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Weak Convergence Theorems for Semigroups of Not Necessarily Continuous Mappings in Banach Spaces

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In this paper, we first introduce a broad semigroup of not necessarily continuous mappings in a Banach space which contains discrete semigroups generated by generalized nonspreading mappings and semigroups of ϕ-nonexpansive mappings. Then we establish a weak convergence theorem of Mann's type iteration for the semigroups of mappings in a Banach space. Using the result, we obtain well-known and new theorems which are connected with weak convergence results in Banach spaces.

Keywords: Attractive point, Banach space, fixed point, generalized nonspreading mapping, invariant mean, weak converegence, nonexpansive semigroup, strongly asymptotically invariant net

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1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and let C be a nonempty subset of H. Kocourek, Takahashi and Yao [19] defined a class of nonlinear

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mappings containing nonexpansive mappings [8], nonspreading mappings [22, 23] and hybrid mappings [28] in a Hilbert space and they proved a fixed point theorem and a mean convergence theorem for the mappings. A mapping $T : C \to C$ is called *generalized hybrid* [19] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$; see also [1]. We call such a mapping (α, β)-*generalized hybrid*. A (1, 0)-generalized hybrid mapping is nonexpansive. It is nonspreading for $\alpha = 2$ and $\beta = 1$. It is hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. In general, nonspreading and hybrid mappings are not continuous. See, for example, [16]. We also know the concept of one-parameter nonexpansive semigroups in a Hilbert space. Let H be a Hilbert space and let C be a nonempty subset of H. Let $S = \mathbb{R}^+ = \{t \in \mathbb{R} : 0 \leq t < \infty\}$. A family $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself is called a *one-parameter nonexpansive semigroup* on C if $\mathcal{S}$ satisfies the following:

(1) $S(t+s)x = S(t)S(s)x, \ \forall x \in C, \ t, s \in \mathbb{R}^+$;
(2) $S(0)x = x, \ \forall x \in C$;
(3) for each $x \in C$, the mapping $t \mapsto S(t)x$ from $\mathbb{R}^+$ into C is continuous;
(2) for each $t \in \mathbb{R}^+$, $S(t)$ is nonexpansive.

Of course, $S(t)$ are continuous. Such one-parameter nonexpansive semigroups are used in the theory of nonlinear evolution equations [6]. Recently, using the concept of means and invariant means, Takahashi, Wong and Yao [30] introduced the concept of semigroups of not necessarily continuous mappings in Hilbert spaces which contains discrete semigroups generated by generalized hybrid mappings and semigroups of nonexpansive mappings. They proved a fixed point theorem and a mean convergence theorem of Baillon's type [4] which generalized simultaneously the results [19] and [5] for generalized hybrid mappings and one-parameter nonexpansive semigroups in a Hilbert space. They also generalized such results to Banach spaces; see [31]. Very recently, using the concept of strongly asymptotically invariant nets, Hussain and Takahashi [12] proved a weak convergence theorem of Mann's type iteration [25] and a strong convergence theorem of Halpern's type iteration [9] in a Hilbert space. It is natural to extend the results of [12] to Banach spaces.

In this paper, we first introduce a broad semigroup of not necessarily continuous mappings in a Banach space which contains discrete semigroups generated by generalized nonspreading mappings [20] and semigroups of ϕ-nonexpansive mappings [31]. Then we prove a weak convergence theorem of Mann's type iteration for the semigroups of mappings. Furthermore, using the result, we obtain well-known and new theorems which are connected with weak convergence results in Banach spaces.

2. Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the weak convergence of $\{x_n\}$ to x and the strong convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. The *duality* mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E, where $\langle x, x^* \rangle$ is the value of $x^* \in E^*$ at $x \in E$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S(E)$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists. In this case, E is called *smooth*. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (1) is attained uniformly for $y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$. It is known that if E uniformly convex, then E is strictly convex and reflexive. Furthermore, we know from [27] that

(i) if E is smooth, then J is single-valued;
(ii) if E is reflexive, then J is onto;
(iii) if E is strictly convex, then J is one-to-one;
(iv) if E is strictly convex, then J is strictly monotone;
(v) if E has a Fréchet differentiable norm, then J is continuous.

Let E be a smooth Banach space and let J be the duality mapping on E. Throughout this paper, define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Observe that, in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2; \tag{2}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle; \tag{3}$$

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \tag{4}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \quad \text{if and only if } x = y. \tag{5}$$

The following lemmas are in Xu [33] and Kamimura and Takahashi [18].

Lemma 2.1 ([33]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|ax + (1 - a)y\|^2 \leq a\|x\|^2 + (1 - a)\|y\|^2 - a(1 - a)g(\|x - y\|)$$

for all $x, y \in B_r$ and $a \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.2 ([18]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \le \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \le r\}$.

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called *generalized nonexpansive* [13] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \le \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \ge 0$. A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [13, 14] for more details. The following results are in Ibaraki and Takahashi [13].

Lemma 2.3 ([13]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.4 ([13]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(*i*) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz\rangle \le 0$ *for all* $y \in C$;
(*ii*) $\phi(Rx, z) + \phi(x, Rx) \le \phi(x, z)$.

In 2007, Kohsaka and Takahashi [21] proved the following results:

Lemma 2.5 ([21]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(*a*) *C is a sunny generalized nonexpansive retract of E;*
(*b*) *C is a generalized nonexpansive retract of E;*
(*c*) *JC is closed and convex.*

Lemma 2.6 ([21]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) $z = Rx$;
(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Inthakon, Dhompongsa and Takahashi [17] obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping in a Banach space; see also Ibaraki and Takahashi [15].

Lemma 2.7 ([17]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 2.5 and 2.7.

Lemma 2.8 ([17]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [27] for the proof of existence of a Banach limit and its other elementary properties.

3. Attractive Point Theorems for Families of Mappings

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $l_s f$ and $r_s f$ as follows:

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s\in S} f(s) \le \mu(f) \le \sup_{s\in S} f(s), \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$. If $S = \mathbb{N}$, an invariant mean on $C(S) = B(S)$ is a Banach limit on l^∞. The following theorem is in [27, Theorem 1.4.5].

Theorem 3.1 ([27]). *Let S be a commutative semitopological semigroup. Then there exists an invariant mean on $C(S)$, i.e., there exists an element $\mu \in C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Let E be a Banach space and let C be a nonempty subset of E. Let S be a semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and $s \mapsto T_s x$ is continuous for each $x \in C$. We denote by $F(\mathcal{S})$ the set of common fixed points of T_s, $s \in S$, i.e.,

$$F(\mathcal{S}) = \cap\{F(T_s) : s \in S\}.$$

The following definition [26] is crucial in the nonlinear ergodic theory of abstract semigroups; see also [10]. Let E be a reflexive Banach space and let E^* be the dual space of E. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique point $z_0 \in \overline{co}\{u(s) : s \in S\}$ such that

$$\mu_s \langle u(s), y^* \rangle = \langle z_0, y^* \rangle, \quad \forall y^* \in E^*. \tag{6}$$

We call such z_0 the *mean vector* of u for μ. In particular, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Putting $u(s) = T_s x$ for all $s \in S$, we have that there exists $z_0 \in E$ such that

$$\mu_s \langle T_s x, y^* \rangle = \langle z_0, y^* \rangle, \quad \forall^* y \in E^*.$$

We denote such z_0 by $T_\mu x$. A net $\{\mu_\alpha\}$ of means on $C(S)$ is said to be *strongly asymptotically invariant* if for each $s \in S$,

$$\|\ell_s^* \mu_\alpha - \mu_\alpha\| \to 0 \quad \text{and} \quad \|r_s^* \mu_\alpha - \mu_\alpha\| \to 0,$$

where ℓ_s^* and r_s^* are the adjoint operators of ℓ_s and r_s, respectively. See [7] and [27] for more details.

Let E be a smooth Banach space and let C be a nonempty subset of E. For a mapping T from C into C, we denote by $A(T)$ the set of attractive points

[24, 29] of T, that is,

$$A(T) = \{u \in E : \phi(u, Tx) \leq \phi(u, x),\ \forall x \in C\}.$$

We know from Lin and Takahashi [24] that $A(T)$ is always closed and convex. Let S be a commutative semitopological semigroup with identity. For a continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings of C into itself, we denote the set $A(\mathcal{S})$ of common attractive points [3, 31] of $\mathcal{S} = \{T_s : s \in S\}$ by

$$A(\mathcal{S}) = \cap\{A(T_t) : t \in S\}.$$

It is obvious from Lin and Takahashi [24] that $A(\mathcal{S})$ is closed and convex. Using the technique developed by Takahashi [26], Takahashi, Wong and Yao [31] also proved the following attractive point theorem for a family of mappings in a Banach space.

Theorem 3.2 ([31]). *Let E be a smooth and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that*

$$\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y)$$

for all $y \in C$ and $t \in S$. Then, $A(\mathcal{S}) = \cap\{A(T_t) : t \in S\}$ is nonempty. In particular, if E is strictly convex and C is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is nonempty.

Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into C. We denote by $B(T)$ the set of *skew-attractive points* [24] of T, i.e.,

$$B(T) = \{z \in E, \phi(Tx, z) \leq \phi(x, z),\ \forall x \in C\}.$$

Lin and Takahashi [24] proved that $B(T)$ is always closed. Using the duality theory of nonlinear mappings [32] and [11], they also proved that $JB(T)$ is closed and convex.

We can also define by $B(\mathcal{S})$ the set of all common skew-attractive points of a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself, i.e., $B(\mathcal{S}) = \cap\{B(T_s) : s \in S\}$. Takahashi, Wong and Yao [31] obtained the following skew-attractive point theorem for semigroups of mappings without continuity in a Banach space.

Theorem 3.3 ([31]). *Let E be a strictly convex and reflexive Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that*

$$\mu_s \phi(T_t y, T_s x) \leq \mu_s \phi(y, T_s x)$$

for all $y \in C$ and $t \in S$. Then, $B(\mathcal{S}) = \cap\{B(T_t) : t \in S\}$ is nonempty. In particular, if C is closed and JC is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is nonempty.

4. Weak convergence theorem

In this section, using the results in Sections 2 and 3, we prove a weak convergence theorem of Mann's type iteration [25] for a commutative family of not necessarily continuous mappings in a Banach space. The following lemma is crucial in the proof of our main theorem.

Lemma 4.1. *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $B(\mathcal{S}) \neq \emptyset$. Let μ be a mean on $C(S)$. Then*

$$\phi(T_\mu x, m) \leq \phi(x, m), \quad \forall x \in C, \ m \in B(\mathcal{S}),$$

where $T_\mu x$ is a mean vector of $\{T_s x : s \in S\}$ and μ.

Proof. Let $x \in C$. Since $B(\mathcal{S})$ is nonempty, $\{T_s x : s \in S\}$ is bounded. Then there exists $T_\mu x \in E$ such that

$$\mu_s \langle T_s x, x^* \rangle = \langle T_\mu x, x^* \rangle, \quad \forall x^* \in E^*.$$

We first have that

$$\begin{aligned} \|T_\mu x\|^2 &= \langle T_\mu x, JT_\mu x \rangle = \mu_t \langle T_t x, JT_\mu x \rangle \\ &\leq \mu_t \big(\|T_t x\| \|T_\mu x\| \big) = \|T_\mu x\| \mu_t \|T_t x\| \end{aligned}$$

and hence $\|T_\mu x\| \leq \mu_t \|T_t x\|$. Then we have

$$\|T_\mu x\|^2 \leq \big(\mu_t \|T_t x\| \big)^2. \tag{7}$$

We also have that

$$\begin{aligned} \big(\mu_t \|T_t x\| \big)^2 &= \big(\mu_t \|T_t x\| \big) \big(\mu_s \|T_s x\| \big) \\ &= \mu_t \big(\mu_s \|T_s x\| \|T_t x\| \big) \\ &\leq \mu_t \left(\mu_s \left(\frac{1}{2} \|T_t x\|^2 + \frac{1}{2} \|T_s x\|^2 \right) \right) \\ &= \mu_t \left(\frac{1}{2} \|T_t x\|^2 + \frac{1}{2} \mu_s \|T_s x\|^2 \right) \\ &= \frac{1}{2} \mu_t \|T_t x\|^2 + \frac{1}{2} \mu_s \|T_s x\|^2 \\ &= \mu_t \|T_t x\|^2. \end{aligned}$$

Using (7), we have that

$$\|T_\mu x\|^2 \le \mu_t \|T_t x\|^2. \tag{8}$$

Thus we have from (8) that for any $w \in B(\mathcal{S})$

$$\begin{aligned}
\phi(T_\mu x, w) &= \|T_\mu x\|^2 - 2\langle T_\mu x, Jw\rangle + \|w\|^2 \\
&\le \mu_t \|T_t x\|^2 - 2\mu_t \langle T_t x, Jw\rangle + \|w\|^2 \\
&= \mu_t \big(\|T_t x\|^2 - 2\langle T_t x, Jw\rangle + \|w\|^2\big) \\
&= \mu_t \phi(T_t x, w) \\
&\le \mu_t \phi(x, w) \\
&= \phi(x, w).
\end{aligned}$$

This completes the proof. □

Using Lemma 4.1, we have the following result.

Lemma 4.2. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $B(\mathcal{S}) \neq \emptyset$. Let $\{\mu_n\}$ be a sequence of means on $C(S)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and let $\{x_n\}$ be a sequence in E generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N}.$$

If $R_{B(\mathcal{S})}$ is a sunny generalized nonexpansive retraction of E onto $B(\mathcal{S})$, then $\{R_{B(\mathcal{S})} x_n\}$ converges strongly to $z \in B(\mathcal{S})$.

Proof. Let $u \in B(\mathcal{S})$. Using Lemma 4.1, we have that

$$\begin{aligned}
\phi(x_{n+1}, u) &= \phi(\alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, u) \\
&\le \alpha_n \phi(x_n, u) + (1-\alpha_n)\phi(T_{\mu_n} x_n, u) \\
&\le \alpha_n \phi(x_n, u) + (1-\alpha_n)\phi(x_n, u) \\
&= \phi(x_n, u).
\end{aligned}$$

So, $\lim_{n\to\infty} \phi(x_n, u)$ exists. Since $\{\phi(x_n, u)\}$ is bounded, $\{x_n\}$ and $\{T_{\mu_n} x_n\}$ are bounded. Define $y_n = R_{B(\mathcal{S})} x_n$ for all $n \in \mathbb{N}$. Since $\phi(x_{n+1}, u) \le \phi(x_n, u)$ for all $u \in B(\mathcal{S})$, we have from $y_n \in B(\mathcal{S})$ that

$$\phi(x_{n+1}, y_n) \le \phi(x_n, y_n). \tag{9}$$

From Lemma 2.4 and (9), we have

$$\begin{aligned}
\phi(x_{n+1}, y_{n+1}) &= \phi(x_{n+1}, R_{B(\mathcal{S})} x_{n+1}) \\
&\le \phi(x_{n+1}, y_n) - \phi(R_{B(\mathcal{S})} x_{n+1}, y_n) \\
&= \phi(x_{n+1}, y_n) - \phi(y_{n+1}, y_n) \\
&\le \phi(x_{n+1}, y_n) \\
&\le \phi(x_n, y_n).
\end{aligned}$$

Then $\phi(x_n, y_n)$ is a convergent sequence. We also have from (9) that for all $m \in \mathbb{N}$

$$\phi(x_{n+m}, y_n) \leq \phi(x_n, y_n).$$

From $y_{n+m} = R_{B(\mathcal{S})}x_{n+m}$ and Lemma 2.4, we have

$$\phi(y_{n+m}, y_n) + \phi(x_{n+m}, y_{n+m}) \leq \phi(x_{n+m}, y_n) \leq \phi(x_n, y_n)$$

and hence

$$\phi(y_{n+m}, y_n) \leq \phi(x_n, y_n) - \phi(x_{n+m}, y_{n+m}).$$

Using Lemma 2.2, we have that

$$g(\|y_{n+m} - y_n\|) \leq \phi(y_{n+m}, y_n) \leq \phi(x_n, y_n) - \phi(x_{n+m}, y_{n+m}),$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function such that $g(0) = 0$. Then, the properties of g yield that $R_{B(\mathcal{S})}x_n$ converges strongly to a point $z \in B(\mathcal{S})$. This completes the proof. $\square$

Now, we can prove the following weak convergence theorem of Mann's type [25] for semigroups of mappings without continuity in a Banach space.

Theorem 4.3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty, closed and convex subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $A(\mathcal{S}) = B(\mathcal{S}) \neq \emptyset$ and let $R_{B(\mathcal{S})}$ be the sunny generalized nonexpansive retraction of E onto $B(\mathcal{S})$. Suppose that*

$$\limsup_{\alpha} \sup_{x,y \in D} (\mu_\alpha)_s(\phi(T_s x, T_t y) - \phi(T_s x, y)) \leq 0, \quad \forall t \in S \tag{10}$$

for every strongly asymptotically invariant net $\{\mu_\alpha\}$ of means on $C(S)$ and every bounded subset D of C. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e., a sequence of means on $C(S)$ such that

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges weakly to a point $z \in F(\mathcal{S})$ and $z = \lim_{n\to\infty} R_{B(\mathcal{S})}x_n$.

Proof. Since $B(\mathcal{S}) \neq \emptyset$, we have that $\{T_s x\}$ is bounded for all $x \in C$. On the other hand, since S is commutative, we have from Theorem 3.1 that there exists an invariant mean on $C(S)$. Let μ be an invariant mean on $C(S)$ and put $\mu_\alpha = \mu$ in (10). Then, we have from (10) that

$$\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y), \quad \forall x, y \in C,\ t \in S. \tag{11}$$

We have from Theorem 3.2 that $F(\mathcal{S})$ is nonempty. Let $x_1 = x \in C$. Since $\{T_s x : s \in S\}$ is bounded for all $x \in C$, $\{T_s x_n : s \in S\}$ is also bounded. So, $T_{\mu_n} x_n$ is well-defined. Using Lemma 4.1, we have that

$$\phi(T_{\mu_n} x_n, w) \leq \phi(x_n, w), \quad \forall w \in B(\mathcal{S}),\ n \in \mathbb{N}. \tag{12}$$

Then we have that for any $w \in B(\mathcal{S})$

$$\begin{aligned} \phi(x_{n+1}, w) &= \phi(\alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, w) \\ &\leq \alpha_n \phi(x_n, w) + (1-\alpha_n)\phi(T_{\mu_n} x_n, w) \\ &\leq \alpha_n \phi(x_n, w) + (1-\alpha_n)\phi(x_n, w) \\ &= \phi(x_n, w) \end{aligned} \tag{13}$$

for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} \phi(x_n, w)$ exists and hence $\{x_n\}$ is bounded. We also have from (12) that $\{T_{\mu_n} x_n\}$ is bounded. From Lemma 2.1, there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and

$$\begin{aligned} \phi(x_{n+1}, w) &= \phi(\alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, w) \\ &= \|\alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n\|^2 - 2\langle \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, Jw\rangle + \|w\|^2 \\ &\leq \alpha_n \|x_n\|^2 + (1-\alpha_n)\|T_{\mu_n} x_n\|^2 - \alpha_n(1-\alpha_n) g(\|x_n - T_{\mu_n} x_n\|) \\ &\qquad - 2\alpha_n \langle x_n, Jw\rangle - 2(1-\alpha_n)\langle T_{\mu_n} x_n, Jw\rangle + \|w\|^2 \\ &= \alpha_n \phi(x_n, w) + (1-\alpha_n)\phi(T_{\mu_n} x_n, w) - \alpha_n(1-\alpha_n) g(\|x_n - T_{\mu_n} x_n\|) \\ &\leq \alpha_n \phi(x_n, w) + (1-\alpha_n)\phi(x_n, w) - \alpha_n(1-\alpha_n) g(\|x_n - T_{\mu_n} x_n\|) \\ &= \phi(x_n, w) - \alpha_n(1-\alpha_n) g(\|x_n - T_{\mu_n} x_n\|). \end{aligned}$$

Thus we have

$$\alpha_n(1-\alpha_n) g(\|T_{\mu_n} x_n - x_n\|) \leq \phi(x_n, z) - \phi(x_{n+1}, z).$$

Since $\lim_{n\to\infty} \phi(x_n, z)$ exists and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have from the properties of g that

$$\|T_{\mu_n} x_n - x_n\| \to 0. \tag{14}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. It follows from (14) that

$$T_{\mu_{n_i}} x_{n_i} \rightharpoonup v. \tag{15}$$

We have from (4) that for $y \in C$ and $s, t \in S$

$$2\langle T_s x_n - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) = \phi(T_s x_n, T_t y) - \phi(T_s x_n, y).$$

Applying μ_n to both sides of the inequality, we have that

$$\begin{aligned} &2(\mu_n)_s \langle T_s x_n - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) \\ &= (\mu_n)_s\big(\phi(T_s x_n, T_t y) - \phi(T_s x_n, y)\big) \end{aligned}$$

and hence

$$\begin{aligned}&2\langle T_{\mu_n}x_n - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y)\\ &= (\mu_n)_s\big(\phi(T_s x_n, T_t y) - \phi(T_s x_n, y)\big)\end{aligned}$$

Since $T_{\mu_{n_i}}x_{n_i} \rightharpoonup v$ and $\limsup_{i\to\infty}(\mu_{n_i})_s\big(\phi(T_s x_{n_i}, T_t y) - \phi(T_s x_{n_i}, y)\big) \leq 0$, we get that

$$2\langle v - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) \leq 0.$$

Since $2\langle v - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) = \phi(v, T_t y) - \phi(v, y)$, we have that

$$\phi(v, T_t y) \leq \phi(v, y), \quad y \in C,\ t \in S. \tag{16}$$

Putting $y = v$, we have $v \in F(T_t)$. Therefore $v \in F(\mathcal{S})$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. We have that $u, v \in F(\mathcal{S})$. We also have from (11) that

$$u, v \in F(\mathcal{S}) = C \cap A(\mathcal{S}) \subset B(\mathcal{S}).$$

Thus we have from (13) that $\lim_{n\to\infty}\phi(x_n, u)$ and $\lim_{n\to\infty}\phi(x_n, v)$ exist. Put

$$a = \lim_{n\to\infty}(\phi(x_n, u) - \phi(x_n, v)).$$

Since $\phi(x_n, u) - \phi(x_n, v) = 2\langle x_n, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$, we have that

$$\begin{aligned}a &= 2\langle u, Jv - Ju\rangle + \|u\|^2 - \|v\|^2;\\ a &= 2\langle v, Jv - Ju\rangle + \|u\|^2 - \|v\|^2.\end{aligned}$$

From these equalities, we obtain

$$\langle u - v, Ju - Jv\rangle = 0.$$

Since J is strictly monotone, it follows that $u = v$; see [27]. Therefore, $\{x_n\}$ converges weakly to an element u of $F(\mathcal{S})$. On the other hand, we know from Lemma 4.2 that $\{R_{B(\mathcal{S})}x_n\}$ converges strongly to $z \in B(\mathcal{S})$. From Lemma 2.4, we also have

$$\langle x_n - R_{B(\mathcal{S})}x_n, JR_{B(\mathcal{S})}x_n - Ju\rangle \geq 0.$$

Since E has a Fréchet differentiable norm, the duality mapping J is norm-to-norm continuous. So, we have $\langle u - z, Jz - Ju\rangle \geq 0$. Since J is monotone, we also have $\langle u - z, Jz - Ju\rangle \leq 0$. So, we have $\langle u - z, Jz - Ju\rangle = 0$. Since E is strictly convex, we have $z = u$. This completes the proof. $\square$

5. Applications

In this section, using Theorem 4.3, we obtain well-known theorems and new theorems which are connected with weak convergence results in Banach spaces. Let E be a smooth Banach space, let C be a nonempty subset of E and let J be the duality mapping from E into E^*. A mapping $T : C \to C$ is called *generalized nonspreading* [20] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Tx,Ty) + (1-\alpha)\phi(x,Ty) + \gamma\{\phi(Ty,Tx) - \phi(Ty,x)\} \\ &\le \beta\phi(Tx,y) + (1-\beta)\phi(x,y) + \delta\{\phi(y,Tx) - \phi(y,x)\} \end{aligned} \tag{17}$$

for all $x, y \in C$. Putting $\alpha = \beta = \gamma = 1$ and $\delta = 0$ in (17), we obtain that

$$\phi(Tx,Ty) + \phi(Ty,Tx) \le \phi(Tx,y) + \phi(Ty,x), \quad \forall x, y \in C.$$

Such a mapping T is *nonspreading* in the sense of Kohsaka and Takahashi [23]. In the case of $\alpha = 1$ and $\beta = \gamma = \delta = 0$ in (17), we obtain that

$$\phi(Tx,Ty) \le \phi(x,y), \quad \forall x, y \in C.$$

Such a mapping T is called *ϕ-nonexpansive*. If E is a Hilbert space, then we have $\phi(x,y) = \|x-y\|^2$ for all $x, y \in E$. Thus we have from (17) that for any $x, y \in C$

$$\begin{aligned} &\alpha \parallel Tx - Ty \parallel^2 + (1-\alpha) \parallel x - Ty \parallel^2 + \gamma(\|Ty - Tx\|^2 - \|Ty - x\|^2) \\ &\le \beta \parallel Tx - y \parallel^2 + (1-\beta) \parallel x - y \parallel^2 + \delta(\|y - Tx\|^2 - \|y - x\|^2) \end{aligned}$$

and hence

$$\begin{aligned} &(\alpha+\gamma) \parallel Tx - Ty \parallel^2 + [1-(\alpha+\gamma)] \parallel x - Ty \parallel^2 \\ &\le (\beta+\delta) \parallel Tx - y \parallel^2 + [1-(\beta+\delta)] \parallel x - y \parallel^2 . \end{aligned}$$

That is, T is a generalized hybrid mapping in the sense of [19]. Using Theorem 3.2, we have the following attractive point theorem for generalized nonspreading mappings in a Banach space which was proved by Lin and Takahashi [24].

Theorem 5.1 ([24]). *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let T be a generalized nonspreading mapping of C into itself. Then the following are equivalent:*

(1) $A(T) \neq \emptyset$;
(2) $\{T^n v_0\}$ *is bounded for some* $v_0 \in C$.

Additionally, if E is strictly convex and C is closed and convex, then the following are equivalent:

(1) $F(T) \neq \emptyset$;
(2) $\{T^n v_0\}$ *is bounded for some* $v_0 \in C$.

Using Theorem 4.3, we first obtain the following weak convergence theorem of Mann's type iteration for generalized nonspreading mappings in a Banach space.

Theorem 5.2. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a generalized nonspreading mapping such that $A(T) = B(T) \neq \emptyset$. Let $R_{B(T)}$ be the sunny generalized nonexpansive retraction of E onto $B(T)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on l^∞, i.e., a sequence of means on l^∞ such that*

$$\|\mu_n - \ell_1^* \mu_n\| \to 0.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in F(T)$, where $z = \lim_{n\to\infty} R_{B(T)} x_n$.

Proof. Let $x \in C$. Since $A(T)$ is nonempty, the sequence $\{T^n x\}$ is bounded. We have from (17) and (3) that for all $x, y \in C$ and $k \in \mathbb{N}$,

$$\begin{aligned}
0 &\leq \beta\phi(T^{k+1}x, y) + (1-\beta)\phi(T^k x, y) + \delta(\phi(y, T^{k+1}x) - \phi(y, T^k x)) \\
&\qquad - \alpha\phi(T^{k+1}x, Ty) - (1-\alpha)\phi(T^k x, Ty) - \gamma(\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)) \\
&= \beta\big\{\phi(T^{k+1}x, Ty) + \phi(Ty, y) + 2\left\langle T^{k+1}x - Ty, JTy - Jy\right\rangle\big\} \\
&\qquad + (1-\beta)\big\{\phi(T^k x, Ty) + \phi(Ty, y) + 2\left\langle T^k x - Ty, JTy - Jy\right\rangle\big\} \\
&\qquad + \delta(\phi(y, T^{k+1}x) - \phi(y, T^k x)) - \alpha\phi(T^{k+1}x, Ty) - (1-\alpha)\phi(T^k x, Ty) \\
&\qquad - \gamma(\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)) \\
&= \phi(Ty, y) + 2\left\langle \beta T^{k+1}x + (1-\beta)T^k x - Ty, JTy - Jy\right\rangle \\
&\qquad + (\beta - \alpha)\big(\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\big) \\
&\qquad + \delta(\phi(y, T^{k+1}x) - \phi(y, T^k x)) - \gamma(\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)) \\
&= \phi(Ty, y) + 2\left\langle T^k x - Ty + \beta(T^{k+1}x - T^k x), JTy - Jy\right\rangle \\
&\qquad + (\beta - \alpha)\big(\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\big) \\
&\qquad + \delta(\phi(y, T^{k+1}x) - \phi(y, T^k x)) - \gamma(\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x))
\end{aligned}$$

and hence

$$\begin{aligned}
&2\langle T^k x - Ty, Jy - JTy\rangle - \phi(Ty, y) \\
&\leq 2\beta\langle T^{k+1}x - T^k x, JTy - Jy\rangle + (\beta - \alpha)\big(\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\big) \\
&\qquad + \delta(\phi(y, T^{k+1}x) - \phi(y, T^k x)) - \gamma(\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x))
\end{aligned}$$

On the other hand, we have from (4) that

$$2\langle T^k x - Ty, Jy - JTy\rangle - \phi(Ty, y) = \phi(T^k x, Ty) - \phi(T^k x, y).$$

So, we have that

$$\begin{aligned}
&\phi(T^k x, Ty) - \phi(T^k x, y) \\
&\leq 2\beta \left\langle T^{k+1}x - T^k x, JTy - Jy\right\rangle + (\beta - \alpha)\big(\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\big) \\
&\qquad + \delta(\phi(y, T^{k+1}x) - \phi(y, T^k x)) - \gamma(\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)).
\end{aligned}$$

If $\{\mu_\alpha\}$ is a strongly asymptotically invariant net of means on $B(\mathbb{N})$, then

$$\begin{aligned}
&(\mu_\alpha)_k\big(\phi(T^k x, Ty) - \phi(T^k x, y)\big) \leq 2\beta(\mu_\alpha)_k \left\langle T^{k+1}x - T^k x, JTy - Jy\right\rangle \\
&\qquad + (\beta - \alpha)\ (\mu_\alpha)_k\big(\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\big) \\
&\qquad + \delta\ (\mu_\alpha)_k(\phi(y, T^{k+1}x) - \phi(y, T^k x)) \\
&\qquad - \gamma\ (\mu_\alpha)_k(\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)) \\
&\leq \|\ell_1^*\mu_\alpha - \mu_\alpha\| \sup_{k\in\mathbb{N}} |\left\langle T^k x, JTy - Jy\right\rangle| \\
&\qquad + |\beta - \alpha|\ \|\ell_1^*\mu_\alpha - \mu_\alpha\| \sup_{k\in\mathbb{N}} \phi(T^k x, Ty) \\
&\qquad + |\delta|\ \|\ell_1^*\mu_\alpha - \mu_\alpha\| \sup_{k\in\mathbb{N}} \phi(y, T^k x)) \\
&\qquad + |\gamma|\ \|\ell_1^*\mu_\alpha - \mu_\alpha\| \sup_{k\in\mathbb{N}} \phi(Ty, T^k x)
\end{aligned}$$

and hence

$$\limsup_{\alpha} \sup_{x,y\in C} (\mu_\alpha)_k(\phi(T^k x, Ty) - \phi(T^k x, y)) \leq 0.$$

Therefore, we have the desired result from Theorem 4.3. □

Using Theorem 5.2, we obtain the following theorem.

Theorem 5.3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \leq \delta$. Assume that $F(T) \neq \emptyset$ and let $R_{F(T)}$ be the sunny generalized nonexpansive retraction of E onto $F(T)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on l^∞, i.e., a sequence of means on l^∞ such that*

$$\|\mu_n - \ell_1^*\mu_n\| \to 0.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n}x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in F(T)$, where $z = \lim_{n\to\infty} R_{F(T)}x_n$.

Proof. We also know from [20] that $\alpha > \beta$ together with $\gamma \leq \delta$ implies that

$$\phi(Tx, u) \leq \phi(x, u)$$

for all $x \in E$ and $u \in F(T)$. Furthermore, we have that $A(T) = B(T) = F(T)$. Therefore, we have the desired result from Theorem 5.2. $\square$

Let E be a smooth Banach space and let C be a nonempty subset of E. Let S be a semitopological semigroup. A continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings on C is a *ϕ-nonexpansive semigroup* on C if each $T_s,\ s \in S$ is ϕ-nonexpansive, i.e., $\phi(T_s x, T_s y) \leq \phi(x, y)$ for all $x, y \in C$. In the case when E is a Hilbert space, a ϕ-nonexpansive semigroup on C is a nonexpansive semigroup on C; see Atsushiba and Takakashi [3]. Using Theorem 3.2, we can also prove an attractive point theorems for ϕ-nonexpansive semigroups in a Banach space; see [31].

Theorem 5.4 ([31]). *Let E be a smooth and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a ϕ-nonexpansive semigroup on C such that $\{T_s z : s \in S\}$ is bounded for some $z \in C$. Then $A(\mathcal{S}) = \cap\{A(T_t) : t \in S\}$ is nonempty. In particular, if E is strictly convex and C is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is nonempty.*

Using Theorem 4.3, we also have the following weak convergence theorem for ϕ-nonexpansive semigroups in a Banach space.

Theorem 5.5. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed and convex subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a ϕ-nonexpansive semigroup on C such that $A(\mathcal{S}) = B(\mathcal{S}) \neq \emptyset$ and let $R_{B(\mathcal{S})}$ be the sunny generalized nonexpansive retraction of E onto $B(\mathcal{S})$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e., a sequence of means on $C(S)$ such that*

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in F(\mathcal{S})$, where $z = \lim_{n\to\infty} R_{B(\mathcal{S})} x_n$.

Proof. Since $\mathcal{S} = \{T_s : s \in S\}$ is a ϕ-nonexpansive semigroup on C, we have that

$$\phi(T_{t+s} x, T_t y) \leq \phi(T_s x, y)$$

for all $x, y \in C$ and $s, t \in S$. Since $A(T)$ is nonempty, $\{T_s x\}$ is bounded for all $x \in C$. Let $u \in B(\mathcal{S})$ and put $M = \{y \in C : \phi(y, u) \leq \phi(x, u)\}$. Then we have $x \in M$, $T_t M \subset M$ for all $t \in S$ and M is bounded, closed and convex. Without loss of generality, we may assume that C is bounded. Since $\mathcal{S} = \{T_t : t \in S\}$ is a ϕ-nonexpansive semigroup on C, we have that for all $x, y \in C$ and $s, t \in S$

$$\begin{aligned}
&\phi(T_s x, T_t y) - \phi(T_s x, y) \\
&= \phi(T_s x, T_t y) - \phi(T_{s+t} x, T_t y) + \phi(T_{s+t} x, T_t y) - \phi(T_s x, y) \\
&\leq \phi(T_s x, T_t y) - \phi(T_{s+t} x, T_t y) + \phi(T_s x, y) - \phi(T_s x, y) \\
&= \phi(T_s x, T_t y) - \phi(T_{s+t} x, T_t y).
\end{aligned}$$

If $\{\mu_\alpha\}$ is a strongly asymptotically invariant net of means on $C(S)$, then

$$\begin{aligned}
&(\mu_\alpha)_s(\phi(T_s x, T_t y) - \phi(T_s x, y)) \\
&\leq (\mu_\alpha)_s(\phi(T_s x, T_t y) - \phi(T_{s+t} x, T_t y)) \\
&= (\mu_\alpha)_s \phi(T_s x, T_t y) - (\ell_t^* \mu_\alpha)_s \phi(T_s x, T_t y)) \\
&\leq \|\mu_\alpha - \ell_t^* \mu_\alpha\| \sup_s \phi(T_s x, T_t y)
\end{aligned}$$

and hence

$$\limsup_\alpha \sup_{x,y \in C} (\mu_\alpha)_s(\phi(T_s x, T_t y) - \phi(T_s x, y)) \leq 0$$

for all $t \in S$. Therefore, we have the desired result from Theorem 4.3. $\square$

As a direct sequence of Theorem 5.5, we have the following weak convergence theorem for commutative semigroups of nonexpansive mappings in a Hilbert space which was proved by Atsushiba and Takakashi [2].

Theorem 5.6 ([2]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H. Let S be a commutative semitopological semigroup with identity and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e., a sequence of means on $C(S)$ such that*

$$\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0, \quad \forall s \in S.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges weakly to a point $z \in F(\mathcal{S})$ and $z = \lim_{n\to\infty} P_{F(\mathcal{S})} x_n$, where $P_{F(\mathcal{S})}$ is the metric projection of H onto $F(\mathcal{S})$.

Proof. Since $\{T_t x : t \in S\}$ is bounded for some $x \in C$, $F(\mathcal{S})$ is nonempty. We also note that $A(\mathcal{S}) = B(\mathcal{S}) = F(\mathcal{S})$. Since $A(\mathcal{S}) = B(\mathcal{S})$ is a nonempty closed convex subset of H, there exists the metric projection of H onto $A(\mathcal{S})$. In a Hilbert space, the metric projection of H onto $A(\mathcal{S})$ is equivalent to the sunny generalized nonexpansive retraction of E onto $A(\mathcal{S})$. Therefore, we have the desired result from Theorem 5.5. □

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