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Gelfand Integral of Multifunctions

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It has been proven by Cascales, Kadets and Rodríguez in J. Convex Analysis 18 (2011) 873–895 that each weak* scalarly integrable multifunction (with respect to a probability measure μ), whose values are compact convex subsets of a conjugate Banach space X^* and the family of support functions determined by X is order bounded in $L_1(\mu)$, is Gelfand integrable in the family of weakly compact convex subsets of X^*. A question has been posed whether a similar result holds true for multifunctions with weakly compact convex values. We prove that the answer is affirmative if X does not contain any isomorphic copy of l_1. If moreover the multifunction is compact valued, then it is Gelfand integrable in the family of compact convex subsets of X^*.

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Introduction.

Throughout this paper Ω is a non-empty set, Σ is a σ-algebra of its subsets and (Ω, Σ, μ) is a complete probability space. X is a Banach space with its dual X^*. If $A \subset X^*$, then $\overline{A}^*$ is the weak* closure of A and $\overline{A}^w$ is its weak closure. $\mathcal{P}(X)$ denotes the collection of all subsets of X and the closed unit ball of X is denoted by $B(X)$. $c(X)$ denotes the collection of all nonempty closed convex subsets of X, $cb(X)$ is the collection of all bounded members of $c(X)$, $cwk(X)$ denotes the family of all weakly compact elements of $cb(X)$ and $ck(X)$ is the collection of all compact members of $cb(X)$. Moreover, $cw^*k(X^*)$ denotes the family of all nonempty convex weak* compact subsets of X^*. For every $C \in c(X)$ the *support function of* C is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) = \sup\{\langle x^*, x\rangle :\ x \in C\}$, for each $x^* \in X^*$. If $C = \emptyset$, $s(\cdot, C)$ is identically $-\infty$. Otherwise $s(\cdot, C)$ does not take value $-\infty$.

A map $\Gamma : \Omega \to c(X)$ is called a *multifunction*. A multifunction Γ is said to be

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scalarly measurable if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable. Γ is said to be *scalarly integrable* if, for every $x^* \in X^*$, the function $s(x^*, \Gamma(\cdot))$ is *integrable.* We set $\mathcal{Z}_\Gamma := \{s(x^*, \Gamma) : \|x^*\| \le 1\}$.

A multifunction $\Gamma : \Omega \to c(X^*)$ is said to be *weak* scalarly measurable* if for every $x \in X$, the map $s(x, \Gamma(\cdot))$ is measurable. We set $\overleftarrow{\mathcal{Z}}_\Gamma := \{s(x, \Gamma) : \|x\| \le 1\}$. $\Gamma : \Omega \to c(X^*)$ is said to be *weak* scalarly integrable* if, for every $x \in X$, the function $s(x, \Gamma(\cdot))$ is *integrable.*

A function $f : \Omega \to X$ is called a *selection of* Γ if, for almost every $\omega \in \Omega$, one has $f(\omega) \in \Gamma(\omega)$.

A function $\gamma : \Omega \to X^*$ is said to be a *weak* quasi-selection* of a multifunction $\Gamma : \Omega \to c(X^*)$ if γ is weak* scalarly measurable and we have $\langle x, \gamma(\omega)\rangle \in \langle x, \Gamma(\omega)\rangle$ for μ-almost every $\omega \in \Omega$ (the exceptional sets depend on x). In particular, $s(x^*, \Gamma - \gamma) \ge 0$ a.e., for each $x^* \in X^*$. The collection of all weak* quasi-selections of Γ will be denoted by $\mathcal{W}^*\mathcal{QS}_\Gamma$.

If moreover $f(\omega) \in \Gamma(\omega)$, for almost every $\omega \in \Omega$, then γ called a *weak* selection* of Γ. We denote the set of all weak* selections of Γ by $\mathcal{W}^*\mathcal{S}_\Gamma$.

One can easily check that a weak* scalarly measurable $\gamma : \Omega \to X^*$ is a weak* quasi-selection of Γ if and only if for every $x \in X$ $\langle x, \gamma\rangle \le s(x, \Gamma)$ a.e.

A family $W \subset L_1(\mu)$ is uniformly integrable if W is bounded in $L_1(\mu)$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(A) < \delta$, then $\sup_{f \in W} \int_A |f|\, d\mu < \varepsilon$.

A map $M : \Sigma \to c(X)$ is additive, if $M(A \cup B) = \overline{M(A) + M(B)}$ for each pair of disjoint elements of Σ. An additive map $M : \Sigma \to c(X^*)$ is called a *weak* multimeasure* if $s(x, M(\cdot))$ is a measure, for every $x \in X$. If M is a point map, then we talk about measure and weak* measure, respectively. If $m : \Sigma \to X^*$ is a weak* measure such that $m(A) \in M(A)$, for every $A \in \Sigma$, then m is called a *weak*-selection* of M. $\mathcal{W}^*\mathcal{S}(M)$ will denote the set of all weak* countably additive selections of M. Costé [4] proved that $\mathcal{W}^*S(M) \neq \emptyset$.

Denote by $\mathcal{C}$ an arbitrary family of non-empty closed convex subsets of X^*. A weak* scalarly integrable multifunction $\Gamma : \Omega \to c(X^*)$ is said to be *Gelfand integrable* in $\mathcal{C}$, if for each set $A \in \Sigma$ there exists a set $M_\Gamma^G(A) \in \mathcal{C}$ such that

$$s(x, M_\Gamma^G(A)) = \int_A s(x, \Gamma)\, d\mu\,, \quad \text{for every } x \in X. \tag{1}$$

It is clear that M_Γ^G is a weak* multimeasure. $(G)\int_E \Gamma\, d\mu := M_\Gamma^G(A)$ is called the Gelfand integral of Γ on A. It follows from [4] that $\mathcal{W}^*S(M_\Gamma^G) \neq \emptyset$.

Denote by $\mathcal{D}$ an arbitrary family of closed convex bounded subsets of X. A scalarly integrable multifunction $\Gamma\colon \Omega \to \mathcal{D}$ is said to be *Pettis integrable* in $\mathcal{D}$, if for each $A \in \Sigma$ there exists a set $M_\Gamma(A) \in \mathcal{D}$ such that

$$s(x^*, M_\Gamma(A)) = \int_A s(x^*, \Gamma)\, d\mu \quad \text{for every } x^* \in X^*. \tag{2}$$

$M_\Gamma(A)$ is called the *Pettis integral of* Γ over A.

It has been proven in [2] and in [17, Theorem 6.7] that each weak* scalarly integrable $\Gamma : \Omega \to cw^*k(X^*)$ is Gelfand integrable in $cw^*k(X^*)$. In view of [17, Proposition 6.5] the Gelfand integral of Γ is a weak* multimeasure of σ-finite variation. It follows that each weak* measure being a selection of the Gelfand integral is of σ-finite variation. But then, it is a classical result (cf. [14, Theorem 11.1] or [15, Theorem 9.1]) that each such weak* measure has a weak* integrable density. Since, as we have already mentioned, $\mathcal{W}^*S(M_\Gamma^G) \neq \emptyset$, we have the following fact (proved also in [2, Theorem 4.5])

If $\Gamma : \Omega \to cw^*k(X^*)$ *is Gelfand integrable in* $cw^*k(X^*)$, *then* $\mathcal{W}^*\mathcal{QS}_\Gamma \neq \emptyset$.

1. The range of Gelfand's integral.

It has been proven in [2, Remark 6.9] that each $ck(X^*)$ valued scalarly integrable multifunction with order bounded $\overleftarrow{\mathcal{Z}}_\Gamma \subset L_1(\mu)$ is Gelfand integrable in $cwk(X^*)$ (The result has been generalized in [17, Theorem 6.10] to $ck(X^*)$ valued multifunction with uniformly integrable $\overleftarrow{\mathcal{Z}}_\Gamma$). A question has been posed in [2] whether the result can be extended to $cwk(X^*)$ valued multifunctions. If $X = l_1$, then the answer is negative (see [2]). But if X does not contain any isomorphic copy of l_1, then the answer turns out to be affirmative (Theorem 1.4) in general. To achieve that result we apply [13, Theorem 3]:

Each X^**-valued weak***-scalarly measurable (integrable) function is scalarly measurable (Pettis integrable) provided* $l_1 \not\subseteq X$ *and* X *is separable.*

It is perhaps worth to mention that [13, Theorem 3] is a consequence of the celebrated result of Odell and Rosenthal [18] saying that if X is separable and $l_1 \not\subseteq X$, then X is weak* sequentially dense in X^{**}.

Lemma 1.1. *Let* $\Gamma : \Omega \to cwk(X^*)$ *be weak** *scalarly measurable multifunction. If* $l_1 \not\subseteq X$ *and* X *is separable, then* Γ *is scalarly measurable.*

Proof. Let us observe that due to Castaing representation (see [3, Theorem III.7] or [20, Proposition 7]) there exists a sequence of weak* scalarly measurable selections $\{f_n : \Omega \to X^* : n \in \mathbb{N}\}$ of Γ such that $\Gamma(\omega) = \overline{\{f_n(\omega) : n \in \mathbb{N}\}}^*$ for every $\omega \in \Omega$. Since $l_1 \not\subseteq X$ and X is separable all the functions f_n are scalarly measurable. Moreover, $\Gamma(\omega) = \overline{\{f_n(\omega) : n \in \mathbb{N}\}}^w$ for every $\omega \in \Omega$, because $\{f_n(\omega) : n \in \mathbb{N}\}$ is weakly relatively compact. Thus, we have for each $x^{**} \in X^{**}$ $s(x^{**}, \Gamma) = \sup_n \langle x^{**}, f_n \rangle$ everywhere and so Γ is scalarly measurable. $\square$

Theorem 1.2. *Let* $\Gamma : \Omega \to cwk(X^*)$ *be a weak** *scalarly integrable multifunction. If* $l_1 \not\subseteq X$ *and* X *is separable, then* Γ *is Pettis integrable in* $cwk(X^*)$.

Proof. Since X is separable, it follows from the Kuratowski and Ryll-Nardzewski selection theorem ([12]) that $\mathcal{W}^*S_\Gamma \neq \emptyset$. Hence according to [1, Theorem 4.2] (or [16, Theorem 4.8]) in order to prove Pettis integrability of Γ, it suffices to show that each scalarly measurable selection of Γ is Pettis integrable. But if $f \in \mathcal{W}^*S_\Gamma$, then f is Gelfand integrable and since $l_1 \not\subseteq X$ and X is separable, it is Pettis integrable ([13]). $\square$

In order to obtain a generalization of Theorem 1.2 to non-separable Banach spaces, we need a simple fact that is certainly known.

Lemma 1.3. *A set $W \subset X^*$ is weakly relatively compact if and only if for each separable subspace Y of X the range of W under the canonical projection of X^* onto Y^* is weakly relatively compact.*

Proof. It is enough to show that if all ranges of W in spaces Y^* are weakly relatively compact, then W is weakly relatively compact. In order to prove the weak relative compactness of W in X^* it suffices to show that for an arbitrary sequence $\{w_i \colon i \in \mathbb{N}\} \subset W$ there exist vectors $v_n \in \mathrm{conv}\{w_i \colon i \geq n\}$ such that the sequence $\langle v_n \rangle_n$ is norm convergent (cf. [11, §24 3(8)]). Let $\{w_i \colon i \in \mathbb{N}\} \subset W$ be arbitrary. According to [5, Lemma VI.8.8] there exists a separable subspace $Y \subset X$ such that the closed linear space Z generated by $\{w_i \colon i \in \mathbb{N}\} \subset W$ is isometrically embedded in Y^*. Let $j : Y \to X$ be the canonical embedding. By the assumption $j^*(W) \supset j^*(\{w_i \colon i \in \mathbb{N}\})$ is weakly relatively compact and so there exist vectors $v_n \in \mathrm{conv}\{w_i \colon i \geq n\}$ such that the sequence $\langle v_n \rangle_n$ is norm convergent in Y^*. But the norm convergence in Y^* coincides on $Z = j^*(Z)$ with the norm convergence in X^*. $\square$

Theorem 1.4. *Let $\Gamma : \Omega \to cwk(X^*)$ be a weak* scalarly integrable multifunction. If $l_1 \not\subseteq X$, then Γ is Gelfand integrable in $cwk(X^*)$.*

Proof. According to Lemma 1.3, a set $W \subset X^*$ is weakly relatively compact provided for each separable subspace Y of X the range of W under canonical projection of X^* onto Y^* is weakly relatively compact.

Let $j : Y \to X$ be the canonical embedding of a separable Banach space Y into X and $M : \Sigma \to cw^*k(X^*)$ be the Gelfand integral of Γ. We have then

$$s(x, M(A)) = \int_A s(x, \Gamma)\, d\mu \quad \text{for every } x \in X \text{ and } A \in \Sigma.$$

In particular, if $x = j(y)$, then

$$\begin{aligned} s(j(y), M(A)) &= s(y, j^*M(A)) \\ \text{and} \quad s(j(y), \Gamma(\omega)) &= s(y, j^*\Gamma(\omega)) \quad \text{for every } \omega \in \Omega. \end{aligned}$$

Hence, $j^*\Gamma$ is weak* scalarly integrable and

$$s(y, j^*M(A)) = \int_A s(y, j^*\Gamma(\omega))\, d\mu.$$

Thus, we may apply Theorem 1.2 obtaining the weak compactness of every set $j^*M(A)$. Thus, $M(A)$ is weakly relatively compact. We should prove yet that it is weakly compact. But $M(A) \in cw^*k(X^*)$ and so

$$M(A) \subset \overline{M(A)}^{w} \subset \overline{M(A)}^{*} = M(A)\,.$$

This completes the proof of the theorem. □

As a corollary from the above theorem we obtain the following result

Proposition 1.5. *Let $\Gamma : \Omega \to cwk(X)$ be scalarly integrable. If $l_1 \not\subseteq X^*$, then Γ is Gelfand integrable in $cwk(X^{**})$.*

Proof. Γ can be treated as a weak* scalarly integrable X^{**}-valued multifunction. Then we apply Theorem 1.4. □

Remark 1.6. Assume that $\operatorname{card} T$ is real measurable. Then $l_1(T)$ is not measure compact and does not have PIP (see [7, page 575]). In particular, there exists a measure space (Ω, Σ, μ) and a scalarly measurable function $f : \Omega \to l_1(T)$ that is not scalarly equivalent to any strongly measurable function (see [6, Proposition 5.4]). Hence, there is a set $E \in \Sigma$ of positive measure such that $f|_E$ is scalarly bounded but not Pettis integrable. This proves that in Theorem 1.2, if X is non-separable (contrary to the separable case), then even uniform integrability of $\mathcal{Z}_\Gamma$ does not guarantee Pettis integrablity of Γ.

To the best of my knowledge no ZFC example of a Banach space X not containing any isomorphic copy of l_1 and such that X^* does not possesses PIP is known. But it is consistent with ZFC to assume that each such an X^* has Lebesgue PIP ([9, Proposition 3C]). □

The rest of the paper is devoted to compact valued Gelfand integrable multifunctions.

Definition 1.7. Following [8] we say that a non-empty set $H \subset X^*$ is an L-set if each $\sigma(X, X^*)$-convergent sequence $\langle x_n \rangle$ is uniformly convergent on H. □

Let us mention that L-sets coincide with $\tau(X^*, X)$ relatively compact sets (see Grothendieck [10, p. 134]). L-sets are also called sometimes X-limited sets.

Lemma 1.8. *If H is a bounded subset of X^*, then $s(\cdot, H)$ is weakly sequentially continuous on X if and only if H is an L-set in X^*.*

Proof. If H is an L-set, the weak sequential continuity of $s(\cdot, H)$ is immediate. Assume now that $s(\cdot, H)$ is weakly sequentially continuous and take $x_n \to 0$ weakly. If $x^* \in H$, then

$$-s(-x_n, H) \le x^*(x_n) \le s(x_n, H) \quad \text{for every } n \in \mathbb{N}.$$

It follows that $x_n \to 0$ uniformly on H. □

Proposition 1.9. *Let $\Gamma : \Omega \to cw^*k(X^*)$ be weak* scalarly integrable and let $T_\Gamma : X \to L_1(\mu)$ be defined by $T_\Gamma(x) = s(x,\Gamma)$. Then T_Γ is sequentially weakly-weakly continuous if and only if Γ is Gelfand integrable in the collection of L-sets.*

Proof. For arbitrary $x \in X$ and $E \in \Sigma$ we have

$$s(x, M_\Gamma^G(E)) = \int_E s(x,\Gamma)\, d\mu = \langle \chi_E, T_\Gamma(x)\rangle. \tag{3}$$

According to Lemma 1.8 $M_\Gamma^G(E)$ is an L-set if and only if $s(\cdot, M_\Gamma^G(E))$ is weakly sequentially continuous. It follows from (3) that $s(\cdot, M_\Gamma^G(E))$ is weakly sequentially continuous if and only if $\langle \chi_E, T_\Gamma(\cdot)\rangle$ is weakly sequentially continuous. $\square$

Theorem 1.10. *Let $\Gamma : \Omega \to ck(X^*)$ be weak* scalarly integrable and $\overleftarrow{\mathcal{Z}}_\Gamma$ be uniformly integrable. Then Γ is Gelfand integrable in the L-sets. If the zero function is a weak* quasi-selection of Γ, then $\bigcup M_\Gamma^G(\Sigma) := \bigcup_{E\in\Sigma} M_\Gamma^G(E)$ is an L-set.*

Proof. Fix $E \in \Sigma$ and let $\langle x_n\rangle$ be weakly convergent to zero. Since each $\Gamma(\omega)$ is compact the sequence $\langle x_n\rangle$ is on $\Gamma(\omega)$ uniformly convergent to zero. By the assumption the sequence $\langle s(x_n,\Gamma)\rangle$ is uniformly integrable, and so, applying the Vitali convergence theorem we obtain

$$s(x_n, M_\Gamma^G(E)) = \int_E s(x_n,\Gamma)\, d\mu \to 0.$$

According to Lemma 1.8 the set $M_\Gamma^G(E)$ is an L-set.

Now let zero function be a weak* quasi-selection of Γ. If $E \in \Sigma$, then $M_\Gamma^G(E) \subset M_\Gamma^G(\Omega)$, because $s(x,\Gamma) \geq 0$ a.e., for every $x \in X$. By the first part of the proof $M_\Gamma^G(\Omega)$ is an L-set. It follows that $\bigcup M_\Gamma^G(\Sigma) \subseteq M_\Gamma^G(\Omega)$ is an L-set. $\square$

I do not know if a similar result for Gelfand integrable functions has been announced somewhere.

The following result is a strengthening of [2, Theorem 6.8] for Banach spaces not containing l_1. In its proof we apply the result of Emmanuele [8]: *if $l_1 \not\subseteq X$, then L-sets are norm relatively compact.* Let us mention that Emmanuele's result follows from Odell's observation that every Dunford-Pettis operator defined on a Banach space not containing l_1 is compact (see [19, p. 377]).

Theorem 1.11. *Let $\Gamma : \Omega \to ck(X^*)$ be weak* scalarly integrable and $\overleftarrow{\mathcal{Z}}_\Gamma$ be uniformly integrable. If X does not contain any isomorphic copy of l_1, then Γ is Gelfand integrable in $ck(X^*)$ and $\bigcup M_\Gamma^G(\Sigma)$ is norm relatively compact.*

Proof. If g is a weak* quasi-selection of Γ and $G\colon = \Gamma - g$, then $(G)\int_E \Gamma\, d\mu \subseteq (G)\int_E G\, d\mu + (G)\int_E g\, d\mu$, for each $E \in \Sigma$. In particular

$$\bigcup M_\Gamma^G(\Sigma) \subseteq \bigcup M_G^G(\Sigma) + \nu_g(\Sigma) \tag{4}$$

where $\nu_g(\Sigma) := \{(G)\int_E g\,d\mu\colon E \in \Sigma\}$. But $l_1 \not\subseteq X$ and so the range of each Gelfand integral is norm relatively compact (cf. [13, Lemma 2]). It follows that $\overleftarrow{\mathcal{Z}}_g$ is uniformly integrable and consequently also $\overleftarrow{\mathcal{Z}}_G$ is uniformly integrable. We may apply now Theorem 1.10 and the result of Emmanuele [8] to obtain the relative compactness of $\bigcup M_G^G(\Sigma)$. The inclusion (4) yields Gelfand integrability of Γ in $ck(X^*)$ and the relative compactness of $\bigcup M_\Gamma^G(\Sigma)$. $\square$

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