible with the multiplication by non-negative scalars, but generally not with the addition. It induces an equivalence relation on $\mathcal{C}_{\mathfrak{V}}(X,E)^*_+$, that is $\nu \sim \mu$ if both $\nu \prec_{\mathfrak{C}} \mu$ and $\mu \prec_{\mathfrak{C}} \nu$. We are particularly interested in minimal elements with respect to this order. (The dual approach considers the reverse order

$$\nu \prec^*_{\mathfrak{C}} \mu \ \text{ if } \mathfrak{q}_{\mathfrak{u}}(\nu) \leq \mathfrak{q}_{\mathfrak{u}}(\mu) \text{ for all } \mathfrak{u} \in -\mathfrak{C} \quad (\text{or } \mathfrak{p}_{\mathfrak{u}}(\mu) \leq \mathfrak{p}_{\mathfrak{u}}(\nu) \ \text{ for all } \mathfrak{u} \in \mathfrak{C})$$

and leads to the study of maximal elements instead.) We proceed to characterize the order $\prec_{\mathfrak{C}}$ in terms of upper and lower envelopes of functions in $\mathfrak{Q}$ and $\mathfrak{P}$, respectively. For this, let $f \in \mathfrak{Q}$ and $\mathfrak{v} \in \mathfrak{V}{\downarrow} \subset \mathfrak{C}$. Since f is homogeneous and bounded above on $\mathfrak{v}^\circ$, there is $\alpha \geq 0$ such that $f(\nu) \leq \alpha \mathfrak{p}_{\mathfrak{v}}(\nu)$ for all $\nu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$, that is $f \leq \alpha \mathfrak{p}_{\mathfrak{v}}$. Since $\alpha \mathfrak{p}_{\mathfrak{v}}(\mu) \leq \alpha$ for all $\mu \in \mathfrak{v}^\circ$, the expression

$$\widehat{f}(\mu) = \inf\{\mathfrak{p}_{\mathfrak{u}}(\mu) \mid \mathfrak{u} \in \mathfrak{C},\ f \leq \mathfrak{p}_{\mathfrak{u}}\}$$

is an element of $\underline{\mathbb{R}}$, bounded above by α whenever $\mu \in \mathfrak{v}^\circ$. Thus $\widehat{f}(\mu) \in \underline{\mathbb{R}}$ for all $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*_+$. Similarly, for $f \in \mathfrak{P}$ one defines its $\overline{\mathbb{R}}$-valued lower envelope on $\mathcal{C}_{\mathfrak{V}}(X,E)^*_+$ by

$$\check{f}(\mu) = \sup\{\mathfrak{q}_{\mathfrak{u}}(\mu) \mid \mathfrak{u} \in -\mathfrak{C},\ \mathfrak{q}_{\mathfrak{u}} \leq f\}.$$

We observe that

$$\begin{aligned} -\widehat{f}(\mu) &= \ -\inf\{\mathfrak{p}_{\mathfrak{u}}(\mu) \mid \mathfrak{u} \in \mathfrak{C},\ f \leq \mathfrak{p}_{\mathfrak{u}}\} \\ &= \ \sup\{\mathfrak{q}_{(-\mathfrak{u})}(\mu) \mid \mathfrak{u} \in \mathfrak{C},\ -f \leq -\mathfrak{q}_{(-\mathfrak{u})}\} = \widetilde{(-f)}(\mu) \end{aligned}$$

for all $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*_+$ and $f \in \mathfrak{Q}$. In this sense, both upper and lower envelopes can be equivalently used in many statements. We shall in the sequel indicate these equivalences in brackets. Moreover,

$$\widehat{(f+g)}\,(\mu) \leq \widehat{f}(\mu) + \widehat{g}(\mu) \quad \text{and} \quad \widehat{(\alpha f)}(\mu) = \alpha \widehat{f}(\mu),$$

as well as

$$\widetilde{(f+g)}\,(\mu) \geq \check{f}(\mu) + \check{g}(\mu) \quad \text{and} \quad \widetilde{(\alpha f)}(\mu) = \alpha \check{f}(\mu),$$

holds for $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*_+$, $\alpha \geq 0$ and $f,g \in \mathfrak{Q}$ or $f,g \in \mathfrak{P}$, respectively.

Lemma 3.3. *Let $\mathfrak{u}$ be a decreasing neighborhood in $\mathcal{C}_{\mathfrak{V}}(X,E)$, let $f \in \mathfrak{Q}$ (or $f \in \mathfrak{P}$) and $\mu \in \mathfrak{u}^\circ$. Then*

$$\widehat{f}(\mu) = \inf\{\mathfrak{p}_{\mathfrak{w}}(\mu) \mid \mathfrak{w} \in \mathfrak{C} \textit{ is } \mathfrak{u}\text{-bounded},\ \textit{and } f \leq \mathfrak{p}_{\mathfrak{w}}\}$$
$$\Big(\textit{or} \quad \check{f}(\mu) = \sup\{\mathfrak{q}_{\mathfrak{w}}(\mu) \mid \mathfrak{w} \in -\mathfrak{C} \textit{ is } (-\mathfrak{u})\text{-bounded},\ \textit{and } \mathfrak{q}_{\mathfrak{w}} \leq f\}\Big)\,.$$

Proof. We shall prove the case for the upper envelope. Let $f \in \mathfrak{Q}$ and $\mu \in \mathfrak{u}^\circ$. Clearly

$$\widehat{f}(\mu) \leq \inf\{\mathfrak{p}_{\mathfrak{w}}(\mu) \mid \mathfrak{w} \in \mathfrak{C} \text{ is } \mathfrak{u}\text{-bounded, and } f \leq \mathfrak{p}_{\mathfrak{w}}\}$$

holds in any case. For the reverse inequality let $\mathfrak{y} = \{h \in \mathcal{C}_{\mathfrak{V}}(X, E) \mid h \geq f\}$. Then

$$f = \inf\{h \mid h \in \mathfrak{y}\} = \mathfrak{q}_{\mathfrak{y}} = -\mathfrak{p}_{(-\mathfrak{y})}$$

by the Hahn-Banach theorem. Now let $\mathfrak{w} \in \mathfrak{C}$ such that $f \leq \mathfrak{p}_{\mathfrak{w}}$, that is $0 \leq \mathfrak{p}_{\mathfrak{w}} + \mathfrak{p}_{(-\mathfrak{y})} = \mathfrak{p}_{\mathfrak{w}-\mathfrak{y}}$. Again by the Hahn-Banach theorem, this implies $0 \in \overline{\mathfrak{w} - \mathfrak{y}}$, the closure of $\mathfrak{w} - \mathfrak{y}$. Given $\varepsilon > 0$ there is $\mathfrak{v} \in \mathfrak{V}\!\downarrow\, \subset \mathfrak{C}$ such that $\mathfrak{v} \subset (\varepsilon/2)\mathfrak{u}$, and we have $0 \in (\mathfrak{w} - \mathfrak{y}) + \mathfrak{v} = \mathfrak{w} - (\mathfrak{y} - \mathfrak{v})$. Hence we find a function $g \in \mathfrak{w} \cap (\mathfrak{y} - \mathfrak{v})$. By (C4) there is a $\mathfrak{v}$-bounded (hence $\mathfrak{u}$-bounded) subset $\mathfrak{w}_0 \in \mathfrak{C}$ of $\mathfrak{w} + \mathfrak{v}$ which also contains g. Thus $g \in \mathfrak{w}_0 \cap (\mathfrak{y} - \mathfrak{v})$, hence $0 \in \mathfrak{w}_0 + (-\mathfrak{y} + \mathfrak{v})$ and $f = -\mathfrak{p}_{(-\mathfrak{y})} \leq \mathfrak{p}_{(\mathfrak{w}_0 + \mathfrak{v})}$. The set $\mathfrak{w}_0 + \mathfrak{v} \in \mathfrak{C}$ is also $\mathfrak{u}$-bounded. From $\mathfrak{w}_0 + \mathfrak{v} \subset \mathfrak{w} + 2\mathfrak{v} \subset \mathfrak{w} + \varepsilon\mathfrak{u}$ and $\mu \in \mathfrak{u}^\circ$, hence $\mathfrak{p}_{\mathfrak{u}}(\mu) \leq 1$, we infer that

$$\mathfrak{p}_{(\mathfrak{w}_0+\mathfrak{v})}(\mu) \leq \mathfrak{p}_{(\mathfrak{w}+\varepsilon\mathfrak{u})}(\mu) = \mathfrak{p}_{\mathfrak{w}}(\mu) + \varepsilon \mathfrak{p}_{\mathfrak{u}}(\mu) \leq \mathfrak{p}_{\mathfrak{w}}(\mu) + \varepsilon.$$

Consequently,

$$\inf\{\mathfrak{p}_{\mathfrak{w}}(\mu) \mid \mathfrak{w} \in \mathfrak{C} \text{ is } \mathfrak{u}\text{-bounded, and } f \leq \mathfrak{p}_{\mathfrak{w}}\} \leq \mathfrak{p}_{\mathfrak{w}}(\mu) + \varepsilon.$$

The latter holds true for all $\mathfrak{w} \in \mathfrak{C}$ such that $f \leq \mathfrak{p}_{\mathfrak{w}}$ and all $\varepsilon > 0$. Our claim follows. $\square$

Lemma 3.4. *For $\nu, \mu \in \mathcal{C}_{\mathfrak{V}}(X, E)^*_+$ the following are equivalent:*

(i) $\nu \prec_{\mathfrak{C}} \mu$.

(ii) $\widehat{f}(\nu) \leq \widehat{f}(\mu)$ *for all* $f \in \mathfrak{Q}$ *(or* $\check{f}(\mu) \leq \check{f}(\nu)$ *for all* $f \in \mathfrak{P}$*).*

(iii) $f(\nu) \leq \widehat{f}(\mu)$ *for all* $f \in \mathcal{C}_{\mathfrak{V}}(X, E)$ *(or* $\check{f}(\mu) \leq f(\nu)$ *for all* $f \in \mathcal{C}_{\mathfrak{V}}(X, E)$*).*

Proof. $(i) \Rightarrow (ii)$: If $\nu \prec_{\mathfrak{C}} \mu$, then

$$\widehat{f}(\nu) = \inf\{\mathfrak{p}_{\mathfrak{u}}(\nu) \mid \mathfrak{u} \in \mathfrak{C},\ f \leq \mathfrak{p}_{\mathfrak{u}}\} \leq \inf\{\mathfrak{p}_{\mathfrak{u}}(\mu) \mid \mathfrak{u} \in \mathfrak{C},\ f \leq \mathfrak{p}_{\mathfrak{u}}\} = \widehat{f}(\mu)$$

for every $f \in \mathfrak{Q}$.

$(ii) \Rightarrow (iii)$ is obvious since $\mathcal{C}_{\mathfrak{V}}(X, E) \subset \mathfrak{Q}$ and $f(\nu) \leq \widehat{f}(\nu)$.

$(iii) \Rightarrow (i)$: Suppose that $f(\nu) \leq \widehat{f}(\mu)$ holds for all $f \in \mathcal{C}_{\mathfrak{V}}(X, E)$ and let $\mathfrak{w} \in \mathfrak{C}$. For every $f \in \mathfrak{w}$ we have $f \leq \mathfrak{p}_{\mathfrak{w}}$, hence

$$f(\nu) \leq \widehat{f}(\mu) = \inf\{\mathfrak{p}_{\mathfrak{u}}(\mu) \mid \mathfrak{u} \in \mathfrak{C},\ f \leq \mathfrak{p}_{\mathfrak{u}}\} \leq \mathfrak{p}_{\mathfrak{w}}(\mu),$$

and therefore $\mathfrak{p}_{\mathfrak{w}}(\nu) = \sup\{f(\nu) \mid f \in \mathfrak{w}\} \leq \mathfrak{p}_{\mathfrak{w}}(\mu)$. $\square$

Proposition 3.5. *For every $f \in \mathfrak{Q}$ (or $f \in \mathfrak{P}$) and $\mu \in \mathcal{C}_{\mathfrak{V}}(X, E)^*_+$ there is $\nu \prec_{\mathfrak{C}} \mu$ such that*

$$f(\nu) = \widehat{f}(\mu) = \max\{f(\vartheta) \mid \vartheta \in \mathcal{C}_{\mathfrak{V}}(X, E)^*,\ \vartheta \prec_{\mathfrak{C}} \mu\}$$

$$\Big(\textit{or}\ \ f(\nu) = \check{f}(\mu) = \min\{f(\vartheta) \mid \vartheta \in \mathcal{C}_{\mathfrak{V}}(X, E)^*,\ \vartheta \prec_{\mathfrak{C}} \mu\}\Big).$$

Proof. We shall prove the case for the lower envelope $\check{f}$ of a function $f \in \mathfrak{P}$. First we observe that $\mathfrak{P}$ is a locally convex cone in the sense of [11], endowed with its pointwise order, if we define neighborhoods $v_{\mathfrak{v}}$, corresponding to $\mathfrak{v} \in \mathfrak{V}{\downarrow}$ on $\mathfrak{P}$ setting

$$g \le h + v_{\mathfrak{v}} \;\text{ for } g, h \in \mathfrak{P}, \text{ if } g(\nu) \le h(\nu) + 1 \text{ for all } \nu \in \mathfrak{v}^{\circ}.$$

Now, corresponding to the fixed $f \in \mathfrak{P}$ and $\mu \in \mathcal{C}_{\mathfrak{V}}(X, E)^*_+$ we define $\overline{\mathbb{R}}$-valued super- and sublinear functionals q and p on the cone $\mathfrak{P}$ setting

$$q(g) = \check{g}(\mu) \quad \text{and} \quad p(\alpha f) = \alpha \check{f}(\mu) \;\text{ for } \alpha \ge 0 \quad \text{and} \quad p(g) = +\infty, \;\text{ else,}$$

respectively, for $g \in \mathfrak{P}$. The Sandwich Theorem (Theorem 3.1 in [11]) for locally convex cones states that there is a monotone $\overline{\mathbb{R}}$-valued continuous linear functional Φ on $\mathfrak{P}$ such that $q \le \Phi \le p$ if and only if there is a neighborhood $v_{\mathfrak{v}}$ such that

$$g \le h + v_{\mathfrak{v}} \;\text{ for } f, g \in \mathfrak{P} \text{ implies that } q(g) \le p(h) + 1.$$

This condition is easily checked in our case. Indeed, choose $\mathfrak{v} \in \mathfrak{V}{\downarrow}$ such that $\mu \in \mathfrak{v}^{\circ}$. For $h \ne \alpha f$ there is nothing to prove. Otherwise, $g \le \alpha f + v_{\mathfrak{v}}$ for some $\alpha \ge 0$ is equivalent to $g \le \alpha f + \mathfrak{p}_{\mathfrak{v}}$. Hence $\mathfrak{q}_{\mathfrak{u}} \lneq g$ for $\mathfrak{u} \in -\mathfrak{C}$ implies that $\mathfrak{q}_{\mathfrak{u}} - \mathfrak{p}_{\mathfrak{v}} = \mathfrak{q}_{\mathfrak{u}} + \mathfrak{q}_{-\mathfrak{v}} = \mathfrak{q}_{(\mathfrak{u}-\mathfrak{v})} \le \alpha f$. Thus $\mathfrak{q}_{\mathfrak{u}} \le \alpha \check{f} + \mathfrak{p}_{\mathfrak{v}}$ since $\mathfrak{u} - \mathfrak{v} \in -\mathfrak{C}$, and therefore $\check{g} \le \alpha \check{f} + \mathfrak{p}_{\mathfrak{v}}$. This yields in particular

$$q(g) = \check{g}(\mu) \le \alpha \check{f}(\mu) + \mathfrak{p}_{\mathfrak{v}}(\mu) \le p(\alpha f) + 1,$$

as claimed. Hence there is a linear functional Φ on $\mathfrak{P}$ such that $q \le \Phi \le p$. Set $\nu(g) = \Phi(g)$ for all $g \in \mathcal{C}_{\mathfrak{V}}(X, E) \subset \mathfrak{P}$. Since $\nu(g) \ge \check{g}(\mu) \ge -1$ for all $g \in \mathfrak{v}$, the linear functional ν is contained in $\mathfrak{v}^{\circ} \subset \mathcal{C}_{\mathfrak{V}}(X, E)^*_+$. Moreover, $g(\nu) \ge \check{g}(\mu)$ for all $g \in \mathcal{C}_{\mathfrak{V}}(X, E)$ yields $\nu \prec_{\mathfrak{C}} \mu$ according to Lemma 3.4. Finally, since the $\overline{\mathbb{R}}$-valued function f is lower semicontinuous and sublinear on $\mathcal{C}_{\mathfrak{V}}(X, E)^*$, we have

$$\begin{aligned} f(\nu) &= \sup\{g(\nu) \mid g \in \mathcal{C}_{\mathfrak{V}}(X, E),\; g \le f\} \\ &= \sup\{\Phi(g) \mid h \in \mathcal{C}_{\mathfrak{V}}(X, E),\; h \le f\} \le \Phi(f) \le \check{f}(\mu). \end{aligned}$$

On the other hand, $\check{f}(\mu) \le \check{f}(\theta) \le f(\theta)$ follows with Lemma 3.4(*ii*) for all $\theta \prec_{\mathfrak{C}} \mu$. This implies $f(\nu) = \check{f}(\mu)$ in particular and completes our argument. □

Note that $f(\nu)$ can take the value $+\infty$ in the preceding line of reasoning.

Lemma 3.6. *For every $\mu \in \mathcal{C}_{\mathfrak{V}}(X, E)^*_+$ the set $\mu{\downarrow}^{\mathfrak{C}} = \{\nu \in \mathcal{C}_{\mathfrak{V}}(X, E)^*_+ \mid \nu \prec_{\mathfrak{C}} \mu\}$ is convex and compact.*

Proof. If $\nu, \omega \prec_{\mathfrak{C}} \mu$ and $0 \le \lambda \le 1$, then $\mathfrak{p}_{\mathfrak{u}}(\lambda\nu + (1-\lambda)\omega) \le \lambda \mathfrak{p}_{\mathfrak{u}}(\nu) + (1-\lambda)\mathfrak{p}_{\mathfrak{u}}(\omega) \le \mathfrak{p}_{\mathfrak{u}}(\mu)$ for all $\mathfrak{u} \in \mathfrak{C}$ by the sublinearity of the functionals $\mathfrak{p}_{\mathfrak{u}}$. Hence $\lambda\nu + (1-\lambda)\omega \prec_{\mathfrak{C}} \mu$, and the set $\mu{\downarrow}^{\mathfrak{C}}$ is seen to be convex. Moreover, $\mu \in \mathfrak{v}^{\circ}$ for some $\mathfrak{v} \in \mathfrak{V}{\downarrow} \subset \mathfrak{C}$ implies that $\mu{\downarrow}^{\mathfrak{C}} \subset \mathfrak{v}^{\circ}$, the latter being a compact subset of $\mathcal{C}_{\mathfrak{V}}(X, E)^*_+$. Finally, since it is the intersection of the inverse images of the intervals $(-\infty, \mathfrak{p}_{\mathfrak{u}}(\mu)]$ under the lower semicontinuous mappings $\mathfrak{p}_{\mathfrak{u}}$, for all $\mathfrak{u} \in \mathfrak{C}$, $\mu{\downarrow}^{\mathfrak{C}}$ is closed, hence compact. □

An element $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)_+^*$ is *minimal* if $\nu \prec_{\mathfrak{C}} \mu$ for $\nu \in \mathcal{C}_{\mathfrak{V}}(X,E)_+^*$ implies that $\nu \sim \mu$. Since $\nu \prec_{\mathfrak{C}} \mu$ if and only if $\lambda\nu \prec_{\mathfrak{C}} \lambda\mu$ for all $\lambda > 0$, minimality of an element $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)_+^*$ implies the minimality of $\lambda\mu$ for all $\lambda > 0$. The zero functional is obviously minimal since $\mu \prec_{\mathfrak{C}} 0$ implies that $\mu \in \varepsilon v^\circ$ for all $\varepsilon > 0$ and $v \in \mathfrak{V}\!\downarrow$, hence $\mu = 0$.

Lemma 3.7. *For every $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)_+^*$ there is a minimal $\bar{\mu} \in \mathcal{C}_{\mathfrak{V}}(X,E)_+^*$ such that $\bar{\mu} \prec_{\mathfrak{C}} \mu$.*

Proof. Let $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)_+^*$. Given a totally ordered subset Ω of $\mu\Downarrow^{\mathfrak{C}}$, the intersection of the sets $\nu\Downarrow^{\mathfrak{C}}$, for all $\nu \in \Omega$, cannot be empty, hence contains a lower bound for Ω in $\mu\Downarrow^{\mathfrak{C}}$. By Zorn's lemma then there is a minimal element in $\mu\Downarrow^{\mathfrak{C}}$. $\square$

Lemma 3.8. *For $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)_+^*$ the following are equivalent:*

(*i*) *μ is minimal.*

(*ii*) *$\widehat{f}(\mu) = \max\{f(\nu) \mid \nu \sim \mu\}$ for all $f \in \mathfrak{Q}$ (or $\check{f}(\mu) = \min\{f(\nu) \mid \nu \sim \mu\}$ for all $f \in \mathfrak{P}$).*

(*iii*) *$\widehat{\mathfrak{q}_{\mathfrak{u}}}(\mu) = \mathfrak{q}_{\mathfrak{u}}(\mu)$ for all $\mathfrak{u} \in -\mathfrak{C}$ (or $\check{\mathfrak{p}_{\mathfrak{u}}}(\mu) = \mathfrak{p}_{\mathfrak{u}}(\mu)$ for all $\mathfrak{u} \in \mathfrak{C}$).*

(*iv*) *For every $\mu \in \mathfrak{C}$ and $\varepsilon > 0$ there is $w \in \mathfrak{C}$ such that $0 \in \overline{\mathfrak{u}+\mathfrak{w}}$ and $\mathfrak{p}_{(u+w)}(\mu) \le \varepsilon$.*

Proof. Clearly (*ii*) implies (*iii*), since $\mathfrak{q}_{\mathfrak{u}} \in \mathfrak{Q}$ for every $\mathfrak{u} \in -\mathfrak{C}$, and $\nu \sim \mu$ implies that $\mathfrak{q}_{\mathfrak{u}}(\nu) = \mathfrak{q}_{\mathfrak{u}}(\mu)$. Now suppose that (*iii*) holds and let $\nu \prec_{\mathfrak{C}} \mu$. Then for every $\mathfrak{u} \in -\mathfrak{C}$ we have

$$\mathfrak{q}_{\mathfrak{u}}(\nu) \le \widehat{\mathfrak{q}_{\mathfrak{u}}}(\nu) \le \widehat{\mathfrak{q}_{\mathfrak{u}}}(\mu) = \mathfrak{q}_{\mathfrak{u}}(\mu)$$

by Lemma 3.4(*ii*). This yields $\mu \prec_{\mathfrak{C}} \nu$, hence the minimality of μ. Furthermore, if μ is minimal and $f \in \mathfrak{Q}$, then $\max\{f(\vartheta) \mid \vartheta \in \mathcal{C}_{\mathfrak{V}}(X,E)_+^*,\ \vartheta \prec_{\mathfrak{C}} \mu\} = \max\{f(\vartheta) \mid \vartheta \in \mathcal{C}_{\mathfrak{V}}(X,E)_+^*,\ \vartheta \sim \mu\} = \widehat{f}(\mu)$ by Proposition 3.5. Therefore (*i*) implies (*ii*) as well. Finally, (*iv*) is just a reformulation of (*iii*), since $\mathfrak{q}_{-\mathfrak{u}} \le \mathfrak{p}_{\mathfrak{w}}$ for $\mathfrak{u}, \mathfrak{w} \in \mathfrak{C}$ is equivalent to $0 \le \mathfrak{p}_u + \mathfrak{p}_{\mathfrak{w}} = \mathfrak{p}_{(\mathfrak{u}+\mathfrak{w})}$, that is $0 \in \overline{\mathfrak{u}+\mathfrak{w}}$, and $\mathfrak{p}_w(\mu) \le \mathfrak{q}_{-\mathfrak{u}}(\mu) + \varepsilon$ is equivalent to $\mathfrak{p}_{(u+w)}(\mu) \le \varepsilon$. $\square$

As in classical Choquet theory we try to characterize minimal positive linear functionals on $\mathcal{C}_{\mathfrak{V}}(X,E)$, that is E_+^*-valued measures on X, in terms of supporting subsets of X. For this, we introduce an analogue to the classical *Choquet boundary.* We set

$$\Delta_{\mathfrak{C}} = \left\{\delta_x^\eta \in \mathcal{C}_{\mathfrak{V}}(X,E)_+^* \mid\ x \in X,\ 0 \ne \eta \in E_+^*,\ \text{such that } \delta_x^\eta \text{ is minimal}\right\}$$

$$\partial_{\mathfrak{C}} = \{x \in X \mid \delta_x^\eta \in \Delta_{\mathfrak{C}} \text{ for some } \eta \in E^*\}.$$

The elements of of $\Delta_{\mathfrak{C}}$ can be identified using the equivalent criteria in Lemma 3.8. Corresponding to a decreasing $\mathcal{C}(X)$-convex neighborhood $\mathfrak{u}$ in $\mathcal{C}_{\mathfrak{V}}(X,E)$ we denote by $\Delta_{\mathfrak{C}}^{\mathfrak{u}}$ and $\partial_{\mathfrak{C}}^{\mathfrak{u}}$ the respective subsets of $\Delta_{\mathfrak{C}}$ and $\partial_{\mathfrak{C}}$ consisting of those point evaluations $\delta_x^\eta \in \Delta_{\mathfrak{C}}$ such that $\mathfrak{p}_{\mathfrak{u}}(\delta_x^\eta) < +\infty$ and their corresponding arguments $x \in \partial_{\mathfrak{C}}$, respectively.

The main tool for advancing the theory is an analogue to the classical case (Proposition I.4.10 in [1]). However, the presence of unbounded and indeed infinity-valued functions defining the Choquet ordering will complicate matters. Let $\mathfrak{u} \in \mathfrak{C}$ be a neighborhood in $\mathcal{C}_{\mathfrak{V}}(X, E)$. Obviously, a function $f \in \mathfrak{P}$ is bounded above on $\mathfrak{u}^\circ$ if $\{h \in \mathcal{C}_{\mathfrak{V}}(X, E) \mid h \le f\} \subset \lambda\mathfrak{u}$ for some $\lambda \ge 0$. Similarly, $f \in \mathfrak{Q}$ is bounded below on $\mathfrak{u}^\circ$ if $\{h \in \mathcal{C}_{\mathfrak{V}}(X, E) \mid h \ge f\} \subset \lambda\mathfrak{u}$ for some $\lambda \le 0$. A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathfrak{P}$ or $\mathfrak{Q}$ is bounded above or below on $\mathfrak{u}^\circ$ if the respective property holds with the same λ for all $n \in \mathbb{N}$.

Theorem 3.9. *Let $\mathfrak{u} \in \mathfrak{C}$ be a decreasing neighborhood in $\mathcal{C}_{\mathfrak{V}}(X, E)$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{Q}$ (or in $\mathfrak{P}$) that is bounded below (or above) on $\mathfrak{u}^\circ$. Let $\alpha \le 0$ (or $\alpha \ge 0$). If*

$$\liminf_{n\in\mathbb{N}} \widehat{f}_n(\delta_x^\eta) \ge \alpha\mathfrak{p}_{\mathfrak{u}}(\delta_x^\eta) \quad \left(or \ \limsup_{n\in\mathbb{N}} \check{f}_n(\delta_x^\eta) \le \alpha\mathfrak{p}_{\mathfrak{u}}(\delta_x^\eta) \right)$$

for all $\delta_x^\eta \in \Delta_{\mathfrak{C}}^{\mathfrak{u}}$, then

$$\liminf_{n\in\mathbb{N}} \widehat{f}_n(\mu) \ge \alpha\mathfrak{p}_{\mathfrak{u}}(\mu) \quad \left(or \ \limsup_{n\in\mathbb{N}} \check{f}_n(\mu) \le \alpha\mathfrak{p}_{\mathfrak{u}}(\mu) \right)$$

for all $\mu \in \mathcal{C}_{\mathfrak{V}}(X, E)^*_+$ such that $\mathfrak{p}_{\mathfrak{u}}(\mu) < +\infty$.

Proof. We will to some extend follow the outlines of the proof of Theorem 2.5 in [9]. Our more general approach will however require some adaptations. We shall verify the case of the lower envelopes. Let $(f_n)_{n\in\mathbb{N}}$, $\mu \in \mathcal{C}_{\mathfrak{V}}(X, E)^*$, $\mathfrak{u} \in \mathfrak{C}$ and $\alpha \ge 0$ be as stated. If $\mathfrak{p}_{\mathfrak{u}}(\mu) = 0$, then $\mu = 0$, and $\check{f}_n(0) \le f_n(0) = 0$ for all $n \in \mathbb{N}$, hence our claim. We may therefore assume that $\mathfrak{p}_{\mathfrak{u}}(\mu) = 1$, that is $\mu \in \mathfrak{u}^\circ$. The sequence $(f_n)_{n\in\mathbb{N}}$ is supposed to be bounded above on $\mathfrak{u}^\circ$ which implies that the sequence $\left(\check{f}_n(\mu)\right)_{n\in\mathbb{N}}$ is bounded above as well. Thus we may use Lemma 3.3 to construct a corresponding sequence $(\mathfrak{w}_n)_{n\in\mathbb{N}}$ of $(-\mathfrak{u})$-bounded sets in $-\mathfrak{C}$ such that

$$\mathfrak{q}_{\mathfrak{w}_n} \le f_n \quad \text{and} \quad \mathfrak{q}_{\mathfrak{w}_n}(\mu) \ge \check{f}_n(\mu) - 1/n.$$

Note that the functions $\mathfrak{q}_{\mathfrak{w}_n}$ are bounded on $\mathfrak{u}^\circ$, since $\mathfrak{w}_n \subset \lambda_n(-\mathfrak{u})\!\uparrow = -\lambda_n\mathfrak{u}\!\downarrow = -\lambda_n\mathfrak{u}$ for some $\lambda_n \ge 0$, hence $\mathfrak{q}_{\mathfrak{w}_n}(\mu) \ge \lambda_n\mathfrak{q}_{(-\mathfrak{u})}(\mu) = -\lambda_n\mathfrak{p}_{\mathfrak{u}}(\mu) \ge -\lambda_n$ for all $\mu \in \mathfrak{u}^\circ$. The sequence $(\mathfrak{q}_{w_n})_{n\in\mathbb{N}}$ is indeed bounded above on $\mathfrak{u}^\circ$, since this property holds for the sequence $(f_n)_{n\in\mathbb{N}}$. As in the proof for the classical case we shall use a representation of $\mathfrak{u}^\circ$ in the space $\mathbb{R}^{\mathbb{N}}$ and apply Choquet's theorem for metrizable compact sets. For this, define $\Phi : \mathfrak{u}^\circ \to \mathbb{R}^{\mathbb{N}}$ by

$$\Phi(\nu) = \left(\mathfrak{q}_{\mathfrak{w}_n}(\nu)\right)_{n\in\mathbb{N}} \quad \text{for } \nu \in \mathfrak{u}^\circ.$$

Since all of the functions $\mathfrak{q}_{\mathfrak{w}_n}$ are bounded on $\mathfrak{u}^\circ$, the sets $\mathfrak{q}_{\mathfrak{w}_n}(\mathfrak{u}^\circ)$ are all bounded in $\mathbb{R}$. The cartesian product of their compact closures is therefore compact in the metric space $\mathbb{R}^{\mathbb{N}}$, and so is the set $K = \overline{\Phi(\mathfrak{u}^\circ)}$, as it is a closed subset of this product. Now let C be the sup-stable point-separating cone of continuous real-valued functions on K, generated by the constants and the projections $\pi_m : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, for all $m \in \mathbb{N}$. Let $\partial_C \subset K$ denote the classical Choquet boundary of C in K (see I.5 in [1]). Recall that ∂_C consists of those points in K whose point evaluation measure is maximal among all probability measures on K with respect to the classical Choquet ordering defined by the functions in C. We shall first establish that for every non-zero $(\alpha_n)_{n\in\mathbb{N}} \in \partial_C$ there is $\delta_x^\eta \in \Delta_{\mathfrak{C}}^{\mathfrak{u}}$ such that $\Phi(\delta_x^\eta) = (\alpha_n)_{n\in\mathbb{N}}$. Indeed, for $(\alpha_n)_{n\in\mathbb{N}} \in \partial_C$ set

$$Z = \{\nu \in \mathfrak{u}^\circ \mid \mathfrak{q}_{\mathfrak{w}_n}(\nu) \geq \alpha_n \text{ for all } n \in \mathbb{N}\}.$$

Since the functions $\mathfrak{q}_{\mathfrak{w}_n}$ are superlinear and upper semicontinuous, Z is a compact convex subset of $\mathfrak{u}^\circ$. Let us verify that it is not empty. There is a net $(\nu_i)_{i\in\mathcal{I}}$ in $\mathfrak{u}^\circ$ such that $\nu_i \to \nu$ in $\mathfrak{u}^\circ$ and $\Phi(\nu_i) \to (\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$. From the upper semicontinuity of the functions $\mathfrak{q}_{\mathfrak{w}_n}$ we infer that

$$\alpha_n = \lim_{i\in\mathcal{I}} (\mathfrak{q}_n(\nu_i)) \leq \mathfrak{q}_{\mathfrak{w}_n}\left(\lim_{i\in\mathcal{I}} \nu_i\right) = \mathfrak{q}_{\mathfrak{w}_n}(\nu)$$

holds for all $n \in \mathbb{N}$, hence $\nu \in Z$. Let $\mathfrak{L}$ be the collection of all non-empty compact convex subsets F of Z with the following property:

> If $\mathfrak{q}_{\mathfrak{w}}(\nu) \leq \lambda\mathfrak{q}_{\mathfrak{w}}(\nu_1) + (1-\lambda)\mathfrak{q}_{\mathfrak{w}}(\nu_2)$ for all $\mathfrak{w} \in -\mathfrak{C}$,
> some $\nu \in F$, $\nu_1, \nu_2 \in \mathfrak{u}^\circ$ and $0 < \lambda < 1$, then $\nu_1, \nu_2 \in F$.

Let us verify that $Z \in \mathfrak{L}$. Let $\mathfrak{q}_{\mathfrak{w}}(\nu) \leq \lambda\mathfrak{q}_{\mathfrak{w}}(\nu_1) + (1-\lambda)\mathfrak{q}_{\mathfrak{w}}(\nu_2)$ for all $\mathfrak{w} \in -\mathfrak{C}$, some $\nu \in F$, and $\nu_1, \nu_2 \in \mathfrak{u}^\circ$. Set $\beta_n = \mathfrak{q}_{\mathfrak{w}_n}(\nu_1)$ and $\gamma_n = \mathfrak{q}_{\mathfrak{w}_n}(\nu_2)$. Then

$$\alpha_n \leq \mathfrak{q}_{\mathfrak{w}_n}(\nu) \leq \lambda\mathfrak{q}_{\mathfrak{w}_n}(\nu_1) + (1-\lambda)\mathfrak{q}_{\mathfrak{w}_n}(\nu_2) = \lambda\beta_n + (1-\lambda)\gamma_n$$

for all $n \in \mathbb{N}$. If $\delta_{(\alpha_n)}, \delta_{(\beta_n)}$ and $\delta_{(\gamma_n)}$ denote the point evaluations on K at the points $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$, respectively, then the above shows that the measure $\lambda\delta_{(\beta_n)} + (1-\lambda)\delta_{(\gamma_n)}$ dominates the measure $\delta_{(\alpha_n)}$ in the classical Choquet ordering generated by C on K, hence $\delta_{(\alpha_n)} = \delta_{(\beta_n)} = \delta_{(\gamma_n)}$, and $\alpha_n = \beta_n = \gamma_n$, since $(\alpha_n)_{n\in\mathbb{N}} \in \partial_C$ is maximal. Thus $\nu_1, \nu_2 \in Z$. Therefore $\mathfrak{L}$ is not empty. Next we observe that every set $F \in \mathfrak{L}$ is decreasing. Indeed, suppose that $\vartheta \prec_{\mathfrak{C}} \nu$ for $\nu \in F$ and $\vartheta \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$. Then $\vartheta \in \mathfrak{u}^\circ$ since $\mathfrak{p}_{\mathfrak{u}}(\vartheta) \leq \mathfrak{p}_u(\nu) = 1$, and $\mathfrak{q}_{\mathfrak{w}}(\nu) \leq \mathfrak{q}_{\mathfrak{w}}(\vartheta) = (1/2)\mathfrak{q}_{\mathfrak{w}}(\vartheta) + (1/2)\mathfrak{q}_{\mathfrak{w}}(\vartheta)$ for all $\mathfrak{w} \in -\mathfrak{C}$, implies that $\vartheta \in F$. Furthermore, every non-zero extreme point ν of F is some point evaluation δ_x^η, because otherwise, according to Corollary 3.2, ν can be expressed as $\nu = \lambda\nu_1 + (1-\lambda)\nu_2$ with mutually singular functionals $\nu_1, \nu_2 \in \mathfrak{u}^\circ$ and $0 < \lambda < 1$. This implies $\nu_1, \nu_2 \in F$ by the definition of $\mathfrak{L}$, a contradiction.

The family $\mathfrak{L}$ is ordered inductively by set inclusion and, following Zorn's lemma, it contains a minimal element F_0. We shall verify that all elements of F_0 are equivalent and minimal. Indeed, let $\mathfrak{w}_0 \in -\mathfrak{C}$ and let

$$G_0 = \{\nu \in F_0 \mid \mathfrak{p}_{\mathfrak{w}_0}(\nu) = \max\{\mathfrak{p}_{\mathfrak{w}_0}(F_0)\}\}.$$

G_0 is a non-empty compact convex subset of F_0 and $\mathfrak{q}_{\mathfrak{w}}(\nu) \leq \lambda\mathfrak{q}_{\mathfrak{w}}(\nu_1) + (1-\lambda)\mathfrak{q}_{\mathfrak{w}}(\nu_2)$ for all $\mathfrak{w} \in -\mathfrak{C}$, some $\nu \in G_0$, $\nu_1, \nu_2 \in \mathfrak{u}^\circ$ and $0 < \lambda < 1$, implies that $\mathfrak{p}_{\mathfrak{w}_0}(\nu_1) = \mathfrak{p}_{\mathfrak{w}_0}(\nu_2) = \max\{\mathfrak{p}_{\mathfrak{w}_0}(F_0)\}$, hence $\nu_1, \nu_2 \in G_0$. The minimality of F_0 implies that $G_0 = F_0$, and therefore all the functionals $\mathfrak{p}_{\mathfrak{w}}$, for $\mathfrak{w} \in -\mathfrak{C}$, are constant on F_0, our claim. They are all minimal because F_0 is decreasing. Since $(\alpha_n)_{n\in\mathbb{N}} \neq 0$ and every non-zero extreme point of F_0 is a point evaluation, hence an element of $\Delta^{\mathfrak{u}}_{\mathfrak{C}}$, we conclude that

$$\Phi^{-1}\left((\alpha_n)_{n\in\mathbb{N}}\right) \cap \Delta^{\mathfrak{u}}_{\mathfrak{C}} \neq \emptyset.$$

Next we choose any $\delta^\eta_x \in \Phi^{-1}\left((\alpha_n)_{n\in\mathbb{N}}\right) \cap \Delta^{\mathfrak{u}}_{\mathfrak{C}}$ and observe that

$$\begin{aligned}\limsup_{m\in\mathbb{N}} \pi_m\left((\alpha_n)_{n\in\mathbb{N}}\right) &= \limsup_{m\in\mathbb{N}} \alpha_m = \limsup_{m\in\mathbb{N}} \mathfrak{q}_{w_m}(\delta^\eta_x) \\ &\leq \limsup_{m\in\mathbb{N}} \widehat{f}_m(\delta^\eta_x) \leq \alpha\mathfrak{p}_{\mathfrak{u}}(\delta^\eta_x) \leq \alpha.\end{aligned}$$

Choquet's theorem for compact metric spaces guarantees the existence of a probability measure θ on K which is supported by ∂C and represents $\Phi(\mu)$, that is

$$\theta(\pi_m) \geq \pi_m\left(\Phi(\mu)\right) = \mathfrak{q}_{w_m}(\mu) \quad \text{for all } m \in \mathbb{N}.$$

The sequence $(\pi_m)_{m\in\mathbb{N}}$ is uniformly bounded above on K, because this holds by assumption for the sequence $(\mathfrak{q}_{w_m})_{m\in\mathbb{N}}$ on $\mathfrak{u}^\circ$. Thus we my use Fatou's lemma for

$$\begin{aligned}\limsup_{m} \check{f}_m(\mu) = \limsup_{m\in\mathbb{N}} \mathfrak{q}_{\mathfrak{w}_m}(\mu) &\leq \limsup_{m\in\mathbb{N}} \theta(\pi_m) \\ &\leq \theta\left(\limsup_{m\in\mathbb{N}} \pi_m\right) \leq \alpha = \alpha\mathfrak{p}_{\mathfrak{u}}(\mu).\end{aligned}$$

This completes our argument. □

The expressions in Theorem 3.9 involving limits and upper (or lower) envelopes of functions can be reformulated using the statements of Proposition 3.5.

Lemma 3.10. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{Q}$ (or in $\mathfrak{P}$). Then*

$$\liminf_{n\in\mathbb{N}} \widehat{f}_n(\mu) = \sup_{\nu\prec_{\mathfrak{C}}\mu}\left\{\liminf_{n\in\mathbb{N}} f_n(\nu)\right\} \quad \left(\text{or } \limsup_{n\in\mathbb{N}} \check{f}_n(\mu) = \inf_{\nu\prec_{\mathfrak{C}}\mu}\left\{\limsup_{n\in\mathbb{N}} f_n(\nu)\right\}\right)$$

holds for all $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*_+$. If $\mathfrak{u} \in \mathfrak{C}$ is a decreasing neighborhood in $\mathcal{C}_{\mathfrak{V}}(X,E)$ and if the functions f_n are continuous and the sequence $(f_n)_{n\in\mathbb{N}}$ is increasing (or decreasing) on $\mathfrak{u}^\circ$, then for every $\mu \in \mathfrak{u}^\circ$ the suprema (or infima) with respect to $\nu \prec_{\mathfrak{C}} \mu$ in the preceding expressions are in fact maxima (or minima).

Proof. We shall prove the case of the upper envelopes. Let us first assume that the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathfrak{Q}$ is increasing and let $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*_+$. Then

$$\begin{aligned}\liminf_{n\in\mathbb{N}} \widehat{f}_n(\mu) &= \sup_{n\in\mathbb{N}} \widehat{f}_n(\mu) = \sup_{n\in\mathbb{N}}\left\{\max_{\nu\prec_{\mathfrak{C}}\mu} f_n(\nu)\right\} \\ &= \sup_{\nu\prec_{\mathfrak{C}}\mu}\left\{\sup_{n\in\mathbb{N}} f_n(\nu)\right\} = \sup_{\nu\prec_{\mathfrak{C}}\mu}\left\{\liminf_{n\in\mathbb{N}} f_n(\nu)\right\}\end{aligned}$$

by Proposition 3.5. If the functions f_n are indeed continuous (and therefore real-valued), we proceed as follows: There are $\nu_n \in \mu\!\downarrow^{\mathfrak{C}} \subset \mathfrak{u}^\circ$ such that $f_n(\nu_n) = \max_{\nu \prec_{\mathfrak{C}} \mu} f_n(\nu)$, and we find a convergent subsequence $\nu_{n_i} \to \nu \in \mu\!\downarrow^{\mathfrak{C}}$ by the compactness of $\mu\!\downarrow^{\mathfrak{C}}$. We claim that

$$\sup_{n\in\mathbb{N}} f_n(\nu) = \sup_{n\in\mathbb{N}} f_n(\nu_n).$$

Indeed, since $f_n(\nu) \leq f_n(\nu_n)$ for all $n \in \mathbb{N}$, we infer that $\sup_{n\in\mathbb{N}} f_n(\nu) \leq \sup_{n\in\mathbb{N}} f_n(\nu_n)$. On the other hand, given $\varepsilon > 0$, there is $i_0 \in \mathbb{N}$ such that $f_1(\nu - \nu_{n_i}) \geq -\varepsilon$ for all $i \geq i_0$. The sequence $(f_n(\nu_n))_{n\in\mathbb{N}}$ is increasing in $\overline{\mathbb{R}}$, and since for each $n \in \mathbb{N}$ we find $n_i \geq n$ for some $i \geq i_0$, using the superlinearity of the f_n we infer that

$$f_n(\nu) \geq f_n(\nu_n) + f_n(\nu - \nu_n),$$

hence

$$f_n(\nu_n) \leq f_{n_i}(\nu_{n_i}) \leq f_{n_i}(\nu) - f_{n_i}(\nu - \nu_{n_i}) \leq f_{n_i}(\nu) - f_1(\nu - \nu_{n_i}) \leq f_{n_i}(\nu) + \varepsilon.$$

Thus

$$\sup_{n\in\mathbb{N}} f_n(\nu_n) \leq \sup_{n\in\mathbb{N}} f_n(\nu) + \varepsilon.$$

This holds for all $\varepsilon > 0$ and yields our claim for increasing sequences. For the general case we set

$$g_n = \inf\{f_m \mid m \geq n\}$$

(the infima are take pointwise on $\mathfrak{u}^\circ$. The functions g_n are again in $\mathfrak{Q}$ and the sequence $(g_n)_{n\in\mathbb{N}}$ is increasing. Thus we may apply the first part of our argument to this sequence, that is

$$\liminf_{n\in\mathbb{N}} \widehat{f}_n(\mu) = \liminf_{n\in\mathbb{N}} \widehat{g}_n(\mu) = \sup_{\nu \prec_{\mathfrak{C}} \mu} \left\{ \sup_{n\in\mathbb{N}} g_n(\nu) \right\} = \sup_{\nu \prec_{\mathfrak{C}} \mu} \left\{ \liminf_{n\in\mathbb{N}} f_n(\nu) \right\},$$

our claim. □

The next corollary relates to the classical Choquet-Bishop-DeLeeuw theorem (see Corollary I.4.12 in [1]). It establishes a localization for the support of the representation measures of minimal linear functionals. Recall that a G_δ subset of X is defined to be a countable intersection of open sets.

Corollary 3.11. *Let $\mathfrak{u} \in \mathfrak{C}$ be a decreasing neighborhood in $\mathcal{C}_{\mathfrak{V}}(X,E)$ and let $\mu \in \mathfrak{u}^\circ$ be minimal. If $A \in \mathfrak{R}$ is a G_δ subset of X disjoint from $\partial^{\mathfrak{u}}_{\mathfrak{C}}$, then there is $V \in \mathcal{V}$ such that*

$$\inf_{\nu\sim\mu} |\nu|_A(V) = 0.$$

Proof. Let $A = \bigcap_{n\in\mathbb{N}} G_n \in \mathfrak{R}$ be disjoint from $\partial^{\mathfrak{u}}_{\mathfrak{C}}$. We may assume that the sequence $(G_n)_{n\in\mathbb{N}}$ of open sets is decreasing and that all of its elements are relatively compact in X, that is contained in some compact subset K of X. Consequently, by the definition of function space neighborhoods in Section 2, there is $V \in \mathcal{V}$ such that $f \in \mathfrak{u}$ for $f \in \mathcal{C}(X)$ whenever $f(x) \in \chi_K(x)\, V$ for all $x \in X$. According to Part (xx) in the proof of Theorem 4.4 in [12] the regularity of μ implies that

$$|\mu|_O(V) = \sup\{\mu(f) \mid f \in \mathcal{E}(X,E) \text{ such that } f(x) \in \chi_K(x)\, V \text{ for all } x \in X\}$$

holds for all open sets $O \in \mathfrak{R}$, where $\mathcal{E}(X,E)$ denotes the subspace of *elementary functions* of $\mathcal{C}_{\mathfrak{V}}(X,E)$ (see 2.3 in [12].) Using this, we observe that the functions $f_n : \mathcal{C}_{\mathfrak{V}}(X,E)^* \to \overline{\mathbb{R}}$, that is

$$f_n(\nu) = |\nu|_{G_n}(V)$$

are both sublinear and lower semicontinuous, hence in $\mathfrak{P}$. Moreover, since all of the sets G_n are contained in K, and therefore all of the functions $f \in \mathcal{E}(X,E)$ involved are in $\mathfrak{u}$, the functions f_n are all bounded above by 1 on $\mathfrak{u}^\circ$. This implies in particular that $|\mu|_{G_n}(V) \le 1$, hence by the regularity of μ we may also assume that $|\mu|_{(G_n \setminus A)}(V) \le 1/n$. Next we apply Theorem 3.9 with $\alpha = 0$. Since $\left(\bigcap_{n\in\mathbb{N}} G_n\right) \cap \partial^{\mathfrak{u}}_{\mathfrak{C}} = \emptyset$, we infer that $\limsup_{n\in\mathbb{N}} \check{f}_n(\delta^\eta_x) \le 0$ for all $\delta^\eta_x \in \Delta^{\mathfrak{u}}_{\mathfrak{C}}$. Thus, using Theorem 3.9 together with Lemma 3.10 we obtain

$$\limsup_{n\in\mathbb{N}} \check{f}_n(\mu) = \inf_{\nu\sim\mu} \left\{ \limsup_{n\in\mathbb{N}} f_n(\nu) \right\} = \inf_{\nu\sim\mu} \left\{ \inf_{n\in\mathbb{N}} |\nu|_{G_n}(V) \right\} = 0.$$

But $|\nu|_{G_n}(V) = |\nu|_A(V) + |\mu|_{(G_n\setminus A)}(V) \le |\nu|_A(V) + 1/n$ implies that $\inf_{n\in\mathbb{N}} |\nu|_{G_n}(V) = |\nu|_A(V)$, hence our claim. □

Corollary 3.12. *Suppose that the functions $\mathfrak{p}_{\mathfrak{w}}$, for $\mathfrak{w} \in \mathfrak{C}$, separate the points of $\mathcal{C}_{\mathfrak{V}}(X,E)^*$. Let $\mathfrak{u} \in \mathfrak{C}$ be a decreasing neighborhood in $\mathcal{C}_{\mathfrak{V}}(X,E)$ and let $\mu \in \mathfrak{u}^\circ$ be minimal. If A is a G_δ subset of X disjoint from $\partial^{\mathfrak{u}}_{\mathfrak{C}}$, then $|\mu|_A = 0$.*

Proof. Let $A = \bigcap_{n\in\mathbb{N}} G_n$ be disjoint from $\partial^{\mathfrak{u}}_{\mathfrak{C}}$. We need to show that $|\mu|_B = 0$ for every $B \in \mathfrak{R}$ such that $B \subset A$. There is a relatively compact open set $O \in \mathfrak{R}$ containing B. Then $C = A \cap O \in \mathfrak{R}$ is a G_δ subset of X and according to Corollary 3.11 $\inf_{\nu\sim\mu} |\nu|_C(V) = 0$ holds for some $V \in \mathcal{V}$. But the equivalence class of μ consists only of μ itself, since the functions in $\mathfrak{C}$ are supposed to separate the points of $\mathcal{C}_{\mathfrak{V}}(X,E)^*$. We infer that $|\mu|_B(V) \le |\mu|_C(V) = 0$, hence $|\mu|_B = 0$ as claimed. □

4. Some special cases and examples

Corollary 3.12, in particular, shows the benefit of having small or indeed singleton equivalence classes in $\mathcal{C}_{\mathfrak{V}}(X,E)^*_+$ while preserving the given Choquet boundary. We shall attempt to achieve this by suitably increasing the given Choquet cone $\mathfrak{C}$. In this context we shall say that a non-empty $C(X)$-convex subset $\mathfrak{u}$ of $\mathcal{C}_{\mathfrak{V}}(X,E)$ is $\mathfrak{C}$-*superharmonic* if $\mathfrak{p}_{\mathfrak{u}}(\mu) \le \mathfrak{p}_{\mathfrak{u}}(\delta^\eta_x)$ whenever $\mu \prec_{\mathfrak{C}} \delta^\eta_x$ for $\mu, \delta^\eta_x \in \mathcal{C}_{\mathfrak{V}}(X,E)^*_+$.

We define *the completion* $\widetilde{\mathfrak{C}}$ of a Choquet cone $\mathfrak{C}$ as the collection of all non-empty C-convex and $\mathfrak{C}$-superharmonic subsets of $\mathcal{C}_{\mathfrak{V}}(X,E)$ which also satisfy the boundedness condition (C4) with respect to $\mathfrak{C}$. Obviously, $\mathfrak{C}\subset\widetilde{\mathfrak{C}}$, and $\widetilde{\mathfrak{C}}$ is itself a Choquet cone. Indeed, Conditions (C1), (C3) and (C4) are evident. Let us verify that $\widetilde{\mathfrak{C}}$ forms a cone. Clearly $\alpha\mathfrak{u}\in\widetilde{\mathfrak{C}}$ whenever $\mathfrak{u}\in\widetilde{\mathfrak{C}}$ and $\alpha\geq 0$. Let $\mathfrak{u},\mathfrak{w}\in\widetilde{\mathfrak{C}}$. The set $\mathfrak{u}+\mathfrak{w}$ is also $\mathcal{C}(X)$-convex and $\mathfrak{C}$-superharmonic, since $\mu\prec_{\mathfrak{C}}\delta_x^\eta$ for $\mu,\delta_x^\eta\in\mathcal{C}_{\mathfrak{V}}(X,E)^*_+$ implies that

$$\mathfrak{p}_{(\mathfrak{u}+\mathfrak{w})}(\mu)=\mathfrak{p}_{\mathfrak{u}}(\mu)+\mathfrak{p}_{\mathfrak{w}}(\mu)\leq\mathfrak{p}_{\mathfrak{u}}(\delta_x^\eta)+\mathfrak{p}_{\mathfrak{w}}(\delta_x^\eta)=\mathfrak{p}_{(\mathfrak{u}+\mathfrak{w})}(\delta_x^\eta).$$

Finally, $\mathfrak{u}+\mathfrak{w}$ satisfies (C4), since for $f+g\in\mathfrak{u}+\mathfrak{w}$ and $\mathfrak{v}\in\mathfrak{V}{\downarrow}$ there are $\mathfrak{v}$-bounded sets $\mathfrak{u}_0,\mathfrak{w}_0\in\mathfrak{C}$ such that $f\in\mathfrak{u}_0\subset\mathfrak{u}+(1/2)\mathfrak{v}$ and $g\in\mathfrak{w}_0\subset\mathfrak{w}+(1/2)\mathfrak{v}$. Thus $f+g\in(\mathfrak{u}_0+\mathfrak{w}_0)\subset(\mathfrak{u}+\mathfrak{w})+\mathfrak{v}$. Hence $\mathfrak{u}+\mathfrak{w}\in\widetilde{\mathfrak{C}}$. The completion $\widetilde{\mathfrak{C}}$ of a Choquet cone $\mathfrak{C}$ serves our purpose since the $\mathfrak{C}$-superharmonicity of its elements guarantees that $\Delta_{\widetilde{\mathfrak{C}}}\subset\Delta_{\mathfrak{C}}$. Indeed, let $\mu\prec_{\mathfrak{C}}\delta_x^\eta$ for $\mu\in\mathcal{C}_{\mathfrak{V}}(X,E)^*_+$ and $\delta_x^\eta\in\Delta_{\widetilde{\mathfrak{C}}}$. Then $\mu\prec_{\widetilde{\mathfrak{C}}}\delta_x^\eta$ by the superharmonicity of the sets in $\widetilde{\mathfrak{C}}$ and therefore $\delta_x^\eta\prec_{\widetilde{\mathfrak{C}}}\mu$, hence $\delta_x^\eta\prec_{\mathfrak{C}}\mu$, by the minimality of δ_x^η.

Proposition 4.1. *Let $\widetilde{\mathfrak{C}}$ be the completion of a Choquet cone $\mathfrak{C}$. If $\mathfrak{u}\in\mathfrak{C}$, then $\mathfrak{u}{\downarrow}\in\widetilde{\mathfrak{C}}$.*

Proof. Clearly, $\mathfrak{u}{\downarrow}$ is $\mathcal{C}(X)$-convex and $\mathfrak{C}$-superharmonic since $\mathfrak{p}_{\mathfrak{u}\downarrow}(\mu)=\mathfrak{p}_{\mathfrak{u}}(\mu)$ for all $\mu\in\mathcal{C}_{\mathfrak{V}}(X,E)^*_+$. For (C4) let $f\in\mathfrak{u}{\downarrow}$, that is $f\leq g$ for some $g\in\mathfrak{u}$. Given $\mathfrak{v}\in\mathfrak{V}{\downarrow}$ there is a $\mathfrak{v}$-bounded set $\mathfrak{u}_0\in\mathfrak{C}$ such that $g\in\mathfrak{u}_0\subset\mathfrak{u}+\mathfrak{v}$, hence $f\in\mathfrak{u}_0\subset\mathfrak{u}+\mathfrak{v}$ since $\mathfrak{u}+\mathfrak{v}$ is decreasing, hence our claim. □

Proposition 4.2. *Let $\widetilde{\mathfrak{C}}$ be the completion of a Choquet cone $\mathfrak{C}$. Let $a\in E_+$ and $\rho_1,\dots\rho_n\in\mathcal{C}(X)$. If all the singleton sets $\{\rho_i\otimes a\}$ are contained in $\mathfrak{C}$, and if $\rho=\bigwedge_{i=1}^n\rho_i$, then $\{\rho\otimes a\}\in\widetilde{\mathfrak{C}}$.*

Proof. We only need to verify that $\{\rho\otimes a\}$ is $\mathfrak{C}$-superharmonic. Indeed, if $\mu\prec_{\mathfrak{C}}\delta_x^\eta$ for $\mu,\delta_x^\eta\in\mathcal{C}_{\mathfrak{V}}(X,E)^*_+$, then

$$\begin{aligned}\mathfrak{p}_{\{\rho\otimes a\}}(\mu)&\leq\bigwedge_{i=1}^n\mathfrak{p}_{\{\rho_i\otimes a\}}(\mu)\leq\bigwedge_{i=1}^n\mathfrak{p}_{\{\rho_i\otimes a\}}(\delta_x^\eta)\\&=\bigwedge_{i=1}^n\rho_i(x)\eta(a)=\rho(x)\eta(a)=\mathfrak{p}_{\{\rho\otimes a\}}(\delta_x^a).\end{aligned}$$ □

Proposition 4.3. *Let $\mathfrak{C}$ be a Choquet cone and let $\mathfrak{D}$ be a family of $\mathfrak{C}$-superharmonic subsets of $\mathcal{C}_{\mathfrak{V}}(X,E)$. If $\mathfrak{w}=\bigcap_{\mathfrak{u}\in\mathfrak{D}}\mathfrak{u}$ is not empty and if $\mathfrak{p}_{\mathfrak{w}}(\delta_x^\eta)=\inf_{\mathfrak{u}\in\mathfrak{D}}\mathfrak{p}_{\mathfrak{u}}(\delta_x^\eta)$ for all $x\in X$ and $\eta\in E^*_+$, then $\mathfrak{w}$ is also $\mathfrak{C}$-superharmonic.*

Proof. Let $\mathfrak{D}$ be as stated and let $\mathfrak{w}=\bigcap_{\mathfrak{u}\in\mathfrak{D}}\mathfrak{u}$ denote the intersection of its elements. Clearly,

$$\mathfrak{p}_{\mathfrak{w}}(\mu)\leq\inf_{\mathfrak{u}\in\mathfrak{D}}\mathfrak{p}_{\mathfrak{u}}(\mu)$$

for every $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*_+$. Let $\mu \prec_{\mathfrak{C}} \delta_x^\eta$ for $\mu, \delta_x^\eta \in \mathcal{C}_{\mathfrak{V}}(X,E)^*_+$. Then

$$\mathfrak{p}_{\mathfrak{w}}(\mu) \le \inf_{\mathfrak{u}\in\mathfrak{D}} \mathfrak{p}_{\mathfrak{u}}(\mu) \le \inf_{\mathfrak{u}\in\mathfrak{D}} \mathfrak{p}_{\mathfrak{u}}(\delta_x^\eta) = \mathfrak{p}_{\mathfrak{w}}(\delta_x^\eta)$$

by our assumption, hence our claim. □

Classical Choquet theory can be recovered in the following way. Let $\mathcal{C}_{\mathfrak{V}}(X,E)$ be a function space such that the neighborhood system $\mathfrak{V}{\downarrow}$ is closed for intersections of finitely many of its elements. Let L be a subspace of $\mathcal{C}_{\mathfrak{V}}(X,E)$. Then

$$\mathfrak{C}_0 = \{f + \mathfrak{v} \mid f \in L,\ \mathfrak{v} \in \mathfrak{V}{\downarrow} \cup \{\{0\}\}\}$$

defines a Choquet cone associated with L. Indeed, conditions (C1), (C2) and (C3) are obvious. For (C4) let $g \in f + \mathfrak{v}$ for $f \in L$ and $\mathfrak{v} \in \mathfrak{V}{\downarrow} \cup \{0\}$, and let $\mathfrak{w} \in \mathfrak{V}{\downarrow}$. There is $\rho \ge 0$ such that $g - f \in \rho\mathfrak{w}$, hence $g \in f + (\mathfrak{v} \cap \rho\mathfrak{w})$. Since the set $\mathfrak{v}_0 = f + (\mathfrak{v} \cap \rho\mathfrak{w}) \in \mathfrak{C}_0$ is $\mathfrak{w}$-bounded and a subset of $f + \mathfrak{v}$, this verifies (C4). In the remainder of this section we shall investigate the classical cases of of real- or complex-valued functions, assuming that $\mathfrak{C}$ is a Choquet cone containing $\mathfrak{C}_0$.

The case that E is $\mathbb{R}$ or $\mathbb{C}$. Let E stand for either $\mathbb{R}$ or $\mathbb{C}$ with the equality as its order. We assume that the subspace L of $\mathcal{C}_{\mathfrak{V}}(X,E)$ is E-linear and that $\mathfrak{C} = \mathfrak{C}_0$. In this case both $\mathcal{C}_{\mathfrak{V}}(X,E)$ and $\mathcal{C}_{\mathfrak{V}}(X,E)^*$ are also vector spaces over E, but the functionals in $\mathcal{C}_{\mathfrak{V}}(X,E)^*$ are always real-valued and real-linear. E is of course its own dual via the evaluation $\eta(z) = \mathfrak{Re}(\eta z)$ for $\eta \in E$ and $z \in E$. Obviously $z\nu \prec_{\mathfrak{C}} z\mu$ whenever $\nu \prec_{\mathfrak{C}} \mu$ for $z \in E$ since $\mathfrak{u} \in \mathfrak{C}$ implies that $z\mathfrak{u} \in \mathfrak{C}$ as well. Consequently, the set $z\mathfrak{u} \subset \mathcal{C}_{\mathfrak{V}}(X,E)$ is $\mathfrak{C}$-superharmonic whenever $\mathfrak{u}$ is $\mathfrak{C}$ -superharmonic. Indeed, let $\mu \prec_{\mathfrak{C}} \delta_x^\eta$ for $\mu, \delta_X^\eta \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$, where $x \in X$, $\eta \in E$, and δ_x^η denotes the point evaluation $f \mapsto \mathfrak{Re}\,(\eta f(x))$. Note that $\delta_x^\eta = \eta\delta_x^1$. This implies $z\mu \prec_{\mathfrak{C}} z\delta_x^\eta$ by the above, hence $\mathfrak{p}_{z\mathfrak{u}}(\mu) = \mathfrak{p}_{\mathfrak{u}}(z\mu) \le \mathfrak{p}_{\mathfrak{u}}(z\delta_x^\eta) = \mathfrak{p}_{z\mathfrak{u}}(\delta_x^\eta)$. We apply Proposition 4.3 to identify further $\mathfrak{C}$-superharmonic subsets of $\mathcal{C}_{\mathfrak{V}}(X,E)$.

Proposition 4.4. *Let E be $\mathbb{R}$ or $\mathbb{C}$. Let $\mathfrak{v} \in \mathfrak{V}$ such that $F_{\mathfrak{v}}(x) = \rho(x)\mathbb{B}$, where $\rho : X \to \overline{\mathbb{R}}$ is a strictly positive lower semicontinuous function. Let $f_1, \ldots, f_n \in L \cap \mathfrak{v}$. Then the set*

$$\mathfrak{w} = \{g \in \mathcal{C}_{\mathfrak{V}}(X,E) \mid |g| \le \rho - \vee_{i=1}^n |f_i|\},$$

is $\mathfrak{C}$-superharmonic.

Proof. Let $\mathfrak{v} \in \mathfrak{V}$ and $f_i \in L \cap \mathfrak{V}$ be as stated. For $f \in \mathfrak{v}$ we denote $\mathfrak{u}_f = \mathfrak{v} - f$ We shall use Proposition 4.3 with the family of $\mathfrak{C}$-superharmonic sets

$$\mathfrak{D} = \{\mathfrak{u}_{zf_i} \mid z \in \mathbb{C},\ |z| = 1,\ i = 1, \ldots, n\}.$$

Since $g \in \mathfrak{u}_{zf_i}$ if and only if $|g + zf_i| \le \rho$, we have $g \in \cap_{|z|=1} \mathfrak{u}_{zf_i}$ if and only if $|g| + |f_i| \le \rho$, hence

$$\mathfrak{w} = \cap_{\mathfrak{u}\in\mathfrak{D}}\, \mathfrak{u} = \{g \in \mathcal{C}_{\mathfrak{V}}(X,E) \mid |g| \le \rho - \vee_{i=1}^n |f_i|\}.$$

Let $\delta_x^\eta \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$, that is $x \in X$ and $\eta \in E$. Then

$$\mathfrak{p}_{\mathfrak{u}_{zf_i}}(\delta_x^\eta) = |\eta|\rho(x) - \mathfrak{Re}\,(\eta z f_i(x)) \quad \text{and} \quad \mathfrak{p}_{\mathfrak{w}}(\delta_x^\eta) = |\eta|\left(\rho(x) - \vee_{i=1}^n |f_i(x)|\right)$$

by the above. Thus

$$\bigwedge_{|z|=1} \mathfrak{p}_{\mathfrak{u}_{zf_i}}(\delta_x^\eta) = |\eta|\rho(x) - |\eta f_i(x)| = |\eta|\left(\rho(x) - |f_i(x)|\right).$$

This yields $\wedge_{\mathfrak{u}\in\mathfrak{D}} \mathfrak{p}_{\mathfrak{u}}(\delta_x^\eta) = \mathfrak{p}_{\mathfrak{w}}(\delta_x^\eta)$ as required for the application of Proposition 4.3. □

The inclusion of this type of $\mathfrak{C}$-superharmonic sets corresponds to the maximization of $|\mu|$ over the cone of the functions $\vee_{i=1}^n |f_i|$, for $f_i \in L$, in classical Choquet theory (see [1] and [9]). If $\rho \in L$ (this corresponds to the requirement that L contains the constants in the classical setting), we can identify some additional $\mathfrak{C}$-superharmonic sets. For $\mathfrak{u} \subset \mathcal{C}_{\mathfrak{V}}(X,E)$ let us denote $\mathfrak{Re}(\mathfrak{u}) = \{\mathfrak{Re}(f) \mid f \in \mathfrak{u}\}$, that is the real parts of the functions in $\mathfrak{u}$.

Proposition 4.5. *Let E be $\mathbb{R}$ or $\mathbb{C}$. Suppose that there is $\mathfrak{v} \in \mathfrak{V}$ such that $F_{\mathfrak{v}}(x) = \rho(x)\mathbb{B}$, where $\rho : X \to \overline{\mathbb{R}}$ is a strictly positive continuous function, and that $\rho \in L$.*

(*i*) *The set $z\mathfrak{Re}(\mathfrak{u})$ is $\mathfrak{C}$-superharmonic whenever $\mathfrak{u}$ is $\mathfrak{C}$-superharmonic and $z \in E$.*

(*ii*) *If $f_1,\ldots,f_n$ and $g_1,\ldots,g_m$ are functions in L such that $F \le G$ for $F = \bigvee_1^n \mathfrak{Re}(f_i)$ and $G = \bigwedge_1^m \mathfrak{Re}(g_k)$, then for every $z \in E$ the set*

$$\mathfrak{w} = z\,\{h \in \mathcal{C}(X) \mid F \le h \le G\}$$

is $\mathfrak{C}$-superharmonic.

Proof. Let $x \in X$, and $\mu \prec_{\mathfrak{C}} \delta_x^1$ for $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$. We claim that $\mu(g) \le 0$ for every function $g \in \mathcal{C}_{\mathfrak{V}}(X,E)$ such that $\mathfrak{Re}(g) \le 0$. Indeed, since the neighborhood $\mathfrak{v} \in \mathfrak{V}$ is absorbing, we may assume that $g \in \mathfrak{v}$, that is $|g| \le \rho$. For $\lambda \ge 2$ consider the set $\mathfrak{w} = \lambda(\mathfrak{v} - \rho) \in \mathfrak{C}$. Since $F_{\mathfrak{w}}(x) = \lambda\rho(x)\,(\mathbb{B} - 1)$, we have

$$\mathfrak{p}_{\mathfrak{w}}(\mu) \le \mathfrak{p}_w(\delta_x^1) = 0,$$

that is $\mu(f) \le 0$ for all $f \in \mathfrak{w}$. We observe that the function $f = -\rho + \sqrt{2\lambda - 1}\,g$ is contained in $\mathfrak{w}$, since it is readily checked that $|f(x) + \lambda\rho(x)| \le \lambda\rho(x)$ holds for all $x \in X$. This yields

$$\mathfrak{u}(f) = -\mu(\rho) + \sqrt{2\lambda - 1}\mu(g) \le 0$$

and $\sqrt{2\lambda - 1}\mu(g) \le \mu(\rho)$. The latter holds for all $\lambda \ge 1$, hence $\mu(g) \le 0$. If $\mathfrak{Re}(g) = 0$, then the same argument with $-g$ in place of g yields $\mu(g) = 0$. Thus

$\mu = \mathfrak{Re}(\mu)$, that is the functional $f \mapsto \mu\left(\mathfrak{Re}(f)\right)$. Consequently, if $\mathfrak{u} \in \mathcal{C}_{\mathfrak{V}}(X,E)$ is a convex set and $z \in E$, we conclude that

$$\begin{aligned} \mathfrak{p}_{z\mathfrak{Re}(\mathfrak{u})}(\mu) &= \sup\left\{\mu\left(z\mathfrak{Re}(f)\right) \mid f \in \mathfrak{u}\right\} \\ &= \begin{cases} \mathfrak{Re}(z)\sup\left\{\mu(f) \mid f \in \mathfrak{u}\right\} = \mathfrak{Re}(z)\mathfrak{p}_{\mathfrak{u}}(\mu), & \text{if } \mathfrak{Re}(z) \geq 0, \\ \mathfrak{Re}(z)\inf\left\{\mu(f) \mid f \in \mathfrak{u}\right\} = -\mathfrak{Re}(z)\mathfrak{p}_{(-\mathfrak{u})}(\mu), & \text{if } \mathfrak{Re}(z) < 0. \end{cases} \end{aligned}$$

This alternative holds for $\mu = \delta_x^1$ as well. We continue arguing part (i). If $\mathfrak{u}$ is $\mathfrak{C}$-superharmonic, so is $-\mathfrak{u}$ by the above and we have both $\mathfrak{p}_{\mathfrak{u}}(\mu) \leq \mathfrak{p}_{\mathfrak{u}}(\delta_x^1)$ and $\mathfrak{p}_{-\mathfrak{u}}(\mu) \leq \mathfrak{p}_{-\mathfrak{u}}(\delta_x^1)$. Thus $\mathfrak{p}_{z\mathfrak{Re}(\mathfrak{u})}(\mu) \leq \mathfrak{p}_{z\mathfrak{Re}(\mathfrak{u})}(\delta_x^1)$ in any case. Finally, let $\mu \prec_{\mathfrak{C}} \delta_x^\eta$ for $0 \neq \eta \in E$. Then $(1/\eta)\mu \prec_{\mathfrak{C}} \delta_x^1$ and therefore

$$\mathfrak{p}_{\mathfrak{Re}(\mathfrak{u})}\left(\mu\right) = \mathfrak{p}_{\eta\mathfrak{Re}(\mathfrak{u})}\left((1/\eta)\mu\right) \leq \mathfrak{p}_{\eta\mathfrak{Re}(\mathfrak{u})}\left(\delta_x^1\right) = \mathfrak{p}_{\mathfrak{Re}(\mathfrak{u})}\left(\delta_x^\eta\right).$$

The set $\mathfrak{Re}(\mu)$ is therefore $\mathfrak{C}$-superharmonic. So are the sets $z\mathfrak{Re}(\mu)$ for all $z \in E$.

For part (ii) let f_i, g_k, and F, G be as stated. Let $\mathfrak{u} = \{h \in \mathcal{C}(X) \mid F \leq h \leq G\}$, and let $\mu \prec_{\mathfrak{C}} \delta_x^1$ for $\mu \in \mathcal{C}_{\mathfrak{V}}(X,E)^*$ and $x \in X$. Then for $z \in E$ we have

$$\mathfrak{p}_{z\mathfrak{u}}(\mathfrak{u}) = \begin{cases} \mathfrak{Re}(z)\mathfrak{p}_{\mathfrak{u}}(\mu) = \mathfrak{Re}(z)\mathfrak{u}(G), & \text{if } \mathfrak{Re}(z) \geq 0, \\ -\mathfrak{Re}(z)\mathfrak{p}_{(-\mathfrak{u})}(\mu) = \mathfrak{Re}(z)\mathfrak{u}(F), & \text{if } \mathfrak{Re}(z) < 0. \end{cases}$$

by the above. The sets $\{\mathfrak{Re}(f_i)\}$ and $\{\mathfrak{Re}(g_k)\}$ are $\mathfrak{C}$-superharmonic by part (i), thus $\mu\left(\mathfrak{Re}(f_i)\right) = \mathfrak{Re}\left(f_i(x)\right)$ and $\mu\left(\mathfrak{Re}(g_k)\right) = \mathfrak{Re}\left(g_k(x)\right)$. This yields $\mu(F) \geq F(x) = \mathfrak{p}_{(-\mathfrak{u})}(\delta_x^1)$ and $\mu(G) \leq G(x) = \mathfrak{p}_{\mathfrak{u}}(\delta_x^1)$. Hence $\mathfrak{p}_{z\mathfrak{u}}(\mathfrak{u}) \leq \mathfrak{p}_{z\mathfrak{u}}(\delta_x^1)$ holds in any case. Now we repeat the argument for part (i) to verify that $\mu \prec_{\mathfrak{C}} \delta_x^\eta$ for $0 \neq \eta \in E$ implies that $\mathfrak{p}_{\mathfrak{u}}(\mathfrak{u}) \leq \mathfrak{p}_{\mathfrak{u}}(\delta_x^\eta)$. The set $\mathfrak{u}$ is therefore $\mathfrak{C}$-superharmonic. □

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