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Extension of Continuous Convex Functions from Subspaces II*

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Given Y a subspace of a topological vector space X, and an open convex set $0 \in A \subset X$, we say that the couple (X, Y) has the CE(A)-property if each continuous convex function on $A \cap Y$ admits a continuous convex extension defined on A.

Using results from our previous paper, we study for given A the relation between the CE(A)-property and the CE(X)-property. As a corollary we obtain that (X, Y) has the CE(A)-property for each A, provided (X, Y) has the CE(X)-property and Y is "conditionally separable". This applies, for instance, if X is locally convex and conditionally separable. Other results concern either the CE(A)-property for sets A of special forms, or the CE(A)-property for each A where X is a normed space with X/Y separable.

In the last section, we point out connections between the CE(X)-property and extendability of certain continuous linear operators. This easily yields a generalization of an extension theorem of Rosenthal, and another result of the same type.

Keywords: Convex function, extension, topological vector space, normed linear space

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1. Introduction

Let X be a topological vector space (t.v.s.), and $Y \subset X$ a subspace. In the present paper, we are interested in extendability of continuous convex functions from Y to X, or from $A \cap Y$ to A where $A \subset X$ is an open convex set intersecting

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Y. More precisely, we study the following properties.

Definition 1.1. Let X be a t.v.s., and $Y \subset X$ a subspace.

(a) Given an open convex set $A \subset X$ containing 0, we say that the couple (X, Y) has the CE(A)-*property* ("convex extension property for A") if each continuous convex function $f : A \cap Y \to \mathbb{R}$ can be extended to a continuous convex function $F : A \to \mathbb{R}$.

(b) We say that (X, Y) has the CE-*property* ("convex extension property") if (X, Y) has the CE(X)-property.

(c) We say that (X, Y) has the SCE-*property* ("strong convex extension property") if (X, Y) has the CE(A)-property for every open convex set $A \subset X$ that contains 0.

Study of the CE-property (though not under this terminology) for Banach spaces started in [2], followed by [12]. In our previous paper [4], we have proved several characterizations for the CE(A)-property (and consequently also for the CE-property) in topological vector spaces; some of them, needed in the sequel, are contained in Theorems 2.1 and 2.2, and Fact 2.3 below. Let us briefly summarize some further results from [4].

- If X is locally convex and X/Y is conditionally separable (see Definition 2.4), then (X, Y) has the CE-property (see [4, Theorem 5.5]). For instance, this always applies if the topology of X is a weak-type topology $\sigma(X, L)$ where L is a subset of the algebraic dual of X (see [4, Corollary 5.7]).
- If X is a Banach space with the separable complementation property and $Y \subset X$ a separable subspace, then (X, Y) has the CE-property (this is an easy combination of the previous point and the fact that (X, Z) has the CE-property whenever Z is a complemented subspace of X; cf. [4, Proposition 4.3(e)]).
- If Y is a separable nonreflexive Banach space, considered as a subspace of ℓ_∞, then (ℓ_∞, Y) fails the CE-property (see [4, Corollary 4.6]; for other examples of couples failing the CE-property, see [3, 4]).
- If X is locally convex and Y is finite-dimensional, then (X, Y) has the SCE-property (see [4, Proposition 4.7(b)]).

In the present paper, we further on develop methods from [4]. In Section 2, we investigate relationship between the CE(A)-property and the CE-property, obtaining as a corollary that, if the subspace Y is conditionally separable, then (X, Y) has the SCE-property if and only if it has the CE-property. Some significant sufficient conditions for the SCE-property are collected in Corollary 3.3. We deduce also some results about the CE(A)-property for open convex sets $0 \in A \subset X$ of some special forms. Section 3 is devoted to a proof of a particular sufficient condition for the SCE-property in normed spaces. In Section 4, we observe that the CE-property can be characterized in terms of extendability of certain continuous linear operators. Together with the proof of [4, Theorem 5.5], this gives generalizations of extension theorems of Rosenthal and Sobczyk, and a similar extension result for operators into c_0-sums of λ-separably injective spaces.

2. Preliminaries

We consider topological vector spaces (t.v.s.) over $\mathbb{R}$ that are not necessarily Hausdorff. Recall that in such spaces the Hahn-Banach theorem still works for separating two convex sets such that one of them is disjoint from the nonempty interior of the other one. Moreover, each such space admits a base of neighborhoods of the origin made of circled (i.e., symmetric and starshaped with respect to 0) open sets. All convex functions we consider are real-valued.

If X is a real topological vector space, we denote by X^* the topological dual of X, usually equipped with the w^*-topology. If Y is a subspace of X then we denote by q the (w^*-w^*-continuous linear) restriction map

$$q\colon X^* \to Y^*\,, \qquad q(x^*) = x^*|_Y\,.$$

It is a well-known fact [7] that if X is a locally convex space and Y is a closed subspace of X, then isomorphically $Y^* = X^*/Y^\perp$ (with Y^*, X^* in their w^*-topologies), and q corresponds to the canonical quotient map.

For $A \subset X$ and $B \subset X^*$, we denote by A° the polar set of A, and by ${}^\circ B$ the prepolar set of B, that is,

$$A^\circ = \{x^* \in X^* : x^*x \le 1\ \forall x \in A\}\,, \qquad {}^\circ B = \{x \in X : x^*x \le 1\ \forall x^* \in B\}\,.$$

Given sets $E, E_1, E_2\ldots,$ the notation $E_n \nearrow E$ means that the sequence $\{E_n\}$ is nondecreasing and its union is the set E. Analogously, $E_n \searrow E$ means that $\{E_n\}$ is nonincreasing and its intersection is E.

Now, we recall some of the results contained in [4].

Theorem 2.1 ([4, Theorem 3.6]). *Let X be a t.v.s., $A \subset X$ an open convex set, $Y \subset X$ a subspace, $0 \in A$. Then the following assertions are equivalent.*

- *(i)* *The couple (X, Y) has the* CE(A)*-property.*
- *(ii)* *For every sequence $\{C_n\}$ of open convex sets in Y such that $C_n \nearrow A \cap Y$, there exists a sequence $\{D_n\}$ of open convex sets in X such that $D_n \nearrow A$, and $D_n \cap Y = C_n$ for each $n \in \mathbb{N}$.*
- *(iii)* *For every sequence $\{B_n\}$ of w^*-compact convex sets in Y^* such that $B_n \searrow q(A^\circ)$ and ${}^\circ B_1$ is a neighborhood of the origin in Y, there exists a sequence $\{E_n\}$ of w^*-compact convex sets in X^* such that $E_n \searrow A^\circ$, ${}^\circ E_1$ is a neighborhood of the origin in X, and $q(E_n) = B_n$ for each $n \in \mathbb{N}$.*

Theorem 2.2 ([4, Corollary 3.7]). *Let Y be a subspace of a t.v.s. X. Then the following assertions are equivalent.*

- *(i)* *The couple (X, Y) has the* CE*-property.*

(*ii*) *For every sequence* $\{C_n\}$ *of open convex sets in* Y *such that* $C_n \nearrow Y$, *there exists a sequence* $\{D_n\}$ *of open convex sets in* X *such that* $D_n \nearrow X$, *and* $D_n \cap Y = C_n$ *for each* $n \in \mathbb{N}$.

(*iii*) *For every sequence* $\{B_n\}$ *of* w^**-compact convex equicontinuous sets in* Y^* *such that* $B_n \searrow \{0\}$, *there exists a sequence* $\{E_n\}$ *of* w^**-compact convex equicontinuous sets in* X^* *such that* $E_n \searrow \{0\}$, *and* $q(E_n) = B_n$ *for each* $n \in \mathbb{N}$.

(*iv*) *For every equicontinuous* w^**-null net* $\{y^*_{n,\gamma}\}_{(n,\gamma)\in\mathbb{N}\times\Gamma}$ *in* Y^*, *there exists an equicontinuous* w^**-null net* $\{x^*_{n,\gamma}\}_{(n,\gamma)\in\mathbb{N}\times\Gamma}$ *in* X^* *such that* $q(x^*_{n,\gamma}) = y^*_{n,\gamma}$ *for each* $(n,\gamma) \in \mathbb{N} \times \Gamma$. *(The nets considered here are directed by the relation:* $(n_1,\gamma_1) \leq (n_2,\gamma_2)$ *iff* $n_1 \leq n_2$.*)*

Fact 2.3 ([4, Observation 3.9]). *The conditions* (*ii*) *and* (*iii*) *in Theorem* 2.1 *are respectively equivalent to the following, formally weaker, conditions.*

(*ii′*) *For every sequence* $\{C_n\}$ *of open convex sets in* Y *such that* $C_n \nearrow A \cap Y$, *there exists a sequence* $\{D_k\}$ *of open convex sets in* X *such that* $D_k \nearrow A$, *and for each* $k \in \mathbb{N}$ *there exists* $n(k) \in \mathbb{N}$ *for which* $D_k \cap Y \subset C_{n(k)}$.

(*iii′*) *For every sequence* $\{B_n\}$ *of* w^**-compact convex sets in* Y^* *such that* $B_n \searrow q(A^\circ)$ *and* ${}^\circ B_1$ *is a neighborhood of the origin in* Y, *there exists a sequence* $\{E_k\}$ *of* w^**-compact convex sets in* X^* *such that* $E_k \searrow A^\circ$, ${}^\circ E_1$ *is a neighborhood of the origin in* X, *and for each* $k \in \mathbb{N}$ *there exists* $n(k) \in \mathbb{N}$ *for which* $q(E_k) \supset B_{n(k)}$.

We shall use the following notion of a conditionally separable set, introduced by us, and some related results.

Definition 2.4 ([4]). Let X be a t.v.s., $A \subset X$. We say that A is *conditionally separable* if for each neighborhood V of 0 there exists a countable set $Q \subset X$ such that $A \subset Q + V$.

Fact 2.5. *The following assertions hold.*

(*i*) *Conditional separability is stable under countable unions, under taking closures, and under passing to subsets. In a metrizable t.v.s., conditional separability is equivalent to separability.*

(*ii*) *A continuous linear image of a conditionally separable set is conditionally separable. In particular, a quotient of a conditionally separable t.v.s. is conditionally separable.*

(*iii*) ([4, Lemma 5.3]). *If* X *is a conditionally separable t.v.s., then for each neighborhood* $V \subset X$ *of the origin there exists a countable set* $Q \subset X$ *that separates the points of* V°.

Theorem 2.6 ([4, Theorem 5.5]). *Let* X *be a locally convex t.v.s.,* $Y \subset X$ *a subspace for which the quotient space* X/Y *is conditionally separable. Then the couple* (X,Y) *has the* CE*-property.*

3. The SCE-property

The following theorem, establishing relations between the CE- and the CE(A)-property, is the core of this section.

Theorem 3.1. *Let X be a t.v.s., $A \subset X$ an open convex set, $Y \subset X$ a subspace, $0 \in A$.*

(a) *Suppose that the points of $q(A^\circ)$ are separated by a countable set $Q \subset Y$. If (X, Y) has the* CE*-property then (X, Y) has the* CE(A)*-property.*

(b) *Suppose that the points of A° are separated by a countable set $P \subset X$. If (X, Y) has the* CE(A)*-property then (X, Y) has the* CE*-property.*

Proof. (a) It suffices to verify the condition (iii') in Fact 2.3. Let $\{B_n\}$ be a sequence of w^*-compact convex sets in Y^* such that $B_n \searrow q(A^\circ)$, and ${}^\circ B_1$ is a neighborhood of the origin in Y. Suppose that $Q = \{y_n\}_n \subset Y$ separates the points of $q(A^\circ)$ and put, for every $m \in \mathbb{N}$,

$$V_m = \{y^* \in Y^* : \ |y^* y_i| \leq 1/m, \ i = 1, \dots, m\}.$$

Let us consider the sets $F_m = V_m \cap (B_m - B_m)$ ($m \in \mathbb{N}$). Then $\{F_n\}$ is a decreasing sequence of w^*-compact convex sets in Y^*.

We claim that ${}^\circ F_1$ is a neighborhood of the origin in Y, to see this we just have to prove that ${}^\circ(B_1 - B_1)$ is a neighborhood of the origin in Y. Let $V \subset {}^\circ B_1$ be a symmetric open set such that $0 \in V$. Then it is easy to see that $V/2 \subset {}^\circ(B_1 - B_1)$, which proves our claim.

Now, we claim that $F_n \searrow \{0\}$. Suppose that $y^* \in \bigcap_m F_m$, then $y^* \in \bigcap_m (B_m - B_m) = q(A^\circ) - q(A^\circ)$. Moreover $y^* \in \bigcap_m V_m$ and hence $y^*|_Q \equiv 0$, but Q separates the points of $q(A^\circ)$ and hence $y^* = 0$. This concludes the proof of the claim.

Since (X, Y) has the CE-property, by Theorem 2.2 there exists a sequence $\{G_m\}$ of w^*-compact convex sets in X^* such that $G_m \searrow \{0\}$, ${}^\circ G_1$ is a neighborhood of the origin in X, and $q(G_m) = F_m$ for each $m \in \mathbb{N}$. Put $E_m = A^\circ + G_m$ ($m \in \mathbb{N}$). Then $\{E_m\}$ is a decreasing sequence of w^*-compact convex sets in X^* such that $E_m \searrow A^\circ$. Moreover, if $V \subset {}^\circ G_1$ is an open subset of X such that $0 \in V$, it is easy to see that $(A \cap V)/2 \subset {}^\circ E_1$, which shows that ${}^\circ E_1$ is a neighborhood of 0 in X.

To conclude the proof, it remains to show that if we fix $m \in \mathbb{N}$ then $q(A^\circ) + F_m = q(E_m) \supset B_k$ for some k. Suppose this is not the case, then for every $k \in \mathbb{N}$ there exists $z_k^* \in B_k \setminus q(E_m)$. Let $\{z_\alpha^*\}_\alpha$ be a w^*-convergent subnet of $\{z_k^*\}$. Then $z_\alpha^* \to z^* \in q(A^\circ)$, moreover for every α we have $z_\alpha^* = z^* + z_\alpha^* - z^*$. Since $\{z_\alpha^*\}$ is a subnet of $\{z_k^*\}$ and $z_k^* \in B_k$, for each $k \in \mathbb{N}$, we have that eventually $z_\alpha^* - z^* \in B_m - B_m$. Moreover, since $z_\alpha^* - z^* \to 0$ in the w^*-topology, eventually $z_\alpha^* - z^* \in V_m$. Then eventually $z_\alpha^* \in q(A^\circ) + F_m = q(E_m)$, a contradiction with the assumption that $z_k^* \in B_k \setminus q(E_m)$, for each $k \in \mathbb{N}$.

(b) It suffices to verify the condition (iii') in Fact 2.3 with $A = X$, and hence $A^\circ = \{0\}$. Let $\{B_n\}$ be a sequence of w^*-compact convex sets in Y^* such that $B_n \searrow \{0\}$ and ${}^\circ B_1$ is a neighborhood of the origin in Y. Suppose that $P = \{x_n\}_n \subset X$ separates the points of A° and put, for every $m \in \mathbb{N}$,

$$V_m = \{x^* \in X^* : \ |x^* x_i| \leq 1/m, \ i = 1, \ldots, m\}.$$

The set $F_m = q(A^\circ) + B_m$ $(m \in \mathbb{N})$ form a decreasing sequence of w^*-compact convex sets in Y^* such that $F_m \searrow q(A^\circ)$. Moreover, if $V \subset {}^\circ B_1$ is an open subset of Y such that $0 \in V$, it is easy to see that $[(A \cap Y) \cap V]/2 \subset {}^\circ F_1$, which shows that ${}^\circ F_1$ is a neighborhood of the origin in Y.

Since (X, Y) has the CE(A)-property, by Theorem 2.1 there exists a sequence $\{G_m\}$ of w^*-compact convex sets in X^* such that $G_m \searrow A^\circ$, ${}^\circ G_1$ is a neighborhood of the origin in X, and $q(G_m) = F_m$ for each $m \in \mathbb{N}$. The sets $E_m = (G_m - G_m) \cap V_m$ $(m \in \mathbb{N})$ form a decreasing sequence of w^*-compact convex sets in X^*.

We claim that ${}^\circ E_1$ is a neighborhood of the origin in X. To prove this, it suffices to show that ${}^\circ(G_1 - G_1)$ is a neighborhood of the origin in X. Let $V \subset {}^\circ G_1$ be a symmetric open set such that $0 \in V$; then it is easy to see that $V/2 \subset {}^\circ(G_1 - G_1)$, and the claim is proved.

Now, we claim that $E_m \searrow \{0\}$. Suppose that $x^* \in \bigcap_m E_m$, then $x^* \in \bigcap_m (G_m - G_m) = A^\circ - A^\circ$. Moreover $x^* \in \bigcap_m V_m$ and hence $x^*|_P \equiv 0$, but P separates the points of A° and hence $x^* = 0$. This proves our claim.

To conclude the proof, it remains to show that if we fix $m \in \mathbb{N}$ then $q(E_m) \supset B_k$ for some k. Suppose this is not the case, then for every $k \in \mathbb{N}$ there exists $z_k^* \in B_k \setminus q(E_m)$. In particular, $z_k^* \to 0$ in the w^*-topology of Y^*. Since $z_k^* \in B_k \subset F_k = q(G_k)$, there exists $x_k^* \in G_k$ such that $x_k^*|_Y = z_k^*$, for each $k \in \mathbb{N}$. Let $\{x_\alpha^*\}$ be a w^*-convergent subnet of $\{x_k^*\}$. Then $x_\alpha^* \to x^* \in A^\circ \cap Y^\perp$. Since $\{x_\alpha^*\}$ is a subnet of $\{x_k^*\}$, and $x_k^* \in G_k$ for each $k \in \mathbb{N}$, we have eventually $x_\alpha^* - x^* \in G_m - G_m$. Moreover, since $x_\alpha^* - x^* \to 0$ in the w^*-topology, eventually $x_\alpha^* - x^* \in V_m$. Then eventually $z_\alpha^* = q(x_\alpha^* - x^*) \in q(E_m)$, a contradiction with the assumption that $z_k^* \in B_k \setminus q(E_m)$, for each $k \in \mathbb{N}$. We are done. □

As an easy corollary we get the following result.

Theorem 3.2. *Let X be a t.v.s. and $Y \subset X$ a subspace which is conditionally separable. Then (X, Y) has the* CE*-property if and only if (X, Y) has the* SCE-*property.*

Proof. Suppose that the couple (X, Y) has the CE-property and that $0 \in A \subset X$ is an open convex set. Since Y is conditionally separable, by Fact 2.5, the points of $q(A^\circ)$ are separated by a countable set in Y. By Theorem 3.1 above, (X, Y) has the CE(A)-property. Since A was arbitrary, it follows that (X, Y) has the SCE-property. The other implication is obvious. □

Let us recall that a Banach space X is said to have the *separable complementation property* if each its separable subspace is contained in a complemented separable subspace. This property is satisfied if, for instance, X is weakly compactly generated, or X is dual with the Radon-Nikodým property, or X is a Banach lattice not containing c_0 (see, e.g., [9]).

Corollary 3.3. *Let X be a t.v.s. and $Y \subset X$ a subspace. Then the couple (X, Y) has the* SCE*-property, provided* at least one *of the following conditions is satisfied.*

(i) *X is locally convex and conditionally separable.*
(ii) *Y is conditionally separable and complemented in X.*
(iii) *X is a Banach space having the separable complementation property, and Y is separable.*
(iv) *The topology of X is a weak-type topology $\sigma(X, L)$ where $L \subset X^*$.*

Proof. Notice that Y is conditionally separable in all four cases (for the case (iv) see [4, Corollary 5.7] or Introduction). By Theorem 3.2, it suffices to show that (X, Y) has the CE-property. Now, the case (i) follows by [4, Theorem 5.5] (see also Introduction) since X/Y is conditionally separable by Fact 2.5. The cases (ii) and (iii) follow from [4, Proposition 4.3], while (iv) follows from [4, Corollary 5.7]. □

Let us conclude this section with two corollaries about particular open convex sets $0 \in A \subset X$ (namely, weakly open sets, and open sets that are intersection of countably many open half-spaces).

Corollary 3.4. *Let X be t.v.s., $Y \subset X$ a subspace, and $0 \in A \subset X$ an open convex set.*

(a) *Suppose that $A \cap Y$ is weakly open in Y. If (X, Y) has the* CE*-property then (X, Y) has the* CE(A)*-property.*
(b) *Suppose that A is weakly open in X. Then (X, Y) has the* CE(A)*-property iff (X, Y) has the* CE*-property.*

Proof. Both statements follow easily from Theorem 3.1 via the following observation: since every t.v.s. Z is conditionally separable in its weak topology (see [4, Observation 5.4]), we can apply Fact 2.5(iii) to conclude that if $0 \in A \subset Z$ is a weakly open convex set then the points of A° are separated by a countable set $Q \subset Z$. □

For the next corollary, we need the following easy observation.

Observation 3.5. Let Z be a normed space and $E \subset Z^*$ a separable set. Then the points of E are separated by a countable set $Q \subset Z$.

Proof. Fix a dense sequence $\{x_n^*\} \subset E - E$. For each n, choose $x_n \in Z$ so that $\|x_n\| = 1$ and $x_n^* x_n \geq \frac{1}{2}\|x_n^*\|$. It is easy to see that the sequence $\{x_n\}$ separates the points of E. $\square$

Corollary 3.6. *Let X be a normed space, $Y \subset X$ a subspace, and $0 \in A \subset X$ an open convex set.*

(*a*) *Suppose that Y is a Banach space and $A \cap Y$ is a countable intersection of open half-spaces of Y. If (X, Y) has the* CE-*property then (X, Y) has the* CE(A)-*property.*

(*b*) *Suppose that X is a Banach space and A is a countable intersection of open half-spaces of X. Then (X, Y) has the* CE(A)-*property iff (X, Y) has the* CE-*property.*

Proof. Both statements can be easily derived (as in Corollary 3.4) from Theorem 3.1 via the following observation. Assume that Z is a Banach space, $\{f_n\}$ is a sequence in Z^* such that the set $A := \bigcap_n f_n^{-1}\big((-\infty, 1)\big)$ is open. Fix an arbitrary $z \in Z$. If $\mathbb{R}^+ z \subset A$ then $\sup z(A^\circ) = 0 = z(0)$. On the other hand, if $\mathbb{R}^+ z \not\subset A$ then there exists $t > 0$ such that $z_1 := tz \in \partial A$; and since A is open, there exists n with $f_n(z_1) = 1 = \sup z_1(A^\circ)$, that is, $f_n(z) = \sup z(A^\circ)$. It follows that the countable set $B = \{f_n\} \cup \{0\}$ is a James boundary for A°. By the Rodé-Godefroy theorem (see [5, Theorem 3.122] or [6, Theorem 5.7, p. 641]), $A^\circ = \overline{\mathrm{conv}}^{\|\cdot\|} B$ and, in particular, A° is norm-separable. By Observation 3.5, the points of A° are separated by a countable set $Q \subset Z$. $\square$

Remark 3.7. Let us show that the proof of Corollary 3.6 does not work for incomplete normed spaces. Let Z be the space of all elements of ℓ_∞ that assume only finitely many values. Then Z is dense in ℓ_∞, and its open ball B_Z^0 can be written as a countable intersection of open halfspaces: $B_Z^0 = \bigcap_n e_n^{-1}\big((-1, 1)\big)$ where $e_n \in Z^* = \ell_\infty^*$ denotes the n-th coordinate functional. However, there exists no countable set $Q \subset Z$ that would separate the points of the polar $(B_Z^0)^\circ = B_{Z^*}$. Indeed, otherwise the weak-type topology $\sigma(Z^*, Q)$, which coincides on B_{Z^*} with the w^*-topology of Z^*, would be metrizable, but this is impossible since Z is not separable.

4. The countable-chain criterion for the SCE-property

In this section, we present a sufficient condition for a couple (X, Y) to have the SCE-property, which can be sometimes used if X is a normed space and the quotient X/Y is separable. The proof will use the fact that if X is complete then the subspace Y is the range of a continuous nonlinear projection of a certain type. Let us start with a definition and a few basic facts.

Definition 4.1. Let X, Y, Z be vector spaces such that $Y \subset X \cap Z$. A mapping $P \colon X \to Z$ is said to be *semi-linear w.r.t.* Y if

$$P(tx + y) = tP(x) + y \quad \text{whenever } t \in \mathbb{R},\ x \in X \text{ and } y \in Y.$$

Notice that every such mapping is homogeneous on X, and coincides on Y with the identity map of Y.

The following lemma is partially contained in [11, Lemma 1]. We present a sketch of proof for the sake of completeness.

Lemma 4.2. *Let Y be a closed subspace of a normed space X, and $P\colon X \to Y$ a mapping which is semi-linear w.r.t. Y.*

- *For each $x \in X$, the decomposition*

$$x = P(x) + \big(x - P(x)\big)$$

  is the unique way of writing x as a sum of an element of Y and an element of $P^{-1}(0)$. We shall shortly write $X = Y \oplus P^{-1}(0)$ in this case.
- *If, moreover, P is continuous then there exists a homeomorphism h of $P^{-1}(0)$ onto X/Y, such that h and h^{-1} preserve boundedness of sets.*

Sketch of proof. The first part is quite easy and is left to the reader. Now, let $q\colon X \to X/Y$ be the quotient map. Then its restriction $h := q|_{P^{-1}(0)}$ is a homeomorphism of $P^{-1}(0)$ onto X/Y by [11, Lemma 1]. Clearly, h is bounded on bounded sets (since q is). The fact that its inverse h^{-1} is bounded on a neighborhood of 0 and homogeneous, easily implies that h^{-1} is bounded on bounded sets. □

The next lemma is an easy known consequence of a version of the Bartle-Graves theorem.

Lemma 4.3. *Let Y be a complete subspace of a normed space X, and $\varepsilon > 0$. Then there exists a mapping $P\colon X \to Y$ which is continuous and semi-linear w.r.t. Y, and satisfies*

$$\|x - P(x)\| \le (1+\varepsilon)\,\mathrm{dist}(x, Y) \quad (x \in X).$$

Proof. Consider Y as a closed subspace of the completion $\widetilde{X}$ of X. By [11, Lemma 2], there exists $P\colon \widetilde{X} \to Y$ with the desired properties. □

We are ready to state the main result of this section. Notice that an analogous result for the CE-property (even for locally convex spaces) follows from results in [4] since X/Y is separable (see Introduction). As usual, if X is a normed space, B_X denotes its closed unit ball.

Theorem 4.4. *Let X be a normed space, and $Y \subset X$ a subspace. Assume that there exists a sequence $Y = Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ of subspaces of X such that $\bigcup_n Y_n$ is dense in X and, for each $n \in \mathbb{N}$, $\dim(Y_n/Y_{n-1}) < \infty$. If each couple (Y_n, Y_{n-1}), $n \in \mathbb{N}$, has the* SCE-*property, then (X, Y) has the* SCE-*property.*

Proof. Let $\widetilde{X}$ be the completion of X, and Z_n the closure of Y_n in $\widetilde{X}$ ($n \in \mathbb{N} \cup \{0\}$). Then $Z_n \supset Z_{n-1}$, $\dim(Z_n/Z_{n-1}) < \infty$ and, by [4, Lemma 4.2], the couple (Z_n, Z_{n-1}) has the SCE-property. Put $Z = \bigcup_{n\in\mathbb{N}} Z_n$. Since X and Z are dense in $\widetilde{X}$, by [4, Lemma 4.2] we just have to prove that the couple (Z, Z_0) has the SCE-property. To prove this it suffices to verify the condition (ii) in Theorem 2.1. So, let $0 \in A \subset Z$ be an open convex set, and $\{C_n\}$ a sequence of open convex sets in Z_0 such that $C_n \nearrow A \cap Z_0$. By homogeneity we can (and do) suppose that $B_{Z_0} \subset C_1$.

Let $P : Z \to Z_0$ be a continuous mapping which is semi-linear w.r.t. Z_0 (see Lemma 4.3), and I the identity map of Z. Denote $\Pi = P^{-1}(0)$ and $Q = I - P$. Given a set $E \subset Z$, we use the notation $B_E = E \cap B_Z$ (which is standard for subspaces). For each $k \in \mathbb{N}$, denote $\Pi_k = \Pi \cap Z_k$. Since the restriction $P_k := P|_{Z_k}$ is semi-linear w.r.t. Z_0, and $P_k^{-1}(0) = \Pi_k$, we can write $Z_k = Z_0 \oplus \Pi_k$ (see Lemma 4.2).

For each $n \in \mathbb{N}$, put $D_n^0 := C_n$. Proceeding inductively, one can find for each $k \in \mathbb{N}$ a sequence $\{D_n^k\}_n$ of open convex sets in Z_k such that $D_n^k \nearrow A \cap Z_k$ and such that $D_n^k \cap Z_{k-1} = D_n^{k-1}$.

We claim that, for some $\delta > 0$, one has

$$\operatorname{conv}(B_{\Pi_k} \cup B_{Z_0}) \supset \delta B_{Z_k} \quad \text{for each } k \in \mathbb{N}. \tag{1}$$

To show this, choose a $\delta \in (0, 1/4)$ such that $\|P(x)\| \leq 1/4$ whenever $\|x\| \leq \delta$. For $x \in \delta B_Z$, we have $P(x) \in \frac{1}{4} B_{Z_0}$ and $Q(x) \in \frac{1}{2} B_\Pi$; thus $x = Q(x) + P(x) \in \frac{1}{2} B_\Pi + \frac{1}{2} B_{Z_0} \subset \operatorname{conv}(B_\Pi \cup B_{Z_0})$, and this implies (1).

Notice that, for each k, the set B_{Π_k} is compact since it is homeomorphic (by Lemma 4.2) to a closed bounded set in the finite-dimensional space Z_k/Z_0. Then there exists $n_1 \in \mathbb{N}$ such that $D_{n_1}^1 \supset B_{\Pi_1}$, moreover, by induction, we can find a sequence of integers $\{n_k\}$ such that $n_{k+1} > n_k$ and $D_{n_k}^k \supset B_{\Pi_k}$ for each $k \in \mathbb{N}$. Let us define $D_k = \operatorname{conv}(\delta B_Z^0 \cup D_{n_k}^k)$ $(k \in \mathbb{N})$, and observe that D_k is an open convex set in Z (cfr. [4, Lemma 2.1]).

Notice that

$$D_{n_{k+1}}^{k+1} \supset D_{n_{k+1}}^{k+1} \cap Z_k = D_{n_{k+1}}^k \supset D_{n_k}^k$$

and

$$A = \bigcup_h (Z_h \cap A) = \bigcup_h \bigcup_{k \geq h} D_{n_k}^h = \bigcup_h D_{n_h}^h \subset \bigcup_h D_h,$$

where the last equality holds since $D_{n_k}^h \subset D_{n_k}^{h'}$ if $h' \geq h$. Since $D_h \subset A$ for each h, it follows that $D_h \nearrow A$.

To conclude the proof, we just have to observe that

$$D_k \cap Z_0 = (D_k \cap Z_k) \cap Z_0 = \operatorname{conv}(D_{n_k}^k \cup \delta B_{Z_k}^0) \cap Z_0 = C_{n_k},$$

where the second equality holds by [4, Lemma 2.1], and the last equality holds since $\delta B_{Z_k}^0 \subset \operatorname{conv}(B_{\Pi_k} \cup B_{Z_0}) \subset D_{n_k}^k$ and $D_{n_k}^k \cap Z_0 = C_{n_k}$. $\square$

As an application, we obtain the following easy corollary.

Corollary 4.5. *Let Y be a subspace of a normed space X such that X/Y is separable. Suppose that the couple $(Y \oplus \mathbb{R}, Y)$ has the* SCE*-property and that Y is isomorphic to its hyperplanes, then the couple (X, Y) has the* SCE*-property.*

5. The CE-property and extension of operators

In this Section, we are going to show that CE-property of a couple (X, Y) is equivalent to extendability of certain continuous linear operators from Y to the whole X. As an application, we obtain versions of theorems by Rosenthal and Sobczyk in the setting of locally convex spaces.

We shall use the following notation from [10]. Given a sequence $\{Z_j\}$ of Banach spaces, we denote by $(Z_j)_{c_0}$ their c_0-sum $(\bigoplus_{j=1}^{\infty} Z_j)_{c_0}$. Moreover, for a fixed Banach space Z, we denote by $c_0(Z)$ the Banach space $(\bigoplus_{j=1}^{\infty} Z)_{c_0}$.

Given $\lambda \geq 1$, a Banach space Z is said to be λ-*(separably) injective* iff for each (separable) Banach space X and each subspace $Y \subset X$, every bounded linear operator $T_0 \colon Y \to Z$ can be extended to a bounded linear operator $T \colon X \to Z$ such that $\|T\| \leq \lambda \|T_0\|$.

Our results are based on the following trivial observation.

Observation 5.1. Let X be a t.v.s., and Γ a set. There exists a canonical one-to-one correspondence between the set of all continuous linear operators $T \colon X \to c_0(\ell_\infty(\Gamma))$ and the set of all equicontinuous w^*-null nets $\{x^*_{n,\gamma}\}_{(n,\gamma)\in\mathbb{N}\times\Gamma}$ in X^*. The correspondence is given by the formula

$$x^*_{n,\gamma} x = [(Tx)(n)](\gamma) \, .$$

Moreover, if $V \subset X$ is a symmetric neighborhood of 0, we have the equivalence

$$T(V) \subset B_{c_0(\ell_\infty(\Gamma))} \quad \Leftrightarrow \quad \{x^*_{n,\gamma}\} \subset V^\circ.$$

Definition 5.2. Let Y be a subspace of a t.v.s. X, and $\mu \geq 1$. We say that the couple (X, Y) has the μ-CE-*property* if for each convex symmetric neighborhood $V \subset X$ of 0 and for each w^*-null net $\{y^*_{n,\gamma}\}$ in Y^* such that $\{y^*_{n,\gamma}\} \subset q(V^\circ)$, there exists a w^*-null net net $\{x^*_{n,\gamma}\}$ in X^* such that $y^*_{n,\gamma} = q(x^*_{n,\gamma})$ for each $(n,\gamma) \in \mathbb{N} \times \Gamma$, and $\{x^*_{n,\gamma}\} \subset \mu V^\circ$.

Now, we are ready for the following two propositions, the second one of which is a quantitative version of the first one. We shall prove only Proposition 5.4 (the proof of Proposition 5.3 goes in the same way by observing that a net $\{x^*_{n,\gamma}\} \subset X^*$ is equicontinuous if and only if the convex symmetric set $V = \{x \in X : |x^*_{n,\gamma} x| \leq 1\}$ is a neighborhood of 0).

Proposition 5.3. *Let Y be a subspace of a t.v.s. X. The following assertions are equivalent.*

(i) *The couple (X, Y) has the* CE*-property.*

(*ii*) *Given a set* Γ, *every continuous linear operator* $T_0\colon Y \to c_0(\ell_\infty(\Gamma))$ *can be extended to a continuous linear operator* $T\colon X \to c_0(\ell_\infty(\Gamma))$.

(*iii*) *Given* $\lambda \geq 1$ *and a sequence* $\{Z_j\}$ *of* λ*-injective Banach spaces, every continuous linear operator* $T_0\colon Y \to (Z_j)_{c_0}$ *can be extended to a continuous linear operator* $T\colon X \to (Z_j)_{c_0}$.

Proposition 5.4. *Let* Y *be a subspace of a t.v.s.* X, *and* $\mu \geq 1$. *The following assertions are equivalent.*

(*i*) *The couple* (X, Y) *has the* μ-CE*-property.*

(*ii*) *Given a set* Γ *and a convex symmetric neighborhood* $V \subset X$ *of* 0, *every continuous linear operator* $T_0\colon Y \to c_0(\ell_\infty(\Gamma))$ *such that* $T_0(V \cap Y) \subset B_{c_0(\ell_\infty(\Gamma))}$, *can be extended to a continuous linear operator* $T\colon X \to c_0(\ell_\infty(\Gamma))$ *such that* $T(V) \subset \mu B_{c_0(\ell_\infty(\Gamma))}$.

(*iii*) *Given* $\lambda \geq 1$, *a sequence* $\{Z_j\}$ *of* λ*-injective Banach spaces and a convex symmetric neighborhood* $V \subset X$ *of* 0, *every continuous linear operator* $T_0\colon Y \to (Z_j)_{c_0}$ *such that* $T_0(V \cap Y) \subset B_{(Z_j)_{c_0}}$ *can be extended to a continuous linear operator* $T\colon X \to (Z_j)_{c_0}$ *such that* $T(V) \subset \lambda\mu B_{(Z_j)_{c_0}}$.

Notice that, in the case that X is a normed space and $V = B_X$, the condition (*ii*) in Proposition 5.4 says that the extension satisfies $\|T\| \leq \mu\|T_0\|$.

Proof. (*i*) $\Leftrightarrow$ (*ii*) follows immediately from Observation 5.1 via the net characterization of the CE-property (Theorem 2.2(*i*+*v*)) and the easy fact that $(V \cap Y)^\circ = q(V^\circ)$.

(*iii*) $\Rightarrow$ (*ii*) is trivial since the space $\ell_\infty(\Gamma)$ is 1-injective.

(*ii*) $\Rightarrow$ (*iii*). Given a sequence $\{Z_j\}$ of λ-injective Banach spaces, fix a set Γ such that $Z_n \subset \ell_\infty(\Gamma)$ for each n. By λ-injectivity, there exist projections $P_n\colon \ell_\infty(\Gamma) \to Z_n$ such that $\|P_n\| \leq \lambda$ ($n \in \mathbb{N}$). Then

$$P\colon \ell_\infty(\Gamma) \to (Z_j)_{c_0}, \quad P((u_n)_n) = (P_n u_n)_n$$

is a projection with $\|P\| \leq \lambda$. Now, if V and T are as in (*iii*), we have $T_0(V \cap Y) \subset B_{(Z_j)_{c_0}} \subset B_{\ell_\infty(\Gamma)}$. By (*ii*), T_0 can be extended to a continuous linear operator $T_1\colon X \to c_0(\ell_\infty(\Gamma))$ such that $T_1(V) \subset \mu B_{c_0(\ell_\infty(\Gamma))}$. Then $T := P \circ T_1\colon X \to (Z_j)_{c_0}$ is the desired extension of T_0. $\square$

The next result is a variant of the above Proposition 5.4(*iii*) for λ-separably injective spaces, under the additional assumption that X/Y is conditionally separable.

Theorem 5.5. *Let* Y *be a subspace of a t.v.s.* X, *and* $\mu \geq 1$. *Assume that the couple* (X, Y) *has the* μ-CE*-property and that* X/Y *is conditionally separable. Then, given* $\lambda \geq 1$, *a sequence* $\{Z_j\}$ *of* λ*-separably injective Banach spaces and a convex symmetric neighborhood* $V \subset X$ *of* 0, *every continuous linear operator* $T_0\colon Y \to (Z_j)_{c_0}$ *such that* $T_0(V \cap Y) \subset B_{(Z_j)_{c_0}}$ *can be extended to a continuous linear operator* $T\colon X \to (Z_j)_{c_0}$ *such that* $T(V) \subset 3\lambda\mu B_{(Z_j)_{c_0}}$.

Proof. Fix a set Γ such that $Z_n \subset \ell_\infty(\Gamma)$ for each n. Since $(Z_j)_{c_0} \subset c_0(\ell_\infty(\Gamma))$, Proposition 5.4 implies that T_0 can be extended to a continuous linear operator $S\colon X \to c_0(\ell_\infty(\Gamma))$ such that $S(V) \subset \mu B_{c_0(\ell_\infty(\Gamma))}$. We can write $S = (S_n)_{n\in\mathbb{N}}$ where $S_n\colon X \to \ell_\infty(\Gamma)$ for each n. Then $E_n := \overline{Z_n + S_n(X)}$ is a closed subspace of $\ell_\infty(\Gamma)$. Write $T_0 = (T_{0n})_{n\in\mathbb{N}}$ where $T_{0n}\colon Y \to Z_n$ for each n.

Fix $n \in \mathbb{N}$. We claim that the quotient E_n/Z_n is separable. To see this, consider the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ Q\ } & X/Y \\ {\scriptstyle S_n}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{S}_n} \\ E_n & \xrightarrow[\ \widetilde{Q}\ ]{} & E_n/Z_n \end{array}$$

where Q and $\widetilde{Q}$ are the quotient maps. The mapping $\widehat{S}_n$ is defined by $\widehat{S}_n\xi = \widetilde{Q}S_n x$ where $x \in Q^{-1}\xi$. This definition does not depend on the choice of x since $S_n(Y) = T_{0n}(Y) \subset Z_n$. Then $\widehat{S}_n$ is a continuous linear operator. Since X/Y is conditionally separable, the normed space $\widehat{S}_n(X/Y)$ is separable (see Fact 2.5). Moreover, $\widehat{S}_n(X/Y) = \widetilde{Q}(S_n(X)) = \widetilde{Q}(Z_n + S_n(X))$ is dense in $\widetilde{Q}(E_n) = E_n/Z_n$. This proves our claim.

Now, since each Z_n is λ-separably injective and E_n/Z_n is separable, [1, Proposition 3.2] implies that there exist bounded linear projections $P_n\colon E_n \to Z_n$ with $\|P_n\| \le 3\lambda$ $(n \in \mathbb{N})$. Then

$$P\colon (E_j)_{c_0} \to (Z_j)_{c_0}\,, \quad P((u_n)_n) = (P_n u_n)_n$$

is a bounded linear projection with $\|P\| \le 3\lambda$. Moreover, since $S(X) \subset (E_j)_{c_0}$ and $S(V) \subset \mu B_{(E_j)_{c_0}}$, we can define the continuous linear operator $T := P \circ S\colon X \to (Z_j)_{c_0}$ which extends T_0 and satisfies $T(V) \subset P(\mu B_{(E_j)_{c_0}}) \subset 3\lambda\mu B_{(Z_j)_{c_0}}$. The proof is complete. $\square$

To apply the previous results, we shall need the following more precise, quantitative version of Theorem 2.6.

Theorem 5.6. *Let Y be a subspace of a t.v.s. X. Assume that X is locally convex and X/Y is conditionally separable. Then the couple (X, Y) has the* 2-CE-*property.*

Proof. In [4, Theorem 5.5], we proved that (X, Y) has the CE-property. But looking more closely at that proof, in fact we proved that, in the condition (iii) in Theorem 2.2, if $V \subset X$ is a convex symmetric neighborhood of 0 such that $B_1 \subset q(V^\circ)$, then the sequence $\{E_n\}$ can be chosen in such a way that $E_1 \subset 2V^\circ$.

Now, it is sufficient to proceed as in the proof of $(iii) \Rightarrow (v)$ in [4, Theorem 3.6]; let us briefly sketch the argument. Let $\{y^*_{n,\gamma}\}$ be a w^*-null net in Y^*, contained in $q(V^\circ)$. The w^*-compact convex sets $B_n := \overline{\mathrm{conv}}^{w^*}\{y^*_{k,\gamma}\}_{k\geq n} \subset Y^*$ ($n \in \mathbb{N}$) satisfy $B_n \searrow \{0\}$ and $B_1 \subset q(V^\circ)$. We already know that there exists a sequence $\{E_n\}$ as in Theorem 2.2(*iii*), satisfying $E_1 \subset 2V^\circ$. For each $(n,\gamma) \in \mathbb{N}\times\Gamma$, since $y^*_{n,\gamma} \in B_n = q(E_n)$, there exists $x^*_{n,\gamma} \in E_n$ with $q(x^*_{n,\gamma}) = y^*_{n,\gamma}$. Then the net $\{x^*_{n,\gamma}\}$ is w^*-null since $E_n \searrow \{0\}$. Moreover, $\{x^*_{n,\gamma}\} \subset E_1 \subset 2V^\circ$, and we are done. □

Now, Proposition 5.4 and Theorems 5.5 and 5.6 give the following extension results.

Corollary 5.7. *Let Y be a subspace of a t.v.s. X. Assume that X is locally convex and X/Y is conditionally separable.*

(a) *Given $\lambda \geq 1$, a sequence $\{Z_j\}$ of λ-injective Banach spaces and a convex symmetric neighborhood $V \subset X$ of 0, every continuous linear operator $T_0\colon Y \to (Z_j)_{c_0}$ such that $T_0(V\cap Y) \subset B_{(Z_j)_{c_0}}$ can be extended to a continuous linear operator $T\colon X \to (Z_j)_{c_0}$ such that $T(V) \subset 2\lambda B_{(Z_j)_{c_0}}$.*

(b) *Every continuous linear operator $T_0\colon Y \to c_0$ such that $T_0(V\cap Y) \subset B_{c_0}$ can be extended to a continuous linear operator $T\colon X \to c_0$ such that $T(V) \subset 2B_{c_0}$.*

(c) *Given $\lambda \geq 1$, a sequence $\{Z_j\}$ of λ-separably injective Banach spaces and a convex symmetric neighborhood $V \subset X$ of 0, every continuous linear operator $T_0\colon Y \to (Z_j)_{c_0}$ such that $T_0(V\cap Y) \subset B_{(Z_j)_{c_0}}$ can be extended to a continuous linear operator $T\colon X \to (Z_j)_{c_0}$ such that $T(V) \subset 6\lambda B_{(Z_j)_{c_0}}$.*

Notice that Banach-space versions of (a), (b), (c) in Corollary 5.7 were already known. In particular: case (a) with $\lambda = 1$ (and X a Banach space) is an extension theorem by Rosenthal [10, Theorem 1.3 and Remark 1.13(ii)]; case (b) follows from (a) and generalizes the well-known Sobczyk's theorem [8, Theorem 2.f.5] (see also [13, Theorem 3.5]); case (c) for Banach spaces X appears, with another constant, in [10, Remark 1.13] (for references to other proofs of this result see the text after the proof of Proposition 3.7 in [1]).

Remark 5.8 (concerning Grothendieck spaces). It is easy to see that a net $\{t_{n,\gamma}\}_{(n,\gamma)\in\mathbb{N}\times\Gamma}$ (directed as in Theorem 2.2(*iv*)) in a topological space converges to a point p if and only if $t_{n,\gamma_n} \to p$ for each sequence $\{\gamma_n\} \subset \Gamma$. It follows that, *for every Grothendieck space X* (that is, a Banach space for which the w^*- and the weak convergence of sequences coincide in the dual), *the couple (X^{**}, X) has the 1-CE-property*; indeed, given a w^*-null net $\{y^*_{n,\gamma}\} \subset X^*$ then $x^*_{n,\gamma} = j(y^*_{n,\gamma})$ works, where $j\colon X^* \to X^{***}$ is the canonical immersion. The class of Grothendieck spaces includes all $L_\infty(\mu)$-spaces, and all $C(K)$-spaces with K compact and Stonean. Our Proposition 5.4 implies the following.

Let X be a Grothendieck space, $\lambda \geq 1$, and $\{Z_j\}$ a sequence of λ-injective Banach spaces. Then every bounded linear operator $T_0\colon X \to (Z_j)_{c_0}$ admits a bounded linear extension $T\colon X^{**} \to (Z_j)_{c_0}$ such that $\|T\| \leq \lambda\|T_0\|$.
In particular, every bounded linear operator $T_0\colon X \to c_0(\ell_\infty(\Gamma))$ admits a bounded

linear extension $T\colon X^{**} \to c_0(\ell_\infty(\Gamma))$ *of the same norm.*

6. Three open problems

Let us conclude our paper with three open problems concerning the SCE-property. Problem 6.1 remains the main open question for us. In view of Corollary 3.3, this question could be answered by solving one of the other two problems. In what follows, Y is a closed subspace of a Banach space X.

Problem 6.1. *Do there exist couples* (X, Y) *having the* CE-*property but not the* SCE-*property?*

Problem 6.2. *If* (X, Y) *has the* SCE-*property, is then* Y *necessarily separable?*

Problem 6.3. *If* $Y = \ell_\infty$ *or* $Y = \ell_2(\Gamma)$ *with* Γ *uncountable, does* $(Y \oplus \mathbb{R}, Y)$ *have the* SCE-*property?*

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