Journal of Convex Analysis
Volume 22 (2015), No. 1, 145–159

Convex Hypersurfaces with Hyperplanar Intersections of Their Homothetic Copies

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Received: October 15, 2013
Accepted: January 27, 2014

Extending a well-known characteristic property of ellipsoids, we describe all convex solids $K \subset \mathbb{R}^n$, possibly unbounded, with the following property: for any vector $z \in \mathbb{R}^n$ and any scalar $\lambda \neq 0$ such that $K \neq z + \lambda K$, the intersection of the boundaries of K and $z + \lambda K$ lies in a hyperplane. This property is related to hyperplanarity of shadow-boundaries of K and central symmetricity of small 2-dimensional sections of K.

Keywords: Besicovitch, body, convex, ellipse, ellipsoid, convex, quadric, section, shadow-boundary, solid.

1991 Mathematics Subject Classification: 52A20

1. Introduction and Main Results

The following fact is well known in analytical geometry: if E is a solid ellipsoid in $\mathbb{R}^n$ and $E' = z + \lambda E$, $\lambda \neq 0$, is its homothetic copy such that $E \neq E'$, then the intersection of their boundaries, $\operatorname{bd} E \cap \operatorname{bd} E'$, lies in a hyperplane (see Proposition 2.3 below for a more general statement). From the results of Gruber [4] (and later of Šaĭdenko [10]) it follows that the above property characterizes solid ellipsoids among all convex bodies in $\mathbb{R}^n$. More precisely, a convex body $K \subset \mathbb{R}^n$ is a solid ellipsoid if and only if, for each nontrivial translate K' of K, the intersection of the boundaries of K and K' lies in a hyperplane.

Our goal here is to extend this characteristic property of ellipsoids to the case of convex quadric hypersurfaces in $\mathbb{R}^n$ (see Theorems 1.1 and 1.2 below). To avoid trivial cases, we assume that $n \geqslant 2$.

In what follows, by a *convex solid* we mean a closed convex set $K \subset \mathbb{R}^n$ which is distinct from the entire space and has nonempty interior (*convex bodies* are compact convex solids). As usual, $\operatorname{bd} K$ and $\operatorname{int} K$ denote, respectively, the boundary and interior of K. Similarly, $\operatorname{rbd} M$ and $\operatorname{rint} M$ stand for the relative boundary and relative interior of a convex set $M \subset \mathbb{R}^n$ of some intermediate

ISSN 0944-6532 / $ 2.50

dimension. By a *plane* of dimension r we mean a translate of an r-dimensional subspace of $\mathbb{R}^n$.

We recall that a *convex hypersurface* (*surface* if $n = 3$, or *curve* if $n = 2$) is the boundary of a convex solid in $\mathbb{R}^n$. This definition includes hyperplanes and pairs of parallel hyperplanes. In a standard way, a *quadric* (or a *second degree surface*) in $\mathbb{R}^n$ is the locus of points $x = (\xi_1, \dots, \xi_n)$ whose coordinates satisfy a quadratic equation

$$\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k + 2\sum_{i=1}^{n} b_i\xi_i + c = 0, \tag{1}$$

where not all scalars a_{ik} are zero. We say that a convex hypersurface $S \subset \mathbb{R}^n$ is a *convex quadric* provided there is a quadric $Q \subset \mathbb{R}^n$ and a component U of $\mathbb{R}^n \setminus Q$ such that U is convex and $S = \operatorname{bd} U$.

As shown in [14, 15], a convex hypersurface $S \subset \mathbb{R}^n$ is a convex quadric if and only if there are Cartesian coordinates $\xi_1, \dots, \xi_n$ for $\mathbb{R}^n$ such that S is the locus of points $x = (\xi_1, \dots, \xi_n)$ satisfying one of the conditions

$$a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, \qquad 1 \leqslant k \leqslant n, \tag{2}$$
$$a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 1,\ \xi_1 \geqslant 0, \qquad 2 \leqslant k \leqslant n, \tag{3}$$
$$a_1\xi_1^2 = 0, \tag{4}$$
$$a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 0,\ \xi_1 \geqslant 0, \qquad 2 \leqslant k \leqslant n, \tag{5}$$
$$a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, \qquad 2 \leqslant k \leqslant n, \tag{6}$$

where all scalars a_i involved are positive. Various characteristic properties of convex quadrics are studied in [12, 13, 18].

The *recession cone* of a convex solid $K \subset \mathbb{R}^n$ is defined by

$$\operatorname{rec} K = \{e \in \mathbb{R}^n : x + \lambda e \in K \text{ whenever } x \in K \text{ and } \lambda \geqslant 0\}.$$

It is known that $\operatorname{rec} K$ is a closed convex cone with apex o (the origin of $\mathbb{R}^n$), and $\operatorname{rec} K \neq \{o\}$ if and only if K is unbounded (see, e.g., [20] for general properties of convex sets). A 1-dimensional subspace $l \subset \mathbb{R}^n$ (as well as any nonzero vector $z \in l$) is called *non-recessional* for K if $l \cap \operatorname{rec} K = \{o\}$. A convex solid $K \subset \mathbb{R}^n$ has 1-dimensional non-recessional subspaces if and only if K is not a halfspace of $\mathbb{R}^n$ (see [16]).

The subspace $\operatorname{lin} K = \operatorname{rec} K \cap (-\operatorname{rec} K)$ is called the *linearity space* of K. Clearly, $\operatorname{lin} K = \{e \in \mathbb{R}^n : x + \lambda e \in K \text{ whenever } x \in K \text{ and } \lambda \in \mathbb{R}\}$. Equivalently, $\operatorname{lin} K = \{e \in \mathbb{R}^n : K = e + K\}$. Furthermore, (*i*) K is line-free (that is, contains no line) if and only if $\operatorname{lin} K = \{o\}$, and (*ii*) $\operatorname{lin} K$ is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ if and only if K is either a halfspace of a slab between a pair of distinct parallel hyperplanes. Any vector $z \in \mathbb{R}^n \setminus \operatorname{lin} K$ will be called a *non-linearity vector* of K.

Given a subspace $L \subset \mathbb{R}^n$ complementary to $\operatorname{lin} K$ (put $L = \mathbb{R}^n$ if $\operatorname{lin} K = \{o\}$), one has $K = \operatorname{lin} K \oplus (K \cap L)$, where $K \cap L$ is a line-free closed convex set such that $\dim(K \cap L) = \dim L$. (The sum of nonempty sets X and Y is called *direct* and denoted $X \oplus Y$ provided X and Y lie in complementary subspaces of $\mathbb{R}^n$.)

Our main results are presented in the following two theorems.

Theorem 1.1. *Given a convex solid $K \subset \mathbb{R}^n$, one has 1) $\Leftrightarrow$ 3) and 2) $\Leftrightarrow$ 4), where:*

1) *for any vector $z \in \mathbb{R}^n$ such that $K \neq z + K$, the set $\operatorname{bd} K \cap \operatorname{bd}(z + K)$ lies in a hyperplane,*
2) *for any vector $z \in \mathbb{R}^n$ and any scalar $\lambda \in (0, 1)$ such that $K \neq z + \lambda K$, the set $\operatorname{bd} K \cap \operatorname{bd}(z + \lambda K)$ lies in a hyperplane,*
3) *K has one of the shapes:*
   (a) *$\operatorname{bd} K$ is a convex quadric,*
   (b_1) *$\dim(\operatorname{lin} K) = n - 2$, and K is the direct sum of $\operatorname{lin} K$ and a 2-dimensional line-free closed convex set M which is either (i) bounded and strictly convex, or (ii) unbounded and contains no pair of parallel halflines in its relative boundary,*
4) *K has one of the shapes: (a) and*
   (b_2) *$\dim(\operatorname{lin} K) = n - 2$, and K is the direct sum of $\operatorname{lin} K$ and a 2-dimensional closed strictly convex set M.*

Theorem 1.2. *Given a convex solid $K \subset \mathbb{R}^n$, one has 1) $\Leftrightarrow$ 3) and 2) $\Leftrightarrow$ 4), where:*

1) *for any vector $z \in \mathbb{R}^n$ such that $K \neq z - K$, the set $\operatorname{bd} K \cap \operatorname{bd}(z - K)$ lies in a hyperplane,*
2) *for any vector $z \in \mathbb{R}^n$ and any scalar $\lambda \in (0, 1)$, the set $\operatorname{bd} K \cap \operatorname{bd}(z - \lambda K)$ lies in a hyperplane,*
3) *K has one of the shapes: (a) and*
   (b_3) *$\dim(\operatorname{lin} K) = n - 2$, and K is the direct sum of $\operatorname{lin} K$ and a 2-dimensional line-free closed convex set M which is either (i) bounded, strictly convex, and centrally symmetric, or (ii) unbounded and contains no pair of parallel halflines in its relative boundary.*
4) *K has one of the shapes: (a) and*
   (b_4) *$\dim(\operatorname{lin} K) = n - 2$, and K is the direct sum of $\operatorname{lin} K$ and a 2-dimensional line-free closed convex set M which is either (i) bounded, strictly convex, and centrally symmetric, or (ii) unbounded.*

Remark 1.3. For a convex solid $K \subset \mathbb{R}^n$, the equality

$$\operatorname{bd} K \cap \operatorname{bd}(z + \lambda K) = z + \lambda\big(\operatorname{bd} K \cap \operatorname{bd}(u + \lambda^{-1} K)\big),$$

where $u = -\lambda^{-1} z$, shows that $\operatorname{bd} K \cap \operatorname{bd}(z + \lambda K)$ lies in a hyperplane if and only if $\operatorname{bd} K \cap \operatorname{bd}(u + \lambda^{-1} K)$ does. Hence one can replace $0 < \lambda < 1$ with $\lambda > 1$ in both theorems above. Furthermore, $K \neq z - \lambda K$ for any $z \in \mathbb{R}^n$ and $0 < \lambda < 1$, as follows from Proposition 2.1.

For the case of convex cones, Condition *1*) of Theorem 1.2 was recently studied by Jerónimo-Castro and McAllister [6], who showed that a line-free closed convex cone $C \subset \mathbb{R}^n$ with apex o is ellipsoidal if and only if, for any point $z \in \operatorname{rint} C$, the set $\operatorname{rbd} C \cap \operatorname{rbd}(z - C)$ lies in a plane of dimension $\dim C - 1$. Goodey [5] proved that convex bodies K_1 and K_2 in $\mathbb{R}^n$ are homothetic solid ellipsoids if and only if for any translate K_2' of K_2 which is distinct from K_1, the set $\operatorname{bd} K_1 \cap \operatorname{bd} K_2'$ lies in a hyperplane. In this regard, we formulate the following problem.

Problem 1.4. *Describe the pairs of convex solids $K_1, K_2 \subset \mathbb{R}^n$ such that, for any translate K_2' of K_2 which is distinct from K_1, the set $\operatorname{bd} K_1 \cap \operatorname{bd} K_2'$ lies in a hyperplane.*

The proof of Theorem 1.1 uses the statement of Kubota [7]: A convex body $K \subset \mathbb{R}^2$ is a solid ellipse if and only if it satisfies the following condition (γ).

(γ) For any ordered line $l \subset \mathbb{R}^2$ there is a scalar $\lambda \in (0, 1)$ such that the locus of points $(1-\lambda)u + \lambda v$, where $[u, v]$ are ordered chords of K in the direction of l, belongs to a line.

We pose a problem, which involves a weaker version of condition (γ). Recall that a parallelogram is inscribed in a convex body $K \subset \mathbb{R}^2$ if all four vertices of the parallelogram belong to $\operatorname{bd} K$.

Problem 1.5. *Is it true that a convex body $K \subset \mathbb{R}^2$ is a solid ellipse if and only if it satisfies the following condition* (Δ)*?*

(Δ) *For any line $l_1 \subset \mathbb{R}^2$ there is another line $l_2 \subset \mathbb{R}^2$ such that each parallelogram inscribed in K, with a pair of sides parallel to l_1, has another pair of sides parallel to l_2.*

As shown in the proof of Theorem 1.1, a convex body $K \subset \mathbb{R}^2$ is a solid ellipse provided certain projective images of K satisfy condition (Δ). We observe that Problem 1.5 deals with an affine version of a result of Besicovitch [1] (see also Danzer [3] and Watson [19] for simplified proofs), which states that a convex body $K \subset \mathbb{R}^2$ is a circular disk if and only if each rectangle with three vertices on $\operatorname{bd} K$ is inscribed in K.

2. Auxiliary Results

In what follows, the set $\langle u, v \rangle = \{(1-\lambda)u + \lambda v : \lambda \in \mathbb{R}\}$ stands for the line through distinct points $u, v \in \mathbb{R}^n$.

Proposition 2.1. *Given a nonempty closed convex set $P \subset \mathbb{R}^n$, the following statements hold.*

1) *For a vector $z \in \mathbb{R}^n$, the equality $P = z + P$ holds if and only if $z \in \operatorname{lin} P$.*

2) *For a vector $z \in \mathbb{R}^n$ and a scalar $\lambda \in (0, 1)$, the equality $P = z + \lambda P$ holds if and only if P is a cone with apex $v = z/(1-\lambda)$.*

3) *For a vector $z \in \mathbb{R}^n$, the equality $P = z - P$ holds if and only if P is symmetric about $z/2$.*

4) *For a vector $z \in \mathbb{R}^n$ and a scalar $\lambda \in (0, 1)$, the equality $P = z - \lambda P$ holds if and only if P is a plane through $z/(1 + \lambda)$.*

Proof. *1)* Since the case $z = o$ is trivial, we may suppose that $z \neq o$. Let $P = z + P$. By induction on k, one has $P = kz + P$ for all integers $k \geqslant 1$. Similarly, from $P = P - z$ it follows that $P = P - kz$ for all integers $k \geqslant 1$. So, given a point $u \in P$, all points $u \pm kz$ belong to P. The convexity of P implies that the whole line $\langle u, u + z\rangle$ lies in P. Hence $z \in \operatorname{lin} P$.

Conversely, if $z \in \operatorname{lin} P$ then the equality $P = z + P$ follows from the definition of $\operatorname{lin} P$.

2) If $P = z + \lambda P$, then $P = v + \lambda(P - v)$, where $v = z/(1 - \lambda)$. Hence $P - v = \lambda(P - v)$, and, by induction on k, it follows that $P - v = \lambda^{\pm k}(P - v)$ for all integers $k \geqslant 1$. Since $\lambda^k \to 0$ and $\lambda^{-k} \to \infty$ as $k \to \infty$, the convexity of $P - v$ implies that the halfline $[o, x\rangle$ lies in $P - v$ for each nonzero vector $x \in P - v$. Hence $P - v$ is a cone with apex o, and P is a cone with apex v.

Conversely, if P is a cone with apex $v = z/(1 - \lambda)$, then $P - v = \lambda(P - v)$, which gives $P = v + \lambda(P - v) = z + \lambda P$.

3) $P = z - P$ if and only if $P - z/2 = z/2 - P$. The latter equality holds if and only if $P - z/2$ is symmetric about o; equivalently, P is symmetric about $z/2$.

4) If $P = z - \lambda P$, then $P = v - \lambda(P - v)$, where $v = z/(1 + \lambda)$. Hence $P - v = \lambda^2(P - v)$, and, by the proved above, $P - v$ is a convex cone with apex o. Then $P - v = -\lambda(P - v) = -(P - v)$, implying that $P - v$ is symmetric about o, which shows that $P - v$ is subspace. Hence P is a plane through v.

Conversely, if P is a plane through $z/(1 + \lambda)$, then $P - z/(1 + \lambda)$ is a subspace. Therefore, $P - z/(1 + \lambda) = -\lambda(P - z/(1 + \lambda))$, or $P = z - \lambda P$. □

Proposition 2.2. *Let $K \subset \mathbb{R}^n$ be a convex solid with $\operatorname{lin} K \neq \{o\}$. If L is a subspace complementary to $\operatorname{lin} K$ and $P = K \cap L$, then, for any vector $z \in \mathbb{R}^n$ and any scalar $\lambda \neq 0$, one has*

$$\operatorname{bd} K \cap \operatorname{bd}(z + \lambda K) = \operatorname{lin} K \oplus \big(\operatorname{rbd} P \cap \operatorname{rbd}(u + \lambda P)\big),$$

where u is the projection of z on L along $\operatorname{lin} K$. Subsequently, the set $\operatorname{bd} K \cap \operatorname{bd}(z + \lambda K)$ lies in a hyperplane $H \subset \mathbb{R}^n$ if and only if $\operatorname{rbd} P \cap \operatorname{rbd}(u + \lambda P)$ lies in the plane $H \cap L$ of dimension $\dim L - 1$.

Proof. Let $z = v + u$, where $v \in \operatorname{lin} K$ and $z \in L$. Clearly, $\mu K = v + \mu K$ for any $v \in \operatorname{lin} K$ and $\mu \neq 0$. So, the first statement of the proposition immediately follows from the fact that $\operatorname{bd} K = \operatorname{lin} K \oplus \operatorname{rbd} P$. Hence any hyperplane H containing $\operatorname{bd} K \cap \operatorname{bd}(z + \lambda K)$ also contains a translate of $\operatorname{lin} K$. Therefore, $H = \operatorname{lin} K \oplus (H \cap L)$. Subsequently, $\operatorname{rbd} P \cap \operatorname{rbd}(u + \lambda P) \subset H \cap L$ and $\dim(H \cap L) = \dim L - 1$. □

Proposition 2.3. *Let $K \subset \mathbb{R}^n$ be a convex solid whose boundary is a convex quadric. Then for any vector $z \in \mathbb{R}^n$ and any scalar $\lambda \neq 0$ such that $K \neq z+\lambda K$, the set* $\mathrm{bd}\, K \cap \mathrm{bd}\,(z+\lambda K)$ *lies in a hyperplane*

Proof. Let $Q \subset \mathbb{R}^n$ be a quadric surface containing $\mathrm{bd}\, K$. Suppose that Q is given by (1). For any vector $z = (\eta_1, \dots, \eta_n)$ and any scalar $\lambda \neq 0$, the set $z + \lambda Q$ is the locus of points $x = (\xi_1, \dots, \xi_n)$ satisfying the equation

$$\sum_{i,k=1}^{n} a_{ik}(\xi_i - \eta_i)(\xi_k - \eta_k) + 2\sum_{i=1}^{n} b_i \lambda(\xi_i - \eta_i) + \lambda^2 c = 0. \tag{7}$$

The set $Q \cap (z + \lambda Q)$ is the solution set (possibly, empty) of the system of equations (1) and (7). Subtracting (7) from (1), we obtain a linear equation of the form

$$\sum_{i=1}^{n} \left(\sum_{k=1}^{n} (a_{ik} + a_{ki})\eta_k + 2b_i(1-\lambda) \right) \xi_i + \delta = 0. \tag{8}$$

Denote by H the solution set of (8). Clearly,

$$T = \mathrm{bd}\, K \cap \mathrm{bd}\,(z + \lambda K) \subset Q \cap (z + \lambda Q) \subset H.$$

If at least one of the coefficients

$$c_i = \sum_{k=1}^{n} (a_{ik} + a_{ki})\eta_k + 2b_i(1-\lambda), \quad 1 \leqslant i \leqslant k,$$

is not zero, then H is a hyperplane, as desired.

First, we consider the case $z \notin \mathrm{lin}\, K$. Choosing suitable Cartesian coordinates, we may suppose that $\mathrm{bd}\, K$ is described by one of the conditions (2)–(6). Clearly,

$$\mathrm{lin}\, K = \{(\xi_1, \dots, \xi_n) : \xi_i = 0,\ 1 \leqslant i \leqslant k\}, \tag{9}$$

with $k = 1$ if $\mathrm{bd}\, K$ is given by (4). From (9) it follows that $z \notin \mathrm{lin}\, K$ if and only if at least one of the coordinates η_i, $1 \leqslant i \leqslant k$, is distinct from zero. In the latter case, $c_i = 2a_i\eta_i \neq 0$, and H is a hyperplane.

Hence it remains to consider the case when $z \in \mathrm{lin}\, K$. Then $z + \lambda K = \lambda K$ for any $\lambda \neq 0$, and the assumption $K \neq z + \lambda K$ becomes $K \neq \lambda K$. As a consequence, $\lambda \neq 1$. Due to the argument of Remark 1.3, one can put $|\lambda| \leqslant 1$. We divide our further consideration into various cases.

Case 1: $0 < \lambda < 1$. According to Proposition 2.1, $K \neq \lambda K$ if and only if K is not a cone with apex o; so, (4) or (5) are excluded. If $\mathrm{bd}\, K$ is given by (2) or by (3), then T is empty, and if $\mathrm{bd}\, K$ is given by (5) or by (6), then $T = \mathrm{lin}\, K$, and T lies in the hyperplane $\xi_1 = 0$.

Case 2: $\lambda = -1$. According to Proposition 2.1, $K \neq -K$ if and only if K is not symmetric about $z/2$. Hence neither (2) nor (4) is possible. If $\mathrm{bd}\, K$ is given by

(3), then $T = \varnothing$. If $\mathrm{bd}\, K$ is given by (5) or by (6), then $T = \mathrm{lin}\, K$, and T lies in the hyperplane $\xi_1 = 0$.

Case 3: $0 < \lambda < -1$. Then T is empty if $\mathrm{bd}\, K$ is given by (2) or by (3). If $\mathrm{bd}\, K$ is given by (4), then T is a hyperplane, described by $\xi_1 = 0$, and if $\mathrm{bd}\, K$ is given by (5) or by (6), then $T = \mathrm{lin}\, K$ and lies in the hyperplane $\xi_1 = 0$. □

If $K \subset \mathbb{R}^n$ is a convex solid and $l \subset \mathbb{R}^n$ a 1-dimensional subspace, then the *parallel shadow-boundary* of K with respect to l, denoted $S_l(K)$, is the subset (possibly, empty) of $\mathrm{bd}\, K$ at which the lines parallel to l support K. Equivalently,

$$S_l(K) = \mathrm{bd}\, K \cap \mathrm{bd}\,(K + l), \tag{10}$$

where $K + l$ is the vector sum of K and l. Similarly, if p is a point in $\mathbb{R}^n \setminus K$, then the *central shadow-boundary* of K with respect to p, denoted $S_p(K)$, is the subset (possibly, empty) of $\mathrm{bd}\, K$ at which the halflines with endpoint p support K. Equivalently,

$$S_p(K) = \mathrm{bd}\, K \cap \mathrm{bd}\, C_p(K), \tag{11}$$

where $C_p(K)$ is the convex cone with apex p generated by K (see the survey [9] for related concepts of illumination by parallel rays and point sources, also [8] for the lower bounds on the numbers of parallel and central shadow-boundaries of convex solids in $\mathbb{R}^n$). As usual, a convex set $C \subset \mathbb{R}^n$ is called a *cone with apex* p if each closed halfline

$$[p, x\rangle = \{p + \lambda(x - p) : \lambda \geqslant 0\}, \quad x \in C \setminus \{p\},$$

lies in C. A convex solid $K \subset \mathbb{R}^n$ distinct from a cone is called *strictly convex* provided its boundary contains no line segments; an n-dimensional closed convex cone $C \subset \mathbb{R}^n$ is called *strictly convex* if it has a unique apex, say p, such that each line segment in $\mathrm{bd}\, C$ belongs to a line through p.

Proposition 2.4 ([16]). *Given a convex solid $K \subset \mathbb{R}^n$ distinct from a halfspace, the following conditions are equivalent:*

1) *for any 1-dimensional non-recessional subspace $l \subset \mathbb{R}^n$ of K, the parallel shadow-boundary $S_l(K)$ lies in a hyperplane,*

2) *K has one of the shapes: (a),*
- (b_5) $\dim(\mathrm{lin}\, K) = n - 2$, *and K is the direct sum of* $\mathrm{lin}\, K$ *and a 2-dimensional line-free closed convex set M which is either (i) bounded and strictly convex, or (ii) unbounded,*
- (c) $\dim(\mathrm{lin}\, K) = n - 3$ *and K is the direct sum of* $\mathrm{lin}\, K$ *and a 3-dimensional closed strictly convex cone C (this is possible if $n \geqslant 3$).* □

Proposition 2.5 ([17]). *Given a convex solid $K \subset \mathbb{R}^n$, the following conditions are equivalent:*

1) *for any point $p \in \mathbb{R}^n \setminus K$, the central shadow-boundary $S_p(K)$ lies in a hyperplane,* *2)* *K has one of the shapes (a) and (b_2) from Theorem* 1.1*, or the shape (c) from Proposition* 2.4*.* □

A section of a convex solid $K \subset \mathbb{R}^n$ by a plane $L \subset \mathbb{R}^n$ is called *proper* provided L meets both $\operatorname{bd} K$ and $\operatorname{int} K$.

Proposition 2.6. *Let $K \subset \mathbb{R}^n$, $n \geqslant 3$, be a line-free convex solid and $\delta > 0$ such that every proper 2-dimensional section of K of diameter less than δ is centrally symmetric. Then $\operatorname{bd} K$ is a convex quadric.*

Proof. The case $n = 3$ immediately derives from the proof of Theorem 2 of Burton's paper [2], which is formulated for the case of convex bodies. Indeed, Burton's proof for $n = 3$ uses the existence of an extreme point x of K (which is true if K is line-free) to show that $\operatorname{bd} K$ is either an elliptic cone with apex x or a strictly convex quadric in $\mathbb{R}^3$ (that is, a paraboloid, an ellipsoid, or a sheet of a hyperboloid of two sheets), subsequently concluding that K is a solid ellipsoid due to the compactness assumption on K.

Suppose that $n \geqslant 4$, and choose a point $p \in \operatorname{int} K$. Let L be a 3-dimensional plane through p, and put $M = K \cap L$. Choose a proper section Γ of M by a 2-dimensional plane $N \subset L$ such that $\operatorname{diam} \Gamma < \delta$. From the hypothesis and the equality

$$\Gamma = M \cap N = (K \cap L) \cap N = K \cap N$$

it follows that Γ is centrally symmetric. By the argument above, the relative boundary of the 3-dimensional closed convex set M is a convex quadric. Since $\operatorname{rbd} M = L \cap \operatorname{bd} K$, all sections of $\operatorname{bd} K$ by 3-dimensional planes through p are 3-dimensional convex quadrics. Therefore, all sections of $\operatorname{bd} K$ by 2-dimensional planes through p are convex quadric curves, and [12] implies that $\operatorname{bd} K$ itself is a convex quadric. □

3. Proof of Theorem 1.1

First, we exclude the trivial case when K is a halfspace or a slab between a pair of parallel hyperplanes. Indeed, in this case K satisfies both conditions *1)* and *3)* of the theorem, and $\operatorname{bd} K$ is a convex quadric, given by (4) or by (2), with $k = 1$, in suitable Cartesian coordinates for $\mathbb{R}^n$. Under this assumption, $\dim(\operatorname{lin} K) \leqslant n-2$, and Proposition 2.2 shows that one can reduce the proof of Theorem 1.1 to the case when K is a line-free convex solid in $\mathbb{R}^n$, $n \geqslant 2$.

1) $\Rightarrow$ *3)*. We state that K satisfies Condition *1)* of Proposition 2.4.

Suppose first that $n = 2$. Then $\operatorname{bd} K \cap \operatorname{bd}(z + K)$ should belong to a line for any choice of a non-recessional vector z of K. It is easy to see that the latter is possible only if K is either (i) bounded and strictly convex, or (ii) unbounded. In either case, $S_l(K)$ belongs to a line.

Assume now that $n \geqslant 3$. Let $l \subset \mathbb{R}^n$ be any 1-dimensional non-recessional subspace of K (such an l exists since K is not a halfspace, see [16]). Choose any nonzero vector $z \in l$. Since K is assumed to be line-free, one has $\operatorname{lin} K = \{o\}$. So, $K \neq z + K$ according to Proposition 2.1. We observe that $S_l(K)$ contains n affinely independent points. Indeed, let L be a hyperplane orthogonal to l. Denote by π the orthogonal projection of $\mathbb{R}^n$ onto L. Because l is non-recessional for K, the set $\pi(K)$ is a line-free closed convex set of dimension $n-1$. If Δ is any $(n-1)$-dimensional simplex inscribed in $\pi(K)$, then its vertices, $u_1, \ldots, u_n$, are affinely independent. Because l is non-recessional for K, each set $(u_i + l) \cap K$, $1 \leqslant i \leqslant n$, is a line segment; denote this segment by $[a_i, c_i]$ (possibly, $a_i = c_i$). Since π is an affine transf! ormation, for any choice of points $u_i' \in [a_i, c_i]$, the set $\{u_1', \ldots, u_n'\}$ should be affinely independent. Denote by v_i the middle point of $[a_i, c_i]$. By the above, the set $\{v_1, \ldots, v_n\}$ lies in $S_l(K)$ and is affinely independent.

We state that all line segments $[a_i, c_i]$, $1 \leqslant i \leqslant n$, are singletons. Indeed, assume for a moment that $[a_1, c_1]$ is not a singleton. Choose a scalar $\varepsilon_0 > 0$ so small that $\operatorname{int} K$ and $\operatorname{int}(\varepsilon_0 z + K)$ meet. By continuity, there is a point

$$w \in \operatorname{int} K \cap \operatorname{int}(\varepsilon z + K) \quad \text{for all } 0 < \varepsilon < \varepsilon_0.$$

Furthermore, there is a scalar $\varepsilon_1 \in (0, \varepsilon_0)$ with the following property: if a line segments $[a_i, c_i]$ is not a singleton, then both points v_i and $\varepsilon z + v_i$ belong to $[a_i, c_i] \cap [\varepsilon z + a_i, \varepsilon z + c_i]$ for all $0 < \varepsilon < \varepsilon_1$. Let P_i be the 2-dimensional plane through $(v_i + l) \cup \{w\}$. Given a positive $\delta > 0$, there is a scalar $\varepsilon_2 \in (0, \min\{\varepsilon_1, \delta/\|z\|\})$ such that for any $0 < \varepsilon < \varepsilon_2$, the plane P_i contains a point $x_i(\varepsilon)$ from the set

$$T_\varepsilon = \operatorname{bd} K \cap \operatorname{bd}(\varepsilon z + K), \quad 1 \leqslant i \leqslant n,$$

satisfying the inequality $\|v_i - x_i(\varepsilon)\| < \delta$. It is easy to see that $x_i(\varepsilon)$ is uniquely defined if $[a_i, c_i] = \{v_i\}$; if $[a_i, c_i]$ is not a singleton, then put $x_i(\varepsilon) = (\varepsilon/2)z + v_i$. Choosing δ sufficiently small, we obtain that the set $\{x_1(\varepsilon), \ldots, x_n(\varepsilon)\}$ is affinely independent for all $0 < \varepsilon < \varepsilon_2$ (as a δ-approximation of the affinely independent set $\{v_1, \ldots, v_n\}$). Hence $H_\varepsilon = \operatorname{aff}\{x_1(\varepsilon), \ldots, x_n(\varepsilon)\}$ is a hyperplane for all $0 < \varepsilon < \varepsilon_2$. By Condition *1*) of the theorem, $T_\varepsilon \subset H_\varepsilon$.

Clearly, H_ε tends to the hyperplane $H = \operatorname{aff}\{v_1, \ldots, v_n\}$ as $\varepsilon \to 0$. Since H is not parallel to l (otherwise $\pi(H)$ would be a plane of dimension $(n-2)$ containing $\{u_1, \ldots, u_n\}$), the hyperplanes H_ε are not parallel to l for all $0 < \varepsilon < \varepsilon_2$. On the other hand, the nontrivial line segment $[a_1, c_1] \cap [\varepsilon z + a_1, \varepsilon z + c_1]$ lies in T_ε if $0 < \varepsilon < \varepsilon_1$; whence it should also lie in H_ε, which is impossible because H_ε is not parallel to l if $0 < \varepsilon < \varepsilon_2$. Therefore, all line segments $[a_i, c_i]$, $1 \leqslant i \leqslant n$, must be singletons.

Finally, we state that $S_l(K) \subset H$. Indeed, let v be a point in $S_l(K)$. Repeating the argument above, we conclude that $(v + l) \cap K = \{v\}$. Hence, for any $\delta > 0$, then plane through $(v + l) \cup \{w\}$ contains a unique point $x(\varepsilon) \in T_\varepsilon$ such that $\|v - x(\varepsilon)\| < \delta$ for all $0 < \varepsilon < \varepsilon_2'$. Since $x(\varepsilon) \in H_\varepsilon$ and $x(\varepsilon) \to v$ as $\varepsilon \to 0$, it follows that $v \in H$. Summing up, $S_l(K) \subset H$, and K satisfies Condition *1*) of Proposition 2.4.

According to Proposition 2.4, K has one of the shapes (a), (b_5), and (c). Because K is assumed to be line-free, either $K = M$ if K has the shape (b_5), or $K = C$ if K has the shape (c). To finalize the proof of *1*) $\Rightarrow$ *3*), it remains to show the following:

1. If M is a 2-dimensional convex set from condition (b_5), then M satisfies condition (b_1).
2. If C is a 3-dimensional closed strictly convex cone from condition (c), then C is a solid elliptic cone.

Suppose first that $K = M$, where M satisfies condition (b_5). Assume for a moment the existence of a pair of parallel halflines h and h' in $\mathrm{bd}\, M$. Given a nonzero vector $z \in L$ in the direction of h, the set $\mathrm{bd}\, M \cap \mathrm{bd}\,(z + M)$ contains the union of halflines $z + h$ and $z + h'$, which does not belong to a line, contrary to Condition *1*) of the theorem. Hence M satisfies condition (b_1).

Suppose next that $K = C$, where C is a cone satisfying condition (c). Translating C on a suitable vertex, we may assume that o is the apex of C. Since any nonzero vector $z \in \mathbb{R}^3$ is a non-linearity vector of C, one has $C \neq z + C$ (see Proposition 2.1). By Condition *1*) of the theorem, the set $\mathrm{bd}\, C \cap \mathrm{bd}\,(z + C)$ lies in a 2-dimensional plane for any choice of a nonzero vectors $z \in \mathbb{R}^3$.

Choose a 2-dimensional subspace $S \subset \mathbb{R}^3$ satisfying the condition $S \cap C = \{o\}$, and let $L \subset \mathbb{R}^2$ be a translate of S which meets $\mathrm{int}\, C$. Then $F = C \cap L$ is a 2-dimensional strictly convex body. Recall that a chord $[x, z]$ of F is an *affine diameter* of F provided there is a pair of parallel lines $l_x, l_z \subset L$ through x and z, respectively, both supporting F; if both l_x and l_z are parallel to a given line $l \subset L$, then $[x, z]$ will be called *generated* by l. Since F is strictly convex, any line $l \subset L$ generates precisely one affine diameter of F. It is known that for any line $l \subset \mathbb{R}^2$ there is an affine diameter of K parallel to l (see, e.g., [11] for general properties of affine diameters of convex bodies).

Lemma 3.1. *F satisfies condition* (Δ) *above. Furthermore, if lines l_1 and l_2 in L satisfy condition* (Δ)*, then l_1* (*respectively, l_2*) *is parallel to the affine diameter of F generated by l_2* (*respectively, generated by l_1*)*.*

Proof. Choose any line $l_1 \subset L$, and denote by m the 1-dimensional subspace of $\mathbb{R}^3$ which is a translate of l_1. The parallel shadow-boundary $S_m(C)$ is the union of two halflines, h_1 and h_2, with common endpoint o. Denote by H the 2-dimensional subspace containing $h_1 \cup h_2$. We state that the lines l_1 and $l_2 = H \cap L$ satisfy condition (Δ). Indeed, choose a chord $[u, v]$ of F parallel to l_1 which is not an affine diameter of F. Then F has another chord, $[u', v']$, which is a translate of $[u, v]$. The nonzero vector $z = v - u$ belongs to m. Clearly, the set $T = \mathrm{bd}\, C \cap \mathrm{bd}\,(z + C)$ lies between the planes H and $z + H$ and contains $\{v, v'\}$. By the argument above, T lies in a plane, N. Since T is an unbounded line-free convex curve which surrounds a translate of the planar cone $C \cap H$, the plane N must be parallel to H. Hence both points $v, v' \in T \cap L$ belong to the line $N \cap L$, which is p! arallel to l_2. Summing up, the sides $[u, u']$ and $[v, v']$ of the parallelogram with vertices u, v, u', v' are parallel to l_2. Thus F satisfies condition (Δ).

If $[u, v]$ tends toward the point $h_1 \cap L$, then $[v, v']$ tends toward the chord $F \cap H$, which is the affine diameter of F generated by l_1. Similarly, $[u, v]$ is parallel to the affine diameter of F generated by l_2. □

Lemma 3.2. *F satisfies condition (γ) above.*

Proof. Let l be an ordered line in L. Denote by $[u, v]$ an ordered affine diameter of F in the direction of l ($[u, v]$ is the longest chord of F in the direction of l). Without loss of generality, we may assume that l contains $[u, v]$. Denote by m the 1-dimensional subspace of $\mathbb{R}^3$ parallel to l, and consider the parallel shadow boundary $S_m(C)$, which consists of two halflines, h and h', with common endpoint o. These halflines meet L at points p and p', respectively, such that $[p, p']$ is the affine diameter of F generated by l. Since F is strictly convex, $[p, p']$ meets $[u, v]$ at a point $w \in (u, v)$. Hence $w = (1 - \lambda)u + \lambda v$ for a suitable scalar $\lambda \in (0, 1)$.

We state that any ordered chord $[\bar{u}, \bar{v}]$ of F in the direction of l meets $[p, p']$ at a point $\bar{w} = (1 - \lambda)\bar{u} + \lambda\bar{v}$. Without loss of generality, we may assume that the point $[\bar{u}, \bar{v}] \cap [p, p']$ lies between p and w. Let L_0 be a plane through $[u, v]$ which cuts C along a bounded set $F_0 = C \cap L_0$. The halflines h and h' meet L_0 at some points p_0 and p_0', respectively. Clearly, $[p_0, p_0']$ is the affine diameter of F_0 generated by l. Denote by u_0 and v_0, respectively, the points at which the halflines $[o, \bar{u}\rangle$ and $[o, \bar{v}\rangle$ meet L_0. Then $[u, v]$ and $[u_0, v_0]$ are parallel. Since $\|\bar{u} - \bar{v}\| < \|u - v\|$ due to strict convexity of F, one can continuously rotate L_0 about $[u, v]$ into a new position such that $[u_0, v_0]$ becomes of the same length as $[u, v]$ (this argument is not possible only if $\bar{u} \in [u, p]$ and $\bar{v} \in [v, p]$, contrary to strict convexity of F). By Lemma 3.1 (with L_0 instead of L), the sides $[u, u_0]$ and $[v, v_0]$ of the parallelogram with vertices u, u_0, v, v_0 should be parallel to the affine diameter $[p_0, p_0']$ of F_0. Hence the parallel chords $[u, v]$ and $[u_0, v_0]$ are divided by $[p_0, p_0']$ is the same ratio $\lambda : 1 - \lambda$. Therefore, the point $w_0 = (1 - \lambda)u_0 + \lambda v_0$ belongs to $[p_0, p_0']$. This argument immediately implies the inclusion $\bar{w} \in [p, p']$, as desired. □

Finally, Kubota's statement (see above) implies that F is a solid ellipse. Hence C is a solid elliptic cone, finalizing the proof of *1)* $\Rightarrow$ *3)*.

3) $\Rightarrow$ *1)*. The shape (a) of K is considered in Proposition 2.3, with $\lambda = 1$. Suppose that K has the shape (b_1). Then $n = 2$ and $K = M$, where M is a 2-dimensional convex set described in (b_1). For any nonzero vector $z \in \mathbb{R}^2$, one has $M \neq z + M$. From the shape of M it easily follows that the set $\mathrm{bd}\, M \cap \mathrm{bd}\,(z + M)$

may be empty, a singleton, a line segment, or a halfline only. In either case, there is a line containing $\mathrm{bd}\, M \cap \mathrm{bd}\,(z + M)$.

2) $\Rightarrow$ *4*). This part of the proof is similar to that of *1*) $\Rightarrow$ *3*). First, it claims that K satisfies Condition *1*) of Proposition 2.5. For this, choose any point $p \in \mathbb{R}^n \setminus K$ and a scalar $\lambda \in (0,1)$. Then $K \neq p + \lambda(K - p) = z + \lambda K$, where $z = (1-\lambda)p$ (see Proposition 2.1). By Condition *2*) of the theorem, the set $T_\lambda = \mathrm{bd}\, K \cap \mathrm{bd}\,(p + \lambda(K - p))$ lies in a hyperplane $H_\lambda \subset \mathbb{R}^n$ for all $\lambda \in (0,1)$.

Suppose first that $n = 2$. Then T_λ should belomg to a line for any choice of $p \in \mathbb{R}^2 \setminus K$ and $\lambda \in (0,1)$, which is possible only if K is either (i) bounded and strictly convex, or (ii) a convex cone. In either case, $S_p(K)$ belongs to a line.

Let $n \geqslant 3$. Since the case $S_p(K) = \varnothing$ trivially holds, we may assume that $S_p(K)$ is nonempty. Centrally projecting K from p on a suitable hyperplane L, choose n affinely independent points, $u_1, \dots, u_n$, in the relative boundary of this projection. The pre-image of u_i in K is a subset B_i of $\mathrm{bd}\, K$, which belongs to the halfline $[p, u_i\rangle$, $1 \leqslant i \leqslant n$. Similarly to the proof of *1*) $\Rightarrow$ *3*), one can show that all sets $B_1, \dots, B_n$ are singletons. Furthermore, if $B_i = \{v_i\}$, then the set $\{v_1, \dots, v_n\}$ is affinely independent. Clearly, $H = \mathrm{aff}\,\{v_1, \dots, v_n\}$. Similarly to the proof of *1*) $\Rightarrow$ *3*), given $\delta > 0$, one can find a scalar $\lambda \in (0,1)$ sufficiently close to 1, and points $x_i(\lambda) \in T_\lambda$ such that $\|v_i - x_i(\lambda)\| < \delta$ for all $i = 1, \dots, n$. Hence the set $\{x_1(\lambda), \dots, x_n(\lambda)\}$ is affin! ely independent for all $\lambda \in (0,1)$ sufficiently close to 1, implying that $H_\lambda = \mathrm{aff}\,\{x_1(\lambda), \dots, x_n(\lambda)\}$. By a compactness argument, H_λ tends to H as $\lambda \to 1$. Similarly to the proof of *1*) $\Rightarrow$ *3*), one has $S_p(K) \subset H$. Summing up, K satisfies Condition *1*) of Proposition 2.5.

According to Proposition 2.5, K has one of the shapes (a), (b_2), and (c). Hence it remains to consider the case when K has the shape (c). Since K is assumed to be line-free, it follows that $n = 3$ and $K = C$, where C is a convex cone satisfying condition (c). We state that C is a solid elliptic cone. Indeed, translating C on a suitable vector, we may assume that o is the apex of C. Then $\lambda C = C$ for all $\lambda > 0$. If $S \subset \mathbb{R}^3$ is a 2-dimensional subspace with the property $S \cap C = \{o\}$, then

$$\mathrm{bd}\, C \cap \mathrm{bd}\,(z + \lambda C) = \mathrm{bd}\, C \cap \mathrm{bd}\,(z + C)$$

for any $z \in S$ and $\lambda \in (0,1)$. Hence the set $\mathrm{bd}\, C \cap \mathrm{bd}\,(z + C)$ lies in a 2-dimensional plane for any choice of a nonzero $z \in S$, and, by the proved in *1*) $\Rightarrow$ *3*), C is a solid elliptic cone, as desired.

The proof of *4*) $\Rightarrow$ *2*) is similar to that of *3*) $\Rightarrow$ *1*).

4. Proof of Theorem 1.2

First, we exclude the trivial cases when K is a halfspace or a slab between a pair of parallel hyperplanes. Indeed, in this case K satisfies both conditions *1*) and *3*) of the theorem, and $\mathrm{bd}\, K$ is a convex quadric, given by (4) or by (2), with $k = 1$, in suitable Cartesian coordinates for $\mathbb{R}^n$. Under this assumption, $\dim(\operatorname{lin} K) \leqslant n-2$, and Proposition 2.2 shows that one can reduce the proof of Theorem 1.2 to the case when K is a line-free convex solid in $\mathbb{R}^n$, $n \geqslant 2$.

1) $\Rightarrow$ *3)*. First, consider the case $n = 2$. By the assumption, the set $\operatorname{bd} K \cap \operatorname{bd}(z-K)$ should belong to a line for any choice of $z \in \mathbb{R}^2$ such that $K \neq z-K$. If K is bounded, then from [5] it follows that $-K$ must be a translate of K, which implies that K is centrally symmetric. Furthermore, $\operatorname{bd} K$ cannot contain a pair of parallel line segments, since otherwise $\operatorname{bd} K \cap \operatorname{bd}(z-K)$ would contain a pair of parallel line segments for a suitable vector $z \in \mathbb{R}^2$ such that $K \neq z-K$. Summing up, K has the shape (b_3).

Suppose now that $n \geqslant 3$. Choose any $\delta > 0$. We state that every proper 2-dimensional section of K of diameter less than δ is centrally symmetric. Indeed, let N be a 2-dimensional plane properly meeting K such that $\operatorname{diam}(N \cap K) < \delta$. Put $F = N \cap K$. Inscribe in F a parallelogram P. Let $V = \{v_1, v_2, v_3, v_4\}$ be the vertex set of P, and u the center of P.

By the assumption, $\operatorname{bd} K \cap \operatorname{bd}(2u-K)$ lies in a hyperplane, H. Since N is the smallest plane containing V, and since $V \subset \operatorname{bd} K \cap \operatorname{bd}(2u-K) \subset H$, one has $u \in N \subset H$. From the equality

$$\begin{aligned}
&\operatorname{bd} K \cap \operatorname{bd}(2u-K) - u = \operatorname{bd}(K-u) \cap \operatorname{bd}(u-K) \\
&= -(\operatorname{bd}(K-u) \cap \operatorname{bd}(u-K)) = u - \operatorname{bd} K \cap \operatorname{bd}(2u-K)
\end{aligned}$$

it follows that $\operatorname{bd} K \cap \operatorname{bd}(2u-K)$ is symmetric about u. Since H contains u, the set $H \cap \operatorname{bd} K \cap \operatorname{bd}(2u-K)$ also is symmetric about u. We state that

$$H \cap \operatorname{bd} K = H \cap \operatorname{bd}(2u-K).$$

Indeed, let l be a line through u not lying in H, and x be any point in $H \cap \operatorname{bd} K$. Consider the halfplane Q bounded by l and containing x. Clearly, the convex curves $Q \cap \operatorname{bd} K$ and $Q \cap \operatorname{bd}(2u-K)$ meet at some point y. By the assumption, y belongs to H; so, y belongs to the halfline $Q \cap H$, containing x. Hence $x = y$, implying the inclusion $H \cap \operatorname{bd} K \subset H \cap \operatorname{bd}(2u-K)$. By a symmetry argument, $H \cap \operatorname{bd}(2u-K) \subset H \cap \operatorname{bd} K$.

Subsequently, each of the sets $H \cap \operatorname{bd} K$ and $H \cap \operatorname{bd}(2u-K)$ is symmetric about u. Hence both $(n-1)$-dimensional sets $H \cap K$ and $H \cap (2u-K)$ are symmetric about u. Therefore, $F = N \cap K = N \cap (H \cap K)$ is symmetric about u, as desired. Finally, Proposition 2.6 shows that $\operatorname{bd} K$ is a convex quadric.

3) $\Rightarrow$ *1)*. The case (a), when $\operatorname{bd} K$ is a convex quadric, is considered in Proposition 2.3, with $\lambda = -1$. Suppose that K has the shape (b_3). Then $n = 2$ and $K = M$, where M is a 2-dimensional convex set described in (b_3). Choose any nonzero vector $z \in \mathbb{R}^2$ such that $M \neq z - M$. From the description of M it easily follows that the set $\operatorname{bd} M \cap \operatorname{bd}(z-M)$ belongs to a line.

2) $\Rightarrow$ *4)*. First, consider the case $n = 2$. By the assumption, the set $T = \operatorname{bd} K \cap \operatorname{bd}(z-\lambda K)$ should belong to a line for any choice of $z \in L$ and $\lambda \in (0,1)$.

If K is bounded, then from Theorems 1 and 2 of [5] it follows that $-\lambda K$ slides freely inside K, which is possible when $\lambda \to 1$ if and only if $-K$ is a translates of K; whence K is centrally symmetric. Furthermore, if $\operatorname{bd} K$ contained a pair of line segments, then suitably choosing a vector $z \in \mathbb{R}^2$ and a $\lambda \in (0,1)$, the set T would consist of a point and a line segment, implying that $\dim T = 2$. Hence M satisfies the conditions of (b_4).

Suppose that $n \geqslant 3$. Choose any $z \in \mathbb{R}^n$ such that $K \neq z - K$. We state that the set $T = \operatorname{bd} K \cap \operatorname{bd}(z-K)$ lies in a hyperplane. Since this statement is trivial if $T = \varnothing$, we may assume that T is nonempty. Furthermore, if $\operatorname{int} K \cap \operatorname{int}(z-K) = \varnothing$, then separation properties of convex sets imply the existence of a hyperplane containing T and separating K and $z-K$. It remains to consider the case when $T \neq \varnothing$ and $\operatorname{int} K \cap \operatorname{int}(z-K) \neq \varnothing$. Then there is a scalar $\lambda_0 \in (0,1)$ such that $\operatorname{bd} K \cap \operatorname{bd}(z - \lambda K) \neq \varnothing$ for all $\lambda \in (\lambda_0, 1)$. From Condition *2)* of the theorem it follows that the set $T_\lambda = \operatorname{bd} K \cap \operatorname{bd}(z - \lambda K)$ lies in a hyperplane H_λ for any choice of $\lambda \in (\lambda_0, 1)$. Since the set T_λ is compact, we easily conclude that T_λ tends to T when $\lambda \to 1$. By a compactness argumen! t, we may assume that H_λ tends to a hyperplane H containing T. By the proved in *1)* $\Rightarrow$ *3)*, the set $\operatorname{bd} K$ is a convex quadric.

4) $\Rightarrow$ *2)*. The case (a), when $\operatorname{bd} K$ is a convex quadric, is considered in Lemma 2.3, with $\lambda \in (-1, 0)$. Suppose that K has the shape (b_4). Then $n = 2$ and $K = M$, where M is a 2-dimensional convex set described in (b_4). Choose any vector $z \in \mathbb{R}^2$ and any scalar $\lambda \in (0,1)$. From the assumption on M in (b_4) it easily follows that the set $\operatorname{bd} M \cap \operatorname{bd}(z - \lambda M)$ may be empty, a singleton, or a line segment. In either case, there is a line containing $\operatorname{bd} M \cap \operatorname{bd}(z - \lambda M)$.

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