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Existence of Many Nonradial Positive Solutions of the Hénon Equation in $\mathbf{R}^{3*}$

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Let B_1 be the open unit ball in $\mathbb{R}^3$ and let $2 < p < 6$. We show that for each $m \in \mathbb{N}$, there exists $\alpha_0 > 0$ such that for each $\alpha \geq \alpha_0$, there exist at least m nonradial positive solutions of

$$-\Delta u = |x|^\alpha |u(x)|^{p-2} u(x) \ \text{ in } B_1,\ u = 0 \text{ on } \partial B_1,$$

which are mutually nonequivalent if $m \geq 2$.

Keywords: Hénon equation, multiplicity of positive solutions, concentration compactness principle, Poincaré's inequalities

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1. Introduction

In this paper, we study the problem

$$\begin{cases} -\Delta u = |x|^\alpha |u(x)|^{p-2} u(x) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1}$$

where $B_r = \{x \in \mathbb{R}^N : |x| < r\}$, $N \geq 2$, $\alpha > 0$ and $p > 2$. The equation was introduced by Hénon [11] to study rotating stellar structures, and many researchers studied it; see [1, 3, 4, 5, 8, 12, 13, 14, 18, 19, 20, 22]. By changing u to $\alpha^{2/(p-2)} u(\alpha x)$, we can see that (1) is equivalent to

$$\begin{cases} -\Delta u = \left|\frac{x}{\alpha}\right|^\alpha |u(x)|^{p-2} u(x) & \text{in } B_\alpha, \\ u = 0 & \text{on } \partial B_\alpha. \end{cases} \tag{2}$$

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Since for each $x \in \mathbb{R}^N$, $|x/\alpha|^\alpha \to 0$ as $\alpha \to \infty$, we can see that problem (2) with sufficiently large α is similar to the problem

$$\begin{cases} -\Delta u = |u(x)|^{p-2}u(x) & \text{in } A_R, \\ u = 0 & \text{on } \partial A_R \end{cases} \tag{3}$$

with sufficiently large R, where $A_R = \{x \in \mathbb{R}^N : R-1 < |x| < R+1\}$. Coffman [7] studied problem (3) in the case of $N = 2$ and $2 < p < \infty$, and he showed that if R is sufficiently large, the least energy solution is not radially symmetric and if R tends to infinity, the number of nonradial positive solutions increase to infinity. He considered the $2\pi/k$-rotation invariant function space with $k \in \mathbb{N}$, and he showed that for each $k \in \mathbb{N}$, if R is sufficiently large, then the least energy level c_k for the problem in the $2\pi/k$-rotation invariant space is strictly less than the least energy level c_∞ for the positive radial solution and it holds $c_1 < \cdots < c_k(< c_\infty)$. Li [15] obtained a similar result for (3) in the case of $N \geq 4$ and $2 < p < 2N/(N-2)$. Moreover, even in the case $2N/(N-2) \leq p < (2N-2)/(N-3)$, he showed that there exist $\lfloor N/2 \rfloor - 1$ nonradial positive solutions. The case $N = 3$ is different from the cases $N = 2$ or $N \geq 4$. Indeed, even if we consider the function space which is odd for one direction and $2\pi/k$-rotation invariant with respect to the direction, the least energy level c_k for the space satisfies $c_2 = c_3 = \cdots$. Byeon [2] overcame this difficulty; see also [6]. Instead of the least energy solutions in the function space, he obtained non local minimal energy solutions, and he showed that even in the case of $N = 3$, for each $k \in \mathbb{N}$, if R is sufficiently large, there exist at least k nonradial positive solutions.

On the other hand, for problem (1), Smets-Willem-Su [22] showed that in the case $N \geq 2$ and subcritical p, if α is large enough then the least energy solution is nonradial; see also the results by Byeon-Wang [4] and Cao-Peng-Yan [5]. Badiale-Serra [1] obtained at least $\lfloor N/2 \rfloor - 1$ nonradial positive solutions in the case that $N \geq 4$ and $p \leq (2N-2)/(N-3)$. Kolonitskii-Nazarov [14] studied the results corresponding to those obtained by Coffman [7] and Li [15] in the case when p is subcritical; see also [21] in the case $N = 2$. However, the case $N = 3$ has not studied yet. Compared with problem (2), problem (3) has the following properties. The Poincaré inequality holds on A_R uniformly for $R > 2$, i.e.,

$$\inf_{R>2} \inf_{u \in \mathcal{D}_0^{1,2}(A_R)\setminus\{0\}} \frac{\int_{A_R} |\nabla u|^2\, dx}{\int_{A_R} |u|^2\, dx} > 0.$$

Each least energy solution u of the natural limit problem for (3)

$$\begin{cases} -\Delta u = |u(x)|^{p-2}u(x) & \text{in } \mathbb{R}^2 \times (-1,1), \\ u = 0 & \text{on } \mathbb{R}^2 \times \{-1,1\} \end{cases}$$

decays exponentially as $\sqrt{x_1^2 + x_2^2} \to \infty$. These properties played important roles in [2]. Although problem (2) does not seem to have such properties, we can show a similar result for problem (1) in the case $N = 3$.

Our result is the following. We say $u, v : B_\alpha \to \mathbb{R}$ are nonequivalent if there does not exist $g \in O(3)$ such that g is not an identity and $u(x) = v(gx)$ for any $x \in B_\alpha$.

Theorem 1.1. *Let $N = 3$ and let $2 < p < 6$. For each $m \in \mathbb{N}$, there exists $\alpha_0 > 0$ such that for each $\alpha \geq \alpha_0$, there exist at least m nonradial positive solutions of (1), which are mutually nonequivalent if $m \geq 2$.*

In order to prove the theorem above, we introduce some variants of the Poincaré inequality. As we mentioned, the case $m = 1$ was studied in [4, 5, 22]. So we prove it for $m \geq 2$ in the following sections.

2. Preliminaries

Let $N \in \mathbb{N}$ with $N \geq 3$. We set $2^* = 2N/(N-2)$ and we denote by $\mathbb{R}^N_-$ the set $\{x = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N : x_N < 0\}$. For each open subset $\Omega \subset \mathbb{R}^N$, we denote by $\mathcal{D}^{1,2}_0(\Omega)$ the completion of $C^\infty_0(\Omega)$ with respect to the inner product $\int_\Omega \nabla u \nabla v \, dx$, and we consider that $\mathcal{D}^{1,2}_0(\Omega) \subset \mathcal{D}^{1,2}_0(\mathbb{R}^N)$ by the zero extension.

Let $2 < p < 2^*$. We set

$$R_p(u) = \frac{\int_{\mathbb{R}^N_-} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^N_-} e^{x_N} |u|^p \, dx\right)^{\frac{2}{p}}} \quad \text{for } u \in \mathcal{D}^{1,2}_0(\mathbb{R}^N_-) \setminus \{0\},$$

$$m_p = \inf\{R_p(u) : u \in \mathcal{D}^{1,2}_0(\mathbb{R}^N_-) \setminus \{0\}\},$$

$$\mathcal{K} = \{u \in \mathcal{D}^{1,2}_0(\mathbb{R}^N_-) \setminus \{0\} : u > 0 \text{ in } \mathbb{R}^N_- \text{ and } R_p(u) = m_p\}.$$

By [5, Theorem 2.1], we know $\mathcal{K} \neq \emptyset$; see also [4]. For $x \in \mathbb{R}^N_-$, we write $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$. For the proof of the following lemma, see [4, Proposition 2.5].

Lemma 2.1. *For each $u \in \mathcal{K}$, there is $y' \in \mathbb{R}^{N-1}$ such that $u(x) = u(|x' - y'|, x_N)$ for each $x = (x', x_N) \in \mathbb{R}^N_-$.*

We set

$$\mathcal{K}_0 = \{u \in \mathcal{K} : u(x) = u(|x'|, x_N) \text{ for } x = (x', x_N) \in \mathbb{R}^N_-\}.$$

The following is a variant of the Poincaré inequality.

Lemma 2.2. *Let $\alpha, \beta > 0$. Then for each $v \in \mathcal{D}^{1,2}_0(B_\alpha)$, there holds*

$$\int_{B_\alpha} \left|\frac{x}{\alpha}\right|^\beta v^2 \, dx \leq \frac{\alpha^2}{(\beta+2)(\beta+N)} \int_{B_\alpha} |\nabla v|^2 \, dx. \tag{4}$$

Proof. We use the polar coordinates $(r,\Theta) \in [0,\infty) \times S^{N-1}$. Let $v \in C_0^\infty(B_\alpha)$. From $v(r,\Theta) = -\int_r^\alpha \frac{\partial v}{\partial r}(t,\Theta)\,dt$, we have

$$|v(r,\Theta)| \le \left(\int_r^\alpha \left|\frac{\partial v}{\partial r}(t,\Theta)\right|^2 t^{N-1}\,dt\right)^{\frac{1}{2}} \left(\frac{1}{N-2}(r^{2-N}-\alpha^{2-N})\right)^{\frac{1}{2}}.$$

Hence we obtain

$$\begin{aligned}
&\int_0^\alpha \left(\frac{r}{\alpha}\right)^\beta v(r,\Theta)^2\, r^{N-1}\,dr \\
&\le \int_0^\alpha \left(\frac{r}{\alpha}\right)^\beta \left(\int_0^\alpha \left|\frac{\partial v}{\partial r}(t,\Theta)\right|^2 t^{N-1}\,dt\right)\left(\frac{1}{N-2}(r^{2-N}-\alpha^{2-N})\right) r^{N-1}\,dr \\
&= \frac{\alpha^2}{(\beta+2)(\beta+N)} \int_0^\alpha \left|\frac{\partial v}{\partial r}(t,\Theta)\right|^2 t^{N-1}\,dt.
\end{aligned}$$

So we see that v satisfies (4). Since $C_0^\infty(B_\alpha)$ is dense in $\mathcal{D}_0^{1,2}(B_\alpha)$, we obtain the conclusion. $\square$

The following is a variant of [17, Lemma I.1]; see also [24, Lemma 1.21]. In the rest of this paper, we denote by $B_r(y)$, the set $\{x \in \mathbb{R}^N : |x-y| < r\}$ for $y \in \mathbb{R}^N$ and $r > 0$, and we use the symbol C for various positive constants independent of α_n and α.

Lemma 2.3. *Let $2 < p < 2^*$, let $\{\alpha_n\} \subset (0,\infty)$ with $\alpha_n \to \infty$, and let $\{u_n\} \subset \mathcal{D}_0^{1,2}(\mathbb{R}^N)$ be a bounded sequence such that $u_n \in \mathcal{D}_0^{1,2}(B_{\alpha_n})$ for each $n \in \mathbb{N}$ and $\sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |x/\alpha_n|^{\alpha_n}|u_n|^p\,dx \to 0$ with $r > 0$. Then $\int_{B_{\alpha_n}} |x/\alpha_n|^{\alpha_n}|u_n|^p\,dx \to 0$.*

Proof. Using Lemma 2.2, we have

$$\begin{aligned}
&\int_{B_{\alpha_n}} \left|\nabla\left(\left|\frac{x}{\alpha_n}\right|^{\frac{\alpha_n}{p}} u_n\right)\right|^2 dx \\
&\le \frac{2}{p^2}\int_{B_{\alpha_n}} \left|\frac{x}{\alpha_n}\right|^{\frac{2\alpha_n}{p}-2} |u_n|^2\,dx + 2\int_{B_{\alpha_n}} \left|\frac{x}{\alpha_n}\right|^{\frac{2\alpha_n}{p}} |\nabla u_n|^2\,dx \\
&\le \frac{5}{2}\int_{B_{\alpha_n}} |\nabla u_n|^2\,dx
\end{aligned}$$

and

$$\int_{B_{\alpha_n}} \left|\left|\frac{x}{\alpha_n}\right|^{\frac{\alpha_n}{p}} u_n\right|^2 dx \le \frac{p^2}{4}\int_{B_{\alpha_n}} |\nabla u_n|^2\,dx.$$

Since we have

$$\begin{aligned}
&\int_{B_r(z)} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^{\frac{2N+4}{N}} dx \\
&\leq \left(\int_{B_r(z)} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^2 dx \right)^{\frac{2}{N}} \left(\int_{B_r(z)} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^{2^*} dx \right)^{\frac{2}{2^*}} \\
&\leq C \left(\sup_{y \in \mathbb{R}^N} \int_{B_r(y)} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^2 dx \right)^{\frac{2}{N}} \\
&\qquad \cdot \int_{B_r(z)} \left(\left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^2 + \left| \nabla \left(\left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right) \right|^2 \right) dx
\end{aligned}$$

for each $z \in \mathbb{R}^N$, we obtain

$$\begin{aligned}
&\int_{R^N} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^{\frac{2N+4}{N}} dx \\
&\leq C \left(\sup_{y \in \mathbb{R}^N} \int_{B_r(y)} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^2 dx \right)^{\frac{2}{N}} \int_{\mathbb{R}^N} \left(\left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^2 + \left| \nabla \left(\left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right) \right|^2 \right) dx \\
&\leq C \left(\sup_{y \in \mathbb{R}^N} \int_{B_r(y)} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^2 dx \right)^{\frac{2}{N}} \int_{B_{\alpha_n}} |\nabla u_n|^2 \, dx.
\end{aligned}$$

From our assumption, we can show

$$\sup_{y \in \mathbb{R}^N} \int_{B_r(y)} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^2 dx \to 0,$$

which and the inequality above yield

$$\int_{B_{\alpha_n}} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^{\frac{2N+4}{N}} dx \to 0.$$

We note that

$$\int_{B_\alpha} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^q \leq \left(\int_{B_\alpha} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^{\frac{2N+4}{N}} \right)^{\delta} \left(\int_{B_\alpha} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^{\gamma} \right)^{1-\delta}$$

for each $q \in (2, 2^*)$ with $q \neq (2N+4)/N$, where

$$\gamma = \begin{cases} 2 & \text{if } q \in (2, (2N+4)/N), \\ 2^* & \text{if } q \in ((2N+4)/N, 2^*) \end{cases} \quad \text{and} \quad \delta = \frac{q - \gamma}{(2N+4)/N - \gamma}.$$

Hence, by Lemma 2.2 and $\int_{B_{\alpha_n}} |u_{\alpha_n}|^{2^*} dx \leq C$, we obtain

$$\int_{B_{\alpha_n}} \left| \left| \frac{x}{\alpha_n} \right|^{\frac{\alpha_n}{p}} u_n \right|^q \to 0 \quad \text{for each } q \in (2, 2^*),$$

which yields the conclusion. □

3. Settings

In the rest of this paper, we consider $N = 3$ and $2 < p < 6$. For each $\alpha > 0$, we set

$$R_{p,\alpha}(u) = \frac{\int_{B_\alpha} |\nabla u|^2 \, dx}{\left(\int_{B_\alpha} \left|\frac{x}{\alpha}\right|^\alpha |u|^p \, dx\right)^{\frac{2}{p}}} \quad \text{for } u \in \mathcal{D}_0^{1,2}(B_\alpha),$$
$$m_{p,\alpha} = \inf\{R_{p,\alpha}(u) : u \in \mathcal{D}_0^{1,2}(B_\alpha) \setminus \{0\}\}.$$

By [5, Remark 4.8], we know the following.

Lemma 3.1. *It holds that $m_{p,\alpha} \to m_p$ as $\alpha \to \infty$.*

Fix $k \in \mathbb{N}$ with $k \geq 2$. We set

$$\widetilde{G}_k = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\pi i/k & -\sin 2\pi i/k \\ 0 & \sin 2\pi i/k & \cos 2\pi i/k \end{pmatrix} : i = 0, 1, \ldots, k-1 \right\},$$

and we denote by G_k, the subgroup of $O(3)$ generated by $\widetilde{G}_k \cup \{\bar{g}\}$, where

$$\bar{g} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

For the sake of simplicity, we write $g(x_1, x_2, x_3)$ instead of $(g(x_1, x_2, x_3)^T)^T$ for $g \in O(3)$ and $(x_1, x_2, x_3) \in \mathbb{R}^3$. We set

$$\mathcal{D}_{0,G_k}^{1,2}(B_\alpha) = \{u \in \mathcal{D}_0^{1,2}(B_\alpha) : u(gx) = u(x) \text{ for each } x \in B_\alpha \text{ and } g \in G_k\}$$

for each $\alpha > 0$.

We fix a constant $\mu \in (0, 1/4)$ and we define $\chi_\alpha : \mathbb{R}^3 \to \mathbb{R}$ by

$$\chi_\alpha(x) = \begin{cases} \alpha^\mu & \text{if } |x_1| > \alpha/2, \\ 0 & \text{if } |x_1| \leq \alpha/2 \end{cases} \quad \text{for } x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We set

$$m_{p,k,\alpha} = \inf_{u \in \Gamma_{p,k,\alpha}} \int_{B_\alpha} |\nabla u|^2 \, dx,$$

where

$$\Gamma_{p,k,\alpha} = \left\{ u \in \mathcal{D}_{0,G_k}^{1,2}(B_\alpha) : \int_{B_\alpha} \left|\frac{x}{\alpha}\right|^\alpha |u|^p \, dx = 1, \int_{B_\alpha} \chi_\alpha(x) \left|\frac{x}{\alpha}\right|^\alpha |u|^p \, dx \leq 1 \right\}.$$

As Byeon did for problem (3) in [2], we will show that if $\alpha \gg 1$ then nonnegative $u_\alpha \in \Gamma_{p,k,\alpha}$ satisfying $\int_{B_\alpha} |\nabla u_\alpha|^2 \, dx = m_{p,k,\alpha}$ is a positive solution of (2).

Lemma 3.2. *There hold*

$$m_p \le \varliminf_{\alpha\to\infty} m_{p,k,\alpha} \quad \textit{and} \quad \varlimsup_{\alpha\to\infty} m_{p,k,\alpha} \le k^{1-\frac{2}{p}} m_p.$$

Proof. Since the left inequality is trivial from Lemma 3.1, we prove the right inequality only. Let $u \in \mathcal{K}_0$ and $\varepsilon > 0$. We can find $v \in C_0^\infty(\mathbb{R}^3_-)$ such that $|R_p(u) - R_p(v)| < \varepsilon$ and $v(x) = v(\sqrt{x_1^2 + x_2^2}, x_3)$ for $x \in \mathbb{R}^3_-$. We set

$$w_\alpha(x_1, x_2, x_3) = v(x_1, x_2, x_3 - \alpha).$$

Since $\operatorname{supp} v$ is a compact subset in $\mathbb{R}^3_-$, we can consider that $w_\alpha \in D_0^{1,2}(B_\alpha)$ for $\alpha \gg 1$. Then we have

$$R_{p,\alpha}(w_\alpha) = \frac{\int_{\mathbb{R}^3_-} |\nabla v|^2\, dx}{\left(\int_{\mathbb{R}^3_-} \left|1 + \frac{2x_3}{\alpha} + \frac{|x|^2}{\alpha^2}\right|^{\alpha/2} v^p\, dx\right)^{2/p}} \to R_p(v) \quad \text{as } \alpha \to \infty.$$

We set

$$\widetilde{w}_\alpha(x_1, x_2, x_3) = \sum_{g \in \widetilde{G}_k} w_\alpha(g(x_1, x_2, x_3)).$$

Since $\operatorname{supp} w_\alpha(g\,\cdot\,) \cap \operatorname{supp} w_\alpha(g'\,\cdot\,) = \emptyset$ for $g, g' \in \widetilde{G}_k$ with $g \neq g'$ and $\alpha \gg 1$, we have

$$\lim_{\alpha\to\infty} R_{p,\alpha}(\widetilde{w}_\alpha) = \lim_{\alpha\to\infty} k^{1-\frac{2}{p}} R_{p,\alpha}(w_\alpha) = k^{1-\frac{2}{p}} R_p(v) \le k^{1-\frac{2}{p}}(R_p(u) + \varepsilon).$$

Since $w_\alpha(x) = 0$ for $\alpha \gg 1$ and $x \in B_\alpha$ with $|x_1| > \alpha/2$, we can infer $\widetilde{w}_\alpha \in \Gamma_{p,k,\alpha}$ for $\alpha \gg 1$. By the arbitrariness of $\varepsilon > 0$, we have shown our assertion. $\square$

For each $\alpha > 0$, we fix $u_\alpha \in \Gamma_{p,k,\alpha}$ such that $R_{p,\alpha}(u_\alpha) = m_{p,k,\alpha}$. Since $|u_\alpha| \in \Gamma_{p,k,\alpha}$ and $\int_{B_\alpha} |\nabla|u_\alpha||^2\, dx = \int_{B_\alpha} |\nabla u_\alpha|^2\, dx$, we may assume $u_\alpha \ge 0$. In Section 5, we will show

$$\int_{B_\alpha} \chi_\alpha(x)\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^p\, dx < 1 \quad \text{for } \alpha \gg 1.$$

Whether this inequality holds or not, for each $\alpha > 0$, we can find $\beta_\alpha, \gamma_\alpha \in \mathbb{R}$ such that

$$-\Delta u_\alpha = \beta_\alpha\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^{p-1} + \gamma_\alpha \chi_\alpha(x)\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^{p-1} \quad \text{in } B_\alpha$$

as Lagrange multipliers. We note $\gamma_\alpha = 0$ in the case $\int_{B_\alpha} \chi_\alpha(x)|x/\alpha|^\alpha u_\alpha^p\, dx < 1$, and by the previous lemma, we can find $C > 1$ such that $1/C \le \beta_\alpha + \gamma_\alpha \le C$ for $\alpha \gg 1$. We also note that $u_\alpha > 0$ in B_α for each $\alpha > 0$ by the maximum principle. Indeed, for example, if $\beta_\alpha \ge 0$ and $\gamma_\alpha < 0$, we have $-\Delta u_\alpha + (-\gamma_\alpha)\chi_\alpha(x)|x/\alpha|^\alpha u_\alpha^{p-1} = \beta_\alpha |x/\alpha|^\alpha u_\alpha^{p-1} \ge 0$.

By a similar calculation as in the proof of [2, Proposition 3.5], we can show the following. For the reader's convenience, we give the proof.

Lemma 3.3. *For each $\alpha > 0$, $\gamma_\alpha \le 0$.*

Proof. Let $\alpha > 0$. In the case $\int_{B_\alpha} \chi_\alpha(x)|x/\alpha|^\alpha u_\alpha^p\,dx < 1$, we know $\gamma_\alpha = 0$. So we consider the case $\int_{B_\alpha} \chi_\alpha(x)|x/\alpha|^\alpha u_\alpha^p\,dx = 1$. For the sake of simplicity, we assume $\alpha \gg 1$. Let $\eta \in C_0^\infty(\mathbb{R}^3) \setminus \{0\}$ be a nonnegative radially symmetric function with $\operatorname{supp}\eta \subset B_1$, and set

$$D(s,t) = \int_{B_\alpha} \left|\frac{x}{\alpha}\right|^\alpha |u_\alpha + t\underline{\eta} - s\bar{\eta}|^p\,dx,$$

where

$$\bar{\eta}(x_1,x_2,x_3) = \eta(x_1+\alpha-1,x_2,x_3) + \eta(x_1-\alpha+1,x_2,x_3)$$

and

$$\underline{\eta}(x_1,x_2,x_3) = \sum_{g\in\widetilde{G}_k} \eta(g(x_1,x_2,x_3+\alpha-1)).$$

From $u_\alpha > 0$ in B_α, we have $\partial D/\partial t(0,0) = \int_{B_\alpha} |x/\alpha|^\alpha u_\alpha^{p-1}\underline{\eta} \ne 0$. By the implicit function theorem, there exist $\varepsilon > 0$ and $t \in C^1(-\varepsilon,\varepsilon)$ such that $t(0) = 0$ and $D(s,t(s)) = 1$ for each $s \in (-\varepsilon,\varepsilon)$. From

$$\frac{d}{ds}\int_{B_\alpha} \chi_\alpha(x)\left|\frac{x}{\alpha}\right|^\alpha |u_\alpha + t(s)\underline{\eta} - s\bar{\eta}|^p\,dx\Bigg|_{s=0} = -p\int_{B_\alpha} \chi_\alpha(x)\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^{p-1}\bar{\eta} < 0,$$

there exists $\delta > 0$ such that $u_\alpha + t(s)\underline{\eta} - s\bar{\eta} \in \Gamma_{p,k,\alpha}$ for $s \in [0,\delta)$. Since

$$0 = \frac{d}{ds}D(s,t(s))\Bigg|_{s=0} = p\int_{B_\alpha} \left|\frac{x}{\alpha}\right|^\alpha u_\alpha^{p-1}(t'(0)\underline{\eta} - \bar{\eta}),$$

we have

$$\begin{aligned} 0 &\le \frac{d}{ds}\int_{B_\alpha} |\nabla(u_\alpha + t(s)\underline{\eta} - s\bar{\eta})|^2\,dx\Bigg|_{s=0} = 2\int_{B_\alpha} \nabla u_\alpha \nabla(t'(0)\underline{\eta} - \bar{\eta})\,dx \\ &= \beta_\alpha \int_{B_\alpha} \left|\frac{x}{\alpha}\right|^\alpha u_\alpha^{p-1}(t'(0)\underline{\eta} - \bar{\eta})\,dx + \gamma_\alpha \int_{B_\alpha} \chi_\alpha(x)\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^{p-1}(t'(0)\underline{\eta} - \bar{\eta})\,dx \\ &= -\gamma_\alpha \int_{B_\alpha} \chi_\alpha(x)\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^{p-1}\bar{\eta}\,dx, \end{aligned}$$

which implies $\gamma_\alpha \le 0$. □

4. Existence of the limit of $\{u_{\alpha_n}\}$

We set

$$B_R^k(x) = \{gy : y \in B_R(x) \text{ and } g \in G_k\} \quad \text{for } x \in \mathbb{R}^3 \text{ and } R > 0.$$

The following is a variant of Lions's concentrated compactness principle; see [16, Lemma 1.1], [23, Lemma 4.3] and [2, Lemma 3.3].

Lemma 4.1. *Let $\{\alpha_n\} \subset (0,\infty)$ with $\alpha_n \to \infty$, and let $\{v_n\}$ be G_k-invariant measurable functions on $\mathbb{R}^3$ which satisfies*

$$\int_{\mathbb{R}^3} \left|\frac{x}{\alpha_n}\right|^{\alpha_n} |v_n|^p \, dx = 1 \quad \text{for each } n \in \mathbb{N}.$$

Then there is a subsequence $\{n_i\}$ of $\{n\}$ which satisfies one of the following three conditions.

(*i*) (*Compactness*) *There exists a sequence $\{x_i\} \subset \mathbb{R}^3$ such that for each $\varepsilon > 0$, there is $R > 0$ such that*

$$\int_{B_R^k(x_i)} \left|\frac{x}{\alpha_{n_i}}\right|^{\alpha_{n_i}} |v_{n_i}|^p \, dx \geq 1 - \varepsilon \quad \text{for } i \gg 1.$$

(*ii*) (*Vanishing*) *For each $R > 0$, there holds*

$$\lim_{i\to\infty} \sup_{x\in\mathbb{R}^3} \int_{B_R^k(x)} \left|\frac{x}{\alpha_{n_i}}\right|^{\alpha_{n_i}} |v_{n_i}|^p \, dx = 0.$$

(*iii*) (*Dichotomy*) *There exists $\lambda \in (0,1)$ such that for each $\varepsilon > 0$, there exist $R > 0$, $\{x_i\} \subset \mathbb{R}^3$ and $\{R_i\} \subset (2R,\infty)$ such that $R_i \to \infty$ and*

$$\left|\lambda - \int_{B_R^k(x_i)} \left|\frac{x}{\alpha_{n_i}}\right|^{\alpha_{n_i}} |v_{n_i}|^p \, dx\right| + \left|1 - \lambda - \int_{\mathbb{R}^3\setminus B_{R_i}^k(x_i)} \left|\frac{x}{\alpha_{n_i}}\right|^{\alpha_{n_i}} |v_{n_i}|^p \, dx\right| \leq \varepsilon$$

for $i \gg 1$.

The following is a direct consequence of Lemma 2.3.

Lemma 4.2. *For each $R > 0$,*

$$\varliminf_{\alpha\to\infty} \sup_{x\in\mathbb{R}^3} \int_{B_R^k(x)} \left|\frac{x}{\alpha}\right|^{\alpha} u_\alpha^p \, dx > 0.$$

The following is another variant of the Poincaré inequality. We can prove it by similar lines as those in the proof of Lemma 2.2.

Lemma 4.3. *For each $\alpha > 0$, $M \in (0,\alpha)$ and $v \in \mathcal{D}_0^{1,2}(B_\alpha)$, there holds*

$$\int_{\{x\in B_\alpha : |x|\geq \alpha - M\}} v^2 \, dx \leq \left(\frac{1}{2}M^2 - \frac{1}{3\alpha}M^3\right) \int_{B_\alpha} |\nabla v|^2 \, dx.$$

For each $\alpha > 0$, we set

$$E_\alpha = \{x \in B_\alpha : \chi_\alpha(x) = 0\} (= \{x \in B_\alpha : |x_1| \leq \alpha/2\}) \quad \text{and} \quad F_\alpha = B_\alpha \setminus E_\alpha.$$

Lemma 4.4. *$\{\beta_\alpha : \alpha \gg 1\}$ is bounded.*

Proof. From Lemma 4.2, there exists $\lambda > 0$, $R > 0$ and $\{x_\alpha\} \subset \mathbb{R}^3$ such that

$$x_\alpha \in B_\alpha \quad \text{and} \quad \int_{B_R^k(x_\alpha)} \left|\frac{x}{\alpha}\right|^\alpha u_\alpha^p \, dx \geq \lambda \quad \text{for } \alpha \gg 1. \tag{6}$$

Without loss of generality, we may assume $B_R^k(x_\alpha) \subset E_\alpha$ for each $\alpha > 0$. We can find $\delta \in (0,1)$ such that

$$\int_{B_{\delta R}^k(x_\alpha)} \left|\frac{x}{\alpha}\right|^\alpha u_\alpha^p \, dx \geq \frac{\lambda}{2} \quad \text{for } \alpha \gg 1.$$

We will show

$$\varlimsup_{\alpha \to \infty} (\alpha - |x_\alpha|) < \infty. \tag{7}$$

If not, we may assume $\alpha - |x_\alpha| \to \infty$ as $\alpha \to \infty$. Then we obtain

$$\int_{B_R^k(x_\alpha)} \left|\frac{x}{\alpha}\right|^\alpha u_\alpha^p \, dx \leq \left(\int_{B_R^k(x_\alpha)} \left|\frac{x}{\alpha}\right|^{\frac{6\alpha}{6-p}} dx \right)^{\frac{6-p}{6}} \left(\int_{\mathbb{R}^3} |u_\alpha|^6 \, dx \right)^{\frac{p}{6}} \to 0$$

as $\alpha \to \infty$, which contradicts (6). Therefore, we have shown (7). Let $\xi_\alpha : \mathbb{R}^3 \to [0,1]$ be a Lipschitz continuous function such that, it is G_k-invariant,

$$|\nabla \xi_\alpha| \leq C \quad \text{and} \quad \xi_\alpha = \begin{cases} 1 & \text{for } x \in B_{\delta R}^k(x_\alpha), \\ 0 & \text{for } x \notin B_R^k(x_\alpha). \end{cases}$$

Using Lemma 4.3 with $M = 2(C_0 + R)$, where $C_0 = \varlimsup_\alpha (\alpha - |x_\alpha|)$, we have

$$\begin{aligned} \frac{\beta_\alpha \lambda}{2} &\leq \beta_\alpha \int_{B_{\delta R}^k(x_\alpha)} \left|\frac{x}{\alpha}\right|^\alpha u_\alpha^p \, dx \leq \beta_\alpha \int_{B_\alpha} \left|\frac{x}{\alpha}\right|^\alpha \xi_\alpha u_\alpha^p \, dx \\ &= \int_{B_\alpha} (\xi_\alpha |\nabla u_\alpha|^2 + u_\alpha \nabla u_\alpha \nabla \xi_\alpha) \, dx \leq C. \end{aligned}$$

Hence, we have shown that $\{\beta_\alpha : \alpha \gg 1\}$ is bounded. □

By the lemma above, we can find $\beta_\infty \in (0, \infty)$ such that

$$\beta_\alpha \leq \beta_\infty \quad \text{for } \alpha \gg 1.$$

By [23, Lemma B.3] and [9, Theorem 8.25], we can infer the following. See also [10, Theorem 4.1].

Lemma 4.5. *There holds*

$$\sup_{\alpha \gg 1} \sup_{x \in B_\alpha} u_\alpha(x) < \infty.$$

Now, we show that compactness occurs as follows.

Proposition 4.6. *For each* $\{\alpha_n\} \subset (0,\infty)$ *with* $\alpha_n \to \infty$*, there exist a subsequence* $\{n_i\}$ *of* $\{n\}$ *and* $\{x_i\} \subset \mathbb{R}^3$ *such that* $x_{i,1} = 0$ *for each* $i \in \mathbb{N}$ *and for each* $\varepsilon > 0$*, there is* $R > 0$ *such that*

$$\int_{B_R^k(x_i)} \left|\frac{x}{\alpha_{n_i}}\right|^{\alpha_{n_i}} |u_{n_i}|^p \, dx \geq 1 - \varepsilon \quad \textit{for } i \gg 1.$$

Proof. We argue indirectly; we assume that dichotomy occurs. Then there exist a subsequence of $\{\alpha_n\}$, still denoted by $\{\alpha_n\}$, and $\lambda_1 \in (0,1)$ such that for each $\varepsilon > 0$, there exist $R > 0$, $\{x_n\} \subset \mathbb{R}^3$ and $\{R_n\} \subset (2R, \infty)$ such that $R_n \to \infty$ and

$$\left|\lambda_1 - \int_{B_R^k(x_n)} \left|\frac{x}{\alpha_n}\right|^{\alpha_n} u_{\alpha_n}^p \, dx\right| + \left|1 - \lambda_1 - \int_{\mathbb{R}^3 \setminus B_{R_n}^k(x_n)} \left|\frac{x}{\alpha_n}\right|^{\alpha_n} u_{\alpha_n}^p \, dx\right| < \varepsilon$$

for $n \gg 1$.

We will show

$$\lambda_1 \geq \left(\frac{m_p}{\beta_\infty}\right)^{\frac{p}{p-2}}. \tag{8}$$

As we have shown (7), we can see that there exists $C_0 > 0$ such that $\alpha_n - |x_n| < C_0$ for each $n \gg 1$. Let $\tau_n : \mathbb{R}^3 \to [0,1]$ be a Lipschitz continuous function such that it is G_k-invariant, $|\nabla \tau_n| \leq C$ and

$$\tau_n(x) = \begin{cases} 1 & \text{if } x \in B_R^k(x_n), \\ 0 & \text{if } x \notin B_{2R}^k(x_n). \end{cases}$$

Since $|x| \geq \alpha_n - C_0 - 2R$ for $x \in B_{2R}^k(x_n)$, we have

$$\begin{aligned}
&\int_{B_{2R}^k(x_n) \setminus B_R^k(x_n)} u_{\alpha_n}^2 \, dx \leq \left|B_{2R}^k(x_n) \setminus B_R^k(x_n)\right|^{\frac{p-2}{p}} \left(\int_{B_{2R}^k(x_n) \setminus B_R^k(x_n)} u_{\alpha_n}^p \, dx\right)^{\frac{2}{p}} \\
&\leq C\left(\sup_{x \in B_{2R}^k(x_n)} \left|\frac{\alpha_n}{x}\right|^{\alpha_n}\right)^{\frac{2}{p}} \left(\int_{B_{2R}^k(x_n) \setminus B_R^k(x_n)} \left|\frac{x}{\alpha_n}\right|^{\alpha_n} u_{\alpha_n}^p \, dx\right)^{\frac{2}{p}} \\
&\leq C\left(1 - \frac{C_0 + 2R}{\alpha_n}\right)^{-\frac{2\alpha_n}{p}} \varepsilon^{\frac{2}{p}}
\end{aligned}$$

for $n \gg 1$. Hence we obtain

$$\begin{aligned}
&m_{p,\alpha_n}(\lambda_1-\varepsilon)^{\frac{2}{p}} \le m_{p,k,\alpha_n}(\lambda_1-\varepsilon)^{\frac{2}{p}} \\
&\le m_{p,k,\alpha_n}\left(\int_{B_{\alpha_n}}\left|\frac{x}{\alpha_n}\right|^{\alpha_n}(\tau_n u_{\alpha_n})^p\right)^{\frac{2}{p}} \le \int_{B_{\alpha_n}}|\nabla(\tau_n u_{\alpha_n})|^2\,dx \\
&\le \int_{B_{\alpha_n}}(\tau_n|\nabla u_{\alpha_n}|^2 + u_{\alpha_n}\nabla\tau_n\nabla u_{\alpha_n})\,dx \\
&\qquad + \int_{B_{\alpha_n}}(-u_{\alpha_n}\nabla\tau_n\nabla u_{\alpha_n} + u_{\alpha_n}^2|\nabla\tau_n|^2 + 2\tau_n u_{\alpha_n}\nabla\tau_n\nabla u_{\alpha_n})\,dx \\
&\le \int_{B_{\alpha_n}}(\beta_{\alpha_n}+\gamma_{\alpha_n}\chi_{\alpha_n})\tau_n\left|\frac{x}{\alpha_n}\right|^{\alpha_n}u_{\alpha_n}^p\,dx + C\int_{B^k_{2R}(x_n)\setminus B^k_R(x_n)}u_{\alpha_n}^2\,dx \\
&\qquad + C\left(\int_{B^k_{2R}(x_n)\setminus B^k_R(x_n)}u_{\alpha_n}^2\,dx\right)^{\frac{1}{2}}\left(\int_{B^k_{2R}(x_n)\setminus B^k_R(x_n)}|\nabla u_{\alpha_n}|^2\,dx\right)^{\frac{1}{2}} \\
&\le \beta_{\alpha_n}(\lambda_1+\varepsilon) + C\left(1-\frac{C_0+2R}{\alpha_n}\right)^{-\frac{2\alpha_n}{p}}\varepsilon^{\frac{2}{p}}
\end{aligned}$$

for $n \gg 1$. Letting $n \to \infty$, we have

$$m_p(\lambda_1-\varepsilon)^{\frac{2}{p}} \le \beta_\infty(\lambda_1+\varepsilon) + Ce^{\frac{2(C_0+2R)}{p}}\varepsilon^{\frac{2}{p}}.$$

Since $\varepsilon > 0$ is arbitrary, we have shown (8).

By (8) and Lemma 4.1, in addition to λ_1, taking a subsequence of $\{\alpha_n\}$ if necessary, we can find that there exist $\lambda_2, \dots, \lambda_m \in (0,1)$ with $\sum_{i=1}^m \lambda_i = 1$ such that for each $\varepsilon > 0$, there exist $R > 0$ and $\{x_n^1\}, \dots, \{x_n^m\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to\infty}\min_{g\in G_k}|x_n^i - gx_n^j| = \infty \quad \text{for } i \ne j$$

and

$$\sum_{i=1}^m\left|\lambda_i - \int_{B^k_R(x_n^i)}\left|\frac{x}{\alpha_n}\right|^{\alpha_n}u_{\alpha_n}^p\,dx\right| < \varepsilon \quad \text{for } n \gg 1.$$

Let $\varepsilon > 0$, and choose R and $\{x_n^1\}, \dots, \{x_n^m\}$ which satisfy the properties above. Without loss of generality, we may assume $x_{n,1}^i = 0$ or $B_{4R}(x_n^i) \cap B_{4R}(\bar{g}x_n^i) = \emptyset$ for each $i = 1, \dots, m$ and $n \gg 1$, where $x_{n,1}^i$ is the first coordinate of x_n^i and $\bar{g}$ is given by (5). By a similar way as we have shown (7), we can show $\alpha_n - |x_n^i| < C$ for $i = 1, \dots, m$ and $n \gg 1$. Let $\tilde{\tau}_n^i : \mathbb{R}^3 \to [0,1]$ be a Lipschitz continuous function such that $\tilde{\tau}_n^i$ is $\bar{g}$-invariant, $|\nabla\tilde{\tau}_n^i| \le 2/R$ and

$$\tilde{\tau}_n^i(x) = \begin{cases} 1 & \text{if } x \in B_R(x_n^i), \\ 0 & \text{if } x \notin B_{2R}(x_n^i). \end{cases}$$

We define $\tau_n^i : \mathbb{R}^3 \to [0,1]$ by

$$\tau_n^i(x) = \sum_{g \in G_k} \tilde{\tau}_n^i(gx) \quad \text{for } x \in \mathbb{R}^3.$$

We can easily see that τ_n^i is G_k-invariant, $|\nabla \tau_n^i| \le 2/R$ and

$$\tau_n^i(x) = \begin{cases} 1 & \text{if } x \in B_R^k(x_n^i), \\ 0 & \text{if } x \notin B_{2R}^k(x_n^i). \end{cases}$$

Noting

$$\begin{aligned} &\sum_{i=1}^m \int_{B_{\alpha_n}} |\nabla(\tau_n^i u_{\alpha_n})|^2 \, dx \\ \le\ & \int_{B_{\alpha_n}} |\nabla u_{\alpha_n}|^2 \, dx + \sum_{i=1}^m \int_{B_{\alpha_n}} \left(|\nabla \tau_n^i|^2 |u_{\alpha_n}|^2 + 2\tau_n^i u_{\alpha_n} \nabla \tau_n^i \nabla u_{\alpha_n} \right) dx \end{aligned}$$

and using a similar argument as before, we have

$$\begin{aligned} \int_{B_{\alpha_n}} |\nabla u_{\alpha_n}|^2 \, dx &\ge \sum_{i=1}^m \int_{B_{\alpha_n}} |\nabla(\tau_n^i u_{\alpha_n})|^2 \, dx - C\varepsilon^{\frac{2}{p}} \\ &= k \sum_{i=1}^m \int_{B_{\alpha_n}} |\nabla(\tilde{\tau}_n^i u_{\alpha_n})|^2 \, dx - C\varepsilon^{\frac{2}{p}} \\ &\ge k m_{p,\alpha_n} \sum_{i=1}^m \left(\int_{B_{\alpha_n}} \left| \frac{x}{\alpha_n} \right|^{\alpha_n} |\tilde{\tau}_n^i u_{\alpha_n}|^p \, dx \right)^{\frac{2}{p}} - C\varepsilon^{\frac{2}{p}} \\ &= k^{1-\frac{2}{p}} m_{p,\alpha_n} \sum_{i=1}^m \left(\int_{B_{\alpha_n}} \left| \frac{x}{\alpha_n} \right|^{\alpha_n} |\tau_n^i u_{\alpha_n}|^p \, dx \right)^{\frac{2}{p}} - C\varepsilon^{\frac{2}{p}} \\ &\ge k^{1-\frac{2}{p}} m_{p,\alpha_n} \sum_{i=1}^m (\lambda_i - \varepsilon)^{\frac{2}{p}} - C\varepsilon^{\frac{2}{p}}. \end{aligned}$$

Letting $n \to \infty$ and using Lemma 3.2, we obtain

$$k^{1-\frac{2}{p}} m_p \ge k^{1-\frac{2}{p}} m_p \sum_{i=1}^m (\lambda_i - \varepsilon)^{\frac{2}{p}} - C\varepsilon^{\frac{2}{p}}.$$

Since $\varepsilon > 0$ is arbitrary, we have $1 \ge \sum_{i=1}^m \lambda_i^{2/p}$, which is a contradiction. Hence, we have shown that dichotomy does not occur. By Lemmas 4.1 and 4.2, taking a subsequence of $\{\alpha_n\}$ if necessary, there exists $\{x_n\} \subset \mathbb{R}^3$ such that for each $\varepsilon > 0$, there is $R > 0$ such that

$$\int_{B_R^k(x_n)} \left| \frac{x}{\alpha_n} \right|^{\alpha_n} |u_n|^p \, dx \ge 1 - \varepsilon \quad \text{for } n \gg 1.$$

If $|x_{n,1}| \to \infty$, where $x_n = (x_{n,1}, x_{n,2}, x_{n,3})$, we can obtain a contradiction as above. So, we may assume $x_{n,1} = 0$ for each $n \in \mathbb{N}$. Thus we have shown our assertion. □

We set

$$h_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad \text{for each } \theta \in \mathbb{R}.$$

Proposition 4.7. *For each $\{\alpha_n\} \subset (0,\infty)$ with $\alpha_n \to \infty$, there exist a subsequence $\{n_i\}$ of $\{n\}$, $\theta \in \mathbb{R}$ and $u \in \mathcal{K}$ such that $\int_{\mathbb{R}^3_-} e^{x_3} u^p\, dx = 1/k$ and*

$$\lim_{i\to\infty} \int_{\mathbb{R}^3} \left| \nabla\Big(u_{\alpha_{n_i}}(x) - \sum_{g \in \widetilde{G}_k} u(h_\theta g x - (0,0,\alpha_n)) \Big) \right|^2 dx = 0. \tag{9}$$

Moreover, there holds

$$\lim_{\alpha\to\infty} \int_{B_\alpha} |\nabla u_\alpha|^2\, dx = k^{1-\frac{2}{p}} m_p. \tag{10}$$

Proof. By Proposition 4.6, there exist a subsequence of $\{\alpha_n\}$, still denoted by $\{\alpha_n\}$, and $\{x_n\} \subset \mathbb{R}^3$ such that $x_{n,1} = 0$ for each $n \in \mathbb{N}$ and for each $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{B^k_R(x_n)} \left| \frac{x}{\alpha_n} \right|^{\alpha_n} u^p_{\alpha_n}\, dx > 1 - \varepsilon \quad \text{for } n \gg 1. \tag{11}$$

Without loss of generality, we may assume $x_n \in B_{\alpha_n}$ for each $n \in \mathbb{N}$. Let $\varepsilon > 0$ and choose $R > 0$ which satisfies (11). We may assume that $\alpha_n - |x_n| \to R_1 \in [0,\infty)$ and $x_n/\alpha_n \to y \in \partial B_1$. We can find $\theta \in \mathbb{R}$ such that $h_\theta y = (0,0,1)$. Taking a subsequence of $\{\alpha_n\}$ if necessary, we can infer that $\{u_{\alpha_n}(h_\theta^{-1}(\,\cdot + (0,0,\alpha_n)))\}$ converges to u weakly in $\mathcal{D}^{1,2}_0(\mathbb{R}^3_-)$. From (11), we have

$$1 \geq k \int_{\{x \in \mathbb{R}^3_- : |x - (0,0,-R_1)| < R\}} e^{x_3} |u|^p\, dx \geq 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we can infer $\int_{\mathbb{R}^3_-} e^{x_3}|u|^p\, dx = 1/k$. By this equality and Lemma 3.2, we have

$$\begin{aligned} 0 &\leq \varlimsup_{n\to\infty} \int_{\mathbb{R}^3} \left| \nabla\Big(u_{\alpha_n}(x) - \sum_{g \in \widetilde{G}_k} u(h_\theta g x - (0,0,\alpha_n)) \Big) \right|^2 dx \\ &= \varlimsup_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_{\alpha_n}|^2\, dx - k \int_{\mathbb{R}^3} |\nabla u|^2\, dx \leq k^{1-\frac{2}{p}} m_p - k \int_{\mathbb{R}^3} |\nabla u|^2\, dx. \end{aligned}$$

Hence we obtain $u \in \mathcal{K}$ and (9). Using the former part of this proposition, we can easily see (10). Therefore we have shown our assertions. □

5. Existence of k positive solutions

In this section, we will show

$$\int_{B_\alpha} \chi_\alpha(x)\Big|\frac{x}{\alpha}\Big|^\alpha u_\alpha^p \, dx < 1 \quad \text{for } \alpha \gg 1. \tag{12}$$

Although it seems to be difficult to show an L^2 boundedness of $\{u_\alpha\}$, we give an L^2 estimation in Lemma 5.1 below, which is important to show (12).

We define $K_\alpha : B_\alpha \times B_\alpha \to (0, \infty]$ by

$$K_\alpha(x,y) = \begin{cases} \dfrac{1}{4\pi}\left(\dfrac{1}{|x-y|} - \dfrac{1}{\left|\frac{\alpha}{|x|}x - \frac{|x|}{\alpha}y\right|}\right) & \text{for } x \neq 0, \\ \dfrac{1}{4\pi}\left(\dfrac{1}{|y|} - \dfrac{1}{\alpha}\right) & \text{for } x = 0, \end{cases}$$

which is the Green's function of $-\Delta$ on B_α. We can easily see

$$K_\alpha(x,y) \le \frac{1}{4\pi|x-y|}. \tag{13}$$

From

$$\left|\frac{\alpha}{|x|}x - \frac{|x|}{\alpha}y\right|^2 - |x-y|^2 = \frac{(\alpha^2 - |x|^2)(\alpha^2 - |y|^2)}{\alpha^2},$$

we can find

$$K_\alpha(x,y) = \begin{cases} \dfrac{(\alpha^2 - |x|^2)(\alpha^2 - |y|^2)}{4\pi\alpha^2|x-y|\left|\frac{\alpha}{|x|}x - \frac{|x|}{\alpha}y\right|\left(\left|\frac{\alpha}{|x|}x - \frac{|x|}{\alpha}y\right| + |x-y|\right)} & \text{for } x \neq 0, \\ \dfrac{\alpha^2 - |y|^2}{4\pi\alpha|y|(\alpha + |y|)} & \text{for } x = 0, \end{cases}$$

which yields

$$K_\alpha(x,y) \le \frac{(\alpha^2 - |x|^2)(\alpha^2 - |y|^2)}{8\pi\alpha^2|x-y|^3}. \tag{14}$$

Since K_α is the Green's function of $-\Delta$ on B_α, we can find

$$u_\alpha(x) \le \beta_\alpha \int_{B_\alpha} K_\alpha(x,y)\Big|\frac{y}{\alpha}\Big|^\alpha u_\alpha(y)^{p-1} \, dy \quad \text{for } x \in B_\alpha. \tag{15}$$

We recall that $\mu \in (0, 1/4)$ is fixed when we define χ_α. We choose $\rho, \sigma \in \mathbb{R}$ such that

$$0 < \rho < \sigma < \frac{2-\mu}{2} \quad \text{and} \quad \frac{5+\mu}{6} + \frac{\rho}{3} < \sigma,$$

and we define a positive constant ν by

$$\nu = \begin{cases} p + 3\sigma - \rho - 4 & \text{in the case } 2 < p < 3, \\ \sigma & \text{in the case } 3 \le p < 6. \end{cases}$$

We can easily see $\max\{3 - 2\nu, 2\sigma\} < 2 - \mu$, which will be used later.

Now, we give an estimate on the L^2-norm of $\{u_\alpha : \alpha \gg 1\}$.

Lemma 5.1. *There holds*

$$\int_{B_\alpha} |u_\alpha|^2 \, dx \le C\alpha^{\max\{3-2\nu, 2\sigma\}} \quad \textit{for } \alpha \gg 1.$$

Proof. Let $|x| \le \alpha - \alpha^\sigma$. We evaluate the integral on the right hand side of the inequality (15) in each of the following three regions in B_α:

(i) $\alpha - \alpha^\rho < |y| < \alpha$,
(ii) $|x - y| > (\alpha - |x|) - \alpha^\rho$ and $|y| < \alpha - \alpha^\rho$,
(iii) $|x - y| < (\alpha - |x|) - \alpha^\rho$.

First, we consider the region (i). In the case $3 \le p < 6$, using $|x - y| > \alpha^\sigma - \alpha^\rho$ for $|y| > \alpha - \alpha^\rho$, (13) and Lemma 2.2, we have

$$\begin{aligned} &\int_{\{y \in B_\alpha : \alpha - \alpha^\rho < |y| < \alpha\}} K_\alpha(x, y) \left|\frac{y}{\alpha}\right|^\alpha u_\alpha(y)^{p-1} \, dy \\ &\le \frac{1}{4\pi(\alpha^\sigma - \alpha^\rho)} \int_{B_\alpha} \left|\frac{y}{\alpha}\right|^\alpha u_\alpha(y)^{p-1} \, dy \le \frac{C}{\alpha^\sigma} = \frac{C}{\alpha^\nu} \end{aligned}$$

for $\alpha \gg 1$. In the case $2 < p < 3$, using (14), Lemma 2.2 and

$$\frac{1}{|x - y|} \le \frac{\alpha - \alpha^\rho}{\alpha^\sigma - \alpha^\rho} \frac{1}{|y|} \quad \text{for } \alpha - \alpha^\rho < |y| < \alpha,$$

we have

$$\begin{aligned} &\int_{\{y \in B_\alpha : \alpha - \alpha^\rho < |y| < \alpha\}} K_\alpha(x, y) \left|\frac{y}{\alpha}\right|^\alpha u_\alpha(y)^{p-1} \, dy \\ &\le \frac{1}{8\pi} \frac{(\alpha^2 - |x|^2)((\alpha^2 - (\alpha - \alpha^\rho)^2)}{\alpha^2} \left(\frac{\alpha - \alpha^\rho}{\alpha^\sigma - \alpha^\rho}\right)^3 \\ &\qquad \cdot \left(\int_{B_\alpha} \left|\frac{y}{\alpha}\right|^\alpha u_\alpha(y)^2 \, dy\right)^{\frac{p-1}{2}} \left(\int_{\{y \in B_\alpha : \alpha - \alpha^\rho < |y| < \alpha\}} |y|^{-\frac{6}{3-p}} \left|\frac{y}{\alpha}\right|^\alpha dy\right)^{\frac{3-p}{2}} \\ &\le C\alpha^{4+\rho-3\sigma} \left(\frac{1}{\alpha^\alpha} \int_0^\alpha r^{\alpha - \frac{6}{3-p} + 2} \, dr\right)^{\frac{3-p}{2}} = \frac{C}{\alpha^{p+3\sigma-\rho-4}} = \frac{C}{\alpha^\nu} \end{aligned}$$

for $\alpha \gg 1$. Next, we consider the regions (ii) and (iii). Noting

$$\lim_{\alpha\to\infty} \alpha^s \left(\frac{\alpha-\alpha^\rho}{\alpha}\right)^\alpha = 0 \quad \text{for each } s > 0$$

and using (13) and Lemma 4.4, we have

$$\begin{aligned}&\int_{\{y\in B_\alpha : |x-y|>(\alpha-|x|)-\alpha^\rho,\, |y|<\alpha-\alpha^\rho\}} K_\alpha(x,y)\left|\frac{y}{\alpha}\right|^\alpha u_\alpha(y)^{p-1}\,dy \\ &\le C\frac{1}{\alpha-|x|-\alpha^\rho}\left(\frac{\alpha-\alpha^\rho}{\alpha}\right)^\alpha \int_0^{\alpha-\alpha^\rho} r^2\,dr \le \frac{C}{\alpha^\nu}\end{aligned}$$

for $\alpha \gg 1$ and

$$\begin{aligned}&\int_{\{y\in B_\alpha : |x-y|<(\alpha-|x|)-\alpha^\rho\}} K_\alpha(x,y)\left|\frac{y}{\alpha}\right|^\alpha u_\alpha(y)^{p-1}\,dy \\ &\le C\left(\frac{\alpha-\alpha^\rho}{\alpha}\right)^\alpha \int_0^{\alpha-|x|-\alpha^\rho} \frac{1}{r}\cdot r^2\,dr \le \frac{C}{\alpha^\nu}\end{aligned}$$

for $\alpha \gg 1$. Thus we have shown

$$\sup_{|x|\le\alpha-\alpha^\sigma} u_\alpha(x) \le \frac{C}{\alpha^\nu} \quad \text{for } \alpha \gg 1.$$

Hence we obtain

$$\int_{\{x\in B_\alpha : |x|\le\alpha-\alpha^\sigma\}} |u_\alpha|^2\,dx \le C\alpha^{3-2\nu} \quad \text{for } \alpha \gg 1.$$

From Lemma 4.3 with $M = \alpha - \alpha^\sigma$, we have

$$\int_{\{x\in B_\alpha : |x|>\alpha-\alpha^\sigma\}} |u_\alpha|^2\,dx \le C\alpha^{2\sigma} \quad \text{for } \alpha \gg 1.$$

So we have shown our assertion. □

We also need the following variant of the Poincaré inequality. We can prove it similarly.

Lemma 5.2. *For each $\alpha > 0$ and $v \in \mathcal{D}_0^{1,2}(B_\alpha)$, there holds*

$$\int_{B_\alpha} \left|\frac{x}{\alpha}\right|^\alpha v^2|x_1|^2\,dx \le \int_{B_\alpha} |\nabla v|^2|x_1|^2\,dx.$$

Now, we show that u_α is a positive solution of (2) for $\alpha \gg 1$.

Proposition 5.3. *There holds*

$$\int_{B_\alpha} \chi_\alpha(x)\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^p\,dx < 1 \quad \textit{for } \alpha \gg 1.$$

Proof. First, we will show that there exists $d > 0$ such that

$$u_\alpha(x) \le (2\beta_\infty)^{-\frac{1}{p-2}} \quad \text{for } x \in B_\alpha \text{ with } |x_1| > d \text{ and } \alpha \gg 1. \tag{16}$$

Indeed, if not, there exist $\{d_n\}, \{\alpha_n\} \subset (0,\infty)$ and $\{x_n\} \subset \mathbb{R}^3$ such that $d_n \to \infty$, $\alpha_n \to \infty$, $x_{n,1} \ge d_n$ for each $n \in \mathbb{N}$ and $u_{\alpha_n}(x_n) > (2\beta_\infty)^{-1/(p-2)}$. By Proposition 4.7, taking a subsequence of $\{\alpha_n\}$ if necessary, we can find $\theta \in \mathbb{R}$ and $u \in \mathcal{K}$ satisfying the properties in the proposition. Then by using [9, Theorem 8.25], we can obtain a contradiction. Thus we have shown (16).

From $-\Delta u_\alpha \le \beta_\alpha |x/\alpha|^\alpha u_\alpha^{p-1}$ in B_α, we have

$$\int_{B_\alpha} \nabla u_\alpha \nabla\left(u_\alpha |x_1|^2\right)\,dx \le \beta_\alpha \int_{B_\alpha}\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^p |x_1|^2\,dx. \tag{17}$$

For the left hand side of the inequality above, we have

$$\begin{aligned}\int_{B_\alpha} \nabla u_\alpha \nabla\left(u_\alpha |x_1|^2\right)\,dx &= \int_{B_\alpha} |\nabla u_\alpha|^2 |x_1|^2\,dx + 2\int_{B_\alpha} \frac{\partial u_\alpha}{\partial x_1} u_\alpha x_1\,dx \\ \ge \int_{B_\alpha} |\nabla u_\alpha|^2|x_1|^2\,dx &- 2\left(\int_{B_\alpha} |\nabla u_\alpha|^2 |x_1|^2\,dx\right)^{\frac12}\left(\int_{B_\alpha} |u_\alpha|^2\,dx\right)^{\frac12}.\end{aligned}$$

For the right hand of the inequality (17), using (16) and Lemma 5.2, we have

$$\begin{aligned}\beta_\alpha \int_{B_\alpha}\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^p |x_1|^2\,dx &\le \beta_\alpha \int_{\{x\in B_\alpha : |x_1| > d\}} + \beta_\alpha \int_{\{x \in B_\alpha : |x_1| \le d\}} \\ \le \frac12 \int_{B_\alpha}\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^2 |x_1|^2\,dx &+ d^2\beta_\infty \int_{B_\alpha}\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^p\,dx \\ \le \frac12 \int_{B_\alpha} |\nabla u_\alpha|^2 |x_1|^2\,dx &+ d^2 \beta_\infty\end{aligned}$$

for $\alpha \gg 1$. So we have

$$\frac12 \int_{B_\alpha} |\nabla u_\alpha|^2 |x_1|^2\,dx - 2\left(\int_{B_\alpha} |\nabla u_\alpha|^2|x_1|^2\,dx\right)^{\frac12}\left(\int_{B_\alpha}|u_\alpha|^2\,dx\right)^{\frac12} - d^2\beta_\infty \le 0$$

for $\alpha \gg 1$, which and Lemma 5.1 yield

$$\int_{B_\alpha} |\nabla u_\alpha|^2 |x_1|^2\,dx \le C\alpha^{\max\{3-2\nu, 2\sigma\}} \quad \text{for } \alpha \gg 1.$$

Using (16), the previous lemma, the inequality above and $\max\{3-2\nu, 2\sigma\} < 2-\mu$, we have

$$\int_{\{x\in B_\alpha:|x_1|>\alpha/2\}} \chi_\alpha(x)\left|\frac{x}{\alpha}\right|^\alpha u_\alpha^p\,dx \le \frac{\alpha^\mu}{2\beta_\infty}\left(\frac{2}{\alpha}\right)^2 \int_{\{x\in B_\alpha:|x_1|>\alpha/2\}} \left|\frac{x}{\alpha}\right|^\alpha u_\alpha^2|x_1|^2\,dx$$
$$\le \frac{2}{\beta_\infty\alpha^{2-\mu}}\int_{B_\alpha}|\nabla u_\alpha|^2|x_1|^2\,dx \le \frac{C}{\alpha^{2-\mu-\max\{3-2\nu,2\sigma\}}} < 1$$

for $\alpha \gg 1$. Hence we have shown our assertion. □

Proof of Theorem. Let $m \in \mathbb{N}$ with $m \ge 2$. Suppose that the conclusion does not hold. Then there exists $\{\alpha_n\} \subset (0,\infty)$ such that $\alpha_n \to \infty$ and for each $n \in \mathbb{N}$, there do not exist m nonequivalent positive solutions of (2) with $\alpha = \alpha_n$. However, by Lemma 3.1, (10) and Proposition 5.3, taking a subsequence of $\{\alpha_n\}$ if necessary, we can find that there exist $\{u_n^1\}, \ldots, \{u_n^m\}$ such that for each $k = 1, \ldots, m$, there hold

(i) for each $n \in \mathbb{N}$, u_n^k is a positive solution of (2) with $\alpha = \alpha_n$,
(ii) $R_{p,\alpha_n}(u_n^k) \to k^{1-2/p}m_p$ as $n \to \infty$.

So there exists $n_0 \in \mathbb{N}$ such that for each $n \ge n_0$, $R_{p,\alpha}(u_n^i) \ne R_{p,\alpha}(u_n^j)$ for $i \ne j$. This is a contradiction. Hence, we have shown our assertion. □

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