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Norms With Infinite Values

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We provide an overview of linear spaces equipped with norms that may take on the value infinity but that otherwise satisfy the properties of ordinary norms. Spaces equipped with such norms fail to be topological vector spaces unless the norm is finite valued. We study the continuous linear transformations between such spaces, extending the theory of conventional linear analysis in often unanticipated ways.

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1. Introduction

Let $\langle X, d\rangle$ be a metric space and let $C(X,\mathbb{R})$ and $C_b(X,\mathbb{R})$ be the vector space of continuous real-valued functions on X and the vector space of bounded continuous real-valued functions, respectively. It is standard to equip the latter with the *supremum norm*

$$||f||_\infty := \sup_{x\in X} |f(x)|.$$

For one thing, we can embed the metric space isometrically into $C_b(X,\mathbb{R})$ so equipped using $x \mapsto d(\cdot,x) - d(\cdot,p)$ where p is a fixed but arbitrary point of X [4, p. 338]. Unless d is bounded, $x \mapsto d(\cdot,x)$ fails to do the job. However, if we allow $||\cdot||_\infty$ to assume infinite values in its definition we do get an isometric embedding into $C(X,\mathbb{R})$. Another advantage of working in this more general context is that we can isometrically embed the nonempty closed subsets of X

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equipped with classical *Hausdorff distance* [1, 9] defined by

$$H_d(A,B) := \max\{\sup_{b\in B} d(b,A), \sup_{a\in A} d(a,B)\}$$

within it. The embedding here is $A \mapsto d(\cdot, A)$, i.e., we have (see, e.g., [1, p. 29])

$$H_d(A,B) = \sup_{x\in X} |d(x,A) - d(x,B)|.$$

Note that Hausdorff distance so defined necessarily involves infinite distances provided d is unbounded, e.g., $H_d(\{p\}, X) = \infty$ for each $p \in X$.

The formula $||f||_\infty := \sup_{x\in X} |f(x)|$ defines a norm on $C(X, \mathbb{R})$ that can assume infinite values when X is noncompact. In the following definition, we denote the origin of our linear space X by 0_X, and we adopt the convention $0 \times \infty = 0$.

Definition 1.1. Let X be a vector space over a field of scalars $\mathbb{F}$ that could be either $\mathbb{C}$ or $\mathbb{R}$. By an *extended norm* on X, we mean a function $||\cdot|| : X \to [0, \infty]$ satisfying the following three properties:

(1) $||x|| = 0$ if and only if $x = 0_X$;
(2) $||\alpha x|| = |\alpha|||x||$ for each $x \in X$ and $\alpha \in \mathbb{F}$;
(3) $||x + y|| \le ||x|| + ||y||$ for each x, y in X.

An extended norm induces an extended metric d on X defined by $d(x, y) = ||x - y||$ and with it, a metrizable topology, having as a base all open metric balls $B_d(p, \alpha) := \{x \in X : ||x-p|| < \alpha\}$ where p runs over X and $\alpha \in (0, \infty)$. But the topology that is obtained while locally convex is not a topological vector space topology, since the neighborhoods of the origin cannot be absorbing provided X has elements of infinite norm (see also Proposition 3.2 below).

An important equivalence relation on such a space is the following: $x \sim y$ provided $||x - y||$ is finite. Each equivalence class $x/\sim$ of $\sim$ is a clopen (= both closed and open) subset of X which we call a *metric component* of X. Each is a translate of the vector subspace $\{x \in X : ||x|| < \infty\}$ which we denote by X_{fin} in the sequel. Evidently, $\langle X_{\text{fin}}, ||\cdot||\rangle$ is a conventional normed linear space and is in particular connected, so the same can be said for each metric component.

It is the purpose of this note to lay bare various aspects of the structure of such spaces, as well as to present a representative set of examples. We discuss a natural class of selectors for their metric components, look at quotient structure in this context, and study continuous linear transformations between extended normed linear spaces. As highlights, we

- introduce and exploit the notion of distance basis;
- present appropriate versions of the uniform boundedness principle, the closed graph theorem, the Banach-Schauder open mapping theorem and the Hahn-Banach theorem in this setting;

- study the operator topology of uniform convergence on bounded subsets which in general is not metrizable and is properly stronger than the operator norm topology;
- characterize those subspaces of a domain space X that can be realized as the preimage of the set of elements of finite norm in a codomain space Y with respect to some element of $\mathbf{B}(X,Y)$;
- make sense out of the curious decoupling in a Cauchy complete space between the existence of a closed complementary subspace to a given closed subspace and the continuity of the corresponding projection onto the subspace.

Connectedness arguments and direct sum decompositions play essential roles in our analysis. There is one principle that applies throughout: the behavior of phenomena on X_{fin} is usually determining.

2. Preliminaries

Let X be a linear space over a field of scalars $\mathbb{F}$ that can either be the real numbers or the complex numbers. We assume that all our spaces contain a vector in addition to the origin. If N is a subspace of X, we call any translate of N a *flat*. The set of all flats $\{x+N : x \in X\}$ determined by a subspace N can be made a vector space denoted by X/N under the well-defined operations $(x+N)+(y+N) := (x+y)+N$ and $\alpha(x+N) := \alpha x + N$. We remind the reader of two particular cases that arise in traditional linear analysis (see, e.g., [15]).

- If P is a semi-norm on X, let $N = \{x : P(x) = 0\}$. Then $||x+N|| := P(x)$ is a well-defined norm on X/N.
- If $\langle X, ||\cdot||\rangle$ is a conventional normed linear space and N is a closed subspace of X, then $||x+N|| := \inf\{||x+n|| : n \in N\}$ defines a so-called *quotient norm* on X/N, which makes the quotient space a Banach space if the original space was.

If M is another subspace of X, we write $X = M \oplus N$ provided each element of X can be written in a unique way as $m+n$ where $m \in M$ and $n \in N$. In this case M and N are called *complementary subspaces*. If N is neither $\{0_X\}$ nor X, we obtain a complementary subspace by completing a basis B_0 for N to a basis B for X, and taking $M = \text{span}(B\backslash B_0)$. If $||\cdot||_1$ and $||\cdot||_2$ are extended norms for M and N respectively, we call $||\cdot|| : X \to [0,\infty]$ defined by

$$||m+n|| := ||m||_1 + ||n||_2, \quad m \in M, n \in N$$

the *direct sum extended norm* on X induced by the two extended subspace norms.

If $\{x_j : j \in \Delta\}$ is a set of distinct vectors in X, by a symbol of the form $\sum_{j\in\Delta} \alpha_j x_j$, we will mean a finite linear combination of $\{x_j : j \in \Delta\}$ where $\alpha_j = 0$ for all but finitely many indices.

We denote the vector space of linear transformations between extended normed spaces X and Y by $\mathbf{L}(X,Y)$. As usual, a *projection* $P \in \mathbf{L}(X,X)$ is an idempotent linear operator.

Within $\mathbf{L}(X,Y)$ lie the continuous linear transformations $\mathbf{B}(X,Y)$ and as a special case, we write X^* for $\mathbf{B}(X,\mathbb{F})$. We will look at the structure of the vector space of such transformations and in particular see that $T \in \mathbf{B}(X,Y)$ if and only if $T|_{X_{\text{fin}}} \in \mathbf{B}(X_{\text{fin}}, Y_{\text{fin}})$ (here $T|_A$ represents the restriction of T to A). The traditional equivalent notions of operator norm carry over to the more inclusive setting.

We close this section with some examples of extended norms.

Example 2.1. A linear space X is called a *discrete extended normed linear space* if it is equipped with the norm

$$||x|| = \begin{cases} 0 & \text{if } x = 0_X \\ \infty & \text{otherwise.} \end{cases}$$

As each open ball is a singleton, the topology is discrete. Conversely, if X_{fin} properly contains $\{0_X\}$, then the topology cannot be discrete, as the relative topology that X_{fin} inherits is not discrete. In a general extended normed linear space, if $||x|| = \infty$, then the relative topology on $\{\alpha x : \alpha \in \mathbb{F}\}$ must be discrete (see Lemma 3.1 below).

Example 2.2. Each subspace E of a linear space X can be realized as X_{fin} for some extended norm $||\cdot||$ on X. If $E = \{0_X\}$, use the discrete extended norm. Otherwise, let $\{b_j : j \in \Delta\}$ be a basis for E. For $x = \sum_{j\in\Delta} \alpha_j b_j$, put $||x|| := \sum_{j\in\Delta} |\alpha_j|$, and let $||x|| = \infty$ otherwise.

Example 2.3. Let ℓ_1 be the vector space of absolutely summable $\mathbb{F}$-sequences and let ℓ_2 be the square summable sequences. Equip $X = \ell_2$ with the extended norm defined by $||x||_1 = \sum_{n=1}^{\infty} |x_n|$. Then $X_{\text{fin}} = \ell_1$, and two ℓ_2-sequences are a finite distance apart if and only if they differ by an ℓ_1-sequence.

Example 2.4. If $\langle X, ||\cdot||_1\rangle$ and $\langle Y, ||\cdot||_2\rangle$ are conventional normed linear spaces, $T \mapsto \sup\{||Tx||_2 : ||x||_1 \leq 1\}$ is an extended norm on $\mathbf{L}(X,Y)$, where $\mathbf{L}(X,Y)_{\text{fin}} = \mathbf{B}(X,Y)$.

Example 2.5. Let $\langle X, d\rangle$ be a metric space. Define $L : C(X,\mathbb{R}) \to [0,\infty]$ by

$$L(f) := \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{d(x_1,x_2)}.$$

If f is Lipschitz continuous, clearly $L(f)$ is its minimal Lipschitz constant; otherwise, $L(f) = \infty$. This is not an extended norm because $L(f) = 0$ only guarantees that f is constant. We get an extended norm that we call the *extended Lipschitz norm* provided we put

$$||f||_L := \max\{|f(x_0)|, L(f)\}$$

where x_0 is a fixed but arbitrary point of X. Recently studied by Beer and Hoffman [2], the same topology results if x_0 is replaced by a different point or the maximum is replaced by a sum. Here, the subspace of elements of finite norm consists of the real-valued Lipschitz functions $\mathrm{Lip}(X)$ on X, which gives rise to a second isometric embedding of $\langle X, d\rangle$ into a Banach space, this time into $\mathrm{Lip}(X)^*$. As one might guess, the embedding is defined in terms of evaluation: $x \mapsto \widehat{x}$ where $\widehat{x}(f) = f(x)$ for $f \in \mathrm{Lip}(X)$ [6]. Actually, the result is valid whether or not the Lipschitz norm is defined by a sum or a maximum.

The extended Lipschitz norm is intrinsic to the interchange of limit and derivative in elementary analysis, as one can prove the following result [2].

Theorem. *Suppose $a < b$ are real numbers. Then $S := \{f \in C([a,b],\mathbb{R}) : \forall x \in (a,b), f'(x)$ exists$\}$ is a closed subspace of $\langle C([a,b],\mathbb{R}), ||\cdot||_L\rangle$. Moreover, if $\langle f_n\rangle$ is a sequence in S and $f \in C([a,b],\mathbb{R})$ with $||f_n - f||_L \to 0$, then $\lim_{n\to\infty} f_n'(x) = f'(x)$ for each x in (a,b).*

From this, one can immediately obtain the standard classical theorem [15, Theorem 7.17].

3. On the structure of extended normed linear spaces

Our first elementary fact has a number of consequences.

Lemma 3.1. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space and suppose $||p|| = \infty$. Then $\alpha p \sim \beta p \Rightarrow \alpha = \beta$.*

Proof. Suppose $\alpha p \sim \beta p$ but $\alpha \neq \beta$. Put $y = (\alpha - \beta)p$; by assumption, $||y|| < \infty$ from which $||p|| = ||(\alpha-\beta)^{-1}y|| < \infty$, a contradiction. □

Proposition 3.2. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space. Then $(\alpha, x) \mapsto \alpha x$ is jointly continuous at (α_0, x_0) if and only if $||x_0|| < \infty$.*

Proof. If $||x_0|| < \infty$, scalar multiplication on any product neighborhood of (α_0, x_0) maps part of $\mathbb{F} \times X_{\text{fin}}$ into X_{fin} and so is jointly continuous at the point. On the other hand, if $||x_0|| = \infty$ and $\alpha \neq \alpha_0$, then by Lemma 3.1, αx_0 is contained in no neighborhood of $\alpha_0 x_0$. □

In contrast, $(x,y) \mapsto x + y$ is globally jointly continuous on $X \times X$, for if $||x - x_0|| < \frac{1}{2}\varepsilon$ and $||y - y_0|| < \frac{1}{2}\varepsilon$, then $||(x+y) - (x_0+y_0)|| < \varepsilon$. This of course implies that translation $x \mapsto x + p$ is a homeomorphism of the space. But $x \mapsto \alpha x$ is as well provided $\alpha \neq 0$! This follows from the fact that such scalar multiplication sets up homeomorphisms between pairs of (clopen) metric components, with the exception of X_{fin} which gets mapped homeomorphically onto itself. The details are left to the reader.

The next proposition speaks to the equivalence of extended norms.

Theorem 3.3. *The extended norms $||\cdot||_1$ and $||\cdot||_2$ determine the same topology on a linear space X if and only if they determine the same subspace of elements of finite norm and the relative topologies on that common subspace coincide.*

Proof. For necessity, suppose for some $p \in X$, we have $||p||_1 = \infty$ while $||p||_2 < \infty$. By Lemma 3.1, the $||\cdot||_1$-relative topology on the line determined by p and 0_X is discrete while the other relative topology is not, and this is impossible. The other condition is obviously necessary. Sufficiency follows from the fact that each ball about a point $p \in X$ lies entirely in its metric component, and there is a natural isometry of X_{fin} onto the component carrying 0_X to p. □

As an application of our last result, we give the following proposition, whose proof is left as an exercise to the reader.

Proposition 3.4. *Suppose $X = M \oplus N$ and $||\cdot||_1$ and $||\cdot||_1'$ are equivalent extended norms on M and $||\cdot||_2$ and $||\cdot||_2'$ are equivalent extended norms on N. Then $||\cdot||$ and $||\cdot||'$ defined on X by*

$$||m+n|| := ||m||_1 + ||n||_2,$$

$$||m+n||' := \max\{||m||_1', ||n||_2'\},$$

are equivalent extended norms on X.

A conventional normed linear space is of course connected, in fact, path connected. This is not so in the setting of extended norms if $X \neq X_{\text{fin}}$. Since a nonempty connected clopen subset of a general topological space must be a connected component, and the metric components of an extended normed linear space are translates of a common connected linear subspace which partition the underlying space, we may state

Theorem 3.5. *The metric components of an extended normed linear space $\langle X, ||\cdot||\rangle$ coincide with the connected components of the space.*

Corollary 3.6. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space and let N be a nonempty linear subspace. Then N is connected if and only $N \subseteq X_{\text{fin}}$.*

Proof. If containment holds, then N is a conventional normed linear space and so is connected. The converse follows from our last theorem, as any connected subspace has 0_X within it, which ensures that the subspace must be contained in $0_X/\sim$. □

Corollary 3.7. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space. Then X is connected if and only if $X = X_{\text{fin}}$.*

We next characterize the clopen subsets of an extended normed linear space $\langle X, ||\cdot||\rangle$, that is, the possible separations of X.

Theorem 3.8. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space and let E be a nonempty subset. Then E is clopen if and only if E is the union of a family of metric components of X.*

Proof. For sufficiency, a union of a family of metric components is surely open, and it is closed as well because the union of a locally finite family of closed sets must be closed [4, Corollary 1.1.12]. For necessity, if E is clopen, we argue that $E = \cup\{x/\sim : x \in E\}$. To see this, let $x \in E$ be arbitrary; since $x/\sim$ is homeomorphic (in fact isometric) with X_{fin}, the flat is connected as well. But then $x/\sim$ must lie entirely in E because $x \in E$, else E would induce a nontrivial separation of the flat with respect to the relative topology. We have shown $E \supseteq \cup\{x/\sim : x \in E\}$, and the reverse inclusion is trivial. □

The following corollary will bear on Theorem 4.15 below.

Corollary 3.9. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space and let N be a linear subspace. The following are equivalent:*

(*1*) *N is clopen;*
(*2*) *N is open;*
(*3*) *$X_{\text{fin}} \subseteq N$.*

Proof. (*1*) $\Rightarrow$ (*2*). This is trivial.

(*2*) $\Rightarrow$ (*3*). If condition (*2*) holds, then N must contain some ball with center 0_X and since N is stable under scalar multiplication, we get $X_{\text{fin}} \subseteq N$.

(*3*) $\Rightarrow$ (*1*). If $X_{\text{fin}} \subseteq N$, then for each $n \in N$, we have $n + X_{\text{fin}} \subseteq N$ as N is closed under addition. Thus, N contains the metric component of each of its elements. Apply the last theorem. □

While an open subspace of an extended normed linear space must be closed, a closed subspace need not be open, e.g., the closed subspace $\{0_X\}$ is not open if $\{0_X\} \neq X_{\text{fin}}$.

Combining Corollary 3.9 with Corollary 3.6, we see that X_{fin} is the only linear subspace of X that is at once connected and open.

Proposition 3.10. *Suppose the extended normed linear space $\langle X, ||\cdot||\rangle$ is separable as a topological space. Then $X = X_{\text{fin}}$.*

Proof. Suppose X were to contain a vector p with infinite norm. By Lemma 3.1 there are uncountably many metric components, and thus an uncountable pairwise disjoint family of open sets. As a result, X can't be separable. □

We next turn to Cauchy completeness.

Proposition 3.11. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space. Then each Cauchy sequence in X with respect to the extended metric induced by the norm is convergent if and only if $\langle X_{\text{fin}}, ||\cdot||\rangle$ is a Banach space.*

Proof. The condition is necessary because X_{fin} is closed. For sufficiency, let $\langle x_n \rangle$ be a Cauchy sequence in X. Then for some $k \in \mathbb{N}$, $\{x_n : n \geq k\}$ is contained in a ball with center x_k, and so $\langle x_n - x_k \rangle$ is a Cauchy sequence in X_{fin} which must converge to some $p \in X_{\text{fin}}$. Thus $\langle x_n \rangle$ converges to $x_k + p$. $\square$

We will henceforth refer to a Cauchy complete extended normed linear space as an *extended Banach space.*

Corollary 3.12. *An extended normed linear space $\langle X, ||\cdot|| \rangle$ is an extended Banach space if and only if each absolutely summable sequence in X is summable.*

Proof. If the space is Cauchy complete, by the last proposition, X_{fin} is a Banach space, and $\sum_{n=1}^{\infty} ||x_n|| < \infty$ forces each x_n to lie in X_{fin} and the classical result [8, p. 191] applies to get convergence of $\sum_{n=1}^{\infty} x_n$. Conversely, the condition implies X_{fin} is a Banach space. $\square$

Theorem 3.13. *Let $\langle X, ||\cdot|| \rangle$ be an extended normed linear space where $X \neq X_{\text{fin}}$ and $X_{\text{fin}} \neq \{0_X\}$. Then the extended norm can be realized as a direct sum extended norm determined by a conventional norm and a discrete extended norm.*

Proof. Let N be a subspace with $X = X_{\text{fin}} \oplus N$. Clearly, the trace of $||\cdot||$ on X_{fin} is a conventional norm and the trace on N is the discrete extended norm. $\square$

The proof of the following elementary lemma is left to the reader.

Lemma 3.14. *Let N be a linear subspace of an extended normed linear space $\langle X, ||\cdot|| \rangle$ and let W be another subspace such that $N = W \oplus (N \cap X_{\text{fin}})$. If w_1 and w_2 are distinct vectors in W, then $||w_1 - w_2|| = \infty$.*

We use this explictly in the proof of the next result.

Proposition 3.15. *A linear subspace N of an extended normed linear space $\langle X, ||\cdot|| \rangle$ is closed if and only if $N \cap X_{\text{fin}}$ is closed.*

Proof. One direction is trivial because X_{fin} is closed. Conversely, suppose $N \cap X_{\text{fin}}$ is closed. If $N \subseteq X_{\text{fin}}$, we are done. Otherwise, let W be a nontrivial linear subspace with $N = W \oplus (N \cap X_{\text{fin}})$. Let p be in the closure of N. Then for each $j \in \mathbb{Z}^+$ there exists $x_j \in N \cap X_{\text{fin}}$ and $w_j \in W$ with $||p - (x_j + w_j)|| < 1/j$. Then if $i \neq j$, then $w_i - w_j$ has finite norm, so by the immediately preceding lemma, $w_1 = w_2 = w_3 = \ldots$. Let w denote their common value; evidently $\langle x_j \rangle$ converges to $p - w$ which puts $p - w$ in $N \cap X_{\text{fin}}$ as required. $\square$

Corollary 3.16. *Each finite dimensional linear subspace of an extended normed linear space is closed.*

Corollary 3.17. *Let $\langle X, ||\cdot|| \rangle$ be an extended normed linear space where $X \neq X_{\text{fin}}$. Let N be a complementary subspace to X_{fin}. Then*

(*1*) *N is a closed subspace;*

(*2*) *N is an open subspace if and only if the extended norm is discrete.*

Proof. Statement (*1*) is immediate from Proposition 3.15. For (*2*), if the extended norm is discrete, then each subset of X is open. The converse follows from Corollary 3.9. $\square$

In conventional Banach space theory, one is interested in whether a given closed linear subspace N has a closed complementary subspace. This is of course equivalent to the existence of a continuous projection onto N [14, p. 126]. To say that this question has attracted a great deal of attention would be an understatement (see, e.g., [5, 10, 12]).

We intend to show that a closed subspace N of an extended normed linear space $\langle X, ||\cdot||\rangle$ has a closed complement provided $N \cap X_{\text{fin}}$ has one in X_{fin}. This will be crucial in the proof of our central result on the existence of a continuous projection presented in the penultimate section of the paper.

We need the following lemma.

Lemma 3.18. *Suppose E_1, E_2 and E_3 are linear subspaces of $\langle X, ||\cdot||\rangle$ where $E_1 \cup E_2 \subseteq X_{\text{fin}}$, $E_3 \cap X_{\text{fin}} = \{0_X\}$ and $E_1 \cap E_2 = \{0_X\}$. Then*

$$E_1 + E_2 + E_3 = E_1 \oplus E_2 \oplus E_3.$$

Proof. Suppose $e_1+e_2+e_3 = 0_X$ where $e_i \in E_i$. Then $e_3 = -(e_1+e_2) \in X_{\text{fin}} \Rightarrow e_3 = 0_X$. Thus $e_1+e_2 = 0_X$ and since $E_1 \cap E_2 = \{0_X\}$, we get $e_1 = e_2 = 0_X$. $\square$

Theorem 3.19. *Let N be a closed linear subspace of $\langle X, ||\cdot||\rangle$ such that there is a closed subspace W of X_{fin} with $X_{\text{fin}} = W \oplus (N \cap X_{\text{fin}})$. Then N has a closed complement M in X with $M \cap X_{\text{fin}} = W$.*

Proof. We may assume that $X \neq X_{\text{fin}}$. Let N_1 be a (closed) subspace such $N = (N \cap X_{\text{fin}}) \oplus N_1$. By our lemma,

$$W + (N \cap X_{\text{fin}}) + N_1 = W \oplus (N \cap X_{\text{fin}}) \oplus N_1.$$

Denote this direct sum by E and let M_1 be a subspace complementary to E. We claim that $M := W + M_1$ does the job.

Since

$$M_1 \cap X_{\text{fin}} = M_1 \cap (W \oplus (N \cap X_{\text{fin}})) \subseteq M_1 \cap E = \{0_X\},$$

each element of M_1 other than 0_X has infinite norm. As a result, $M \cap X_{\text{fin}} = W$, so that by Proposition 3.15, the subspace M is closed.

Next, let $p \in X$ be arbitrary; there exist unique $m_1 \in M_1$ and $e \in E$ such that $p = m_1 + e$. But e can be written uniquely as $w + n$ where $w \in W$ and $n \in N$. Thus $p = (m_1 + w) + n \in M + N$ and the representation is unique. $\square$

If a vector space X can be written as a direct sum of subspaces $M \oplus N$ where N is nontrivial, then $n \mapsto n + M$ is a linear bijection from N to X/M [8, p. 2], and as a result, if B is a basis for N, then $\{b + M : b \in B\}$ is a basis for X/M. In particular, the linear combinations of elements of B select representatives from the flats that are the translates of M. At the risk of some redundancy, we explore what this means when $M = X_{\text{fin}}$. In view of the next result, it would seem reasonable to call such a complementary basis a *distance basis* for X.

Theorem 3.20. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space where $X \neq X_{\text{fin}}$. For a collection $\{b_j : j \in \Delta\}$ of distinct vectors in X, the following conditions are equivalent:*

(*1*) $X = X_{\text{fin}} \oplus \text{span}(\{b_j : j \in \Delta\})$ *and* $\{b_j : j \in \Delta\}$ *is a linearly independent subset of* X*;*
(*2*) *for each* $p \in X$*, there is a unique linear combination* $\sum_{j\in\Delta} \alpha_j b_j$ *with* $||p - \sum_{j\in\Delta} \alpha_j b_j|| < \infty$*;*
(*3*) $\{b_j : j \in \Delta\}$ *is a maximal set of vectors in* X *such that* $||\sum_{j\in\Delta} \alpha_j b_j|| < \infty \Rightarrow \forall j \in \Delta, \alpha_j = 0$.

Proof. (*1*) $\Rightarrow$ (*2*). By the direct sum decomposition of condition (*1*), there exists unique $y \in X_{\text{fin}}$ and $w \in \text{span}(\{b_j : j \in \Delta\})$ with $p - w = y$. But w can be written uniquely as a linear combination of $\{b_j : j \in \Delta\}$ by linear independence.

(*2*) $\Rightarrow$ (*3*). Suppose (*2*) holds, and $||\sum_{j\in\Delta} \alpha_j b_j|| < \infty$. Then with $p = 0_X$ in condition (*2*), the uniqueness ensures that for all $j \in \Delta, \alpha_j = 0$. For maximality, suppose $p \notin \{b_j : j \in \Delta\}$. Then by (*2*), there is a linear combination of elements of $p \cup \{b_j : j \in \Delta\}$ of finite norm whose coefficients are not all zero.

(*3*) $\Rightarrow$ (*1*). Obviously, $\{b_j : j \in \Delta\}$ is linearly independent because $\sum_{j\in\Delta} \alpha_j b_j = 0_X \Rightarrow ||\sum_{j\in\Delta} \alpha_j b_j|| < \infty$. We next show that

$$X = X_{\text{fin}} + \text{span}(\{b_j : j \in \Delta\}).$$

Suppose $p \notin \text{span}(\{b_j : j \in \Delta\})$. By maximality in condition (*3*) there exists some linear combination of $\{p\} \cup \{b_j : j \in \Delta\}$ of finite norm where the coefficient of p is nonzero. Thus for some $w \in X_{\text{fin}}$ and $\alpha \neq 0$, we may write

$$\alpha p + \sum_{j\in\Delta} \alpha_j b_j = w,$$

and solving for p places p in $X_{\text{fin}} + \text{span}(\{b_j : j \in \Delta\})$.

To see that the sum is direct, suppose for w_1, w_2 of finite norm, we have

$$w_1 + \sum_{j\in\Delta} \alpha_j b_j = w_2 + \sum_{j\in\Delta} \beta_j b_j.$$

Then $||\sum_{j\in\Delta}(\alpha_j - \beta)b_j|| < \infty$ so that by (*3*) for all $j, \alpha_j = \beta_j$ and the representation is unique. □

We now turn our attention to quotient spaces. If $\langle X, ||\cdot||\rangle$ is an extended normed linear space and N is a closed subspace, then the standard argument employed in the conventional setting shows that $||x+N|| := \inf\{||x+n|| : n \in N\}$ defines an extended norm on X/N. The standard argument can be used to show that if X is Cauchy complete, then so is the quotient. We next address these two questions:

- when is the extended quotient norm a conventional norm?
- when is the quotient space discrete?

Answers to both questions fall immediately out of our next result.

Theorem 3.21. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space and let N be a closed subspace. Then the flats of finite extended quotient norm consist of $\{x+N : x \in X_{\text{fin}}\}$.*

Proof. Suppose $x+N$ has finite extended quotient norm. Then by definition, there exists $n \in N$ with $x+n \in X_{\text{fin}}$, and clearly $(x+n)+N = x+N$. Conversely, if $||x|| < \infty$, then $||x+N|| \leq ||x+0_X|| < \infty$. □

Corollary 3.22. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space and let N be a closed subspace. Then the extended quotient norm just assumes finite values if and only if $X = X_{\text{fin}} + N$.*

Proof. The condition says that each element p of X belongs to a flat of the form $x+N$ for some $x \in X_{\text{fin}}$. Thus

$$||p+N|| = ||x+N|| \leq ||x|| < \infty.$$

Conversely, suppose the quotient extended norm is a conventional norm. Fix $p \in X$; $||p+N|| < \infty$ means there exists $x \in X_{\text{fin}}$ and $n \in N$ with $p+(-n) = x$. □

Corollary 3.23. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space and let N be a closed subspace. Then the extended quotient norm is the discrete norm if and only if $X_{\text{fin}} \subseteq N$.*

Proof. Sufficiency follows immediately from the Theorem 3.21. Conversely, suppose we have a discrete extended quotient norm. Thus whenever $x \notin N$, we have $||x+N|| = \infty$. In particular, $||x|| = \infty$, i.e., $x \notin X_{\text{fin}}$. □

4. Continuous Linear Transformations

Our first result gives various characterizations of continuity, one of which involves the preservation of boundedness. In the context of extended normed linear spaces, we need a more inclusive notion of boundedness, if we wish our bounded sets to contain the singletons, be stable under finite unions, and be a hereditary family, that is, to form a *bornology* of subsets (see, e.g., [3, 7, 17]). Following

[2], we give the following definition used there in the setting of the extended Lipschitz norm to characterize compactness.

Definition 4.1. Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space. We call a subset A of X *bounded* if there exists $\{x_1, x_2, \ldots, x_n\} \subseteq X$ and $\{\mu_1, \mu_2, \ldots, \mu_n\} \subseteq (0, \infty)$ with $A \subseteq \cup_{i=1}^n B_d(x_i, \mu_i)$.

Obviously, each totally bounded set as usually understood [4, p. 332] is bounded in our sense. In a conventional normed linear space, a set lying in a finite union of balls lies in a single ball, i.e., our definition reduces to the usual one. The proof of the following alternate description of bounded set is left to the reader.

Proposition 4.2. *A subset A of an extended normed linear space $\langle X, ||\cdot||\rangle$ is bounded if and only if A hits at most finitely many metric components $x_1/\sim$, $x_2/\sim, \ldots, x_n/\sim$ and for some $\mu > 0$, $A \subseteq \cup_{j=1}^n B_d(x_j, \mu)$.*

One key message of our next result is that continuity for a linear transformation means that it maps bounded sets into bounded sets. Another is that its continuity is determined by its behavior restricted to X_{fin}.

Theorem 4.3. *Let $T : \langle X, ||\cdot||_1\rangle \to \langle Y, ||\cdot||_2\rangle$ be a linear transformation between extended normed linear spaces. The following conditions are equivalent:*

(*1*) $T \in \boldsymbol{B}(X, Y)$;
(*2*) $T|_{X_{\text{fin}}} \in \boldsymbol{B}(X_{\text{fin}}, Y_{\text{fin}})$;
(*3*) *T is continuous at 0_X;*
(*4*) *T maps bounded subsets of X into bounded subsets of Y;*
(*5*) *T maps sequences in X convergent to 0_X into bounded subsets of Y.*

Proof. (*1*) $\Rightarrow$ (*3*). This is trivial.

(*3*) $\Rightarrow$ (*4*). By continuity at the origin, choose $\delta > 0$ such that $||x||_1 < \delta \Rightarrow ||Tx||_2 < 1$. Using linearity, T maps each ball $B_d(p, \alpha)$ into $B_d(Tp, \frac{\alpha}{\delta})$, so T maps each bounded set as we have defined it into a bounded set.

(*4*) $\Rightarrow$ (*5*). If $\lim_{n\to\infty} x_n = 0_X$, then all but finitely many terms lie in a single ball, so that $\{x_n : n \in \mathbb{Z}^+\}$ lies in a finite union of balls.

(*5*) $\Rightarrow$ (*2*). Suppose T satisfies condition (*5*). Let $x \in X$ with $0 < ||x||_1 < \infty$ be arbitrary. If $||Tx||_2 = \infty$, then by Lemma 3.1, T maps the terms of $\langle \frac{1}{n}x\rangle$ to a set that fails to lie in finitely many metric components, let alone in a finite union of balls. This shows that T maps X_{fin} into Y_{fin}. Condition (*5*) for a transformation between conventional normed linear spaces is obviously enough to ensure continuity.

(*2*) $\Rightarrow$ (*1*). By ordinary linear analysis the restriction is Lipschitz continuous. If λ is a positive Lipschitz constant for the restriction, then anywhere in X we have $||x - w||_1 < \frac{\varepsilon}{\lambda} \Rightarrow ||Tx - Tw||_2 < \varepsilon$. Thus, T is continuous on X. $\square$

Corollary 4.4. *Let $\langle X, ||\cdot||_1\rangle$ and $\langle Y, ||\cdot||_2\rangle$ be extended normed linear spaces, and let $S \in \boldsymbol{B}(X_{\text{fin}}, Y_{\text{fin}})$. Suppose T in $\boldsymbol{L}(X, Y)$ and T extends S. Then T belongs to $\boldsymbol{B}(X, Y)$.*

It is worthwhile gaining an alternative understanding of this corollary. The values of an extension T of S throughout X are fully determined once the extension's values on some distance basis are given. In particular, prescribing the values of T on a distance basis $\{b_j : j \in \Delta\}$ dictates the value of T at the unique point $p_x \in \text{span}(\{b_j : j \in \Delta\})$ lying in each metric component $x/\sim$. From this, the values of T at all other points of $x/\sim$ are determined by S and $T(p_x)$, and continuity of T on the open set $x/\sim$ immediately follows from continuity of S.

Corollary 4.5. *Two extended norms on a linear space X are equivalent if and only if they determine the same families of bounded sets.*

Proof. The two topologies coincide if and only if the identity linear transformation is bicontinuous. □

Corollary 4.6. *Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space and let $\phi \in \boldsymbol{L}(X, \mathbb{F})$ have closed kernel. Then $\phi \in X^*$.*

Proof. Evidently, $\text{Ker}(\phi|_{X_{\text{fin}}}) = \text{Ker}(\phi) \cap X_{\text{fin}}$, so $\phi|_{X_{\text{fin}}}$ is continuous. □

There is another (weaker) possible way to define boundedness for subsets of an extended normed linear space that yields a bornology and reduces to the standard notion for conventional spaces [2]: E is *weakly bounded* if its intersection with each metric component is contained in a ball. Condition (4) of Theorem 4.3 does not force continuity if boundedness is replaced by weak boundedness.

Example 4.7. Let $||\cdot||_1$ be the Euclidean norm on $\mathbb{R}^2$ and let $||\cdot||_2$ be the discrete extended norm. Since the metric components of $\langle \mathbb{R}^2, ||\cdot||_2\rangle$ are singletons, each subset is weakly bounded. As a result, the identity map from $\langle \mathbb{R}^2, ||\cdot||_1\rangle$ to $\langle \mathbb{R}^2, ||\cdot||_2\rangle$ while mapping weakly bounded sets to weakly bounded sets is not continuous.

The last example brings to light a notable fact: a linear transformation between finite dimensional extended normed linear spaces need not be continuous! But a linear functional ϕ on an extended finite dimensional normed linear space obviously must be (see Corollary 3.16).

If T is a linear transformation between extended normed linear spaces X and Y, we put $||T||_{\text{op}} := \sup\{||Tx||_2 : ||x||_1 \leq 1\}$. Of course, $||T||_{\text{op}} = ||T|_{X_{\text{fin}}}||_{\text{op}}$, and so by Theorem 4.3, $||T||_{\text{op}} < \infty$ if and only if $T \in \mathbf{B}(X, Y)$. As a special case, unless $X_{\text{fin}} = \{0_X\}$, any projection operator onto a subspace that contains X_{fin} must have norm one. Notice also that if X is equipped with the discrete extended norm, then $||T||_{\text{op}} = 0$ for any linear transformation T defined on X. Thus, with our convention regarding $0 \times \infty$ in mind, the inequality $||Tx||_2 \leq ||T||_{\text{op}}||x||_1$

can fail. On the other hand, if $T \in \mathbf{L}(X,Y)$ and $||T||_{\rm op} > 0$, then the inequality is valid.

In view of Corollary 4.4, $||\cdot||_{\rm op}$ is a not a norm on $\mathbf{B}(X,Y)$ if $X_{\rm fin} \neq X$; rather, it is a seminorm. However, we will still call it the *operator norm* in the sequel. The abuse of language aside, our definition does not seem quite legitimate in that it totally ignores inputs of infinite norm. To put things right, we reconcile our definition with another possible one: the minimal Lipschitz constant over the entire space.

Theorem 4.8. *Let* $T : \langle X, ||\cdot||_1\rangle \to \langle Y, ||\cdot||_2\rangle$ *be a linear transformation between extended normed linear spaces. Then* $||T||_{\rm op} = \inf\{\lambda > 0 : \forall (x,w) \in X \times X,\ ||Tx - Tw||_2 \le \lambda ||x-w||_1\}$.

Proof. If $||T||_{op} = \infty$, then $\forall \mu > 0, \exists x \in X$ with $||x||_1 \le 1$ but $||Tx||_2 > \mu$, and so $||Tx - T0_X||_2 > \mu ||x - 0_X||_1$. Thus the infimum is ∞ as well. Otherwise, T maps $X_{\rm fin}$ into $Y_{\rm fin}$. As $||u||_1 = \infty \Rightarrow ||Tu||_2 \le \lambda ||u||_1$ for each positive λ, we compute

$$\begin{aligned} & \inf\{\lambda > 0 : \forall (x,w) \in X \times X,\ ||Tx - Tw||_2 \le \lambda ||x-w||_1\} \\ = \ & \inf\{\lambda > 0 : \text{whenever } ||x-w||_1 < \infty,\ ||Tx - Tw||_2 \le \lambda ||x-w||_1\} \\ = \ & \inf\{\lambda > 0 : \forall u \in X_{\rm fin},\ ||Tu||_2 \le \lambda ||u||_1\}. \end{aligned}$$

As is well-known [8, p. 62], this last expression agrees with $||T|_{X_{\rm fin}}||_{\rm op}$. □

We now give an expected result regarding evaluation functionals on X^*.

Proposition 4.9. *Let* $\langle X, ||\cdot||\rangle$ *be an extended normed linear space. For each* $x \in X$, *define* $\widehat{x} \in \boldsymbol{L}(X^*, \mathbb{F})$ *by* $\widehat{x}(\phi) = \phi(x)$. *Then*

$$||x|| = \sup\{|\phi(x)| : \phi \in X^* \text{ and } ||\phi||_{\rm op} \le 1\}.$$

Proof. We consider two cases for x: (i) $x \in X_{\rm fin}$, and (ii) $||x|| = \infty$. From conventional linear analysis, in case (i)

$$\begin{aligned} ||x|| = \ & \sup\{|\psi(x)| : \psi \in X^*_{\rm fin} \text{ and } ||\psi||_{\rm op} \le 1\} \\ = \ & \sup\{|\phi(x)| : \phi \in X^* \text{ and } ||\phi||_{\rm op} \le 1\} \end{aligned}$$

because each member of X^* is an extension of a member of $X^*_{\rm fin}$ with the same operator norm.

In case (ii), for each $n \in \mathbb{Z}^+$, there exists $\phi \in \mathbf{L}(X, \mathbb{F})$ taking each element of $X_{\rm fin}$ to 0 and x to n. By definition, $||\phi||_{\rm op} = 0$ while $|\phi(x)| = n$. Thus the supremum on the right is is ∞ as well. □

As mentioned earlier, our definition of $||\cdot||_{\rm op}$ produces only a seminorm on $\mathbf{B}(X,Y)$. As indicated in the Preliminaries, on an appropriate quotient of $\mathbf{B}(X,Y)$, it induces a norm. We have the following summary result whose proof is left to the reader.

Theorem 4.10. *Let $\langle X, ||\cdot||_1\rangle$ and $\langle Y, ||\cdot||_2\rangle$ be extended normed linear spaces. Put $\boldsymbol{N} := \{T \in \boldsymbol{B}(X,Y) : T$ restricted to X_{fin} is the zero transformation$\}$. Let N be a subspace complementary to X_{fin}, and for each $S \in \boldsymbol{B}(X_{\text{fin}}, Y_{\text{fin}})$ define $\overline{S} \in \boldsymbol{B}(X,Y)$ by $\overline{S}(m+n) = S(m)$ for $m \in X_{\text{fin}}$ and $n \in N$. Then $S \mapsto \overline{S} + \boldsymbol{N}$ is an isometric isomorphism of $\boldsymbol{B}(X_{\text{fin}}, Y_{\text{fin}})$ onto $\boldsymbol{B}(X,Y)/\boldsymbol{N}$.*

Convergence of a sequence or net of continuous linear transformations with respect to the operator norm topology obviously does not ensure pointwise convergence since one has no control over the transformations outside of X_{fin}. Ideally, one would like to have uniform convergence on bounded subsets as in the conventional setting. This is obtained if we also demand pointwise convergence. More precisely, we have the following result.

Theorem 4.11. *Let $\langle X, ||\cdot||_1\rangle$ and $\langle Y, ||\cdot||_2\rangle$ be extended normed linear spaces with $X \neq X_{\text{fin}}$ and let T be a continuous linear transformation from X to Y. Suppose $\langle T_\lambda\rangle$ is a net in $\boldsymbol{B}(X,Y)$ satisfying $\lim_\lambda ||T_\lambda - T||_{\text{op}} = 0$. The following conditions are equivalent:*

(*1*) $\langle T_\lambda\rangle$ *converges pointwise to T on some distance basis for X;*
(*2*) $\langle T_\lambda\rangle$ *converges pointwise to T;*
(*3*) $\langle T_\lambda\rangle$ *converges uniformly to T on each open ball;*
(*4*) $\langle T_\lambda\rangle$ *converges uniformly to T on bounded subsets of X.*

Proof. The string of implications (*3*) $\Rightarrow$ (*4*) $\Rightarrow$ (*2*) $\Rightarrow$ (*1*) is obvious, so it remains to prove (*1*) $\Rightarrow$ (*3*). If we have pointwise convergence on some distance basis $\{b_j : j \in \Delta\}$, then we have pointwise convergence on its linear span. Let $B_d(p,\alpha)$ be an open ball in X. By condition (*2*) of Theorem 4.3, we can find $w \in \text{span}(\{b_j : j \in \Delta\})$ and $\beta > 0$ such that $||p - w||_1 < \beta$. As a result $B_d(p,\alpha) \subseteq B_d(w, \alpha+\beta)$, and since we have $\lim_\lambda T_\lambda w = Tw$ and uniform convergence on $\{x : ||x||_1 < \alpha+\beta\}$ we have by linearity uniform convergence on $B_d(w, \alpha+\beta)$. □

If $\langle T_\lambda\rangle$ converges uniformly on bounded subsets of X to T, then in particular this holds for bounded subsets of X_{fin} so that by conventional linear analysis, $\lim_\lambda ||T_\lambda - T||_{\text{op}} = 0$. To summarize: operator norm convergence plus pointwise convergence is equivalent to uniform convergence on bounded subsets of X.

The topology of uniform convergence on bounded sets τ_{ucb} is determined by the uniformity $\mathscr{U}_{ucb}$ on $\mathbf{B}(X,Y)$ having as a base all sets of the form

$$[B;\varepsilon] := \{(T_1,T_2) : \forall x \in B,\ ||T_1x - T_2x||_2 < \varepsilon\},\quad B \text{ bounded},\ \varepsilon > 0.$$

Unless $X = X_{\text{fin}}$, there is no countable cofinal family of bounded subsets with respect to inclusion (because there are uncountably many distinct metric components); so, the uniformity cannot be metrizable [18, p. 257]. For this reason, the last result was stated in terms of nets rather than sequences.

When the space $\langle Y, ||\cdot||_2\rangle$ is Cauchy complete, the uniformity $\mathscr{U}_{ucb}$ is a complete uniformity [18, p. 260], i.e., each Cauchy net with respect to $\mathscr{U}_{ucb}$ is τ_{ucb}-convergent to some continuous linear transformation. We leave verification of this statement to the reader, but indicate the key points in the argument:

- if $\langle T_\lambda\rangle$ is $\mathscr{U}_{ucb}$-Cauchy, then for each $x \in X$, $\langle T_\lambda x\rangle$ is a Cauchy net in Y;
- each Cauchy net in Y is convergent (see, e.g., [18, p. 261]);
- a pointwise limit of a net of linear transformations is again linear;
- if a net of continuous functions converges uniformly on bounded subsets, then the limit is continuous.

If $T \in \mathbf{B}(X, Y)$ where both $X_{\text{fin}} \neq X$ and $Y_{\text{fin}} \neq Y$, then $T(X\backslash X_{\text{fin}})$ can be (i) contained in Y_{fin} , or (ii) contained in $Y\backslash Y_{\text{fin}}$, or (iii) intersect both. Since the zero transformation satisfies (i), we just give concrete examples of the other two.

Example 4.12. Define a norm one projection operator T on $\langle C([0,\infty),\mathbb{R}), ||\cdot||_\infty\rangle$ by

$$(Tf)(x) = \begin{cases} f(x) & \text{if } 1 \leq x \\ f(1) & \text{otherwise} \end{cases}$$

Evidently, T maps each element of infinite norm to another element of infinite norm.

Here is an example of a different flavor where $T(X\backslash X_{\text{fin}})$ hits both sets.

Example 4.13. Our transformation T will be defined on $\langle C([0,\infty),\mathbb{R}), ||\cdot||_L\rangle$ with distinguished base point $x_0 = 0$ and take values in $\langle C([0,\infty),\mathbb{R}), ||\cdot||_1\rangle$ where $||f||_1 := \int_0^\infty |f(x)|\, dx$. Define T by $(Tf)(x) = e^{-x}f(x)$. If $||f||_L \leq 1$, then for all $x \geq 0$, we have

$$-1 - x \leq f(x) \leq 1 + x,$$

and so $||T||_1 = \int_0^\infty (1+x)e^{-x}\, dx = 2$.

Notice that T maps certain elements of infinite extended Lipschitz norm to values with finite norm, e.g., $f(x) = x^2$, whereas it sends others to values with infinite norm, e.g., $f(x) = e^x$.

This discussion bring us to the following definition.

Definition 4.14. Let $T : \langle X, ||\cdot||_1\rangle \to \langle Y, ||\cdot||_2\rangle$ be a continuous linear transformation By the *effective domain* of T we mean $\text{dom}(T) := \{x \in X : ||Tx||_2 < \infty\} = T^{-1}(Y_{\text{fin}})$.

Our usage is inspired by similar usage by the unilateral analysis community (see, e.g., [1, 11, 13]); more precisely, our effective domain is the effective domain in their understanding of the proper lower semicontinuous convex function $x \mapsto ||Tx||_2$.

By continuity, the effective domain of T at least contains X_{fin}. Our next result characterizes those subsets of X that can so serve. Recall that if M is a subspace of a vector space X, then any two complementary subspaces have the same dimension as they are both linearly isomorphic to X/M. This common dimension is called the *codimension* of M, and is denoted by $\text{codim}(M)$.

Theorem 4.15. *Let $\langle X, ||\cdot||_1\rangle$ and $\langle Y, ||\cdot||_2\rangle$ be extended normed linear spaces where $Y \neq Y_{\text{fin}}$ and let $E \subseteq X$ be nonempty. Then $E = \text{dom}(T)$ for some $T \in \boldsymbol{B}(X, Y)$ if and only if E is a linear subspace of X containing X_{fin} with $\text{codim}(E) \leq \text{codim}(Y_{\text{fin}})$.*

Proof. Starting with necessity, clearly $\text{dom}(T)$ is a subspace containing X_{fin}. Suppose $\text{codim}(E) > \text{codim}(Y_{\text{fin}})$. Let B be a basis for a subspace N complementary to E; then each element of B is sent to a vector with infinite norm. Clearly, $\{Tb : b \in B\}$ must be linearly independent, else T would map some $n \in N$ other than the origin into Y_{fin} which is impossible. The dimensionality restriction now implies $T(N) \cap Y_{\text{fin}}$ properly contains $\{0_Y\}$, and again we have a contradiction.

We turn to sufficiency. If $E = X$, we can use the transformation that sends each vector to the origin. Otherwise, let B_1 complete a basis of E to a basis for X and let B_2 be a distance basis for Y. Clearly, $X = E \oplus \text{span}(B_1)$. By the assumption on codimensions, there is an injection $\phi : B_1 \to B_2$. Let T send each element of E to 0_Y and extend ϕ linearly on $\text{span}(B_1)$. By condition *(3)* of Theorem 3.20, T maps each nontrivial linear combination of elements of B_1 to a vector of infinite norm, and as a result $E = \text{dom}(T)$. Finally, T is continuous restricted to E and maps E into Y_{fin}, so X_{fin} inherits these properties. By Corollary 4.4, T does the job. $\square$

Notice that there is no need to specify that the subspace E be either or closed or open in the hypotheses of Theorem 4.15, as by Corollary 3.9, these are both guaranteed by the inclusion $X_{\text{fin}} \subseteq E$.

5. The big four

We first consider the uniform boundedness principle for extended normed linear spaces. If both domain and codomain contain elements of infinite norm, then no pointwise boundedness condition over the entire domain can be necessary for uniform boundedness for a family of transformations.

Example 5.1. Suppose $\langle X, ||\cdot||_1\rangle$ contains x_0 with $||x_0||_1 = \infty$ and $\langle Y, ||\cdot||_2\rangle$ contains y_0 with $0 < ||y_0||_2 < \infty$. Let B be a distance basis for X containing x_0, and for each $n \in \mathbb{Z}^+$ let $T_n \in \mathbf{B}(X, Y)$ be the unique linear transformation (1) mapping x_0 to ny_0 and (2) mapping $X_{\text{fin}} \cup \{b \in B : b \neq x_0\}$ to $\{0_Y\}$. Then

$\forall n \in \mathbb{Z}^+, ||T_n||_{\text{op}} = 0$ while $\{T_n x_0 : n \in \mathbb{Z}^+\}$ can lie in no weakly bounded subset of Y.

Theorem 5.2. *Let $\langle X, ||\cdot||_1\rangle$ and $\langle Y, ||\cdot||_2\rangle$ be extended normed linear spaces where $||\cdot||_1$ is Cauchy complete. Let $\Omega \subseteq \boldsymbol{B}(X,Y)$, and suppose there exists some nonempty open set V such that for each $x \in V, \{Tx : T \in \Omega\}$ lies in some common bounded set B_x of Y. Then there exists $\alpha \in \mathbb{R}$ such that $\forall T \in \Omega, ||T||_{\text{op}} < \alpha$.*

Proof. Since X has a compatible complete metric, namely

$$\rho(x,w) = \min\{1, ||x-w||_1\},$$

and each open subspace of a completely metrizable space is again completely metrizable [4, p. 342], we are free to apply Baire's theorem within V.

Without loss of generality, we may assume that V is an open ball with center p, thus lying in the metric component $p/\sim$. Now B_p only intersects finitely many metric components of Y, say $\{y_1/\sim, y_2/\sim, \ldots, y_n/\sim\}$. As a result, if $T \in \Omega$, choosing $i \leq n$ with $Tp \in y_i/\sim$, we have $T(p/\sim) \subseteq y_i/\sim$ because by Theorem 3.5, $y_i/\sim$ is a connected component of Y. In particular we have $T(V) \subseteq y_i/\sim$.

For each $i \leq n$, put $\Omega_i := \{T \in \Omega : T(V) \subseteq y_i/\sim\}$. Since $\{\Omega_i : i \leq n\}$ partitions Ω, we will be done if we can show

$$\sup\{||T||_{\text{op}} : T \in \Omega_i\} < \infty \quad \text{for } i = 1, 2, 3, \ldots, n.$$

Fix $i \leq n$; by the pointwise boundedness assumption, $\forall x \in V, \exists n_x \in \mathbb{Z}^+$ such that $\forall T \in \Omega_i, T(x) \in \{y \in Y : ||y - y_i||_2 \leq n_x\}$ where n_x can be chosen independent of T. For each positive integer n, let E_n be the relatively closed subset of V defined by

$$E_n := \{x \in V : \forall T \in \Omega_i,\ ||Tx - y_i||_2 \leq n\}.$$

By Baire's theorem applied in $V, \exists n \in \mathbb{Z}^+, \exists x \in V, \exists \delta > 0$ with $B_d(x,\delta) \subseteq E_n$. This means that each $T \in \Omega_i$ maps $B_d(x,\delta)$ into $\{y \in Y : ||y - y_i||_2 \leq n\}$. It is a routine computation to now verify that for all $T \in \Omega_i$, $||T||_{\text{op}} \leq \frac{2n}{\delta}$. □

Obviously, the usual statement of the uniform boundedness principle [8, p. 134] falls out as a special case of Theorem 5.2. But the usual statement is not sufficiently general for our extended context, as it is only applicable to a subset of $\mathbf{B}(X,Y)$ each member of which takes values only in Y_{fin}. We also note that if the open set V in the statement of Theorem 5.2 contains X_{fin}, then the hypotheses of the classical result are satisfied for $\{T|_{X_{\text{fin}}} : T \in \Omega\}$ which gives uniform boundedness of the family of restrictions and thus uniform boundedness of the family itself.

Next comes the closed graph theorem.

Theorem 5.3. *Let* $\langle X, ||\cdot||_1\rangle$ *and* $\langle Y, ||\cdot||_2\rangle$ *be extended Banach spaces and let* $T \in \boldsymbol{L}(X,Y)$ *have closed graph and map* X_{fin} *into* Y_{fin}. *Then* T *is continuous.*

Proof. Since X_{fin} is closed, the restriction of T to X_{fin} has closed graph as well, as its graph is the intersection of $\pi_1^{-1}(X_{\text{fin}})$ with the graph of T. Since it maps the Banach space X_{fin} into the Banach space Y_{fin}, the result follows from the classical closed graph theorem [8, p. 140] and Corollary 4.4. □

Example 5.4. Let $\langle Y, ||\cdot||_2\rangle$ be an extended Banach space where $Y \neq Y_{\text{fin}}$, and let y_0 be a vector of infinite norm. Equip $\mathbb{F}$ with the Euclidean norm and define $T : \mathbb{F} \to Y$ by $T(\alpha) = \alpha y_0$. Then the graph of T is closed because it is uniformly discrete set with respect to the product norm $\max\{|\cdot|, ||\cdot||_2\}$, but T is not continuous.

We proceed to an extension of the Banach-Schauder open mapping theorem.

Theorem 5.5. *Let* $\langle X, ||\cdot||_1\rangle$ *and* $\langle Y, ||\cdot||_2\rangle$ *be extended Banach spaces and let* $T \in \boldsymbol{B}(X,Y)$ *map* X_{fin} *onto* Y_{fin}. *Then* T *maps open subsets of* X *to open subsets of* Y.

Proof. We need only show that T maps each open ball $B_d(p,\alpha)$ in X onto an open subset of Y. Since $T|_{X_{\text{fin}}}$ satisfies the hypotheses of the classical open mapping theorem, T maps $B_d(0_X,\alpha)$ onto to some relatively open subset V of Y_{fin}, and since Y_{fin} is itself an open set, V is open in Y. Evidently, T maps $B_d(p,\alpha)$ onto $T(p)+V$ which is open because $y \mapsto y+T(p)$ is a homeomorphism of Y. □

Surjectivity of T in the last theorem is an inadequate hypothesis.

Example 5.6. Returning to the spaces of Example 4.7, the identity map from $\langle \mathbb{R}^2, ||\cdot||_2\rangle$ to $\langle \mathbb{R}^2, ||\cdot||_1\rangle$ is surjective and continuous but fails to be open.

We close this section with an unremarkable extension of the Hahn-Banach theorem to our context.

Theorem 5.7. *Let* $\langle X, ||\cdot||\rangle$ *be an extended normed linear space with nontrivial proper subspace* M, *and suppose* $\phi \in M^*$. *Then there is a continuous linear extension* $\widehat{\phi} \in X^*$ *such that* $||\widehat{\phi}||_{\text{op}} = ||\phi||_{\text{op}}$.

Proof. Put $M_1 = M \cap X_{\text{fin}} = M_{\text{fin}}$; by definition, $||\phi|_{M_1}||_{\text{op}} = ||\phi||_{\text{op}}$. Choose a subspace M_2 such that $M_1 \oplus M_2 = M$, noting that $M_2 \cap X_{\text{fin}} = \{0_X\}$. Finally choose a subspace N with $(M + X_{\text{fin}}) \oplus N = X$. By the conventional result [8, p. 20], let ψ be a Hahn-Banach extension of $\phi|_{M_1}$ to X_{fin}. The desired (well-defined) norm-preserving $\widehat{\phi}$ is the unique linear functional that (i) agrees with ψ on X_{fin}, (ii) agrees with ϕ on M_2, and (iii) is zero on N. □

Our Hahn-Banach theorem may be restated as a sandwich-extension theorem as follows: whenever $||\phi||_{\text{op}} \in (0,\infty)$, ϕ has a linear extension $\widehat{\phi}$ such that for all $x \in X$, $-||\phi||_{\text{op}}||x||_1 \leq \widehat{\phi}(x) \leq ||\phi||_{\text{op}}||x||_1$. One may ask: can one always do the same if $x \mapsto ||\phi||_{\text{op}}||x||_1$ is replaced by a proper lower semicontinuous sublinear functional $g : X \to [0,\infty]$? But as Simons [16, p. 116] has observed, even when $X = \mathbb{R}^2$ equipped with the Euclidean norm, such a linear extension may not be possible if $x \mapsto ||\phi||_{\text{op}}||x||_1$ is replaced by a continuous nonnegative real-valued sublinear functional g such that ϕ is sandwiched between $-g$ and g on M.

6. More on projections

The proof of the following result is standard and is left to the reader.

Proposition 6.1. *Suppose P is a continuous projection operator on the extended normed linear space $\langle X, ||\cdot||\rangle$. Then both $P(X)$ and* Ker(P) *are closed and $X = P(X) \oplus \text{Ker}(P)$.*

It is a little disconcerting that a projection mapping on a Cauchy complete extended normed linear space onto a closed subspace M as determined by a closed complement N need not be continuous.

Example 6.2. Let $X = \mathbb{R}^2$ be equipped with the Cauchy complete extended norm defined by

$$||x|| = \begin{cases} |\alpha| & \text{if } x = (\alpha, 0) \\ \infty & \text{otherwise.} \end{cases}$$

Let $M = \{(\alpha, -\alpha) : \alpha \in \mathbb{R}\}$ and let $N = \{(\alpha,\alpha) : \alpha \in \mathbb{R}\}$. As $M \cap X_{\text{fin}} = N \cap X_{\text{fin}} = \{0_X\}$, by Proposition 3.15 M and N are both closed subspaces. Clearly $X = M \oplus N$. Define $P : X \to M$ by $P(m+n) = m$ where $m \in M, n \in N$. Since

$$(\alpha, 0) = \frac{\alpha}{2}(1,-1) + \frac{\alpha}{2}(1,1) \in M + N,$$

P is injective on X_{fin} whose topology is the usual topology of line. By Lemma 3.1 the topology of M is discrete, and so the projection is not continuous on the horizontal axis.

Example 6.2 leads to the next definition.

Definition 6.3. Let M and N be complementary closed subspaces of the extended normed linear space $\langle X, ||\cdot||\rangle$. We say that M is a *projection complement* of N if $m + n \mapsto m$ where $m \in M$ and $n \in N$ is continuous.

Note that if M is a projection complement of N, then N is a projection complement of M. Returning to Example 6.2, note also that $\{(\alpha, 0) : \alpha \in \mathbb{R}\}$ is a projection complement of the subspace N while M is not.

Example 6.4. Let $\langle X, ||\cdot||\rangle$ be an extended normed linear space where $X \neq X_{\text{fin}}$. Let N be a (closed) subspace complementary to X_{fin}, and put $M = X_{\text{fin}}$. For each $m \in M$ and $n \in N$, we have $||m|| \leq ||m+n||$ so that M is a projection complement of N. Notice that $m + n \mapsto m$ is actually an open mapping as the ball $B_d(m+n, \alpha)$ is mapped onto $B_d(m, \alpha)$. In conventional linear analysis, a continuous projection onto a proper subspace M cannot be an open mapping, else M would be clopen. Finally, note that $m \mapsto m + N$ is an isometric isomorphism of X_{fin} onto X/N.

The proof in the conventional setting of the following result goes through without modification.

Proposition 6.5. *Suppose for an extended normed linear space $\langle X, ||\cdot||\rangle$ that N, M are closed complementary linear subspaces. Then M is a projection complement of N if and only if $(m, n) \mapsto m + n$ is a homeomorphism of $M \times N$ equipped with the product topology onto X.*

Proposition 6.6. *Let N be a finite dimensional subspace of an extended normed linear space $\langle X, ||\cdot||\rangle$. Then N has a projection complement.*

Proof. If $N = \{0_X\}$, this is trivial. Otherwise, let $\{b_1, b_2, \ldots, b_n\}$ be a basis for the closed subspace N. By our extended Hahn-Banach theorem, for each $j \leq n$ let ϕ_j be a continuous linear functional mapping b_j to 1 and b_i to 0 for $i \neq j$. Then

$$x \mapsto \phi_1(x)b_1 + \phi_2(x)b_2 + \cdots + \phi_n(x)b_n$$

is a continuous projection on X with range N. $\square$

Proposition 6.7. *Let N be a closed subspace of finite codimension of an extended normed linear space $\langle X, ||\cdot||\rangle$. Then N has a projection complement.*

Proof. This does not require the Hahn-Banach theorem and goes through the quotient space X/N. Starting with a basis $\{b_1, b_2, \ldots, b_n\}$ for the complementary subspace, consider the unique (continuous) linear functional ϕ on X/N mapping $b_j + N$ to one and $b_i + N$ to zero for $i \neq j$. Then compose each ϕ_j with the norm one quotient map $x \mapsto x + N$ and carry on as before. $\square$

We next present two definitive results in the context of Cauchy complete spaces regarding the existence of a projection complement.

Theorem 6.8. *Let N, M be closed complementary linear subspaces of an extended Banach space $\langle X, ||\cdot||\rangle$. The following conditions are equivalent:*

(*1*) *M is a projection complement of N;*
(*2*) $X_{\text{fin}} = (M \cap X_{\text{fin}}) \oplus (N \cap X_{\text{fin}})$.

Proof. Suppose the projection $P : X \to X$ defined by $P(m + n) = m$ is continuous. Let $x \in X_{\text{fin}}$ be arbitrary; we can write $x = m + n$ where $m \in M$

and $n \in N$. By Theorem 4.3, $P|_{X_{\text{fin}}}$ maps X_{fin} into itself, so $||m|| < \infty$ from which $||n|| \leq ||x|| + ||m|| < \infty$ as well.

Conversely, suppose $X_{\text{fin}} = (M \cap X_{\text{fin}}) \oplus (N \cap X_{\text{fin}})$. Since X_{fin} is a conventional Banach space,

$$m + n \mapsto m \qquad m \in M \cap X_{\text{fin}},\ n \in N \cap X_{\text{fin}}$$

is continuous. By Theorem 4.3, P is continuous. □

Theorem 6.9. *Let N be a closed linear subspace of an extended Banach space $\langle X, || \cdot || \rangle$. Then N has a projection complement if and only if there is a closed subspace W of X_{fin} with $X_{\text{fin}} = W \oplus (N \cap X_{\text{fin}})$.*

Proof. Necessity is a consequence of Theorem 6.8, while sufficiency follows from Theorem 3.19 and Theorem 6.8. □

With Example 6.2 in mind, we pose this open question: if a closed subspace N has a closed complement, must it have a projection complement as well?

Let N be a proper closed linear subspace of an extended normed linear space $\langle X, || \cdot || \rangle$. If $X_{\text{fin}} \subseteq N$ and M is a complementary subspace to N, then $|| \cdot ||$ restricted to M is the discrete extended norm and so each linear transformation on M has operator norm zero. That possibility aside, one is interested in the existence of a complementary (closed) subspace M such that the projection map $m + n \mapsto m$, where $m \in M$ and $n \in N$, has operator norm one. Equivalently, this would give a complementary subspace M such that $m \mapsto m + N$ is an isometric isomorphism of M onto X/N.

Theorem 6.10. *Let N be a proper closed linear subspace of an extended normed linear space $\langle X, || \cdot || \rangle$ where $X_{\text{fin}} \nsubseteq N$. Then N has a (closed) complementary subspace M such that $m+n \mapsto m$ is a norm one projection if and only if $N \cap X_{\text{fin}}$ has the same properties relative to X_{fin}.*

Proof. Suppose $X = M \oplus N$ where for each $m \in M$ and $n \in N$ we have $||m|| \leq ||m+n||$. Since $m+n \mapsto m$ is nonexpansive, the projection is continuous, and by Theorem 6.8

$$X_{\text{fin}} = (M \cap X_{\text{fin}}) \oplus (N \cap X_{\text{fin}}).$$

That $||m+n|| \geq ||m||$ within X_{fin} is obvious, and this implies the projection has norm one because $M \cap X_{\text{fin}}$ is nontrivial. The converse follows from Theorem 3.19, noting that the norm of the projection restricted to X_{fin} determines its norm as an operator on X. □

Example 6.11. On $X = \mathbb{R}^5$ consider the extended norm defined by

$$||x|| = \begin{cases} \sqrt{\alpha^2 + \beta^2} & \text{if } x = (\alpha, \beta, 0, 0, 0) \\ \infty & \text{otherwise.} \end{cases}$$

Evidently $X_{\text{fin}} = \{(\alpha, \beta, 0, 0, 0) : (\alpha, \beta) \in \mathbb{R}^2\}$. Next consider the subspace

$$N := \text{span}((1, 0, 0, 0, 0), (0, 0, 2, 3, -1), (0, 0, -5, 0, 1)).$$

Note that $X_{\text{fin}} \cap N = \{(\alpha, 0, 0, 0, 0) : \alpha \in \mathbb{R}\}$, and this has a complement of the desired form in X_{fin}, namely $\{(0, \beta, 0, 0, 0) : \beta \in \mathbb{R}\}$. Thus by Theorem 6.10, N has a complement M such that $\forall m \in M,\ \forall n \in N,\ ||m|| \leq ||m + n||$.

7. On line segments

Suppose x and w are distinct points of an extended normed linear space $\langle X, ||\cdot||\rangle$; then the line segment joining x to w is

$$\text{seg}(\{x, w\}) := \{\alpha w + (1 - \alpha)x : 0 \leq \alpha \leq 1\}.$$

If $||x - w|| < \infty$, then $\alpha \mapsto \alpha w + (1 - \alpha)x$ is continuous so that the segment is compact with respect to the norm topology. But if $||x-w|| = \infty$, then by Lemma 3.1, the relative topology is discrete and the segment fails to be compact. The purpose of this short section is to show that the segment nevertheless is weakly compact.

This presents an opportunity to use the deep Alexander subbase theorem [18, p. 129], of which the Tychonoff theorem on compactness of products is a trivial consequence.

Alexander subbase theorem. Let $\mathscr{A}$ be a subbase of a topological space $\langle X, \tau\rangle$ such that each open cover of X by some subfamily of $\mathscr{A}$ has a finite subcover. Then $\langle X, \tau\rangle$ is itself compact.

The subbase we intend to use for the weak topology is $\{\phi^{-1}((\mu, \infty)) : \phi \in X^*,\ \mu \in \mathbb{R}\}$.

Proposition 7.1. *Suppose x and w are distinct points of the extended normed linear space $\langle X, ||\cdot||\rangle$. Then* seg$(\{x, w\})$ *is weakly compact.*

Proof. Let $\mathscr{A}_0 \subseteq \{\phi^{-1}((\mu, \infty)) : \phi \in X^*,\ \mu \in \mathbb{R}\}$ be a cover of the segment. If $\{x, w\}$ is contained in any member of $\mathscr{A}_0$, we are done. Otherwise, for each $\alpha \in [0, 1]$, choose $\phi_\alpha \in X^*$ and $\mu_\alpha \in \mathbb{R}$ with $\phi_\alpha^{-1}((\mu_\alpha, \infty)) \in \mathscr{A}_0$ and

$$\phi_\alpha(\alpha w + (1 - \alpha)x) > \mu_\alpha.$$

Now either $\phi_\alpha(x) \leq \mu_\alpha$ or $\phi_\alpha(w) \leq \mu_\alpha$. In the first case, there is $\beta < \alpha$ in [0,1] such that

$$\gamma \in (\beta, 1] \Rightarrow \phi_\alpha(\gamma w + (1 - \gamma)x) > \mu_\alpha.$$

In the second, there is $\beta > \alpha$ in [0,1] such that

$$\gamma \in [0, \beta) \Rightarrow \phi_\alpha(\gamma w + (1 - \gamma)x) > \mu_\alpha.$$

This sets up a relatively open cover of [0,1] by half-open intervals, a finite subcover of which leads to a finite subcover of $\mathscr{A}_0$ of seg$(\{x, w\})$. □

An alternative proof can be given by showing that the straight-line path $\alpha \mapsto \alpha w + (1-\alpha)x$ is continuous when X is equipped with the weak topology, which also gives path connectedness of the space.

Suppose $||x - w|| = \infty$; although $\mathrm{seg}(\{x, w\})$ is weakly compact, it is not bounded. However, it is weakly bounded. While one can show that each weakly compact subset of X is weakly bounded, it is outside the scope of this paper to consider weak topologies more generally. We confine ourselves to our one result.

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