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Stability Results of Diameter Two Properties

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A Banach space has the diameter two property if every (nonempty) weakly open set of its unit ball has diameter two. We prove that this property is stable under finite sums, whenever an absolute norm is considered in the product space, improving some previous results. Recently Abrahamsen, Lima and Nygaard defined the so-called *strong diameter two property*, i.e. every convex combination of slices in the unit ball has diameter two.

We show that the strong diameter two property is never stable under (non-trivial) ℓ_p-sums. As a consequence, both diameter two properties are not equivalent. We also prove that any Banach space whose centralizer is infinite-dimensional has the strong diameter two property, solving an open problem. This result can be applied to infinite-dimensional C^*-algebras and L_1-preduals, for instance.

Keywords: Banach space, weakly open set, slice, absolute norm, diameter two property, Radon-Nikodým property

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1. Introduction

The Radon-Nikodým property (RNP) implies the abundance of slices of the unit ball with diameter arbitrarily small. The same parallel phenomena happens for

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the convex point of continuity property (CPCP) (resp. strong regularity (SR)) and weakly open sets of the unit ball (resp. convex combinations of relatively weakly open subsets) (see [12], [13] and [14] for background). These three classical properties are purely isomorphic. But some classical Banach spaces do not have the above properties. In fact every weakly open set relatively to the unit ball of $C(K)$ has diameter two whenever K is an infinite compact Hausdorff topological space. Clearly in such cases the maximum possible value of the diameter is 2. So there are three isometric properties that corresponds to the absence of the above mentioned isomorphic properties in the strongest possible way. The first two properties have been studied by a number of authors (see for instance [18], [9] and [7]). We recall these notions.

A Banach space X has the *slice diameter two property*, respectively *diameter two property, strong diameter two property*, if every slice, respectively nonempty relatively weakly open subset, nonempty convex combination of relatively weakly open subsets, of the unit ball has diameter 2. Let us remark that the above properties do not depend on the scalar field.

It is clear that diameter two property implies slice diameter two property. Also strong diameter two property implies diameter two property. This last assertion is a consequence of [13, Lemma 2.1], where it is proved that every nonempty relatively weakly open subset of the unit ball in a Banach space has to contain some convex combination of slices.

A lot of attention has been devoted to the slice diameter two property and the diameter two property since several classical Banach spaces failing the Radon-Nikodým property have them (see for instance [1, 3, 4, 6, 7, 8, 9, 17, 18, 20]). However, there are only a few stability results of those properties. The aim of this note is to study the stability of diameter two properties for product spaces. So we assume that two Banach spaces have some diameter two property and pose the following question: Does the product of both spaces have the same diameter two property for some norm?

We prove in Theorem 2.4 that every absolute norm in a product space has diameter two property, whenever the factors satisfy this property, which improves the result obtained in [1] for p-norms. However we prove that the ℓ_p-sum of two nontrivial Banach spaces never has the strong diameter two property ($1 < p < +\infty$) (Theorem 3.2). Since the diameter two property is stable under any ℓ_p-sums then both diameter two properties are not equivalent. This solves an open problem posed in [1]. However we still do not know if the slice diameter two property implies the diameter two property.

We also show that for any $p \geq 1$ the space $L_p(\mu, X)$ satisfies the diameter two property for every finite measure μ whenever X also has the same property and $L_p(\mu)$ is nonzero (Proposition 2.6).

Finally, we prove that every Banach space whose centralizer is infinite-dimensional satisfies the strong diameter two property, solving an open problem posed in [1]. As a consequence we deduce that infinite-dimensional C^*-algebras and L_1-preduals satisfy the strong diameter two property (Corollary 3.6).

We would like to point out some other problems in relation with the three diameter two properties. It is an usual problem to change the isometric character of a property in Banach spaces by an isomorphic behavior: What Banach spaces admit an equivalent norm satisfying one of the diameter two properties? More precisely, is it true that every Banach space failing RNP, can be equivalently renormed to satisfy the slice diameter two property? (see [19] and [21]). It is known that not every Banach space failing the RNP admits an equivalent norm satisfying the diameter two property (see [1, Introduction]).

2. Stability of diameter two property

Firstly we introduce some notation and recall some basic notions. Throughout the paper, X, Y and Z denote **real** Banach spaces. For a set $B \subset X$, $\operatorname{co} B$ (respect. $\operatorname{aco} B$) is the *convex hull* (respect. the *absolutely convex hull*) of B. As usual, X^* denotes the topological dual of X; B_X and S_X stand for the closed unit ball and the unit sphere of the Banach space X, respectively. If $C \subset X$ is a (non-empty) bounded set, $x^* \in X^* \backslash \{0\}$ and $\alpha > 0$ the subset $S(C, x^*, \alpha)$ given by

$$S(C, x^*, \alpha) := \{x \in C : \ x^*(x) > \sup x^*(C) - \alpha\},$$

is called a *slice* of C. Notice that we can always assume that $x^* \in S_{X^*}$.

For a bounded set $C \subset X$, an element $x_0 \in C$ is *strongly exposed* if there is some $x_0^* \in X^*$ such that $x_0^*(x_0) = \sup x_0^*(C)$ and $\inf_{\alpha>0} \operatorname{diam} S(C, x_0^*, \alpha) = 0$. In such case we will say that the functional x_0^* *strongly exposes* the subset C at x_0.

As it looks like at a first glance, the slice diameter two property is an isometric property. This is a consequence of the next lemma, whose proof is standard and can be found, for instance in [2, p. 373].

Lemma 2.1. *Let X be a Banach space and $x_0 \in X \backslash B_X$. Let Y be the space X (as a subset), endowed with the norm whose closed unit ball is given by $B := \operatorname{aco}(B_X \cup \{x_0\})$. Then x_0 is a strongly exposed point of B. Hence Y does not have the slice diameter two property.*

Now we provide the following example that shows the lack of stability of the diameter two property under products. Hence not every norm on $Z = X \times Y$ that induces the product topology preserves the diameter two property, even if we assume that the restriction of this to any space also enjoys the same property.

Example 2.2. Let B be the closed unit ball of the space $c_0 \oplus_1 c_0$ and $C = \operatorname{co}(B \cup \{(e_1, e_1)\} \cup \{(-e_1, -e_1)\})$, where e_1 is the first vector of the usual (Schauder) basis of c_0. It is clear that C is the unit ball of some norm on the space $Y = c_0 \times c_0$. It is immediate that this norm induces the product topology on Y. Also it holds that the restriction of this norm to any of the factors is the usual norm of c_0

and so, both have the diameter two property. However, in view of Lemma 2.1, the space Y, endowed with this norm, fails the diameter two property.

In spite of the previous example, we will prove a stability result of the diameter two property for products under certain assumption. To this purpose we will use the elementary fact that every nonempty relatively weakly open subset of the closed unit ball of any infinite-dimensional Banach space always contains an element in the unit sphere. Also we need to recall the following notion:

Definition 2.3. If $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are Banach spaces, and $Z = X \times Y$ is the product space, a norm $\|\cdot\|$ in Z is said to be *absolute* if there is a function $f : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ such that

$$\|(x, y)\|_f = f(\|x\|_X, \|y\|_Y), \quad \forall x \in X,\ y \in Y\,. \tag{1}$$

The absolute norm is *normalized* if $f(1, 0) = 1 = f(0, 1)$.

It is immediate to check that in case that the equality (1) gives a norm in Z, the function f can be extended to a norm $|\ |$ on $\mathbb{R}^2$ satisfying $|(r, s)| = f(|r|, |s|)$ for every pair of real numbers (r, s). We also recall that the norm $|\ |_f$ is absolute on $\mathbb{R}^2$ if, and only if, the function f is increasing in each variable, that is,

$$0 \le r \le s,\ 0 \le t \le u \quad \Rightarrow \quad f(r, t) \le f(s, u)$$

(see [11, Lemma 21.2]).

Clearly the usual norm of the ℓ_p-sum of two Banach spaces is an absolute norm for every $1 \le p \le \infty$.

Theorem 2.4. *Let X and Y be Banach spaces with the diameter two property and $Z := X \times Y$. If $\|\cdot\|$ is an absolute norm on Z, then $(Z, \|\cdot\|)$ has the diameter two property. The analogous statement for the slice diameter two property also holds.*

Proof. We begin by proving the first statement. So assume that X and Y have the diameter two property.

Let W be a weakly open subset of Z such that $W \cap B_Z \neq \varnothing$ and $\varepsilon > 0$. By assumption X and Y are infinite-dimensional spaces, so Z is also infinite-dimensional. So we can choose $(x_0, y_0) \in W \cap S_Z$. Hence there are weakly open subsets U and V of X and Y, respectively, such that $x_0 \in U$, $y_0 \in V$ and $(U, V) \subset W$. As a consequence, $U \cap \|x_0\|B_X$ is a nonempty relatively weakly open subset of $\|x_0\|B_X$ since $x_0 \in U \cap \|x_0\|B_X$. The same argument can be applied to $V \cap \|y_0\|B_Y$ and the element y_0.

If $(x, y) \in (U \cap \|x_0\|B_X, V \cap \|y_0\|B_Y)$, it is clear that $(x, y) \in (U, V) \subset W$ and $\|x\| \le \|x_0\|$, $\|y\| \le \|y_0\|$. As the norm $\|\cdot\|$ is absolute, we deduce that $\|(x, y)\| \le \|(x_0, y_0)\| = 1$. As a consequence we have that

$$(U \cap \|x_0\|B_X, V \cap \|y_0\|B_Y) \subset W \cap B_Z. \tag{2}$$

Since we are assuming that X has the diameter two property, every nonempty weakly open set of rB_X has diameter $2r$ for every nonnegative real number r. By assumption we can choose $x_1,\ x_2 \in U \cap \|x_0\|B_X,\ y_1,\ y_2 \in V \cap \|y_0\|B_Y$ such that $\|x_1 - x_2\| \geq (2-\varepsilon)\|x_0\|$ and $\|y_1 - y_2\| \geq (2-\varepsilon)\|y_0\|$. Since the norm of Z is absolute we have

$$\|(x_1 - x_2, y_1 - y_2)\| \geq (2-\varepsilon)\|(x_0, y_0)\| = 2 - \varepsilon.$$

Now in view of (2) we know that $(x_1, y_1),\ (x_2, y_2) \in W \cap B_Z$ and so, $\operatorname{diam}(W \cap B_Z) \geq \|(x_1 - x_2, y_1 - y_2)\| \geq 2 - \varepsilon$. Since ε is any positive real number, we conclude that $\operatorname{diam}(W \cap B_Z) = 2$. We proved that the previous condition is satisfied for any weakly open set $W \subset Z$ satisfying $W \cap B_Z \neq \varnothing$, and so Z has the diameter two property.

Now assume that X and Y have the slice diameter two property. Fix any $\varepsilon > 0$. Let S be a slice of B_Z, hence there is an element $z^* \in Z^*$ and $\alpha > 0$ such that $S = S(B_Z, z^*, \alpha)$. In order to prove the slice diameter two property, changing the functional and the positive real number α, if needed, we can assume that $z^* \in S_{Z^*}$. Also in view of Bishop-Phelps Theorem, we can assume that z^* attains its norm at an element $(x_0, y_0) \in S_Z$. Since $Z = X \times Y$, z^* can be identified with a pair $(x^*, y^*) \in X^* \times Y^*$.

Clearly we have that $(x_0, y_0) \in S(B_Z, z^*, \alpha)$. Since the norm of Z is absolute and $1 = z^*(x_0, y_0) = x^*(x_0) + y^*(y_0)$ it is also clear that

$$(x, y) \in S(B_Z, z^*, \alpha) \quad \forall x \in S_1, y \in S_2, \tag{3}$$

where S_1 and S_2 are the slices of $\|x_0\|B_X$ and $\|y_0\|B_Y$, respectively, given by

$$S_1 = \left\{x \in \|x_0\|B_X : x^*(x) > x^*(x_0) - \frac{\alpha}{2}\right\}$$

and

$$S_2 = \left\{y \in \|y_0\|B_Y : y^*(y) > y^*(y_0) - \frac{\alpha}{2}\right\}.$$

By assumption we have that $\operatorname{diam} S_1 = 2\|x_0\|$ and $\operatorname{diam} S_2 = 2\|y_0\|$. Now we can finish the proof by using the same argument as the one used for the diameter two property, but replacing condition (2) by (3). $\square$

The above result covers, for example, the ℓ_p-sum of two spaces $(1 \leq p \leq \infty)$, obtained in [1, Theorem 3.2]. Indeed, for the maximum norm, it suffices to assume that one of the spaces involved satisfies the diameter two property in order to get the conclusion (see [6, Lemma 2.2]). Indeed the argument used above proves this fact. So the converse of Theorem 2.4 for the diameter two property is not true even for the ℓ_∞-sum. We do not know if the ℓ_∞-sum is the only exception where the converse does not hold. At least for the rest of the ℓ_p-sums we have a converse.

Proposition 2.5. *Let X and Y be Banach spaces and assume that X is infinite-dimensional. Let f be an normalized absolute norm on $\mathbb{R}^2$ satisfying that $(1,0)$ is an extreme point of the unit ball of $(\mathbb{R}^2, f)$. Suppose that $Z := X \times Y$, endowed with $\| \|_f$ has the diameter two property. Then X has the diameter two property. The analogous statement also holds for the slice diameter two property.*

As a consequence, if a normalized absolute norm f on $\mathbb{R}^2$ satisfies that $(0,1)$ and $(1,0)$ are extreme points in the unit ball of $(\mathbb{R}^2, f)$ and X and Y are infinite-dimensional spaces, then the diameter 2 property is inherited from Z to X and Y. The slice diameter two property of Z also implies that X and Y have the same property.

Proof. Since we are assuming that f is an absolute norm on $\mathbb{R}^2$ and $(1,0)$ is an extreme point of the unit ball of $(\mathbb{R}^2, f)$, then this element is an strongly exposed point of the unit ball. Indeed the functional $e_1^* \in (\mathbb{R}^2, f)^*$ given by

$$(x, y) \mapsto x$$

belongs to the unit sphere of the dual space and strongly exposes the unit ball of $(\mathbb{R}^2, f)$ at $(1,0)$. That is, for every $\varepsilon > 0$ there is $\delta > 0$ satisfying

$$(r, s) \in \mathbb{R}^2, \quad f(r, s) \leq 1, \ 1 - \delta < r \quad \Rightarrow \quad |s| < \varepsilon \,. \tag{4}$$

We prove the first statement by contradiction. So assume that U is a nonempty weakly open subset relative to B_X with diameter less than 2. Let us choose $0 < \varepsilon_0 < \frac{2-\operatorname{diam}(U)}{2}$ and let δ_0 be the positive real number satisfying condition (4) for $\varepsilon = \varepsilon_0$.

By assumption U contains an element $x_0 \in S_X$. Let us choose $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 1$. Then there is a finite set F, a subset of functionals $\{x_i^* : i \in F\}$ and a subset of real numbers $\{t_i : i \in F\}$ such that

$$\varnothing \neq W = \{x \in X : x_0^*(x) > 1 - \delta_0\} \cap \{x \in B_X : x_i^*(x) > t_i, \forall i \in F\} \subset U \,.$$

Consider now the subset V given by

$$\begin{aligned} V &:= \{(x, y) \in B_Z : x \in W\} \\ &= \{(x, y) \in B_Z : x_0^*(x) > 1 - \delta_0 \text{ and } x_i^*(x) > t_i, \forall i \in F\} \,. \end{aligned}$$

It is clear that V is a non-empty weakly open subset in B_Z.

Every element $(x, y) \in V$ satisfies

$$1 - \delta_0 < x_0^*(x) \leq \|x\| = \|(x, 0)\| \leq \|(x, y)\| = f(\|x\|, \|y\|) \leq 1$$

and in view of (4) we obtain that

$$\|y\| < \varepsilon_0, \quad \forall (x, y) \in V \,.$$

As a consequence, for any elements (x, y) and $(x', y') \in V$ we have that

$$\begin{aligned}\|(x,y)-(x',y')\| &\le \|(x-x',0)\|+\|(0,y-y')\| \\ &= \|x-x'\|+\|y-y'\| \le \operatorname{diam}(U)+2\varepsilon_0 < 2\,.\end{aligned}$$

We proved that $\operatorname{diam} V \le \operatorname{diam}(U)+2\varepsilon_0 < 2$, which contradicts the assumption. Hence X has the diameter two property.

Now assume that Z has the slice diameter two property, the norm of Z is a normalized absolute norm associated to f and $(1, 0)$ is an extreme point of the unit ball of $(\mathbb{R}^2, f)$. We will show that X also has the slice diameter two property. Let $S = S(B_X, x^*, \alpha)$ be a slice of the unit ball of X. We can assume that $x^* \in S_{X^*}$. Fix $\varepsilon > 0$. Assume that $0 < \delta < \frac{\alpha}{2}$ satisfies (4). By Bishop-Phelps Theorem there is $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ such that

$$x_0^*(x_0) = 1 \quad \text{and} \quad \|x_0^* - x^*\| < \frac{\alpha}{2}.$$

Since $0 < \delta < \frac{\alpha}{2}$ it is immediate that we have

$$S\left(B_X, x_0^*, \delta\right) \subset S\left(B_X, x_0^*, \frac{\alpha}{2}\right) \subset S\left(B_X, x^*, \alpha\right). \tag{5}$$

Now for every $(x, y) \in S(B_Z, (x_0^*, 0), \delta)$, by using that Z has the absolute norm associated to f and $f(1, 0) = 1$, we have that

$$\|x\| \ge x_0^*(x) > \sup(x_0^*, 0)(B_Z) - \delta = \sup x_0^*(B_X) - \delta = 1 - \delta\,.$$

In view of (4) we have that $\|y\| < \varepsilon$ and so

$$\|(x,y)-(x,0)\|_f = \|(0,y)\|_f = \|y\| f(0,1) = \|y\| < \varepsilon\,.$$

Since we also know that $x \in S(B_X, x_0^*, \delta)$, by using (5), the above argument, the fact that the norm of Z is absolute, the equation $f(1, 0) = 1$, and the slice diameter two property of Z we deduce that

$$\operatorname{diam} S(B_X, x^*, \alpha) \ge \operatorname{diam} S(B_X, x_0^*, \delta) \ge \operatorname{diam} S(B_Z, (x_0^*, 0), \delta) - 2\varepsilon = 2 - 2\varepsilon\,.$$

We proved that X has the slice diameter two property as wanted.

The two last statements of the proposition clearly follows from the first part. □

Let us notice that the above result can be applied, for instance, to any ℓ_p-sum for $1 \le p < \infty$. However, the conclusion fails for the ℓ_∞-sum of Banach spaces. For this norm the assumption is satisfied whenever one of the spaces has the diameter two property [6, Lemma 2.2].

In view of Theorem 2.4, it seems natural to study if the diameter two property is satisfied by the space $L_p(\mu, X)$ whenever X also has this property. Let us recall that for a finite measure μ and a Banach space X, $L_p(\mu, X)$ stands for the Banach space of p-Bochner integrable functions. In the case $p = \infty$, it was proved in [6, Theorem 2.13 i)] that $L_\infty(\mu, X)$ has diameter two property if, and only if, either $L_\infty(\mu)$ is infinite-dimensional or X has diameter two property.

Proposition 2.6. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. If X is a Banach space with the diameter two property, then $L_p(\mu, X)$ also has the diameter two property for every $1 \leq p < \infty$ whenever $L_p(\mu) \neq \{0\}$.*

Proof. Let W be a weakly open set in $L_p(\mu, X)$ such that $W \cap B_{L_p(\mu,X)} \neq \varnothing$ and fix $\varepsilon > 0$.

By the assumptions, the space $L_p(\mu, X)$ is infinite-dimensional and so there is an element $f_0 \in W \cap S_{L_p(\mu,X)}$. We can assume that f_0 is a simple function. So f_0 can be written as $f_0 = \sum_{i=1}^n x_i \chi_{A_i}$ for some $n \in \mathbb{N}$, $x_i \in X \setminus \{0\}$ and pairwise disjoint measurable subsets A_i $(1 \leq i \leq n)$. Since $\|f_0\| = 1$, we have that $\sum_{i=1}^n \|x_i\|^p \mu(A_i) = 1$.

Since the weak topology is a linear topology, there are weakly open subsets W_i in $L_p(\mu, X)$ $(1 \leq i \leq n)$ satisfying

$$x_i \chi_{A_i} \in W_i, \quad \forall 1 \leq i \leq n \text{ and } \sum_{i=1}^n W_i \subset W \,. \tag{6}$$

Since for every subset $A \in \mathcal{A}$, the mapping from X into $L_p(\mu, X)$ given by $x \mapsto x\chi_A$ is linear and continuous, it is also w-continuous. Hence for every $1 \leq i \leq n$ there is a weakly open subsets V_i in X such that $x_i \in V_i$ and also satisfies

$$x\chi_{A_i} \in W_i \quad \text{whenever } x \in V_i. \tag{7}$$

Since we are assuming that X has the diameter 2 property, we know that $\operatorname{diam}(V_i \cap \|x_i\| B_X) = 2\|x_i\|$ for each i. Hence for each $1 \leq i \leq n$ there are elements y_i, z_i in $V_i \cap \|x_i\| B_X$ such that $\|y_i - z_i\| \geq (2 - \varepsilon)\|x_i\|$.

In view of (7) and (6), we have that the mappings g and h given by

$$g = \sum_{i=1}^n y_i \chi_{A_i}, \qquad h = \sum_{i=1}^n z_i \chi_{A_i}$$

belong to W. Also it is clear that $\|g\|, \|h\| \leq \|f_0\| = 1$. Hence we obtain that

$$\begin{aligned}
\operatorname{diam}(W \cap B_{L_p(\mu,X)}) \geq \|g - h\| &= \left(\sum_{i=1}^n \|y_i - z_i\|^p \chi_{A_i} \right)^{1/p} \\
&\geq (2 - \varepsilon) \left(\sum_{i=1}^n \|x_i\|^p \chi_{A_i} \right)^{1/p} \\
&\geq (2 - \varepsilon)\|f_0\| = 2 - \varepsilon \,.
\end{aligned}$$

We conclude that $\operatorname{diam}(W \cap B_{L_p(\mu,X)}) = 2$ as we wanted. $\square$

3. Strong diameter two property

We will show that the strong diameter two property does not have the same stability properties that the slice diameter two property. As a consequence of this difference we will obtain that both properties are not equivalent.

First we check that the strong diameter two property is stable under ℓ_1-sums. The proof of this fact is essentially contained in [6, Lemma 2.1] and stated in [1, Theorem 2.7]. In fact for non-trivial ℓ_1-sums, we obtain a characterization and include the whole proof for the sake of completeness.

Proposition 3.1. *Let X and Y be (non-trivial) Banach spaces. Then the space $X \oplus_1 Y$ has the strong diameter two property if, and only if, X and Y has the same property.*

Proof. We write $Z = X \oplus_1 Y$ and so $Z^* = X^* \oplus_\infty Y^*$.

First we prove the necessary condition. So assume that Z has the strong diameter two property and we will show that X also has it.

In order to prove that fact, we will check that every slice of B_X produces a slice of B_Z close to the first one (consider X embedded into Z in a natural way). Let $S = S(B_X, x^*, \alpha)$ be a slice of B_X and fix $0 < \eta < \alpha$. We can assume that $x^* \in S_{X^*}$. If $(x, y) \in S(B_Z, (x^*, 0), \eta)$, then $x^*(x) > 1 - \eta > 1 - \alpha$ and so $\|x\| > 1 - \eta$. Thus $\|y\| < \eta$. As a consequence, $\|(x, y) - (x, 0)\| < \eta$. We checked that

$$S(B_Z, (x^*, 0), \eta) \subset S(B_X, x^*, \alpha) \times \eta B_Y. \tag{8}$$

By the previous inclusion, if C is a convex combination of slices of B_X, for $\eta > 0$ small enough we deduce that

$$\inf\{\operatorname{diam}(U) : U \text{ is a convex combination of slices of } B_Z\} \leq \operatorname{diam} C + 2\eta. \tag{9}$$

By assumption Z has the strong diameter two property, so we obtain that $\operatorname{diam} C = 2$. Hence X has the strong diameter two property.

In order to prove the sufficient condition, let us notice that for every $(x^*, y^*) \in S_{Z^*}$ and $\alpha > 0$ we clearly have that

$$x^* \in S_{X^*} \quad \Rightarrow \quad S(B_X, x^*, \alpha) \times \{0\} \subset S(B_Z, (x^*, y^*), \alpha) \, . \tag{10}$$

Let C be a convex combination of slices of B_Z and $\varepsilon > 0$. So there is a finite set F, positive real numbers $\{t_i : i \in F\}$ with $\sum_{i \in F} t_i = 1$ and a subset $\{S_i : i \in F\}$ of slices of B_Z such that $C = \sum_{i \in F} t_i S_i$. By using (10) and also that $Z^* = X^* \oplus_\infty Y^*$, we obtain that there are pairwise disjoint subsets I and J of F satisfying $F = I \cup J$ and also two subsets of slices $\{R_i : i \in I\}$ and $\{T_j : j \in J\}$ of B_X and B_Y, respectively, such that

$$R_i \times \{0\} \subset S_i, \ \forall i \in I \quad \text{and} \quad \{0\} \times T_j \subset S_j, \ \forall j \in J. \tag{11}$$

Assume that $\alpha_I := \sum_{i\in I} t_i$ and $\alpha_J := \sum_{j\in J} t_j$ are positive numbers (otherwise the proof is simpler). By assumption the sets $\sum_{i\in I} \frac{t_i}{\alpha_I} R_i$ and $\sum_{i\in J} \frac{t_j}{\alpha_J} T_j$ have diameter two, since they are convex combinations of slices of B_X and B_Y, respectively. It follows that there are two pair of elements $(x_1, x_2) \in X \times X$ and $(y_1, y_2) \in Y \times Y$ satisfying

$$x_1, x_2 \in \sum_{i\in I} t_i R_i, \quad \|x_1 - x_2\| > (2-\varepsilon)\alpha_I,$$

$$\text{and} \quad y_1, y_2 \in \sum_{j\in J} t_j T_j, \quad \|y_1 - y_2\| > (2-\varepsilon)\alpha_J.$$

In view of (11) it follows that

$$(x_k, y_k) \in \left(\sum_{i\in I} t_i R_i, \sum_{j\in J} t_j T_j\right) \subset \sum_{i\in F} t_i S_i \ \text{ for } k = 1, 2.$$

As a consequence, we deduce that

$$\begin{aligned}\operatorname{diam} C = \ \operatorname{diam}\left(\sum_{i\in F} t_i S_i\right) &\geq \|(x_1, y_1) - (x_2, y_2)\| \\ &= \|x_1 - x_2\| + \|y_1 - y_2\| > (\alpha_I + \alpha_J)(2-\varepsilon) = 2 - \varepsilon.\end{aligned}$$

Since ε is any positive number, we proved that $\operatorname{diam} C = 2$. Since this holds for every convex combination of slices C of B_Z, Z has the strong diameter two property. $\square$

If a space satisfies the strong diameter two property, then the ℓ_∞-sum of this one and any other Banach space also satisfies the same property [1, Proposition 4.6].

Up to now, every known stability property for the diameter two property also holds for the strong diameter two property. However we aim to show that the strong diameter two property is far from being stable under products endowed with absolute norms. Before that it is convenient to recall the following notion.

A Banach space X is *uniformly convex* if for every $\varepsilon > 0$ there is $\delta > 0$ satisfying

$$x, y \in B_X, \ \|x + y\| > 2 - 2\delta \quad \Rightarrow \quad \|x - y\| < \varepsilon.$$

This is equivalent to the following condition:

$$\{x_n\}, \{y_n\} \in B_X, \left\{\frac{\|x_n + y_n\|}{2}\right\} \to 1 \quad \Rightarrow \quad \{\|x_n - y_n\|\} \to 0.$$

It is well known (and immediate) that every uniformly convex space is strictly convex, that is, every element in the unit sphere in an extreme point of the unit ball. Let us notice that for finite-dimensional spaces both properties are equivalent.

Theorem 3.2. *Let X and Y be nonzero Banach spaces. Assume that $|\ |$ is an absolute norm on $\mathbb{R}^2$ such that $|(1,1)| < 2$ and the elements $(1,0)$ and $(0,1)$ are extreme points in the closed unit ball. Let $Z = X \oplus_f Y$, where f is the restriction of $|\ |$ to $\left(\mathbb{R}_0^+\right)^2$. Then there are slices S_1 and S_2 in B_Z such that the set $\frac{S_1+S_2}{2}$ has diameter strictly less than 2. As a consequence, Z does not have the strong diameter two property.*

Proof. Since we asume that $f(1,1) = |(1,1)| < 2$, we choose $\varepsilon_0 > 0$ such that $2\varepsilon_0 < 2 - f(1,1)$. By using that the elements $(1,0)$ and $(0,1)$ are extreme points in the unit ball of $(\mathbb{R}^2, |\ |)$, we can argue as in the proof of Proposition 2.5 and so for ε_0 there is $\delta_0 > 0$ satisfying

$$(r,s) \in \left(\mathbb{R}_0^+\right)^2, \quad f(r,s) \le 1, \ 1 - \delta_0 < r \quad \Rightarrow \quad s < \varepsilon_0 \,. \tag{12}$$

and also

$$(r,s) \in \left(\mathbb{R}_0^+\right)^2, \quad f(r,s) \le 1, \ 1 - \delta_0 < s \quad \Rightarrow \quad r < \varepsilon_0 \,. \tag{13}$$

By assumption, there are elements $(x^*, y^*) \in S_{X^*} \times S_{Y^*}$. By using that the absolute norm is normalized, we obtain that $(x^*, 0)$ and $(0, y^*)$ belong to S_{Z^*}. We take $S_1 = S(B_Z, (x^*, 0), \delta_0)$ and $S_2 = S(B_Y, (0, y^*), \delta_0)$ and we will show that the subset $\frac{S_1+S_2}{2} \subset cB_Z$ for $c = \frac{f(1,1)}{2} + \varepsilon_0 < 1$ and consequently diam $\frac{S_1+S_2}{2} \le 2c < 2$ as wanted.

In order to check that, let us choose elements $(x_1, y_1) \in S_1$ and $(x_2, y_2) \in S_2$. Hence

$$(x_1, y_1) \in B_Z, \quad 1 - \delta_0 < x^*(x_1) \le \|x_1\|,$$

and also

$$(x_2, y_2) \in B_Z, \quad 1 - \delta_0 < y^*(y_2) \le \|y_2\|.$$

Since the norm on Z is absolute, in view of (12) we have that

$$f(\|x_1\|, \|y_1\|) \le 1, \ \|x_1\| > 1 - \delta_0 \quad \Rightarrow \|y_1\| \le \varepsilon_0.$$

By using the same argument and (13) we also deduce

$$f(\|x_2\|, \|y_2\|) \le 1, \ \|y_2\| > 1 - \delta_0 \quad \Rightarrow \quad \|x_2\| \le \varepsilon_0.$$

Since the norm of Z is absolute and normalized, in view of the previous inequalities we clearly deduce that

$$\begin{aligned}
&\|(x_1, y_1) + (x_2, y_2)\| = \|(x_1 + x_2, y_1 + y_2)\| \\
&\le \|(x_1, y_2)\| + \|x_2\| + \|y_1\| \\
&\le f(\|x_1\|, \|y_2\|) + 2\varepsilon_0 \\
&\le f(1,1) + 2\varepsilon_0 = 2c,
\end{aligned}$$

as we wanted. $\square$

Let us remark that for $p \neq 1$ the $\| \|_p$-norm on $\mathbb{R}^2$ clearly satisfies the assumptions of the absolute norm in Theorem 3.2. By Theorem 2.4 we have that the space $c_0 \oplus_2 c_0$ has the diameter two property. However, in view of the above result, it fails the strong diameter two property. As a consequence, both properties are not equivalent. This answers in the negative one of the open questions posed in [1, p. 11]. The referee kindly let us know that Haller, Langemets and Põldvere also showed that both diameter two properties are not equivalent (see also [15, Theorem 1]).

In spite of the fact that strong diameter two property and diameter two property are not equivalent, there are certain classes of spaces enjoying the diameter two property for which we next prove that, in fact, the strong diameter two property is satisfied. We need some notions in order to present this class.

We recall that a *function module* is a triple $(K, (X_t)_{t\in K}, X)$, where K is a non-empty compact Hausdorff topological space (called the base space), $(X_t)_{t\in K}$ a family of Banach spaces, and X a closed $\mathcal{C}(K)$-submodule of the $\mathcal{C}(K)$-module $\prod_{t\in K}^{\infty} X_t$ (the ℓ_∞-sum of the spaces X_t) satisfying the following conditions:

(1) For every $x \in X$, the function $t \longrightarrow \|x(t)\|$ from K to $\mathbb{R}$ is upper semicontinuous.
(2) For every $t \in K$, we have $X_t = \{x(t) : x \in X\}$.
(3) The set $\{t \in K : X_t \neq 0\}$ is dense in K.

In fact it is known that X is embedded into $\prod_{t\in K}^{\infty} X_t$, that is,

$$\|x\| := \sup_{t\in K} \|x(t)\|, \quad \forall x \in X\,.$$

We refer to [10] for notation and background on function modules.

Let X be a Banach space and $L(X)$ the space of all bounded and linear operators on X. By a *multiplier* on X we mean an element $T \in L(X)$ such that every extreme point of B_{X^*} becomes an eigenvector for T^*. Thus, given a multiplier T on X, and an extreme point p of B_{X^*}, there exists a unique scalar $a_T(p)$ satisfying $T^*(p) = a_T(p)p$. The *centralizer* of X (denoted by $Z(X)$) is defined as the set of those multipliers T on X such that there exists a multiplier S on X satisfying $a_S(p) = \overline{a_T(p)}$ for every extreme point p of B_{X^*}. Thus, if X a real Banach space, then $Z(X)$ coincides with the set of all multipliers on X. In any case, $Z(X)$ is a closed subalgebra of $L(X)$ isometrically isomorphic to $\mathcal{C}(K_X)$, for some compact Hausdorff topological space K_X (see [10, Proposition 3.10]). Moreover X can be seen as a function module whose base space is precisely K_X, and such that the elements of $Z(X)$ are precisely the operators of multiplication by the elements of $\mathcal{C}(K_X)$ ([10, Theorem 4.14]).

Next result improves [9, Lemma 2.3], where it is proved the diameter two property under the same assumption.

Proposition 3.3. *Let X be a Banach space and assume that $Z(X)$ is infinite-dimensional. Then X has the strong diameter two property.*

Proof. Let k be a natural number, $S_i = S(B_X, x_i^*, \delta_i)$ a slice of B_X, and α_i a positive real number for each $1 \le i \le k$ such that $\sum_{i=1}^k \alpha_i = 1$. We choose $x_i \in S_i$ for every $1 \le i \le k$ and write $x_0 := \sum_{i=1}^k \alpha_i x_i$, which is an element in $\sum_{i=1}^n \alpha_i S_i$. For each $1 \le i \le k$, we define $S_i^{**} := S\left(B_{X^{**}}, x_i^*, \delta_i\right)$. By assumption, $Z(X)$ is infinite-dimensional, so $Z(X^{**})$ is also infinite-dimensional by [16, Corollary I.3.15]. By a previous remark, we can assume that X^{**} is a function module with base space equal to some compact $K := K_{X^{**}}$, and such that $Z(X^{**})$ coincide with the set of operators of multiplication by elements of $\mathcal{C}(K)$. Since $Z(X^{**})$ is infinite-dimensional, K is infinite. Hence there is a sequence $\{O_n\}$ of nonempty pairwise disjoint open subsets of K. For each $n \in \mathbb{N}$, we choose $t_n \in O_n$, and apply Urysohn's lemma to pick f_n in $\mathcal{C}(K)$ with $0 \le f_n \le 1$, $f_n(t_n) = 1$ and $f_n(t) = 0$ whenever $t \in K \setminus O_n$. Since the bounded sequence $\{f_n\}$ converges pointwise to 0, it converges weakly to 0 in $\mathcal{C}(K)$, and hence for every $x^{**} \in X^{**}$, $\{(1 - f_n)x^{**}\}$ converges weakly to x^{**} in X^{**}. Let e^{**} be an extreme point of $B_{X^{**}}$. We define the sequences $\{x_n^{**}\}$ and $\{y_n^{**}\}$ in X^{**} by

$$x_n^{**} = \sum_{i=1}^k \alpha_i (1 - f_n) x_i + f_n e^{**},$$

and

$$y_n^{**} = \sum_{i=1}^k \alpha_i (1 - f_n) x_i - f_n e^{**}.$$

Since for every $1 \le i \le k$, $x_i \in B_X$, $e^{**} \in B_{X^{**}}$ and $0 \le f_n \le 1$ for each n, for every $t \in K$ we have that

$$\| \left((1 - f_n)x_i \pm f_n e^{**}\right)(t) \| \le 1.$$

Hence $x_n^{**}, y_n^{**} \in B_{X^{**}}$ for every n.

It is clear that for every $1 \le i \le k$, the sequence $\{(1 - f_n)x_i \pm f_n e^{**}\}$ converges weakly to x_i in X^{**}. Bearing in mind that $x_i \in S_i \subset S_i^{**}$ and S_i^{**} is a weakly open set in the unit ball, we deduce that for some n it is satisfied that $x_n^{**}, y_n^{**} \in \sum_{i=1}^k \alpha_i S_i^{**}$.

On the other hand, we know that there is an element t_n in K satisfying $f_n(t_n) = 1$. Since e^{**} is an extreme point of $B_{X^{**}}$, by [9, Lemma 2.1], we have that $\|e^{**}(t)\| = 1$ for every $t \in K$. So it is immediate that

$$\|x_n^{**} - y_n^{**}\| = 2\|f_n e^{**}\| \ge 2\|f_n(t_n) e^{**}(t_n)\| = 2\|e^{**}(t_n)\| = 2.$$

We showed that $\operatorname{diam}\left(\sum_{i=1}^k \alpha_i S_i^{**}\right) = 2$. By Goldstine's Theorem, every S_i is w^*-dense in S_i^{**}, so $\sum_{i=1}^k \alpha_i S_i$ is w^*-dense in $\sum_{i=1}^k \alpha_i S_i^{**}$. By the w^*-lower semicontinuity of the norm it follows that

$$2 = \operatorname{diam}\left(\sum_{i=1}^k \alpha_i S_i^{**}\right) \le \operatorname{diam}\left(\sum_{i=1}^k \alpha_i S_i\right) \le 2,$$

so diam $\left(\sum_{i=1}^k \alpha_i S_i = 2\right)$ as we wanted. $\square$

Given a Banach space X, we consider the increasing sequence of its even duals

$$X \subseteq X^{**} \subseteq X^{(4} \subseteq \cdots \subseteq X^{(2n} \subseteq \cdots.$$

Since every Banach space is isometrically embedded into its second dual, we can define $X^{(\infty}$ as the completion of the normed space $\bigcup_{n=0}^{\infty} X^{(2n}$.

It is known that for every T in $Z(X)$, T^{**} lies in $Z(X^{**})$ [16, Corollary I.3.15]. Since T^{**} is an isometric extension of T, there exists a natural embedding from $L(X)$ into $L(X^{(\infty})$. The image under this embedding of an operator $T \in L(X)$ is the operator such that its restriction to $X^{(2m}$ is obtained by transposing T $2m$ times. Indeed, it is known that the image of $Z(X)$ under this embedding is contained in $Z(X^{(\infty})$ (see [9, Proposition 4.3.]).

Next result solves an open question posed in [1, p. 451], and also improves [9, Corollary 4.2]. However its argument follows essentially the same idea of the one in the proof of the analogous result for the diameter two property in [9].

Theorem 3.4. *Let X be a Banach space and assume that $Z(X^{(\infty})$ is infinite-dimensional. Then X has the strong diameter two property.*

Proof. Let $k \in \mathbb{N}$, $S_i = S(B_X, x_i^*, \alpha_i)$ be a slice of B_X and t_i a positive real number for each $i \le k$ satisfying $\sum_{i=1}^k t_i = 1$.

Firstly we notice the existence of a natural embedding $f \mapsto \widetilde{f}$ from X^* to $(X^{(\infty})^*$. Indeed, let f be an element in X^*. Given $u \in \bigcup_{n=0}^{\infty} X^{(2n}$, there exists $m \in \mathbb{N}$ such that u belongs to $X^{(2m}$, so that, regarding f as an element of $(X^*)^{(2m} = (X^{(2m})^*$, the symbol $f(u)$ has a meaning which does not depend on m. In this way we are provided with a natural extension of f to $\bigcup_{n=0}^{\infty} X^{(2n}$, which extends uniquely by continuity to $X^{(\infty}$, giving rise to an element $\widetilde{f}$ of $(X^{(\infty})^*$.

Now we fix any $i \le k$. We define a slice of $B_{X^{(\infty}}$ as follows $S_i^{(\infty} := S\left(B_{X^{(\infty}}, \widetilde{x}_i^*, \alpha_i\right)$. For every natural number m, consider the subset

$$S_i^{(2m} := S_i^{(\infty} \cap X^{(2m} = \{z \in B_{X^{(2m}} : x_i^*(z) > 1 - \alpha_i\}.$$

Let ε be any positive real number. By Proposition 3.3, we have that $\sum_{i=1}^k t_i S_i^{(\infty}$ has diameter 2. Since the linear space $\bigcup_{n=0}^{\infty} X^{(2n}$ is (norm) dense in $X^{(\infty}$, we deduce that $\bigcup_{n=0}^{\infty} S_i^{(2n}$ is also norm (dense) in $S_i^{(\infty}$. Since the previous sequence of subsets is increasing for every $i \le k$, we have that there is some natural number m such that

$$\operatorname{diam}\left(\sum_{i=1}^k t_i S_i^{(2m}\right) > \operatorname{diam}\left(\sum_{i=1}^k t_i S_i^{(\infty}\right) - \varepsilon = 2 - \varepsilon\,.$$

Now we can proceed as in the last part of the proof of Proposition 3.3, by using the w^*-denseness of the slice of the form $S_i(B_{X^{(2m-2}}, x_i^*, \alpha_i)$ in $S_i(B_{X^{(2m}}, x_i^*, \alpha_i)$. If we do that we obtain that

$$\operatorname{diam}\left(\sum_{i=1}^{k} t_i S_i^{(2m-2}\right) \geq \operatorname{diam}\left(\sum_{i=1}^{k} t_i S_i^{(2m}\right) > 2-\varepsilon.$$

After a finite number of steps, we deduce that $\operatorname{diam}\left(\sum_{i=1}^{k} t_i S_i\right) \geq 2-\varepsilon$. Since ε is any positive number we conclude that $\operatorname{diam}\left(\sum_{i=1}^{k} t_i S_i\right) = 2$. $\square$

For every infinite compact topological space K, the space $\mathcal{C}(K,(X,\tau))$ has infinite dimensional centralizer, whenever X is a nonzero Banach space and τ is a vector space topology on X that contains the w-topology and weaker than the norm topology (see [9, Proposition 3.2]).

Now we will provide classes of spaces where the previous results can be applied.

Corollary 3.5. *For every infinite compact topological space K, the space $\mathcal{C}(K,(X,\tau))$ has the strong diameter 2 property, whenever X is a nonzero Banach space and τ is a vector space topology on X that contains the weak-topology of X and is coarser than the norm topology .*

Under the assumptions of the above result, the space $\mathcal{C}(K,(X,\tau))$ has infinite dimensional centralizer (see [9, Proposition 3.2]). So the statement follows from Proposition 3.3.

In [3, Proposition 3.3], it is proved that every non reflexive Banach space X such that X^* is L-embedded satisfies that $Z(X^{(\infty})$ is infinite-dimensional. It is known that the dual of a real or complex C^*-algebra is a L-embedded space (see [7, Proposition 2.2] and [5, Proposition 3.4]).

Corollary 3.6. *Let X be a non reflexive Banach space such that X^* is L-embedded. Then X has the strong diameter two property. In particular, every real or complex infinite dimensional C^*-algebra, and every infinite-dimensional predual of an L_1-space, has the strong diameter two property.*

For a Banach space X, an *L-projection* on X is a (linear) projection $P : X \longrightarrow X$ satisfying $\|x\| = \|P(x)\| + \|x - P(x)\|$ for every $x \in X$. In such a case, we will say that the subspace $P(X)$ is an *L-summand* of X. Let us notice that the composition of two L-projections on X is an L-projection [10, Proposition 1.7], so the closed linear subspace of $L(X)$ generated by all L-projections on X is a subalgebra of $L(X)$, the space of all bounded and linear operators on X. This algebra, denoted by $C(X)$, is called the *Cunningham algebra* of X. It is known that $C(X)$ is linearly isometric to $Z(X^*)$ ([10, Theorems 5.7 and 5.9]). For instance, any infinite-dimensional space $L_1(\mu)$ satisfies that its Cunningham algebra is infinite-dimensional. By [16, Lemma VI.1.1], for every nontrivial

Banach spaces X and Y, the centralizer of $L(X,Y)$, the space of all bounded and linear operators from X to Y, is infinite dimensional whenever either $C(X)$ is infinite dimensional or $Z(Y)$ is infinite dimensional. As a consequence we obtain the following result:

Corollary 3.7. *Let X be a nonzero Banach space. Then the following spaces have the strong diameter 2 property:*

a) $L(L_1(\mu), X)$ *whenever* $L_1(\mu)$ *is infinite-dimensional.*

b) $L(X, C(K))$, *for any infinite compact topological space* K.

References

[1] T. A. Abrahansen, V. Lima, O. Nygaard: Remarks on diameter two properties, J. Convex Analysis 20 (2013) 439–452.

[2] Y. A. Abramovich, C. D. Aliprantis: Problems in Operator Theory, Graduate Studies in Mathematics 51, American Mathematical Society, Providence (2002).

[3] M. D. Acosta, J. Becerra Guerrero: Weakly open sets in the unit ball of some Banach spaces and the centralizer, J. Funct. Anal. 259 (2010) 842–856.

[4] M. D. Acosta, A. Kamińska: Weak neighborhoods and the Daugavet property of the interpolation spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$, Indiana Univ. Math. J. 57 (2008) 77–96.

[5] T. J. Barton, R. M. Timoney: W^*-continuity of Jordan triple products and applications, Math. Scand. 59 (1986) 177–191.

[6] J. Becerra-Guerrero, G. López-Pérez: Relatively weakly open subsets of the unit ball in functions spaces, J. Math. Anal. Appl. 315(2) (2006) 544–554.

[7] J. Becerra Guerrero, G. López Pérez, A. M. Peralta, A. Rodríguez-Palacios: Relatively weakly open sets in closed balls of Banach spaces, and real JB^*-triples of finite rank, Math. Ann. 330 (2004) 45–58.

[8] J. Becerra-Guerrero, G. López-Pérez, A. Rodríguez-Palacios: Relatively weakly open sets in closed balls of C^*-algebras, J. London Math. Soc. 68 (2003) 753–761.

[9] J. Becerra Guerrero, A. Rodríguez-Palacios: Relatively weakly open sets in closed balls of Banach spaces, and the centralizer, Math. Z. 262 (2009) 557–570.

[10] E. Behrends: M-Structure and the Banach-Stone Theorem, Lecture Notes in Math. 736, Springer, Berlin (1979).

[11] F. F. Bonsall, J. Duncan: Numerical Ranges II, London Mathematical Society Lecture Notes Series 10, Cambridge University Press, New York (1973).

[12] R. D. Bourgin: Geometric Aspects of Convex Sets with the Radon-Nikodým Property, Lecture Notes in Mathematics 993, Springer, Berlin (1983).

[13] N. Ghoussoub, G. Godefroy, B. Maurey, W. Schachermayer: Some topological and geometrical structures in Banach spaces, Mem. Amer. Math. Soc. 378 (1987) 116 p.

[14] N. Ghoussoub, B. Maurey, W. Schachermayer: Geometrical implications of certain infinite-dimensional decomposition, Trans. Amer. Math. Soc. 317 (1990) 541–584.

[15] R. Haller, J. Langmets: Two remarks on diameter 2 properties, Proc. Est. Acad. Sci. 63 (2014) 2–7.

[16] P. Harmand, D. Werner, W. Werner: M-ideals in Banach Spaces and Banach Algebras, Lecture Notes in Math. 1547, Springer, Berlin (1993).

[17] G. López-Pérez: The big slice phenomena in M-embedded and L-embedded spaces, Proc. Amer. Math. Soc. 134 (2005) 273–282.

[18] O. Nygaard, D. Werner: Slices in the unit ball of a uniform algebra, Arch. Math. 76 (2001) 441–444.

[19] W. Schachermayer, A. Sersouri, E. Werner: Moduli of nondentability and the Radon-Nikodým property in Banach spaces, Israel J. Math. 65(3) (1989) 225–257.

[20] R. V. Shvydkoy: Geometric aspects of the Daugavet property, J. Funct. Anal. 176 (2000) 198–212.

[21] E. Werner: Nondentable solid subsets in Banach lattices failing RNP. Applications to renormings, Proc. Amer. Math. Soc. 107(3) (1989) 611–620.