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Nash Equilibrium and the Legendre Transform in Optimal Stopping Games with One Dimensional Diffusions

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We show that the value function of an optimal stopping game driven by a one dimensional diffusion can be characterised using the extension of the Legendre transform introduced in [19]. It is shown that under certain integrability conditions, a Nash equilibrium of the optimal stopping game can be derived from this extension of the Legendre transform. This result is an analytical complement to the results in [19] where the 'duality' between a concave biconjugate which is modified to remain below an upper barrier and a convex biconjugate which is modified to remain above a lower barrier is proven by appealing to the probabilistic result in [18]. The main contribution of this paper is to show that, for optimal stopping games driven by a one dimensional diffusion, the semiharmonic characterisation of the value function may be proven using only results from convex analysis.

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1. Introduction

This paper examines the connection between convex analysis and optimal stopping of one dimensional diffusions. The connection between the study of superharmonic functions and optimal stopping problems dates back to Dynkin [7] where the solution to an optimal stopping problem of the type

$$V(x) = \sup_{\tau} E_x \left[G(X_\tau) \right]$$

(where X is a Markov process) was first characterised as the smallest superharmonic majorant of the gains function G. When X is a one-dimensional diffusion, the corresponding superharmonic functions can been characterised in terms of a generalised type of concavity (see [10] pp. 115). This so called 'superharmonic characterisation' of the value function is discussed in further detail in

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[20] Chapter IV Section 9, to illustrate the properties of the solution to certain free-boundary problems associated with optimal stopping problems. More recently, the change of time and scale technique underpinning the superharmonic characterisation was 're-introduced' in [6]. Furthermore, the connection between the superharmonic characterisation and convex analysis has been highlighted in [19].

The minimax version of this problem is referred to as an optimal stopping (or Dynkin) game. The sup-player selects a stopping time τ with the aim of maximising the functional $R_x(\tau,\sigma) = E_x\left[G(X_\tau)\mathbb{I}_{[\tau\le\sigma]} + H(X_\sigma)\mathbb{I}_{[\tau>\sigma]}\right]$ while the inf-player selects a stopping time σ with the aim of minimising the same functional. Optimal stopping games are explained in more detail in Section 2 below. A variant of this optimal stopping game was first studied by Dynkin [9] using martingale based methods. These problems have also been approached via variational inequalities in [2] and [3]. General conditions which ensure that such a saddle point exists have been studied in [11] and [12]. Optimal stopping games have been applied to solve problems in finance, see for example, [1], [13], [14], [15], [16] or [17].

In [18], the superharmonic characterisation of the solution to optimal stopping problems has been extended to optimal stopping games where it is shown that the value function is a 'semiharmonic' function. The value function is shown to be the smallest function which is superharmonic and dominates G on the set $\{V < H\}$ as well as the largest function which is subharmonic and minorises H on the set $\{V > G\}$. It is shown in [18] that a single function with both these properties exists if and only if the optimal stopping game exhibits a saddle point. When the underlying Markov process is a one dimensional diffusion, these dual problems have been formulated in terms of an extension of the Legendre transform in [19].

The purpose of this paper is to establish the 'semiharmonic characterisation' of the value function of optimal stopping games driven by one dimensional diffusions using purely analytical techniques. This requires us to modify some of the basic objects of convex analysis to account for the additional constraint on the domain of the smallest superharmonic majorant (resp. largest subharmonic minorant). The next section formally introduces optimal stopping games and sign-posts the contents of the rest of this paper.

2. Semiharmonic characterisation

Consider a stochastic basis $\left(\Omega,\mathcal{F},\mathbb{F},\{P_x\}_{x\in I}\right)$ supporting a one dimensional diffusion $X = (X_t)_{t\ge0}$ with $X_0 = x \in I \subseteq \mathbb{R}$ under P_x. The level passage times of X will be denoted by $T_y = \inf\{t\ge 0 \mid X_t = y\}$. The state space I of X is an open interval $I = (a,b)$ and the boundaries of I are natural, i.e. $P_x(T_a<\infty) = P_x(T_b<\infty) = 0$ for all $x\in I$. The diffusion X is also assumed to be regular in the sense that for all $x \in I$, $P_x(T_y<\infty) > 0$ for some $y\in(x,b)$.

For a given discount rate $r > 0$, a Borel measurable function $U : I \to \mathbb{R}$ is r-superharmonic with respect to X if

$$U(x) \geq E_x\left[e^{-r\tau} U(X_\tau)\, \mathbb{I}_{[\tau<\infty]}\right]$$

for all $x \in I$ and all stopping times τ. The function U is r-subharmonic with respect to X if $U(x) \leq E_x\left[e^{-r\tau} U(X_\tau)\, \mathbb{I}_{[\tau<\infty]}\right]$ for all $x \in I$ and all stopping times τ. Moreover, U is referred to as r-harmonic, if it is both r-superharmonic and r-subharmonic.

The generator of X is denoted $\mathbb{L}_X$ and under some additional regularity conditions (see [4] Section 4.6) can be expressed as

$$\mathbb{L}_X f(x) = \mu(x) \frac{d}{dx} f(x) + D(x) \frac{d^2}{dx^2} f(x)$$

for $x \in I$ where μ is the drift and $D > 0$ is the diffusion coefficient of X. Take a constant $r > 0$ and consider the ODE

$$\mathbb{L}_X f(x) = r f(x) \tag{1}$$

for $x \in I$. This ODE has two linearly independent positive solutions, denoted φ and ψ. As the boundaries of I are natural, φ and ψ may be taken such that φ is increasing and ψ is decreasing with

$$\varphi(a+) = \psi(b-) = 0, \qquad \varphi(b-) = \psi(a+) = +\infty.$$

The functions φ and ψ are shown in [23] V.50 to be continuous, strictly monotone and strictly convex. Furthermore, for $y \in I$, the process $(e^{-r(t\wedge T_y)}\varphi(X_{t\wedge T_y}))_{t\geq 0}$ is a $(P_x, \mathcal{F}_{t\wedge T_y})$-martingale for all $y > x$ while $(e^{-r(t\wedge T_y)}\psi(X_{t\wedge T_y}))_{t\geq 0}$ is a $(P_x, \mathcal{F}_{t\wedge T_y})$-martingale for all $y \leq x$. For $x, y \in I$ the Laplace transforms of the first passage times may be expressed as

$$E_x\left[e^{-rT_y}\mathbb{I}_{[T_y<\infty]}\right] = \begin{cases} \frac{\psi(x)}{\psi(y)} & \text{for } x > y, \\ \frac{\varphi(x)}{\varphi(y)} & \text{for } x \leq y. \end{cases}$$

Definition 2.1. Let $J : I \to \mathbb{R}$ be a monotone function. A Borel measurable function $f : I \to \mathbb{R}$ is J-concave if

$$f(x) \geq f(y) \frac{J(z) - J(x)}{J(z) - J(y)} + f(z) \frac{J(x) - J(y)}{J(z) - J(y)}.$$

for $x \in [y, z] \subseteq I$. The J-derivative of f is denoted

$$\frac{d}{dJ} f(x) := \lim_{y \downarrow x} \frac{f(y) - f(x)}{J(y) - J(x)}.$$

Define a pair of strictly increasing functions, $F : I \to (0, +\infty)$ and $\widetilde{F} : I \to (-\infty, 0)$ using

$$F(x) = \frac{\varphi(x)}{\psi(x)}, \qquad \widetilde{F}(x) = -\frac{\psi(x)}{\varphi(x)} = -\frac{1}{F(x)}. \tag{2}$$

It is well-known (see [8] Theorem 16.4) that a Borel measurable function U is r-superharmonic if and only if U/ψ is F-concave, or equivalently, U/φ is $\widetilde{F}$-concave. For $a < y \leq x \leq z < b$ the Laplace transforms of the exit times from the open set $(y, z) \subset I$ are

$$\begin{aligned} E_x\left[e^{-rT_y}\mathbb{I}_{[T_y<T_z]}\right] &= \frac{\psi(x)}{\psi(y)}\frac{F(z)-F(x)}{F(z)-F(y)} = \frac{\varphi(x)}{\varphi(y)}\frac{\widetilde{F}(z)-\widetilde{F}(x)}{\widetilde{F}(z)-\widetilde{F}(y)}, \\ E_x\left[e^{-rT_z}\mathbb{I}_{[T_y>T_z]}\right] &= \frac{\psi(x)}{\psi(z)}\frac{F(x)-F(y)}{F(z)-F(y)} = \frac{\varphi(x)}{\varphi(z)}\frac{\widetilde{F}(x)-\widetilde{F}(y)}{\widetilde{F}(z)-\widetilde{F}(y)}. \end{aligned} \tag{3}$$

Take a gains function $G : I \to \mathbb{R}$ which is upper semi-continuous and such that for all $x \in I$,

$$P_x\left(\lim_{t\to\infty} e^{-rt}G(X_t) = c\right) = 1, \qquad E_x\left[\sup_{t\geq 0}\left|e^{-rt}G(X_t)\right|\right] < +\infty \tag{4}$$

for some $c \in \mathbb{R}$. Section 3 examines the solution to the discounted optimal stopping problem

$$V(x) = \sup_{\tau} E_x\left[e^{-r\tau}G(X_\tau)\right] \tag{5}$$

for $x \in I$. Theorem 3.2 in [19] shows that when the value function V defined in (5) coincides with the solution to the 'dual problem'

$$\hat{V} = \inf_{U\in \mathrm{Sup}(G]} U \tag{6}$$

where

$$\mathrm{Sup}(G] := \{U : I \to [G, +\infty) \mid U \text{ is continuous and } r\text{-superharmonic}\}$$

the function V can be expressed in terms of the concave biconjugate of $W := (G/\psi) \circ F^{-1}$. The observation that when a finite optimal stopping time in (5) exists, the solutions to (5) and (6) coincide is often referred to as the 'superharmonic characterisation' of the value function (see Theorem 2.4 in [20] for a probabilistic proof). Theorem 3.2 is the converse of Theorem 3.2 in [19] in the sense that function V is shown to be related to the concave biconjugate of W without assuming that V solves (6). Furthermore, Theorem 3.2 provides a new proof that (5) and (6) coincide.

The remainder of this section introduces optimal stopping games and the semiharmonic characterisation of the solution to such games as well as outlining the contents of the rest of this paper. Take a pair of gains functions G, H that are continuous and for all $x \in I$ satisfy:

$$G(x) \leq H(x), \qquad E_x\left[\sup_{t\geq 0}\left|e^{-rt}G(X_t)\right|\right] < +\infty, \qquad E_x\left[\sup_{t\geq 0}\left|e^{-rt}H(X_t)\right|\right] < +\infty \tag{7}$$

and

$$P_x\left(\lim_{t\to\infty} e^{-rt}G(X_t) = \lim_{t\to\infty} e^{-rt}H(X_t)\right) = 1. \tag{8}$$

Section 5 studies the solution to the infinite time horizon optimal stopping game with lower value $\underline{V}$ defined as

$$\underline{V}(x) = \sup_{\tau}\inf_{\sigma} E_x\left[e^{-r(\tau\wedge\sigma)}\left(G(X_\tau)\mathbb{I}_{[\tau\leq\sigma]} + H(X_\sigma)\mathbb{I}_{[\tau>\sigma]}\right)\right], \tag{9}$$

and upper value $\overline{V}$ defined as

$$\overline{V}(x) = \inf_{\sigma}\sup_{\tau} E_x\left[e^{-r(\tau\wedge\sigma)}\left(G(X_\tau)\mathbb{I}_{[\tau\leq\sigma]} + H(X_\sigma)\mathbb{I}_{[\tau>\sigma]}\right)\right]. \tag{10}$$

where G and H satisfy the assumptions outlined above. For ease of notation, the objective function is denoted

$$R_x(\tau,\sigma) := E_x\left[e^{-r(\tau\wedge\sigma)}\left(G(X_\tau)\mathbb{I}_{[\tau\leq\sigma]} + H(X_\sigma)\mathbb{I}_{[\tau>\sigma]}\right)\right]. \tag{11}$$

The optimal stopping game has a Stackelberg equilibrium if the upper and lower values coincide, i.e. $\underline{V}(x) = \overline{V}(x) =: V(x)$ for all $x \in I$. In [11] and [12] it is shown via probabilistic means that this game exhibits a Stackelberg equilibrium when both (7) and (8) hold. The assumptions in [12] are slightly more general, in which case the Stackelberg equilibrium is determined by how the objective function is specified at the natural boundaries.

A saddle point is a pair of stopping times (τ^*, σ^*) such that for any other stopping times τ, σ

$$R_x(\tau,\sigma^*) \leq R_x(\tau^*,\sigma^*) \leq R_x(\tau^*,\sigma) \quad \forall\, x \in I.$$

The optimal stopping game exhibits a Nash equilibrium if the game has a saddle point. In particular, existence of a Nash equilibrium implies a Stakelberg equilibrium exists but the converse is not necessarily true. The result in [11] (which applies to more general processes) shows that, under the assumptions (7) and (8), the optimal stopping game described above has a Nash equilibrium.

Introduce a pair of dual problems

$$\hat{V} := \inf_{U\in\mathrm{Sup}[G,H)} U, \qquad \check{V} := \sup_{U\in\mathrm{Sub}(G,H]} U \tag{12}$$

where the admissible sets of functions are

$$\begin{aligned}
&\operatorname{Sup}[G,H) \\
&= \{U : I \to [G,H] \mid U \text{ is continuous and } r\text{-superharmonic on } \{U < H\}\}, \\
&\operatorname{Sub}(G,H] \\
&= \{U : I \to [G,H] \mid U \text{ is continuous and } r\text{-subharmonic on } \{U > G\}\}.
\end{aligned}$$

Any function $U \in \operatorname{Sup}[G,H) \cup \operatorname{Sub}(G,H]$ is referred to as r-semiharmonic. It has been shown in [18] Theorem 2.1 that when (7) and (8) hold $\hat{V} = \check{V}$ and that the value of the dual problems coincides with the value of the optimal stopping game with upper and lower values (9)–(10) if and only if the optimal stopping game has a Nash equilibrium. The joint solution to the dual problems (12) is referred to in [18] as the semiharmonic characterisation of the value function.

When the driving process X is a one dimensional diffusion absorbed upon exit from a compact set, it has been shown in [19] that the solutions to the dual problems (12) can be expressed in terms of an extension of the Legendre transform. It then follows from Theorem 2.1 in [18] that the Nash and Stackelberg equilibrium of the optimal stopping game (9)–(10) can be expressed in terms of this extension of the Legendre transform.

The main contribution of Section 4 is to establish equality between the two extensions of the Legendre transform introduced in [19] without reference to the probabilistic results in [18] and [11]. In particular, we construct an extension of the concave biconjugate of $W^G := (G/\psi) \circ F^{-1}$ of the form

$$(W^G)^{**}_H(y) = \inf_{z \in \mathbb{R}} \sup_{y \in \mathcal{A}^x_G(z)} (z(x-y) + W^G(y))$$

for $y > 0$ where the set mapping $z \mapsto \mathcal{A}^x_G(z)$ is defined in such a way as to ensure that $(W^G)^{**}_H(y) \leq (H/\psi) \circ F^{-1}(y) =: W_H(y)$ for all $y > 0$. Similarly, we construct an extension of the convex biconjugate of W_H of the form

$$(W_H)^G_{**}(y) = \sup_{z \in \mathbb{R}} \inf_{y \in \mathcal{A}^H_x(z)} (z(x-y) + W_H(y))$$

for $y > 0$ where the set mapping $z \mapsto \mathcal{A}^H_x(z)$ is defined in such a way as to ensure that $(W_H)^G_{**}(y) \geq W^G(y)$ for all $y > 0$. In particular, Theorem 4.7 is a converse to [19] Theorem 4.1 as it shows that $(W^G)^{**}_H(y) = (W_H)^G_{**}(y)$ for all $y > 0$ using a purely analytical approach.

Section 5 establishes that under the assumptions (7) and (8) the infinite time horizon optimal stopping game (9)–(10) has both a Stackelberg and Nash equilibrium. In Theorem 5.1 the optimal stopping game (9)–(10) is shown to have a Stackelberg equilibrium which can be expressed in terms of the extensions of the Legendre transform introduced in Section 4. It is then shown that the functions $(G)^{**}_H, (H)^{**}_{*,*} : I \to [G,H]$ defined as

$$(G)^{**}_H(x) = (W^G)^{**}_H(F(x))\psi(x), \qquad (H)^G_{**}(x) = (W_H)^G_{**}(F(x))\psi(x)$$

are r-semiharmonic and solve the dual problems (12). Consequently, it follows from the results in Section 4 that the solutions to the dual problems (12) coincide. Finally in Theorem 5.5 it is shown that the F-concave/F-convex structure of $(W^G)^{**}_H$ can be used to characterise a Nash equilibrium of the optimal stopping game (9)–(10).

3. Optimal stopping using the Legendre transformation

Before proceeding to solve the optimal stopping problem (5), we shall first recall the definition and some properties of the concave biconjugate. Let $f : \mathrm{dom}\,(f) \to \mathbb{R}$ be a finite, measurable function on the domain $\mathrm{dom}\,(f) \subseteq [-\infty, +\infty]$. The concave conjugate of f, denoted f_*, is defined for $c \in \mathbb{R}$ as

$$f_*\,(c) = \inf_{x \in \mathrm{dom}(f)} \left(cx - f\,(x)\right).$$

The concave biconjugate of f is defined as

$$f_{**}\,(x) = \inf_{y \in \mathrm{dom}(f^*)} \left(xy - f_*\,(y)\right) = \inf_{y \in \mathrm{dom}(f^*)} \sup_{c \in \mathrm{dom}(f)} \left(y\,(x-c) + f\,(c)\right). \qquad (13)$$

The epigraph of a function f is the set of all points above the graph of f, that is

$$\mathrm{epi}\,(f) := \left\{(x,\mu) \in \mathbb{R}^2 \mid \mu \geq f\,(x)\right\}.$$

The convex hull of the set $\mathrm{epi}\,(f)$ is the intersection of all convex sets containing $\mathrm{epi}\,(f)$ and is denoted $\mathrm{conv}\,(f)$. With a slight abuse of notation, let

$$\mathrm{conv}\,(f)\,(x) := \inf\left\{y \in \mathbb{R} \mid\, (x,y) \in \mathrm{conv}\,(f)\right\},$$

then $-f_{**}$ is the upper semi-continuous modification of $\mathrm{conv}\,(-f)\,(x)$, see [22] Theorem 12.2 and Corollary 12.1.1. A constant $c \in \mathbb{R}$ is a subgradient of the function f at $x \in \mathbb{R}$ if

$$f\,(z) \geq f\,(x) + c\,(z-x) \quad \forall z \in \mathbb{R}.$$

The set of all such subgradients, denoted

$$\partial f\,(x) := \left\{c \in \mathbb{R} \mid f\,(y) - f\,(x) \leq c(y-x)\ \forall y \in \mathbb{R}\right\} \qquad (14)$$

is referred to as the subdifferential. The function f is referred to as concave at $x \in \mathbb{R}$ when $\partial f\,(x) \neq \emptyset$. When f is concave and differentiable at x then $\partial f\,(x) = \{f'\,(x)\}$.

The next lemma characterises the set upon which f coincides with f_{**}. The proof illustrates how $x \mapsto f_{**}\,(x)$ can be constructed using a spike variation.

Lemma 3.1. *Any function $f : \mathrm{dom}(f) \to \mathbb{R}$ coincides with its concave biconjugate on the set*

$$\left\{x \in \mathbb{R} \mid f_{**}\,(x) = f\,(x)\right\} = \left\{x \in \mathbb{R} \mid \partial f\,(x) \neq \emptyset\right\}. \qquad (15)$$

Proof. Suppose that $\partial f(x) \neq \emptyset$ and take $c \in \partial f(x)$ then it follows from the definition (14) that

$$f_*(c) := \inf_{y\in\mathbb{R}} (cy - f(y)) = cx - f(x).$$

Consequently, f_{**} can be written as

$$f_{**}(x) = \inf_{c\in\mathbb{R}} (c(x-x) + f(x)) = f(x).$$

so $\{x \in \mathbb{R} \mid \partial f(x) \neq \emptyset\} \subseteq \{x \in \mathbb{R} \mid f_{**}(x) = f(x)\}$. To show this inclusion holds with equality assume that $\partial f(x) = \emptyset$ and for a fixed $c \in \mathbb{R}$ let

$$\varepsilon(x;c) := \inf\{\varepsilon \geq 0 \mid f(y) + c(x-y) \leq f(x) + \varepsilon \ \forall y \in \mathbb{R}\}.$$

It follows that

$$-f_*(c) = \sup_{y\in\mathbb{R}} (f(y) - cy) = f(x) + \varepsilon(x;c) - cx,$$

and f_{**} can be written as

$$f_{**}(x) = \inf_{c\in\mathbb{R}} (cx - f_*(c)) = \inf_{c\in\mathbb{R}} (f(x) + \varepsilon(x;c)). \tag{16}$$

Let

$$\varepsilon^*(x) := \inf\{\varepsilon \geq 0 \mid \exists c \in \mathbb{R} \text{ s.t. } f(y) + c(x-y) \leq f(x) + \varepsilon \ \forall y \in \mathbb{R}\} \tag{17}$$

so that $\{x \in \mathbb{R} \mid \varepsilon^*(x) > 0\} = \{x \in \mathbb{R} \mid \partial f(x) = \emptyset\}$. Hence we may conclude from (16) that when $\partial f(x) = \emptyset$, $f_{**}(x) = f(x) + \varepsilon^*(x) > f(x)$. $\square$

Let $W : (0,\infty) \to \mathbb{R}$ be defined as

$$W(y) := \left(\frac{G}{\psi}\right) \circ (F)^{-1}(y) \tag{18}$$

where F was defined in (2). Similarly, let $\widetilde{W} : (-\infty, 0) \to \mathbb{R}$ be defined via

$$\widetilde{W}(y) := \left(\frac{G}{\varphi}\right) \circ (\widetilde{F})^{-1}(y) \tag{19}$$

The next result is the main result in this section and it shows that the value function V defined in (5) is such that V/ψ coincides with $W_{**} \circ F$ where W_{**} is the concave biconjugate of the function W. The value function V is also such that V/φ coincides with $\widetilde{W}_{**} \circ \widetilde{F}$ where $\widetilde{W}_{**}$ is the concave biconjugate of the function $\widetilde{W}$. The case that both boundaries are absorbing has been handled by showing that $W_{**}(F(x))\psi(x)$ solves (6) in [19] Theorem 3.2.

Theorem 3.2. *Assume that* $G : I \to \mathbb{R}$ *is an upper-semicontinuous function satisfying the assumption* (4) *and such that* $W(0+) = \widetilde{W}(0-) = 0$*. Consider the stopping problem* (5)*, then*

$$V(x) = W_{**}(F(x))\,\psi(x) = \widetilde{W}_{**}(\widetilde{F}(x))\varphi(x)$$

for all $x \in I$*. The stopping time which attains the supremum in* (5) *is* $\tau^* = T_{a^*} \wedge T_{b^*}$ *where*

$$\begin{aligned} a^* &= \sup\{c \in (a,x] \,|\, W(F(c)) = W_{**}(F(c))\}, \\ b^* &= \inf\{c \in [x,b) \,|\, W(F(c)) = W_{**}(F(c))\}, \end{aligned} \tag{20}$$

and we use the convention that $\sup\emptyset = a$ *and* $\inf\emptyset = b$*. Furthermore* V *solves* (6)*.*

Proof. It is assumed that $W(0+) = \widetilde{W}(0-) = 0$ without loss of generality as when this is not true it can be achieved by subtracting the constant c introduced in (4) from the value function V. Let $(b_n)_{n\geq 1}$ be a sequence of real numbers such $(a, b_n] \subset (a, b)$, $b_n \leq b_{n+1}$ for all $n \geq 1$ and $(a, b_n] \to (a, b)$ as $n \to +\infty$. Fix $x \in (a, b)$ then for some sufficiently large N, $x \in (a, b_n]$ for all $n \geq N$. Let $X^n_t := X_{t \wedge T_{b_n}}$ and consider the following family of optimal stopping problems

$$V^n(x) = \sup_\tau E_x\left[e^{-r\tau} G(X^n_\tau)\right] \tag{21}$$

where G satisfies (4). Consider the sets of functions

$$\mathrm{Sup}^n(G) = \{U : (a, b_n] \to [G, +\infty) \,|\, U \,\text{is continuous and}\, r\text{−superharmonic}\}$$

for each $n \geq 1$. Take any $U_1, U_2 \in \mathrm{Sup}^n(G)$, then U_1/ψ and U_2/ψ are F-concave on $(a, b_n]$ (see [8] Theorem 16.4). Since U_1/ψ and U_2/ψ are F-concave, the functions $W_1 = (U_1/\psi)\circ F^{-1}$ and $W_2 = (U_2/\psi)\circ F^{-1}$ are concave. Consequently, $W_3 = W_1 \wedge W_2$ is a concave function and reversing this argument implies $U_3 = U_1 \wedge U_2 \in \mathrm{Sup}^n(G)$. Hence the set $\mathrm{Sup}^n(G)$ is downwards directed so $\widetilde{V}^n := \inf_{U \in \mathrm{Sup}^n(G)} U$ exists. In fact this infimum is attained since, when X is a one dimensional diffusion, all r-superharmonic functions are continuous (see [20] Sections 9.3.3 and 9.3.4 replacing S with F). This shows that the conditions of [20] Theorem 2.17 hold so if an optimal stopping time exists it is of the form $\tau^*_n = \inf\{t \geq 0 \,|\, X_t \in \mathcal{D}^n\}$ where $\mathcal{D}^n := \{x \in I \,|\, G(x) = \widetilde{V}^n(x)\}$. The set $\mathcal{D}$ is a closed set and $\mathcal{C}^n := \{x \in I \,|\, G(x) < \widetilde{V}^n(x)\}$ is open since G is upper-semicontinuous and $\widetilde{V}^n$ is continuous on $(a, b_n]$. Hence, for each $n \geq 1$ there is a collection of disjoint open intervals $((x_i, y_i))_{i\geq 1}$ such that $\mathcal{C}^n = \bigcup_{i\geq 1}(x_i, y_i)$. Suppose that $X_0 \in (x_i, y_i)$ for some $i \geq 1$, as the paths $t \mapsto X_t$ are continuous, $T_{x_i} \wedge T_{y_i} \leq T_{x_j} \wedge T_{y_j}$ for all $j \neq i$. Consequently we can restrict attention to candidate stopping times which are exit times from open intervals.

If an optimal stopping time for (21) exists we may write the optimal stopping time for (21) as $\tau^*_n = T_{a^*_n} \wedge T_{b^*_n}$ for some $a \leq a^*_n \leq x \leq b^*_n \leq b_n$. Furthermore, each $\tau_n^* \leq T_{b_n}$, $\tau_n^* \leq \tau_{n+1}^*$ for all $n \geq 1$ and $\lim_{n\to+\infty} \tau_n^* = \tau^*$ P_x-a.s. for all $x \in I$. For all $n \geq 1$, $X^n_{\tau_n^*} = X_{\tau_n^*}$ and the process X is left-continuous over stopping times so $\lim_{n\to+\infty} X^n_{\tau_n^*} = X_{\tau^*}$ P_x-a.s for all $x \in I$. The assumption (4) implies that $(e^{-rt}G(X_t))_{t\geq 0}$ is uniformly integrable so it follows that

$$\lim_{n\to\infty} V^n(x) = \lim_{n\to\infty} E_x[e^{-r\tau_n^*}G(X^n_{\tau_n^*})] = V(x).$$

The process $(e^{-r(t\wedge T_{b_n})}\varphi(X_t^n))_{t\geq 0}$ is a $(P_x, \mathcal{F}_{t\wedge T_{b_n}})$-martingale for each $b_n > x$ so applying the optional sampling theorem yields $E_x[e^{-r\tau}\varphi(X_\tau^n)] = \varphi(x)$ for all $x \in (a, b_n]$ and all $\tau \leq T_{b_n}$. Hence for an arbitrary $c \in \mathbb{R}$

$$V^n(x) = \sup_\tau E_x\left[e^{-r\tau}G(X_\tau^n)\right] = c\varphi(x) + \sup_\tau E_x\left[e^{-r\tau}\left(G(X_\tau^n) - c\varphi(X_\tau^n)\right)\right].$$

Thus,

$$V^n(x) = \inf_{c\in\mathbb{R}}\Big(c\varphi(x) - \inf_\tau E_x\left[e^{-r\tau}\left(c\varphi(X_\tau^n) - G(X_\tau^n)\right)\right]\Big). \tag{22}$$

The next step is to expand the right hand side of this expression in such a way that it converges to the concave biconjugate of W as $n \to +\infty$. Using (3), the inner infimum in (22) can be written as

$$\begin{aligned}
&\inf_{a<y\leq x\leq z\leq b_n} E_x\left[e^{-r(T_y\wedge T_z)}(c\varphi(X^n_{T_y\wedge T_z}) - G(X^n_{T_y\wedge T_z}))\right]\\
&= \inf_{a<y\leq x\leq z\leq b_n}\Bigg(\frac{c\varphi(y) - G(y)}{\psi(y)}\frac{F(z) - F(x)}{F(z) - F(y)}\\
&\qquad + \frac{c\varphi(z) - G(z)}{\psi(z)}\frac{F(x) - F(y)}{F(z) - F(y)}\Bigg)\psi(x).
\end{aligned}$$

We claim that the right hand side of this expression converges to

$$\inf_{y\in(a,b)}\left(\frac{c\varphi(y) - G(y)}{\psi(y)}\right)\psi(x) = \inf_{y'>0}(cy' - W(y'))\psi(x) =: W_*(c)\,\psi(x)$$

as $n \to \infty$. To this end, take

$$\begin{aligned}
z^+ &= \sup\{z \in I \;\mid W_*(c)\,\psi(z) = c\varphi(z) - G(z)\},\\
y^- &= \inf\{y \in I \;\mid W_*(c)\,\psi(y) = c\varphi(y) - G(y)\}
\end{aligned}$$

with the convention that $\sup\emptyset = a$ and $\inf\emptyset = b$. In general for all $a \leq y \leq x \leq z \leq b_n$, $W_*(c) \leq \lambda(cy - W(y)) + (1-\lambda)(cz - W(z))$ for all $\lambda \in [0,1]$. In particular, taking $\lambda = (F(z) - F(x))/(F(z) - F(y))$ we obtain

$$W_*(c) \leq \frac{c\varphi(y) - G(y)}{\psi(y)}\frac{F(z) - F(x)}{F(z) - F(y)} + \frac{c\varphi(z) - G(z)}{\psi(z)}\frac{F(x) - F(y)}{F(z) - F(y)}. \tag{23}$$

When $y^- \leq x \leq z^+$ (23) holds with equality for $y = y^-$ and $z = z^+$. When $y^- \geq x$, let $(y_m)_{m\geq 1}$ be such that $y_m \leq y^-$ and $\lim_{m\to\infty} y_m = a$. For $y = y_m$ and $z = y^-$ the inequality in (23) is strict, however, since we assumed $W(0+) = 0$ it follows from the definition of F that the as $m \to \infty$ the right hand side of (23) converges to $W_*(c)$. When $z^+ \leq x$ the inequality is strict as

$$\begin{aligned} &\frac{c\varphi(y) - G(y)}{\psi(y)} \frac{F(z^+) - F(x)}{F(z^+) - F(y)} + \frac{c\varphi(z^+) - G(z^+)}{\psi(z^+)} \frac{F(x) - F(y)}{F(z^+) - F(y)} \\ &> \frac{c\varphi(z^+) - G(z^+)}{\psi(z^+)} \end{aligned} \tag{24}$$

for all $y \in [x, b_n]$. We may switch to the other ratio of the fundamental solutions using (3) on the left hand side of (24) and use that $\varphi(y) \to +\infty$ as $y \uparrow b$ to conclude

$$\begin{aligned} &\lim_{n\to\infty} \inf_{y\in[x,b_n]} \left(\frac{c\varphi(y) - G(y)}{\varphi(y)} \frac{\widetilde{F}(z^+) - \widetilde{F}(x)}{\widetilde{F}(z^+) - \widetilde{F}(y)} + \frac{c\varphi(z^+) - G(z^+)}{\varphi(z^+)} \frac{\widetilde{F}(x) - \widetilde{F}(y)}{\widetilde{F}(z^+) - \widetilde{F}(y)} \right) \\ &= W_*(c). \end{aligned}$$

Hence letting $n \to \infty$ on both sides of (22) yields

$$V(x) = \inf_{c\in\mathbb{R}} \left(c\frac{\varphi(x)}{\psi(x)} + W_*(c) \right) \psi(x).$$

The argument using the other ratio of fundamental solutions follows analogously. The statement about the optimal stopping time follows from the form of the value function provided as the stopping region $\mathcal{D}$ and continuation region $\mathcal{C}$ for (5) are

$$\begin{aligned} \mathcal{C} &= \{x \in I \mid V(x) > G(x)\} = \{x \in I \mid W_{**}(F(x))\psi(x) > G(x)\}, \\ \mathcal{D} &= \{x \in I \mid V(x) = G(x)\} = \{x \in I \mid W_{**}(F(x))\psi(x) = G(x)\}. \end{aligned}$$

The stopping time $\tau^* := \{t \geq 0 \mid X_t \in \mathcal{D}\} = T_{a^*} \wedge T_{b^*}$ attains the supremum in (5) although it may occur that $P_x(\tau^* < +\infty) < 1$. Finally, $W_{**} = -\mathrm{cl}(\mathrm{conv}(-W))$ is the smallest concave function dominating W, so $V(x) = W_{**}(F(x))\psi(x)$ is the smallest r-superharmonic function dominating the gains function $G(x)$ and hence $V(x)$ solves (6). □

In Theorem 3.2 it has been shown that when $W(0+) = \widetilde{W}(0-) = 0$ the value function of (5) coincides with the solution to (6). Whether the optimal stopping time is obtained in finite time depends both on the function G and the behaviour of the diffusion X_t as $t \to \infty$. In Theorem 3.2 we can relax the assumptions (4) and instead impose that $|W(0+)| < \infty$ and $|\widetilde{W}(0-)| < \infty$ which is the assumption used in [6]. The case that $W(0+) = +\infty$ is degenerate as $V(x) \geq \lim_{c\downarrow a} E_x[G(X_{T_c})e^{-rT_c}\mathbb{I}_{T_c<\infty}] = \lim_{c\downarrow 0} W(c)\psi(x) = W(0+)\psi(x) = \infty$.

Remark 3.3. The assumption in Theorem 3.2 that $W(0+) = \widetilde{W}(0-) = 0$ ensures that $V(a+) = V(b-) = 0$. In [6] and [12] this condition is imposed by extending all Borel measurable functions F on I onto $[a, b]$ by setting $F(X_\tau(\omega)) =$

0 on $\{\tau = +\infty\}$. If we use this convention and the weaker assumption that $|W(0+)| < \infty$ and $|\widetilde{W}(0-)| < \infty$ rather than (4) the optimal stopping time may not exist. To see this, suppose that a sequence $(a_n)_{n\geq 1}$ such that $a_n \downarrow a$ as $n \to \infty$ is optimising in the sense that $E_x[G(X_{T_{a_n}})e^{-rT_{a_n}}] \leq E_x[G(X_{T_{a_{n+1}}})e^{-rT_{a_{n+1}}}]$ and $V(x) = \lim_{n\to\infty} E_x[G(X_{T_{a_n}})e^{-rT_{a_n}}] = G(a+)\psi(x)/\psi(a+) = W(0+)$. When $W(0+) > 0$, this limit is not obtained as $\tau^* = \lim_{n\to\infty} T_{a_n} = \infty$ P_x-a.s. for all $x \in I$ so $E_x[G(X_{\tau^*})e^{-r\tau^*}] = 0$.

For fixed $x \in I$ let

$$l^x(c,p) = \sup\left\{ y \leq x \,\middle|\, \left(\frac{G}{\psi}\right)(y) - \frac{p}{\psi(x)} = c\left(F(y) - F(x)\right) \right\},$$
$$r^x(c,p) = \inf\left\{ y \geq x \,\middle|\, \left(\frac{G}{\psi}\right)(y) - \frac{p}{\psi(x)} = c\left(F(y) - F(x)\right) \right\},$$

with the usual convention that $\sup\emptyset = -\infty$ and $\inf\emptyset = +\infty$. For a given $x \in I$ consider the sets

$$A_1(x) = \{p \in [G(x), +\infty) \mid \exists\, c \in \mathbb{R} \text{ s.t. } (l^x(c,p), r^x(c,p)) \subset I\}, \tag{25}$$
$$A_2(x) = \{p \in [G(x), +\infty) \mid \exists\, c \in \mathbb{R} \text{ s.t. } r^x(c,p) = -l^x(c,p) = +\infty\}.$$

In Lemma 3.1 it was shown that $x \mapsto W_{**}(x)$ can be constructed by minimising over the 'F-tangents' of the form $y \mapsto p/\psi(x) + c(F(y) - F(x))$ which strictly dominate W on $\mathrm{dom}(W)$, i.e. $W_{**}(x) = \inf A_2(x)$. The next corollary provides a short proof of the 'dual interpretation' provided in [19] which claims that $x \mapsto W_{**}(x)$ can be also constructed by maximising over the F-tangents which intercept W on both sides of the point x in $\mathrm{dom}(W)$, i.e. $W_{**}(x) = \sup A_1(x)$. The latter formulation facilitates the construction of a maximising sequence of stopping times.

Corollary 3.4. *Let*

$$\widehat{\varepsilon}(x;c) := \sup\{\varepsilon \geq 0 \mid a \leq l^x(c, G(x)+\varepsilon) \leq x \leq r^x(c, G(x)+\varepsilon) \leq b\}$$

and set $A_c(x) := [F(l^x(c, G(x)+\widehat{\varepsilon}(x;c))), F(r^x(c, G(x)+\widehat{\varepsilon}(x;c)))]$, *then the value function of the optimal stopping problem* (5) *can be represented as*

$$V(x) = \sup_{c\in\mathbb{R}} \sup_{y\in A_c(x)} \left(c\left(F(x) - y\right) + W(y)\right)\psi(x).$$

Moreover, $V(x) = \sup A_1(x)$ *for each* $x \in I$ *where* $A_1(x)$ *is the set defined in* (25).

Proof. For the first statement, note that for each $c \in \mathbb{R}$, by definition

$$\frac{G(x) + \widehat{\varepsilon}(x;c)}{\psi(x)} + c\left(y - F(x)\right) \geq W(y)$$

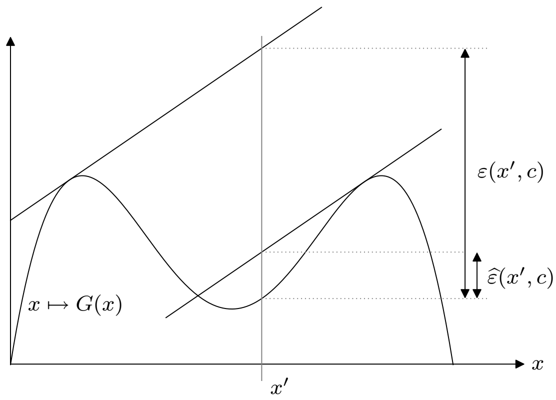

Figure 3.1: This diagram illustrates $\widehat{\varepsilon}(x;c)$ and $\varepsilon(x;c)$ for a fixed c in the case that $r = 0$ and X is a standard Brownian motion (i.e. $\psi \equiv 1$ and $\varphi(x) = x$ hence $F(x) = x$)

for $y \in A_c(x)$ and equality holds for $y = F\left(l^x\left(c, G(x) + \widehat{\varepsilon}(x;c)\right)\right)$. Hence,

$$\sup_{y\in A_c(x)} \left(W(y) - cy\right) = W(F(x)) + \frac{\widehat{\varepsilon}(x;c)}{\psi(x)} - cF(x)$$

and it follows that

$$\sup_{c\in\mathbb{R}} \sup_{y\in A_c(x)} \left(c\left(F(x) - y\right) + W(y)\right)\psi(x) = \sup_{c\in\mathbb{R}} \left(G(x) + \widehat{\varepsilon}(x;c)\right)$$

Let

$$\varepsilon(x;c) := \inf\left\{\varepsilon \geq 0 \;\middle|\; \left(\frac{G}{\psi}\right)(y) + c\left(F(x) - y\right) \leq \frac{G(x) + \varepsilon}{\psi(x)} \quad \forall y \in \mathbb{R}\right\}.$$

For each $c \in \mathbb{R}$, $\varepsilon(x;c) \geq \widehat{\varepsilon}(x;c)$ as illustrated in Figure 3.1 but to avoid contradicting their definitions $\inf_{c\in\mathbb{R}} \varepsilon(x;c) = \sup_{c\in\mathbb{R}} \widehat{\varepsilon}(x;c)$ and hence the result follows from Lemma 3.1. Furthermore, it follows from the definition of the set $A_1(x)$ that $\sup A_1(x) = G(x) + \sup_{c\in\mathbb{R}} \widehat{\varepsilon}(x;c) = W_{**}(F(x))\psi(x) = V(x)$. $\square$

The next corollary shows that Theorem 3.2 is capable of handling the case when one (or more) of the boundaries is absorbing.

Corollary 3.5. *Take $x \in (\alpha, \beta) \subset (a, b)$ and consider the stopped diffusion $X_t^{\alpha,\beta} := X_{t\wedge T_{\alpha,\beta}}$ where $T_{\alpha,\beta} = T_\alpha \wedge T_\beta$ and the corresponding optimal stopping problem*

$$V_{\alpha,\beta}(x) = \sup_\tau E_x\left[e^{-r\tau} G(X_\tau^{\alpha,\beta})\right] \tag{26}$$

for $G : (a, b) \to \mathbb{R}$ satisfying the assumptions in Theorem 3.2. Then

$$V_{\alpha,\beta}(x) = (W_{[\alpha,\beta]})_{**}(F(x))\psi(x) = (\widetilde{W}_{[\alpha,\beta]})_{**}(\widetilde{F}(x))\varphi(x)$$

where

$$W_{[\alpha,\beta]}(y) := W(y) - \delta(y \,|\, F^{-1}(y) \in I \setminus [\alpha,\beta]),$$
$$\widetilde{W}_{[\alpha,\beta]}(y) := \widetilde{W}(y) - \delta(y \,|\, \widetilde{F}^{-1}(y) \in I \setminus [\alpha,\beta]).$$

Moreover, the stopping time which attains the supremum in (5) *is* $\tau = T_{a^*} \wedge T_{b^*}$ *where*

$$a^* = \sup\{y \in [\alpha, x] \,|\, W_{[\alpha,\beta]}(F(y)) = (W_{[\alpha,\beta]})_{**}(F(y))\},$$
$$b^* = \inf\{y \in [x, \beta] \,|\, W_{[\alpha,\beta]}(F(y)) = (W_{[\alpha,\beta]})_{**}(F(y))\}.$$

Proof. An optimal stopping time for a problem of (26) exists (see [20] Chapter 1 Theorem 2.7) and is the exit time from a open set containing x so we may write $\tau^* = T_{a^*} \wedge T_{b^*}$ for some $\alpha \leq a^* \leq x \leq b^* \leq \beta$. Suppose that $\beta < b$ then as shown in the proof of Theorem 3.2, the value function of the optimal stopping problem (26) can be written as

$$V_{\alpha,\beta}(x) = \inf_{c \in \mathbb{R}} \left(c\varphi(x) - R_c(x)\right). \tag{27}$$

where

$$R_c(x) := \inf_{a \leq y \leq x \leq z \leq b} \left(\frac{c\varphi(y) - \widetilde{G}(y)}{\psi(y)} \frac{F(z) - F(x)}{F(z) - F(y)} + \frac{c\varphi(z) - \widetilde{G}(z)}{\psi(z)} \frac{F(x) - F(y)}{F(z) - F(y)} \right) \psi(x)$$

and

$$\widetilde{G}(y) := G(\alpha)\,\mathbb{I}_{(a,\alpha]} + G(y)\,\mathbb{I}_{(\alpha,b)} + G(\beta)\,\mathbb{I}_{[\beta,b)}.$$

As φ is r-harmonic on $[y, b)$ for all $y > a$, it follows that for any $x \in I$

$$\begin{aligned}
\frac{c\varphi(y) - \widetilde{G}(y)}{\psi(y)}\psi(x) &= E_x[e^{-rT_y}(c\varphi(X_{T_y}) - \widetilde{G}(X_{T_y}))] \\
&= c\varphi(x) - E_x[e^{-rT_y}\widetilde{G}(X_{T_y})] \\
&> c\varphi(x) - G(\alpha)\,E_x\left[e^{-rT_\alpha}\right] = \frac{c\varphi(\alpha) - G(\alpha)}{\psi(\alpha)}\psi(x)
\end{aligned}$$

for all $y \in (a, \alpha)$. Likewise for $y \in (\beta, b)$,

$$\frac{c\varphi(y) - \widetilde{G}(y)}{\psi(y)}\psi(x) > c\varphi(x) - G(\beta)\,E_x\left[e^{-rT_\beta}\right] = \frac{c\varphi(\alpha) - G(\beta)}{\psi(\beta)}\psi(x).$$

Hence, an argument similar to that presented in Theorem 3.2 can be used to show that

$$\begin{aligned}
R_c(x) &= \inf_{y \in (a,b)} \left(\frac{c\varphi(y) - \widetilde{G}(y)}{\psi(y)} \right) \psi(x) \\
&= \inf_{y \in [\alpha,\beta]} \left(\frac{c\varphi(y) - G(y)}{\psi(y)} \right) \psi(x) =: W_*(c).
\end{aligned}$$

Thus (27) reads

$$\begin{aligned} V_{\alpha,\beta}(x) &= \inf_{c\in\mathbb{R}}\left(c\varphi(x) - \inf_{y\in[\alpha,\beta]}\left(\frac{c\varphi(y)-G(y)}{\psi(y)}\psi(x)\right)\right) \\ &= \left(W_{[\alpha,\beta]}\right)_{**}(F(x))\,\psi(x). \end{aligned}$$

The case $\beta = b$ can be handled using an approximating sequence of domains as in Theorem 3.2. The argument using the other ratio of fundamental solutions follows analogously. □

This section concludes with a basic example, the perpetual American put option.

Example 3.6. Take $\sigma > 0$, $r > 0$ and define X to be the solution to $dX_t = rX_t\,dt + \sigma X_t\,dW_t$ with $X_0 = x > 0$ where W is a standard Brownian motion. Consider the perpetual American put option, the gains function of which is $G(x) = (K-x)^+$ for a strike price $K > 0$. The risk neutral value of this option is

$$V(x) := \sup_{\tau} E_x\left[e^{-r\tau}(K - X_\tau)^+\right]. \tag{28}$$

The infinitesimal generator associated with the geometric Brownian motion X is

$$\mathbb{L}_X u := rx\frac{du}{dx} + \frac{1}{2}\sigma^2 x^2\frac{d^2u}{dx^2}$$

and the ODE $\mathbb{L}_X u = ru$ has two fundamental solutions $\varphi(x) = x$ and $\psi(x) = x^{-2r/\sigma^2}$. Let $-\widetilde{F}(x) = (\psi/\varphi)(x) = x^{-1/\alpha}$ where $\alpha := 1/(1+2r/\sigma^2) \in [0,1]$. The rescaled gains function associated with the problem (28) is

$$\widetilde{G}(y) := (Ky^\alpha - 1)^+.$$

The function $y \mapsto \widetilde{G}(y)$ has $\widetilde{G}''(y) \le 0$ for all $y \in \mathbb{R}_+$ and hence to find $y \mapsto \widetilde{G}_{**}(y)$ we need only to find $y \ge 0$ such that

$$\widetilde{G}(y) - \widetilde{G}'(y)\,y = 0. \tag{29}$$

The unique solution to (29) is $y^* = 1/(K(1-\alpha))^{1/\alpha}$ and

$$\widetilde{G}_{**}(y) = \begin{cases} \widetilde{G}'(y^*)\,y & \text{for } y \in [0,y^*], \\ \widetilde{G}(y) & \text{for } y > y^*. \end{cases}$$

Applying Theorem 3.2,

$$V(x) = \widetilde{G}_{**}(-\widetilde{F}(x))\varphi(x) = \begin{cases} \frac{\sigma^2}{2r}\left(\frac{K}{(1+\sigma^2/2r)}\right)^{1+2r/\sigma^2} x^{-2r/\sigma^2} & \text{for } x \ge x^* \\ K - x & \text{for } x \in [0,x^*] \end{cases} \tag{30}$$

where $x^* = (y^*)^\alpha = K/(1+\sigma^2/2r)$. Figure 3.2 illustrates the original and transformed payoff functions, the concave biconjugate of the transformed payoff and the corresponding value function.

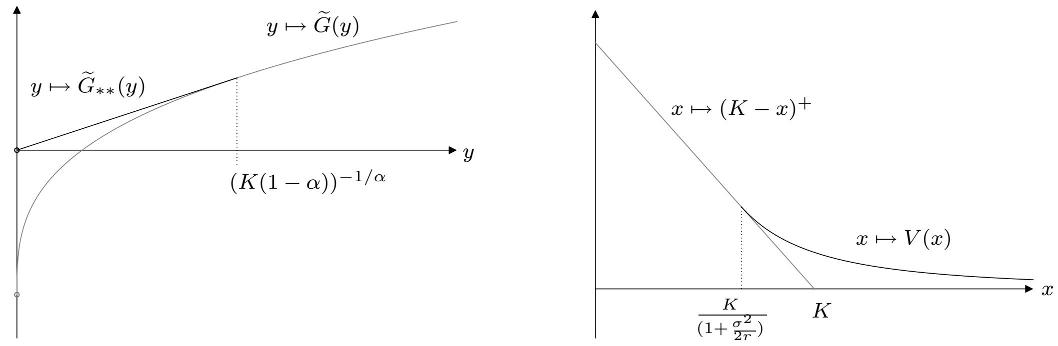

Figure 3.2: In the left figure, the transformed payoff function $\widetilde{G}$ is drawn along with its concave biconjugate. The original payoff $G(x) = (K-x)^+$ and the corresponding value function V is in the figure on the right hand side.

Moreover, the continuation and stopping region for the problem (28) are

$$\begin{aligned} \mathcal{D} &:= \{x \in \mathbb{R}_+ \mid V(x) = G(x)\} = [0, x^*], \\ \mathcal{C} &:= \{x \in \mathbb{R}_+ \mid V(x) > G(x)\} = (x^*, +\infty). \end{aligned}$$

which coincides with the established solution (see for example [20] Section 25.1).

4. An extension of the Legendre transform

The purpose of this section is to extend the Legendre transform so that the observations about optimal stopping in the previous section can be applied to optimal stopping games. The function G represents the payoff of the maximising agent while the function H represents the payoff to the minimising agent. The two continuous gains functions G, H are such that $G(x) \leq H(x)$ for all $x \in I$ and satisfy the assumptions (7) and (8) unless otherwise stated.

Inspired by the transformation used in Theorem 3.2 introduce a pair of rescaled gains functions $W^G : (0,\infty) \to \mathbb{R}$ and $W_H : (0,\infty) \to \mathbb{R}$ defined via

$$W^G(y) := \left(\frac{G}{\psi}\right) \circ F^{-1}(y) \tag{31}$$

and

$$W_H(y) := \left(\frac{H}{\psi}\right) \circ F^{-1}(y). \tag{32}$$

Under the assumptions (7) and (8) we have

$$\lim_{x\downarrow 0}(W_H(x) - W^G(x)) = \lim_{x\uparrow\infty}(W_H(x) - W^G(x)) = 0. \tag{33}$$

This assumption can be relaxed a little as discussed in Remark 4.3 below. These definitions could equally well be formulated with respect to the other ratio of the fundamental solutions but for clarity we shall focus only on these two expressions.

The aim of this section is to define and describe a version of the convex biconjugate of the function W_H which is modified to ensure that it remains inside $\mathrm{epi}(W^G)$. At the same time, we define a version of the concave biconjugate of the function W^G which is modified to ensure that it remains inside $\mathrm{cl}(\mathbb{R}^2 \setminus \mathrm{epi}(W_H))$. This is achieved by defining an extension of the ε-sub/superdifferential typically used in convex analysis.

The main result in this section is Theorem 4.7 which is the purely analytical version of [19] Theorem 4.1. Theorem 4.7 shows that the convex biconjugate respecting the lower barrier W_H and the concave biconjugate respecting the upper barrier W^G coincide. This 'duality' result naturally follows from the spike-variations we use to construct these extensions of the Legendre transform. Moreover, the duality between these two extensions of the Legendre transform are used in Section 5 to provide a new (purely analytical) proof that the optimal stopping game (9)–(10) exhibits both a Stackelberg and a Nash equilibrium.

For a given function $f : \mathbb{R}_+ \to \mathbb{R}$, let

$$\begin{aligned} l_f^x(c,p) &= \sup\{z \le x \mid f(z) - p = c(z-x)\}, \\ r_f^x(c,p) &= \inf\{y \ge x \mid f(y) - p = c(y-x)\}, \end{aligned}$$

with the convention that $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$. For ease of notation, let $l_G^x(c,p) := l_{W^G}^x(c,p)$ and $l_H^x(c,p) := l_{W_H}^x(c,p)$. The point $l_G^x(c, W_H(x))$ (resp. $r_G^x(c, W_H(x))$) is the last (resp. first) time the line passing through $(x, W_H(x))$ with slope c intercepts W^G before (resp. after) x.

Define the superdifferential of W_H in the presence of the lower boundary W^G as

$$\begin{aligned} &\partial^G H(x) \qquad (34)\\ &:= \{c \in \mathbb{R} \mid W_H(y) - W_H(x) \ge c(y-x)\ \forall y \in (l_G^x(c, W_H(x)), r_G^x(c, W_H(x)))\}. \end{aligned}$$

If the tangent $y \mapsto W_H(x) + c(y-x)$ minorises $y \mapsto W_H(y)$ prior to intercepting the lower boundary $y \mapsto W^G(y)$, then we refer to c as a 'supergradient of W_H in the presence of W^G at x', i.e. $c \in \partial^G H(x)$, as shown in Figure 4.1. Similarly, the subdifferential of W^G in the presence of the upper boundary W_H is

$$\begin{aligned} &\partial^H G(x) \qquad (35)\\ &:= \left\{c \in \mathbb{R} \,\middle|\, W^G(y) - W^G(x) \le c(y-x)\ \forall y \in (l_H^x(c, W^G(x)), r_H^x(c, W^G(x)))\right\}. \end{aligned}$$

If the tangent $y \mapsto W^G(x) + c(y-x)$ majorises $y \mapsto W^G(y)$ prior to intercepting the upper boundary $y \mapsto W_H(y)$, then we refer to c as a 'subgradient of G in the presence of H at x', i.e. $c \in \partial^H G(x)$, as illustrated in Figure 4.1.

The δ-superdifferential of W_H in the presence of the lower boundary W^G is defined as

$$\begin{aligned} \partial_\delta^G H(x) := \{c \in \mathbb{R} \mid\ & W_H(y) - W_H(x) + \delta \ge c(y-x) \\ & \forall y \in (l_G^x(c, W_H(x) - \delta), r_G^x(c, W_H(x) - \delta))\}. \qquad (36) \end{aligned}$$

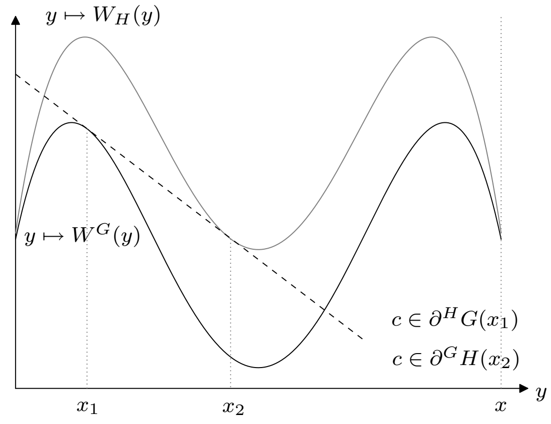

Figure 4.1: In this figure $W^G(y) = W_H(y)$ for all $y \geq x_3$. The slope of the dashed black line, c, is a supergradient of W^G in the presence of W_H at x_1 and a subgradient of W_H in the presence of W^G at x_2.

When c is a δ-supergradient of W_H in the presence of the lower boundary W^G, i.e. $c \in \partial^G_\delta H(x)$, it is possible to draw a line with slope c through the point $(x, W_H(x) - \delta)$ which minorises $y \mapsto W_H(y)$ prior to intercepting the lower boundary $y \mapsto W^G(y)$. For example $0 \in \partial^G_\delta G(x_2)$ in Figure 4.2. Similarly, the ε-subdifferential of W^G in the presence of the upper boundary W_H is defined as

$$\partial^H_\varepsilon G(x) := \big\{ c \in \mathbb{R} \,\big|\, W^G(y) - W^G(x) - \varepsilon \leq c(y - x) \\ \forall y \in (l^x_H(c, W^G(x) + \varepsilon), r^x_H(c, W^G(x) + \varepsilon)) \big\}\,. \tag{37}$$

When c is an ε-subgradient of W^G in the presence of the upper boundary W_H, i.e. $c \in \partial^H_\varepsilon G(x)$, it is possible to draw a line with slope c through the point $(x, W^G(x) + \varepsilon)$ which dominates $y \mapsto W^G(y)$ prior to intercepting the upper boundary $y \mapsto W_H(y)$. For example $0 \in \partial^H_\varepsilon G(x_1)$ in Figure 4.2.

Let

$$\varepsilon^*_c(x) := \inf \big\{ \varepsilon \in [0, W_H(x) - W^G(x)] \,\big|\, c \in \partial^H_\varepsilon G(x) \big\} \tag{38}$$

and

$$\delta^*_c(x) := \inf \big\{ \delta \in [0, W_H(x) - W^G(x)] \,\big|\, c \in \partial^G_\delta H(x) \big\} \tag{39}$$

which are the smallest spike variations that can be made in W^G, resp. W_H at x such that $c \in \partial^H_\varepsilon G(x)$, resp. $c \in \partial^G_\delta H(x)$. The quantities (38) and (39) are illustrated in Figure 4.3.

Remark 4.1. It follows from these definitions that for all $\delta > \delta^*_c(x)$

$$W_H(y) > W_H(x) - \delta + c(y - x) \quad \forall\, y \in [l^x_G(c, W_H(x) - \delta), r^x_G(c, W_H(x) - \delta)].$$

Whereas, for $\delta < \delta^*_c(x)$ there exists $x' \in [l^x_G(c, W_H(x) - \delta), r^x_G(c, W_H(x) - \delta)]$ such that $W_H(x') < W_H(x) - \delta + c(x' - x)$. These two statements imply that either:

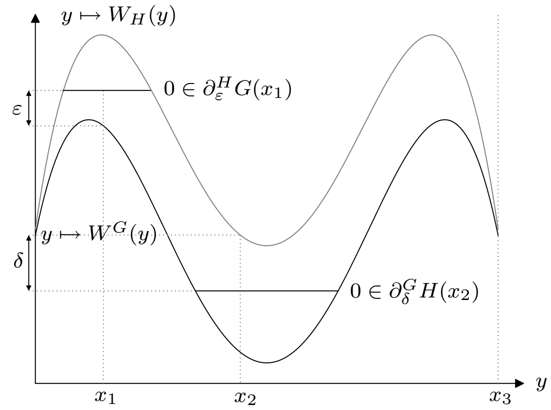

Figure 4.2: $c = 0$ is a ε-subgradient of W^G in the presence of W_H at x_1 and a δ-supergradient of W_H in the presence of W^G at x_2.

(a) There exists $z \in [l_G^x(c, W_H(x) - \delta_c^*(x)), r_G^x(c, W_H(x) - \delta_c^*(x))]$ such that $c \in \partial^G H(z)$ and/or

(b) $\delta_c^*(z) = W_H(z) - W^G(z)$ for $z = l_G^x(c, W_H(x) - \delta_c^*(x))$ and/or $z = r_G^x(c, W_H(x) - \delta_c^*(x))$.

The top left panel of Figure 4.3 shows a point where only condition (a) holds, whereas the bottom left panel illustrates a situation where only condition (b) holds. Similarly, for all $\varepsilon > \varepsilon_c^*(x)$

$$W^G(y) > W^G(x) + \varepsilon + c(y - x) \quad \forall\, y \in [l_H^x(c, W^G(x) + \varepsilon), r_H^x(c, W^G(x) + \varepsilon)].$$

Whereas, for $\varepsilon < \varepsilon_c^*(x)$ there exists $x' \in [l_H^x(c, W^G(x) - \varepsilon), r_H^x(c, W^G(x) - \varepsilon)]$ such that $W^G(x') < W^G(x) + \varepsilon + c(x' - x)$. These two statements imply that either:

(a) There exists $z \in [l_H^x(c, W^G(x) + \varepsilon_c^*(x)), r_H^x(c, W^G(x) + \varepsilon_c^*(x))]$ such that $c \in \partial^H G(z)$ and/or

(b) $\delta_c^*(z) = W_H(z) - W^G(z)$ for $z = l_G^x(c, W_H(x) - \delta_c^*(x))$ and/or $z = r_G^x(c, W_H(x) - \delta_c^*(x))$.

The top right panel of Figure 4.3 shows a point where only condition (a) holds, whereas the bottom right panel illustrates a situation where only condition (b) holds.

Furthermore, let

$$\varepsilon^*(x) := \inf_{c \in \mathbb{R}} \varepsilon_c^*(x), \qquad \delta^*(x) := \inf_{c \in \mathbb{R}} \delta_c^*(x) \tag{40}$$

which are the smallest spike variations that can be made in W^G (resp. W_H) at x such that the set $\partial_\varepsilon^H G(x)$ (resp. $\partial_\delta^G H(x)$) is non-empty. The first result in this section shows that there is 'no gap' between the minimum spike variation in W_H downwards admitting a δ-supergradient and the minimum spike variation in W^G upwards admitting an ε-subgradient.

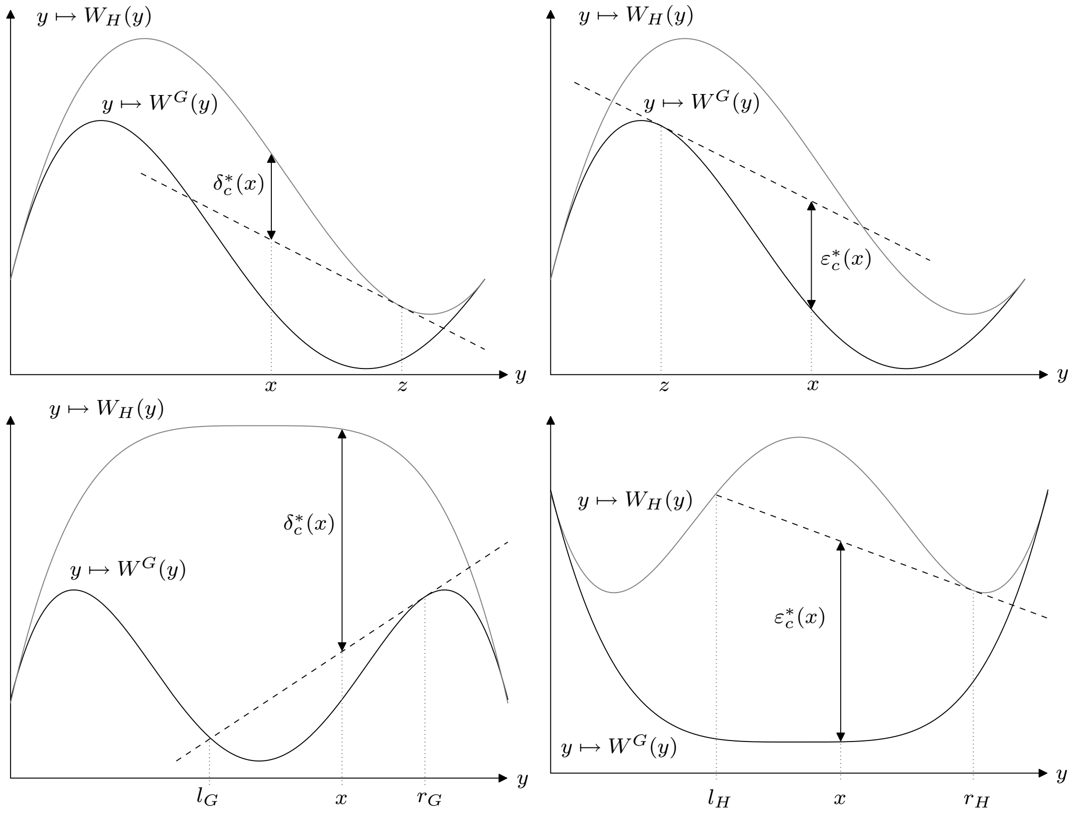

Figure 4.3: The two figures on the left illustrate $\delta_c^*(x)$ and the two figures on the right illustrate $\varepsilon_c^*(x)$. For ease of notation, in the bottom left figure $l_G := l_G^x(c, W_H(x) - \delta_c^*(x))$ and $r_G := r_G^x(c, W_H(x) - \delta_c^*(x))$ and in the bottom right figure $l_H := l_H^x(c, W^G(x) + \varepsilon_c^*(x))$ and $r_H := r_H^x(c, W^G(x) + \varepsilon_c^*(x))$.

Lemma 4.2. *Suppose that* $W^G : (0, \infty) \to \mathbb{R}$ *and* $W_H : (0, \infty) \to \mathbb{R}$ *are as defined in* (31) *and* (32) *and that* (33) *holds, then: for all* $x > 0$ *and* $c \in \mathbb{R}$

$$W^G(x) + \varepsilon_c^*(x) \geq W_H(x) - \delta_c^*(x) \tag{41}$$

and

$$W^G(x) + \varepsilon^*(x) = W_H(x) - \delta^*(x). \tag{42}$$

Proof. Fix $x > 0$, $c \in \mathbb{R}$ and take $\delta > \delta_c^*(x)$. It follows from the definition of $\delta_c^*(x)$ that

$$W_H(y) > W_H(x) - \delta + c(y - x) \quad \forall\, y \in [l_G^x(c, W_H(x) - \delta), r_G^x(c, W_H(x) - \delta)]$$

and

$$[l_G^x(c, W_H(x) - \delta), r_G^x(c, W_H(x) - \delta)] \subset [l_H^x(c, W_H(x) - \delta), r_H^x(c, W_H(x) - \delta)] \tag{43}$$

where the inequality and inclusion are strict to avoid contradicting that $\delta > \delta_c^*(x)$. Due to the properties of the line $y \mapsto W_H(x) - \delta_c^*(x) + c(y - x)$ described in Remark 4.1, it follows from the inclusion (43) and the continuity of G that

$$\exists\, y \in [l_H^x(c, W_H(x)-\delta), r_H^x(c, W_H(x)-\delta)] \quad \text{s.t. } W_H(x)-\delta+c(y-x) < W^G(y) \tag{44}$$

Let $\varepsilon' := W_H(x) - W^G(x) - \delta$ so that (44) is equivalent to

$$\exists\, y \in [l_H^x(c, W_H(x)-\delta), r_H^x(c, W_H(x)-\delta)] \quad \text{s.t. } W^G(x)-\varepsilon'+c(y-x) < W^G(y).$$

It follows from the definition of $\varepsilon_c^*(x)$ that $\varepsilon'(x) < \varepsilon_c^*(x)$ or equivalently

$$W_H(x) - \delta < W^G(x) + \varepsilon_c^*(x) \quad \forall\, \delta > \delta_c^*(x). \tag{45}$$

Take any sequence of real numbers $(\delta_n)_{n\geq 1}$ such that $\delta_n \in (\delta_c^*(x), W_H(x) - W^G(x))$ for all $n \geq 1$ and $\lim_{n\to\infty} \delta_n = \delta_c^*(x)$ then (45) shows that $W_H(x) - \delta_n < W^G(x) + \varepsilon_c^*(x)$ for all $n \geq 1$. Taking the limit as $n \to \infty$ of both sides of this inequality we obtain (41).

Suppose that for some $c \in \mathbb{R}$ that $W^G(x) + \varepsilon_c^*(x) = W_H(x) - \delta_c^*(x)$. When the assumption (33) holds the line $g(y) := W^G(x) + \varepsilon_c^*(x) + c(y-x)$ has both sets of properties described in Remark 4.1 so is the only line passing through (x, p) for some $p \in (W^G(x), W_H(x))$ which both dominates $y \mapsto W^G(y)$ on $[l_H^x(c,p), r_H^x(c,p)]$ and minorises $y \mapsto W_H(y)$ on $[l_G^x(c,p), r_G^x(c,p)]$. When (33) fails, $y \mapsto g(y)$ may not be the only line with these properties and (42) may not hold, this case is examined further in Remark 4.3 below. Consequently, it follows from the continuity of G and H and the assumption (33) that for any $c' \neq c$ the line $y \mapsto h(y) = W^G(x) + \varepsilon_c^*(x) + c'(y-x)$ must either satisfy both

$$\begin{aligned} &l_G^x(c', W_H(x) - \delta_c^*(x)) < l_H^x(c', W_H(x) - \delta_c^*(x)),\\ &r_G^x(c', W_H(x) - \delta_c^*(x)) < r_H^x(c', W_H(x) - \delta_c^*(x)), \end{aligned} \tag{46}$$

or

$$\begin{aligned} &l_G^x(c', W_H(x) - \delta_c^*(x)) > l_H^x(c', W_H(x) - \delta_c^*(x)),\\ &r_G^x(c', W_H(x) - \delta_c^*(x)) > r_H^x(c', W_H(x) - \delta_c^*(x)). \end{aligned} \tag{47}$$

In the right panel of Figure 4.4 the line marked (b) has the properties (46) whereas in the left panel the line marked (b) has the properties (47). When (33) fails one (or more) of the inequalities in both of these statements can hold with equality which invalidates the following argument. Next the line $y \mapsto h(y)$ is shifted upwards so that it passes through $(x, W_H(x) - \delta)$ for some $\delta \in [0, \delta_c^*(x)]$. As is illustrated by the lines marked (c) in Figure 4.4 the properties (47) imply that

$$l_H^x(c', H(x) - \delta) \in [l_H^x(c', H(x) - \delta_{c'}(x)), x]$$

and hence $l_H^x(c', H(x) - \delta) > l_G^x(c', H(x) - \delta)$. Whereas if (46) holds then

$$r_H^x(c', H(x) - \delta) \in [x, r_H^x(c', H(x) - \delta_{c'}(x))].$$

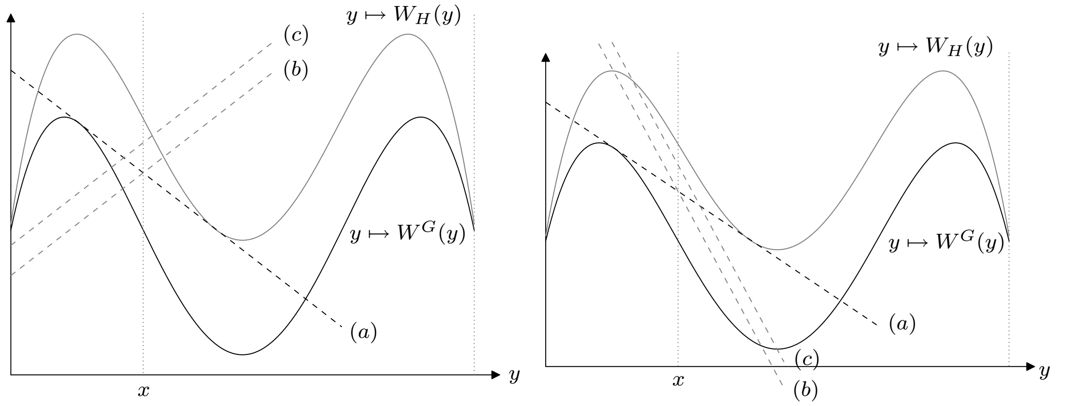

Figure 4.4: These figures illustrate the properties used in the proof of Lemma 4.2.

which implies that $r_H^x(c', H(x)-\delta) < r_G^x(c', H(x)-\delta)$. In both cases $c' \notin \partial_\delta^G H(x)$ for arbitrary $\delta < \delta_c^*(c)$. Moreover, as c' was arbitrary we have shown that $\partial_\delta^G H(x) = \emptyset$ for all $\delta < \delta_c^*(c)$. We may conclude that when for some $c \in \mathbb{R}$ equality holds in (41) that $\delta_c^*(x) = \delta^*(x)$. A symmetric argument can be used to show that when equality holds for some $c \in \mathbb{R}$ equality holds in (41) that $\varepsilon_c^*(x) = \varepsilon^*(x)$ from which we deduce (42) holds. □

Remark 4.3. In the previous lemma it is essential to assume that (33) holds. When this is not the case it may be the case that $W^G(x)+\varepsilon^*(x) < W_H(x)+\delta^*(x)$ as is illustrated in Figure 4.5. Suppose that as in the illustration $W^G(0+) < W_H(0+)$ and there is a $x > 0$ such that for some $c' \in \mathbb{R}$, $c' \in \partial^H G(x)$, $l_G^x(c', W^G(x)) = l_H^x(c', W^G(x)) = 0$ and $W^G(0+) < W^G(x) - c'x < W_H(0+)$. Take $a \in (W^G(x) - c'x, W_H(0+))$ such that $\exists b < c'$ with the properties: (i) $a + bx \in (W^G(x), W_H(x))$, (ii) $r_G^x(b, a + bx) < r_H^x(b, a + bx)$ and (iii) $b \in \partial^H G(r_G^x(b, a+bx))$. We have chosen a, b and x such that $\delta_b^*(x) = W_H(x)-(a+bx)$ hence $\delta^*(x) \leq W_H(x) - (a + bx) < W_H(x) - W^G(x)$. However, by construction $\partial^H G(x) \neq \emptyset$ so $\varepsilon^*(x) = 0$ which implies

$$W^G(x) + \varepsilon^*(x) < W_H(x) - \delta^*(x),$$

so in this case (42) fails.

Remark 4.4. The assumption that (33) holds in Lemma 4.2 can be relaxed in the following way. When $W^G(0+) < W_H(0+)$, take any $w_0 \in [W^G(0+), W_H(0+)]$ and extend W^G (respectively W_H) onto $[0, \infty)$ by allowing W^G to be multivalued at zero taking all values $[W^G(0+), w_0]$ (respectively $[w_0, W_H(0+)]$). If we use this extension of W^G and W_H and the intervals in the definitions of $\partial_\varepsilon^G H(x)$ and $\partial_H^\delta G(x)$ are taken to be closed (as apposed to open) then (42) in Lemma 4.2 holds without the need to assume (33) as the problem discussed in Remark 4.3 can not occur. However, the choice of w_0 not only completely determines $\delta^*(0+)$ and $\varepsilon^*(0+)$ but the values of the functions $x \mapsto \delta^*(x)$ and $x \mapsto \varepsilon^*(x)$ on $(0, z)$ for some $z \geq 0$.

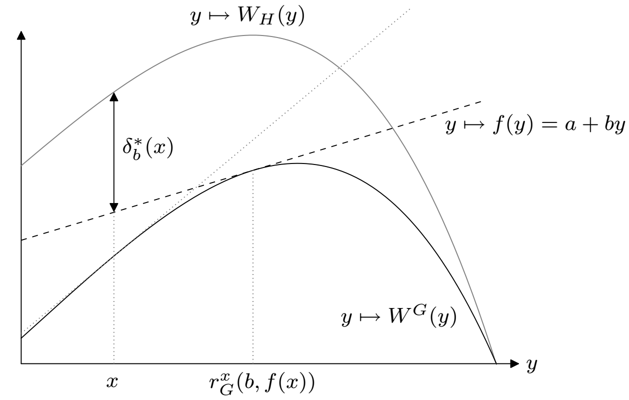

Figure 4.5: When $W_H(0+) > W^G(0+)$ it may be the case that condition (42) in Lemma 4.2 fails.

For a given $x > 0$, define two sets of admissible neighborhoods using

$$\widehat{\mathcal{A}}_G^x(c) := \bigcup_{p \in [W^G(x), W_H(x)]} \{y > 0 \mid l_H^x(c,p) \leq l_G^x(c,p) \leq y \leq r_G^x(c,p) \leq r_H^x(c,p)\}, \tag{48}$$

$$\widehat{\mathcal{A}}_x^H(c) := \bigcup_{p \in [W^G(x), W_H(x)]} \{y > 0 \mid l_G^x(c,p) \leq l_H^x(c,p) \leq y \leq r_H^x(c,p) \leq r_G^x(c,p)\}. \tag{49}$$

These admissible neighborhoods are illustrated in Figure 4.6. The next result describes a pair of functions which coincide with $y \mapsto W^G(y) + \varepsilon^*(y)$ and $y \mapsto W_H(y) - \delta^*(y)$.

Proposition 4.5. *For $x > 0$, define a pair of functions*

$$(W^G)_H^{**}(x) := \sup_{z \in \mathbb{R}} \sup_{y \in \widehat{\mathcal{A}}_G^x(z)} \left(z(x-y) + W^G(y)\right), \tag{50}$$

$$(W_H)_{**}^G(x) := \inf_{z \in \mathbb{R}} \inf_{y \in \widehat{\mathcal{A}}_x^H(z)} \left(z(x-y) + W_H(y)\right), \tag{51}$$

then $(W^G)_H^{**}(x) = W_H(x) - \delta^*(x)$ and $(W_H)_{**}^G(x) = W^G(x) + \varepsilon^*(x)$.

Proof. Fix a $x > 0$ and $c \in \mathbb{R}$, it follows from the definition of $\delta_c^*(x)$ and the continuity of G that the set (48) satisfies

$$\widehat{\mathcal{A}}_G^x(c) = [l_G^x(c, W_H(x) - \delta_c^*(x)), r_G^x(c, W_H(x) - \delta_c^*(x))].$$

By definition $c \in \partial_\delta^G H(x)$ for all $\delta > \delta_c^*(x)$, which is equivalent to

$$W_H(x) - \delta_c^*(x) + c(y - x) \geq W^G(y) \quad \forall\, y \in \widehat{\mathcal{A}}_G^x(c) \tag{52}$$

and equality holds in (52) for $y = l_G^x(c, W_H(x) - \delta_c^*(x))$ and $y = r_G^x(c, W_H(x) - \delta_c^*(x))$. Hence,

$$\sup_{y \in \widehat{\mathcal{A}}_G^x(c)} (W^G(y) - cy) = W^G(r_G^x(c, W_H(x) - \delta_c^*(x))) - cr_G^x(c, W_H(x) - \delta_c^*(x))$$

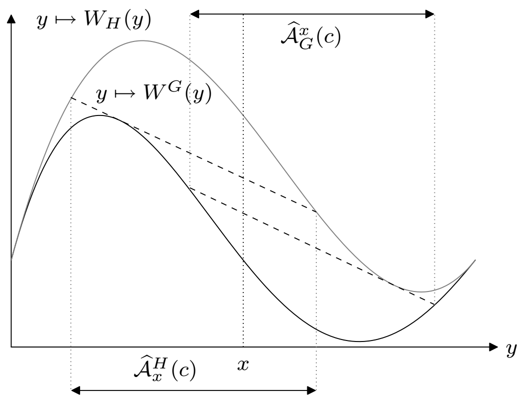

Figure 4.6: The sets $\widehat{\mathcal{A}}^x_G(c)$ and $\widehat{\mathcal{A}}^H_x(c)$ are defined in (48) and (49) are marked when c is the slope of the pair of parallel lines.

and

$$\begin{aligned}(W^G)^{**}_H(x) &:= \sup_{c\in\mathbb{R}} \sup_{y\in\widehat{\mathcal{A}}^x_G(c)} \big(c(x-y)+W^G(y)\big)\\ &= \sup_{c\in\mathbb{R}} \big(c(x-r^x_G(c,W_H(x)-\delta^*_c(x)))+W^G(r^x_G(c,W_H(x)-\delta^*_c(x)))\big)\\ &= \sup_{c\in\mathbb{R}} (W_H(x)-\delta^*_c(x)) = W_H(x)-\delta^*(x).\end{aligned}$$

Thus we have shown that $(W^G)^{**}_H(x) = W_H(x)-\delta^*(x)$ for all $x>0$. A symmetric argument can be used to show that $(W_H)^G_{**}(x) = W^G(x)+\varepsilon^*(x)$. □

The previous result holds without the need to assume that (33) holds. At first glance, the notation used in the previous result may appear to be the wrong way around but when (33) holds Lemma 4.2 implies that $(W^G)^{**}_H(x) = W^G(x)+\varepsilon^*(x)$ and $(W_H)^G_{**}(x) = W_H(x)-\delta^*(x)$.

Remark 4.6. For a given $x>0$ consider the sets

$$\begin{aligned}A_1(x) &:= \left\{p\in[W^G(x),W_H(x)] \,\middle|\, \exists\, c\in\mathbb{R} \text{ s.t. } [l^x_G(c,p),r^x_G(c,p)]\subseteq\widehat{\mathcal{A}}^x_G(c)\right\},\\ A_2(x) &:= \left\{p\in[W^G(x),W_H(x)] \,\middle|\, \exists\, c\in\mathbb{R} \text{ s.t. } [l^x_H(c,p),r^x_H(c,p)]\subseteq\widehat{\mathcal{A}}^H_x(c)\right\}.\end{aligned}$$

The 'dual interpretation' provided in [19] illustrates that $(W^G)^{**}_H(x)$ can be constructed by maximising over functions of the form $y\mapsto p+c(y-x)$ which hit W^G before W_H, i.e. $(W^G)^{**}_H(x) = \sup A_1(x)$ as shown in Proposition 4.5 or by minimising over the functions of the form $y\mapsto p+c(y-x)$ which hit W_H before W^G i.e. $(W^G)^{**}_H(x) = \inf A_2(x)$.

In view of the previous remark, Proposition 4.5 can be viewed as a preliminary version of the next result which shows that the functions (50) and (51) can be

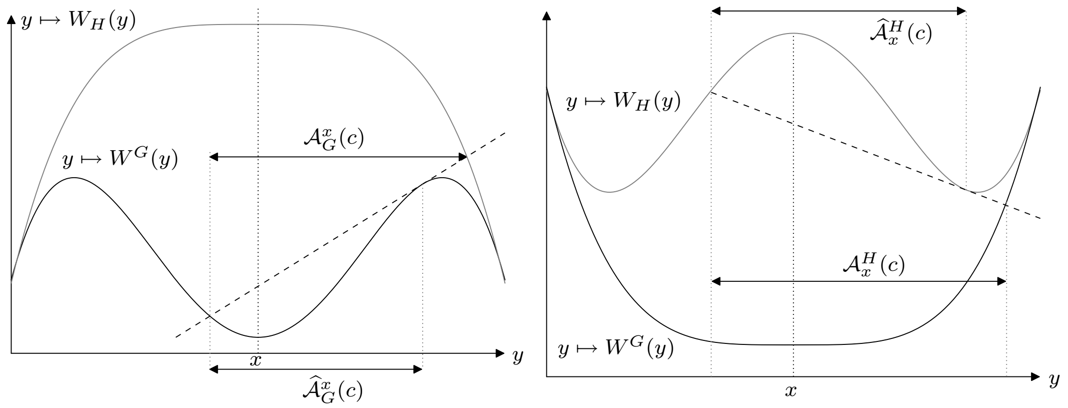

Figure 4.7: The left figure shows a situation when $\widehat{\mathcal{A}}_G^x(c) \subset \mathcal{A}_G^x(c)$ whereas the figure on the right shows a situation when $\widehat{\mathcal{A}}_x^H(c) \subset \mathcal{A}_x^H(c)$.

viewed as an extension of the concave biconjugate of $y \mapsto W^G(y)$ and the convex biconjugate of $y \mapsto W_H(y)$. To this end, let

$$\begin{aligned} L^x(c,p) &= \sup\{z \le x \,|\, p + c(z-x) \notin \mathrm{cl}(\mathrm{epi}(W^G) \setminus \mathrm{epi}(W_H))\}, \\ R^x(c,p) &= \inf\{y \ge x \,|\, p + c(y-x) \notin \mathrm{cl}(\mathrm{epi}(W^G) \setminus \mathrm{epi}(W_H))\}. \end{aligned}$$

For a given $x > 0$, let $p(x) = W_H(x) - W^G(x)$ and define two sets of admissible neighborhoods which are larger than (48) and (49) using

$$\begin{aligned} &\mathcal{A}_G^x(c) \\ :=& \bigcup_{\delta \in [0,p(x)]} \bigcup_{c \in \partial_\delta^G H(x)} \{y \in (0,\infty) \,|\, L^x(c, W_H(x) - \delta) \le y \le R^x(c, W_H(x) - \delta)\}, \end{aligned} \tag{53}$$

$$\begin{aligned} &\mathcal{A}_x^H(c) \\ :=& \bigcup_{\varepsilon \in [0,p(x)]} \bigcup_{c \in \partial_\varepsilon^H G(x)} \{y \in (0,\infty) \,|\, L^x(c, W^G(x) + \varepsilon) \le y \le R^x(c, W^G(x) + \varepsilon)\}. \end{aligned} \tag{54}$$

These admissible neighborhoods are illustrated in Figure 4.7. These neighborhoods coincide with those used in [19] and are used in the next result which is an analytical complement to [19] Theorem 4.1.

Theorem 4.7. *Suppose that* (33) *holds, then the functions* $x \mapsto (W^G)_H^{**}(x)$ *and* $x \mapsto (W_H)_{**}^G(x)$ *defined in* (50) *and* (51) *satisfy*

$$(W^G)_H^{**}(x) = \inf_{z \in \mathbb{R}} \sup_{y \in \mathcal{A}_G^x(z)} \left(z(x-y) + W^G(y)\right), \tag{55}$$

$$(W_H)_{**}^G(x) = \sup_{z \in \mathbb{R}} \inf_{y \in \mathcal{A}_x^H(z)} \left(z(x-y) + W_H(y)\right). \tag{56}$$

for all $x > 0$. *Moreover,* $(W^G)_H^{**}(x) = (W_H)_{**}^G(x)$ *for all* $x > 0$ *and*

$$\begin{aligned} \left\{x \in (0,\infty) \,\middle|\, (W^G)_H^{**}(x) = W^G(x)\right\} &= \left\{x \in (0,\infty) \,\middle|\, \partial^H G(x) \ne \emptyset\right\}, \\ \left\{x \in (0,\infty) \,\middle|\, (W^G)_H^{**}(x) = W_H(x)\right\} &= \left\{x \in (0,\infty) \,\middle|\, \partial^G H(x) \ne \emptyset\right\}. \end{aligned}$$

Proof. Fix $x > 0$ and $c \in \mathbb{R}$ and define

$$(W^G)^*_H(c) := \inf_{y \in \mathcal{A}^H_x(c)} (cy - W^G(y)).$$

By definition, the line $f(y) := W^G(x) + \varepsilon^*_c(x) + c(y - x)$ dominates $y \mapsto W^G(y)$ on $\widehat{\mathcal{A}}^H_x(c) = [l^x_H(c, W^G(x)), r^x_H(c, W^G(x))] \subseteq \mathcal{A}^H_x(c)$. If there exists $z \in \mathcal{A}^H_x(c) \setminus \widehat{\mathcal{A}}^H_x(c)$ such that $f(z) = W^G(z)$ then either $c \in \partial^H G(z)$ or z is on the boundary of $\mathcal{A}^H_x(c)$, i.e. $z = L^x_H(c, W^G(x))$ or $z = R^x_H(c, W^G(x))$. Thus we may conclude that $(W^G)^*_H(c) = cx' - W^G(x')$ for some x' such that

$$c(y - x') + W^G(x') = W^G(x) + \varepsilon^*_c(x).$$

Hence

$$\inf_{z \in \mathbb{R}} \sup_{y \in \mathcal{A}^x_G(z)} \big(z(x - y) + W^G(y)\big) = \inf_{z \in \mathbb{R}} (W^G(x) - \varepsilon^*_z(x))$$

so it follows from the definition of $\varepsilon^*(x)$ and Proposition 4.5 that

$$\inf_{z \in \mathbb{R}} \sup_{y \in \mathcal{A}^x_G(z)} \big(z(x - y) + W^G(y)\big) = W^G(x) + \varepsilon^*(x) = (W^G)^{**}_H(x).$$

A symmetric argument can be used to show that

$$\inf_{z \in \mathbb{R}} \sup_{y \in \mathcal{A}^x_G(z)} \big(z(x - y) + W^G(y)\big) = W_H(x) - \delta^*(x) = (W_H)^G_{**}(x).$$

It follows from Lemma 4.2 that $(W^G)^{**}_H(x) = (W_H)^G_{**}(x)$ and the final statement follows from the definition of $\varepsilon^*(x)$ and $\delta^*(x)$. $\square$

Theorem 4.7 can be reformulated in terms of the F-concavity (resp. F-convexity) of the functions G and H determining W^G and W_H. To see this observe that it follows from the definitions of $y \mapsto W^G(y)$ and $y \mapsto W_H(y)$ in (31) and (32) that

$$\begin{aligned}
(G)^{**}_H(x) &:= (W^G)^{**}_H(F(x))\psi(x) \\
&= \inf_{z \in \mathbb{R}} \sup_{F(y) \in \mathcal{A}^x_G(z)} \left(z(F(x) - F(y)) + \left(\frac{G}{\psi}\right)(y) \right) \psi(x), \qquad (57)\\
(H)^G_{**}(x) &:= (W_H)^G_{**}(F(x))\psi(x) \\
&= \sup_{z \in \mathbb{R}} \inf_{F(y) \in \mathcal{A}^H_x(z)} \left(z(F(x) - F(y)) + \left(\frac{H}{\psi}\right)(y) \right) \psi(x), \qquad (58)
\end{aligned}$$

which are the modified F-concave/F-convex biconjugates used in [19]. It is shown in the next section that (57)–(58) are r-semiharmonic functions which solve the dual problems (12).

We began constructing an extension of the convex/concave biconjugates of W^G and W_H due to the fact that G is r-superharmonic with respect to the diffusion

X if and only if G/ψ is F-concave or equivalently W^G is concave. For studying optimal stopping games, we shall also use that G is r-superharmonic with respect to the diffusion X if and only if G/φ is $\widetilde{F}$-concave or equivalently $(G/\varphi)\circ(-\widetilde{F})^{-1}$ is concave on $(0,\infty)$. With this in mind, Theorem (4.7) can be formulated with respect to the other ratio of the fundamental solutions.

Corollary 4.8. *Let $\widetilde{W^G} := (G/\varphi)\circ(-\widetilde{F})^{-1}$ and $\widetilde{W}_H := (H/\varphi)\circ(-\widetilde{F})^{-1}$. Assuming that (7)–(8) hold. Then for all $x > 0$*

$$\begin{aligned}(\widetilde{W^G})^{**}_H(x) :=& \inf_{z\in\mathbb{R}} \sup_{y\in\widetilde{\mathcal{A}}^x_G(z)} \Big(z(x-y)+\widetilde{W^G}(y)\Big)\\ =& \sup_{z\in\mathbb{R}} \inf_{y\in\widetilde{\mathcal{A}}^H_x(z)} \Big(z(x-y)+\widetilde{W}_H(y)\Big) =: (\widetilde{W}_H)^G_{**}(x).\end{aligned}$$

where the sets $\widetilde{\mathcal{A}}^x_G(z)$, $\widetilde{\mathcal{A}}^H_x(z)$ are defined as in (53) and (54) replacing W^G, W_H with $\widetilde{W^G}$, $\widetilde{W}_G$.

Proof. In this section we have not used any properties of the diffusion X, nor the specific form of the functions W^G, W_H. Consequently, the previous Theorem holds for any two continuous functions $g,h:(0,\infty)\to\mathbb{R}$ such that $g\le h$ and $\lim_{x\downarrow 0} g(x)-h(x) = \lim_{x\uparrow\infty} g(x)-h(x) = 0$. In, particular, replacing W^G, W_H with $\widetilde{W^G}$, $\widetilde{W}_G$ does not alter the result. □

It was observed in Remark 4.4 that the assumption (8) may be relaxed by extending the functions W^G, W_H onto $\mathbb{R}_+$ in a manner which ensures that $W^G(x)+\varepsilon^*(x) = W_H(x)-\delta^*(x)$ for all $x>0$. This approach can be used to relax the assumptions of Theorem 4.7.

Corollary 4.9. *Take $w_0 \in [W^G(0+), W_H(0+)]$ and extend the functions W^G, W_H onto $\mathbb{R}_+$ by setting $W^G(0) = [W^G(0+), w_0]$ and $W_H(0) = [w_0, W_H(0+)]$. Suppose that the intervals in the definitions (36)–(37) are take to be closed (as apposed to open) then*

$$\begin{aligned}(W^G)^{**}_H(x) =& \inf_{z\in\mathbb{R}} \sup_{y\in\mathcal{A}^x_G(z)} \big(z(x-y)+W^G(y)\big)\\ =& \sup_{z\in\mathbb{R}} \inf_{y\in\mathcal{A}^H_x(z)} \big(z(x-y)+W_H(y)\big) = (W_H)^G_{**}(x)\end{aligned}$$

for all $x\ge 0$.

Proof. Under these assumptions, Remark 4.4 shows that Lemma 4.2 holds. Furthermore, under these assumptions the functions $(W^G)^{**}_H$ and $(W_H)^G_{**}$ defined in (51)–(50) are defined on $\mathbb{R}_+$ and $(W^G)^{**}_H(0) = (W_H)^G_{**}(0) = w_0$. Thus with this extended definition, the corollary follows from Theorem 4.7. □

Remark 4.10. The function $y\mapsto (W^G)^{**}_H(y)$ can be viewed as the 'concave biconjugate of W^G in the presence of $y\mapsto W_H(y)$' (or constrained to remain within $\mathrm{cl}(\mathbb{R}^2\setminus \mathrm{epi}(W_H))$) for the following three reasons:

1. Theorem 4.7 illustrates that the function $y \mapsto (W^G)^{**}_H(y)$ can be expressed as
$$(W^G)^{**}_H(x) = \inf_{z\in\mathbb{R}}\left(zx - \inf_{y\in\mathbb{R}}\left(zy - W^G(y) + \delta(y\,|\,\mathbb{R}_+\setminus\mathcal{A}^x_G(z))\right)\right),$$
where $\delta(y\,|\,\mathbb{R}_+\setminus\mathcal{A}^x_G(z))$ is the 'characteristic function' of the set $\mathbb{R}_+\setminus\mathcal{A}^x_G(z)$, i.e. $\delta(y\,|\,\mathbb{R}_+\setminus\mathcal{A}^x_G(z)) = 0$ if $y\in\mathcal{A}^x_G(z)$ and $\delta(y\,|\,\mathbb{R}_+\setminus\mathcal{A}^x_G(z)) = \infty$ otherwise. The difference with the standard concave biconjugate is that this characteristic function depends on the choice of z and x. However, this dependence seems essential to ensure that $\operatorname{epi}(W_H)\subseteq\operatorname{epi}((W^G)^{**}_H)$.
2. For any measurable function $f:(0,\infty)\to\mathbb{R}$, the superdifferential of f is as defined in (14) and the ε-superdifferential is defined as
$$\partial_\varepsilon f(x) = \{c\in\mathbb{R}\,|\,f(y)-f(x)-\varepsilon\geq c(y-x)\ \forall y\in(0,\infty)\}.$$
Hence $f_{**}(x)-f(x) = \inf\{\varepsilon\geq 0\,|\,\partial_\varepsilon f(x)\neq\emptyset\}$. This definition of a ε-superdifferential coincides with the definition (37) when W_H is taken to be the multivalued function
$$W_H(y) = \begin{cases}\left[W^G(0),+\infty\right] & y=0,\\ \infty & y\in(0,+\infty),\\ \left[W^G(\infty),+\infty\right] & y=+\infty.\end{cases}$$
Moreover, when (33) holds, it follows from Theorem 4.7 that $(W^G)^{**}_H(x) - W^G(x) = \varepsilon^*(x)$ (where $\varepsilon^*(x)$ is as defined in (40)) so $y\mapsto(W^G)^{**}_H(y)$ and $y\mapsto f_{**}(y)$ can both be characterised in terms of a smallest spike variation.
3. In Corollary 3.4 it was shown that the concave biconjugate of $y\mapsto(G/\psi)\circ F^{-1}(y)$ is such that $((G/\psi)\circ F^{-1})_{**}(x) = \sup A_1(x)$ where the set $A_1(x)$ is defined in (25). This characterisation of the concave biconjugate uses the same method of construction as is used in Proposition 4.5 to define $y\mapsto(W^G)^{**}_H(y)$.

The function $y\mapsto(W_H)^G_{**}(y)$ will be referred to as the 'convex biconjugate of W_G in the presence of $y\mapsto W^H(y)$ (or constrained to remain within $\operatorname{epi}(W^G)$) for directly similar reasons.

5. Nash equilibrium in optimal stopping games

In this section Theorem 4.7 is applied to show that the infinite time horizon optimal stopping game (9)–(10) exhibits both a Stackelberg and a Nash equilibrium. The first theorem of this section shows that under the assumptions (7) and (8) upper and lower values of the optimal stopping game (9) and (10) satisfy $\underline{V}\geq(G)^{**}_H$ and $\overline{V}\leq(H)^G_{**}$ respectively. Consequently, it follows from Theorem 4.7 that the game has a value, namely $\overline{V}=\underline{V}=(G)^{**}_H$ hence the game has a Stackelberg equilibrium. In fact the game has a value without the need to assume the boundary condition (8) holds, but we need to account for the fact that when both minimising and maximising agent select the same stopping time $R_x(\tau,\tau) = E_x[e^{-r\tau}G(X_\tau)]$.

Theorem 5.1. *Suppose that* (7) *holds and consider the optimal stopping game* (9)–(10)*. The optimal stopping game has a Stackelberg equilibrium , i.e. the games has a value* V *which can be represented as*

$$V(x) = (G)_H^{**}(x) = (W^G)_H^{**}(F(x))\psi(x)$$

for all $x \in I$*. Moreover, when* (8) *holds, it follows that* $V = (G)_H^{**} = (H)_{**}^G$ *without the need to extend* W^G *and* W_H *onto* $\mathbb{R}_+$*.*

Proof. To accomodate the asymmetry in the objective function discussed above, extend the functions W^G, W_H onto $\mathbb{R}_+$ by setting $W^G(0) := W^G(0+)$ and $W_H(0) = [W^G(0+), W_H(0+)]$. As noted in Corollary 4.9, these extended definitions of W^G, W_H ensure that $(W_H)_{**}^G(0) = (W^G)_H^{**}(0) = W^G(0+) = G(a+)/\varphi(a+)$. Let $F(a) := F(a+) = 0$ and $\varphi(a) := \varphi(a+) = +\infty$ then

$$(H)_{**}^G(x) := (W_H)_{**}^G(F(x))\varphi(x), \qquad (G)_H^{**}(x) := (W^G)_H^{**}(F(x))\varphi(x),$$

for all $x \in (a,b)$ and $(H)_{**}^G(a) = (G)_H^{**}(a) = G(a+)$. It follows from Corollary 4.9 (or Theorem 4.7 if (8) holds) that it is sufficient to show that

$$(H)_{**}^G \leq \underline{V} \leq \overline{V} \leq (G)_H^{**} \tag{59}$$

for all $x \in I$ where $\underline{V}$ and $\overline{V}$ were defined in (9) and (10) respectively. The second inequality follows from the definitions of $\underline{V}$ and $\overline{V}$.

For the first inequality in (59) let $\widehat{\mathcal{A}}_x = \bigcup_{z\in\mathbb{Z}} \widehat{\mathcal{A}}_G^x(z)$ and $\widehat{\tau} = \inf\{t \geq 0 \,|\, X_t \notin \widehat{\mathcal{A}}_x\}$. Let $a' = \inf \widehat{\mathcal{A}}_x$ and $b' = \sup \widehat{\mathcal{A}}_x$ and suppose that $b' < b$. It follows that

$$\underline{V}(x) \geq \inf_{\sigma \leq \widehat{\tau}} E_x\left[e^{-r\sigma}\widehat{H}(X_\sigma)\right] =: V^-(x).$$

where $\widehat{H}(y) := H(y)\,\mathbb{I}_{(a',b')} + G(y)\,\mathbb{I}_{[y=a']\cup[y=b']}$. The stopping time which attains this infimum is of the form $\widehat{\sigma} = T_{a^*} \wedge T_{b^*}$ for some $a' \leq a^* \leq x \leq b^* \leq b'$. As we have assumed that $\sup \widehat{\mathcal{A}}_x < b$ the process $\left(e^{-r(t\wedge\widehat{\sigma})}\varphi(X_{t\wedge\widehat{\sigma}})\right)_{t\geq 0}$ is a $(P_x, \mathcal{F}_{t\wedge\widehat{\sigma}})$-martingale. The optional sampling theorem implies that $E_y\left[e^{-r\sigma}\varphi(X_\sigma)\right] = \varphi(y)$ for all $y \in \widehat{\mathcal{A}}_x$ and all $\sigma \leq \widehat{\tau}$. Hence for arbitrary $c \in \mathbb{R}$,

$$V^-(x) = \inf_{\sigma\leq\widehat{\tau}} E_x\left[e^{-r\sigma}\widehat{H}(X_\sigma)\right] = c\varphi(x) + \inf_{\sigma\leq\widehat{\tau}} E_x\left[e^{-r\sigma}(\widehat{H}(X_\sigma) - c\varphi(X_\sigma))\right]. \tag{60}$$

Since (60) holds for all $c \in \mathbb{R}$, it follows that

$$V^-(x) = \sup_{c\in\mathbb{R}}\left(c\varphi(x) + \inf_{\sigma\leq\widehat{\tau}} E_x\left[e^{-r\sigma}(\widehat{H}(X_\sigma) - c\varphi(X_\sigma))\right]\right). \tag{61}$$

The optimal stopping problem on the right hand side of (61) can be expressed as

$$\begin{aligned}R_c(x) &:= \inf_{\sigma\leq\widehat{\tau}} E_x\left[e^{-r\sigma}(\widehat{H}(X_\sigma) - c\varphi(X_\sigma))\right] \\ &= \inf_{a'\leq y\leq x\leq z\leq b'}\left(\frac{\widehat{H}(y)-c\varphi(y)}{\psi(y)}\frac{F(z)-F(x)}{F(z)-F(y)}\right. \\ &\qquad\left.+\frac{\widehat{H}(z)-c\varphi(z)}{\psi(z)}\frac{F(x)-F(y)}{F(z)-F(y)}\right)\psi(x) \\ &\geq \inf_{y\in\widehat{\mathcal{A}}_x}\left(\frac{\widehat{H}(y)-c\varphi(y)}{\psi(y)}\right)\psi(x).\end{aligned}$$

Hence, from (61) we obtain

$$V^-(x) \geq \sup_{c\in\mathbb{R}}\left(c\frac{\varphi(x)}{\psi(x)} + \inf_{y\in\widehat{\mathcal{A}}_x}\left(\frac{\widehat{H}(y)-c\varphi(y)}{\psi(y)}\right)\right)\psi(x).$$

Let

$$\underline{c}(x) = \inf\{c\in\ \mathbb{R} \mid \inf\widehat{\mathcal{A}}_x = \inf\widehat{\mathcal{A}}_G^x(c)\},$$
$$\overline{c}(x) = \sup\{c\in\ \mathbb{R} \mid \sup\widehat{\mathcal{A}}_x = \sup\widehat{\mathcal{A}}_G^x(c)\}.$$

Recall that $\delta_c^*(x)$ and $\delta^*(x)$ were defined in (39) and (40). Due to the definition of $\widehat{\mathcal{A}}_x$ it follows that for all $c\in\mathbb{R}$

$$cF(x) + \inf_{y\in\widehat{\mathcal{A}}_x}\left(\frac{\widehat{H}(y)-c\varphi(y)}{\psi(y)}\right) \leq \frac{H(x)}{\psi(x)} - \delta_c^*(F(x)).$$

and equality holds for all $c\in[\underline{c}(x),\overline{c}(x)]$. In particular, it follows from the definition of $\widehat{\mathcal{A}}_x$ that $\delta_c^*(F(x)) = \delta^*(F(x))$ for some $c\in[\underline{c}(x),\overline{c}(x)]$ so we may conclude that

$$V^-(x) \geq H(x) - \delta^*(F(x))\psi(x) = (W_H)_{**}^G(F(x))\psi(x) = (H)_{**}^G(x)$$

which is the first inequality in (59). The case that $b' = b$ can be handled by using an approximating sequence of domains as in Theorem 3.2.

For the final inequality in (59), let $\mathcal{A}_x = \bigcup_{z\in\mathbb{Z}}\widehat{\mathcal{A}}_x^H(z)$ and let $\widehat{\sigma} = \inf\{t\geq 0 \mid X_t\notin\mathcal{A}_x\}$. Let $a' = \inf\mathcal{A}_x$ and $b' = \sup\mathcal{A}_x$ and suppose that $b' < b$. There is a natural asymmetry in the value function defined in (11) as $R_x(\tau,\tau) = E_x[e^{-r\tau}G(X_\tau)]$. Thus

$$\begin{aligned}\overline{V}(x) &= \inf_\sigma\sup_\tau E_x\left[e^{-r(\tau\wedge\sigma)}\left(G(X_\tau)\mathbb{I}_{[\tau\leq\sigma]} + H(X_\sigma)\mathbb{I}_{[\tau>\sigma]}\right)\right] \\ &\leq \sup_{\tau\leq\widehat{\sigma}} E_x\left[e^{-r\tau}G(X_\tau)\right] =: V^+(x).\end{aligned}$$

A similar argument to that used above can be used to show that

$$V^{+}(x) \leq G(x) - \varepsilon^{*}(F(x))\psi(x) = (W^{G})_{H}^{**}(F(x))\psi(x) = (G)_{**}^{H}(x).$$

which is the final inequality in (59). The case that $b' = b$ can be handled by using an approximating sequence of domains as in Theorem 3.2.

The extension of W^G and W_H onto $\mathbb{R}_+$ mentioned at the beginning of proof is only necessary to exclude the case that $W^G(0+) = (W^G)_H^{**}(0+) < (W_H)_{**}^G(0+) = W_H(0+)$. When $(W^G)_H^{**}(0+) < (W_H)_{**}^G(0+)$ it follows from the definitions (57)–(58) that $(G)_H^{**}(a+) < (H)_{**}^G(a+)$. However, when (8) holds $W^G(0+) = W_H(0+)$ so this case can not occur. $\square$

Remark 5.2. In the previous Theorem the assumption (7) has only been used to rule out the degenerate case that $\limsup_{y\downarrow a}(G/\psi)(y) = +\infty$ (resp. $\limsup_{y\uparrow b} (G/\psi)(y) = +\infty$) as in this case the optimal stopping time for the maximising agent is not finite and $\overline{V}(x) = \underline{V}(x) = +\infty$ for all $x \in I$. In this case the functions $(G)_H^{**}$ and $(H)_{**}^G$ are no longer related to the value function of the optimal stopping game.

In Theorem 5.1 the functions H and G are defined on the entire domain of the one dimensional diffusion X. Theorem 5.1 can handle the case that X is absorbed upon exit from a set $[c,d] \in I$ by setting $G(x) = H(x)$ for all $x \in I \setminus [c,d]$. The value function in Theorem 5.1 can be thought of as an elastic cord which is tied to $\limsup_{y\downarrow 0} W^G(y)$ and pulled towards infinity between the two obstacles W^G and W_H. On a compact domain it is the shortest path between these two obstacles as shown in [19].

The next two lemmata examine some properties of the value of the game identified in Theorem 5.1 which will be used to prove that the game with upper- and lower-value (9)–(10) has a Nash equilibrium. The first lemma shows that the value of the game is r-semiharmonic, that is $(G)_H^{**} \in \mathrm{Sup}\,[G,H) \cap \mathrm{Sub}\,(G,H]$ where

$$\begin{aligned} &\mathrm{Sup}\,[G,H) \\ &= \{U : I \to [G,H] \,|\, U \text{ is continuous and } r\text{-superharmonic on } \{U < H\}\}, \\ &\mathrm{Sub}\,(G,H] \\ &= \{U : I \to [G,H] \,|\, U \text{ is continuous and } r\text{-subharmonic on } \{U > G\}\}, \end{aligned}$$

are the admissible sets in the dual problems defined in (12). The second lemma shows that the r-harmonic functions $(G)_H^{**}$ and $(H)_{**}^G$ solve the dual problems (12).

Lemma 5.3. *The concave biconjugate of W^G in the presence of the upper barrier W_H defined in* (55) *is such that the associated function $(G)_H^{**}$ defined in* (57) *is in the admissible sets for the dual problems, i.e.*

$$(G)_H^{**} \in \mathrm{Sup}\,[G,H) \cap \mathrm{Sub}\,(G,H]\,. \tag{62}$$

Moreover, the convex biconjugate of W_H in the presence of the lower barrier W_G defined in (56) is such that the associated function $(H)_{**}^G$ defined in (58) is in the admissible sets for the dual problems, i.e.

$$(H)_{**}^G \in \mathrm{Sup}\,[G, H) \cap \mathrm{Sub}\,(G, H]\,.$$

Proof. First we show that $(G)_H^{**} \in \mathrm{Sup}\,[G, H)$. Take $x \in I$ such that $(G)_H^{**}(x) < H(x)$ and let

$$\begin{aligned} z_-(x) &:= \sup\{\, y \le x \mid (G)_H^{**}(y) = H(y)\}\,; \\ z_+(x) &:= \inf\{\, y \ge x \mid (G)_H^{**}(y) = H(y)\}\,. \end{aligned} \tag{63}$$

Let $\widetilde{G}(y) := G(y)\mathbb{I}_{(z_-(x),z_+(x))} + H(x)\mathbb{I}_{[y=z_-(x)]\cup[y=z_+(x)]}$ and define a function $\widetilde{W}$ using

$$\widetilde{W}(y, x) := \left(\frac{\widetilde{G}}{\psi}\right) \circ F^{-1}(y).$$

For $z \in [F(z_-(x)), F(z_+(x))] =: J^x$ and $c \in \mathbb{R}$ let

$$\widehat{\varepsilon}_c(z) := \inf\left\{\varepsilon \ge 0 \;\middle|\; \widetilde{W}(y, x) \le W^G(z) + \varepsilon + c(y - z)\ \forall y \in J^x\right\}.$$

For fixed $z \in J^x$, $c \mapsto \widehat{\varepsilon}_c(z)$ this continuous. Define $\widehat{\varepsilon}(z) := \inf_{c\in\mathbb{R}} \widehat{\varepsilon}_c(z)$ which is attained at some $\widehat{c} \in \mathbb{R}$. We claim that $\widehat{\varepsilon}(z) = \varepsilon^*(z)$ for all $z \in J^x$ where $\varepsilon^*(z)$ is defined in (38). To prove this first suppose that

$$\widehat{\varepsilon}(z) = \widehat{\varepsilon}_{\widehat{c}}(z) < \inf_{c\in\mathbb{R}} \varepsilon_c^*(z) = \varepsilon^*(z). \tag{64}$$

By definition

$$h(y, z) := W^G(z) + \widehat{\varepsilon}(z) + \widehat{c}(y - z) \ge \widetilde{W}(y, x) \quad \forall y \in J^x$$

and in particular $h(F(z_-(x)), F(x))\psi(z_-(x)) \ge H(z_-(x))$ and $h(F(z_+(x)), F(x))\psi(z_+(x)) \ge H(z_+(x))$. Thus the continuity of $y \mapsto (H/\psi)(y)$ implies that for all $z \in J^x$

$$[l_H^z(\widehat{c}, W^G(z) + \widehat{\varepsilon}(z)), r_H^z(\widehat{c}, W^G(x) + \widehat{\varepsilon}(z))] \subseteq [z_-(x), z_+(x)] = J^x$$

hence $\widehat{c} \in \partial^H G_{\widehat{\varepsilon}(z)}(z)$ which contradicts (64) so $\widehat{\varepsilon}(z) \ge \varepsilon^*(z)$ for all $z \in J^x$.

Suppose that $\widehat{\varepsilon}(z) > \varepsilon^*(z)$ and take $c' \in \partial^H_{\varepsilon^*(z)} G(z)$ so that by assumption

$$h(y) := W^G(z) + \varepsilon^*(z) + c'(y - z) < W^G(y)$$

for some $y \in J^x \setminus [l_H^z(\widehat{c}, W^G(z) + \varepsilon^*(z)), r_H^z(\widehat{c}, W^G(x) + \varepsilon^*(z))]$. Let $\hat{c} = \inf\{c \in \mathbb{R} \mid c \in \partial^G H(F(z_+(x)))\}$ then either: (i)

$$\begin{aligned} &W_H(r_H^z(c', W^G(z) + \varepsilon^*(z))) \\ &\ge W_H(F(z_+(x))) + \hat{c}(r_H^z(c', W^G(z) + \varepsilon^*(z)) - F(z_+(x))) \end{aligned}$$

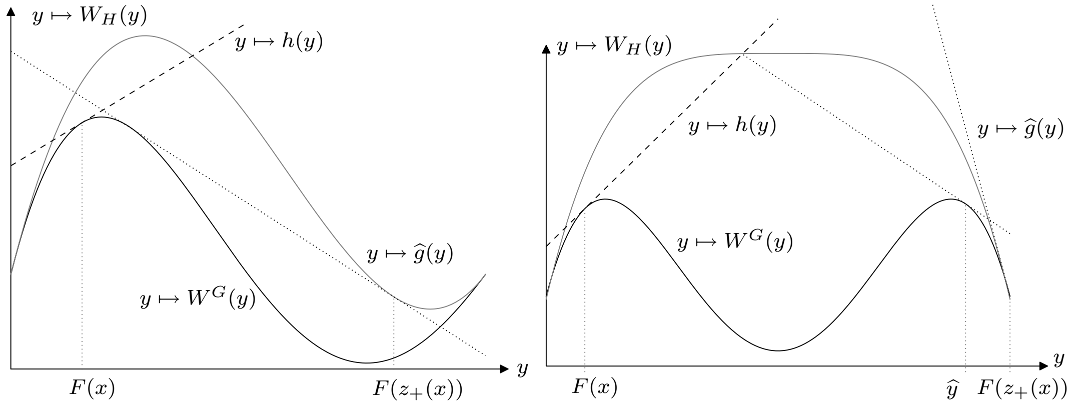

Figure 5.1: In the figure on the left the line h intercepts W_H at a point above the line $\widehat{g}$ whereas in the the figure on the right this is not the case. However, in the figure on the right h intercepts W_H at the same point as the tangent to W^G at $\widehat{y}$ intercepts W_H.

which implies that

$$h(y) > W_H(F(z_+(x))) + \hat{c}(y + \varepsilon^*(z)) - F(z_+(x))) \geq W^G(y)$$

for all $y \in [r_H^z(c', W^G(z)+\varepsilon^*(z)), F(z_+(z))]$; or (ii) there exists $\hat{y} \in [r_H^z(c', W^G(z)+\varepsilon^*(z)), F(z_+(z))]$ and $b \in \mathbb{R}$ such that

$$r_H^z(c', W^G(z) + \varepsilon^*(z)) = l_H^{\hat{y}}(b, W^G(\hat{y}))$$

and hence $h(y) > W^G(\hat{y}) + b(y - \hat{y}) \geq W^G(y)$ for all $y \in [r_H^z(c', W^G(z) + \varepsilon^*(z)), F(z_+(z))]$. These two cases are illustrated in Figure 5.1. We may conclude that $h(y) \geq W^G(y)$ for all $y \in [r_H^z(c', W^G(z)+\varepsilon^*(z)), F(z_+(z))]$. A similar argument can be used to show that $h(y) \geq W^G(y)$ for all $y \in [F(z_-(z)), l_H^z(c', W^G(z)+\varepsilon^*(z))]$. We may conclude that $\widehat{\varepsilon}(z) = \varepsilon^*(z)$ for all $z \in J^x$.

Using Lemma 3.1, for all $z \in J^x$ the concave biconjugate of $\widetilde{W}$ may be written as

$$\begin{aligned}\widetilde{W}_{**}(z; x) &= \inf_{c \in \mathbb{R}} \big(W^G(z) + \widehat{\varepsilon}_c(z)\big) \\ &= W^G(z) + \varepsilon^*(z) = (W^G)_H^{**}(z).\end{aligned}$$

By definition $z \mapsto \widetilde{W}_{**}(z; x)$ is concave on J^x which implies that $x \mapsto (G)_H^{**}(x)$ is F-concave on $[z_-(x), z_+(x)]$. Consequently, $(G)_H^{**}$ is F-superharmonic on $[z_-(x), z_+(x)]$ and since x was arbitrarily chosen $(G)_H^{**}$ is F-superharmonic on each connected section of $\{x \in I \,|\, (G)_H^{**}(x) < H(x)\}$. The concave biconjugate $y \mapsto \widetilde{W}_{**}(y; x)$ is continuous on J^x (see [22] Theorem 10.1) and

$$\begin{aligned}\widetilde{W}_{**}(F(z_-(x); x)\psi(z_-(x)) &= H(z_-(x)), \\ \widetilde{W}_{**}(F(z_+(x); x)\psi(z_+(x)) &= H(z_+(x))\end{aligned}$$

so the continuity of $x \mapsto H(x)$ implies that $x \mapsto (G)^{**}_H(x)$ is continuous. We conclude that $(G)^{**}_H \in \mathrm{Sup}\,[G,H)$ as required.

Let $\widetilde{H}(x) = H(x)\mathbb{I}_{(a,b)} + G(x)\mathbb{I}_{[x=a]\cup[x=b]}$ then by reversing the roles of G and H and replacing H with $\widetilde{H}$ in the arguments above it can be shown that $(\widetilde{H})^G_{**} \in \mathrm{Sub}(G,\widetilde{H}] \subset \mathrm{Sub}\,(G,H]$. The first part of the lemma then follows from Theorem 4.7 as $(\widetilde{H})^G_{**} = (G)^{**}_{\widetilde{H}} = (G)^{**}_H$. The second part can be derived using a symmetric argument. □

As $(G)^{**}_H \in \mathrm{Sup}\,(G,H]$ from the definition of the dual problems $\hat{V} \le (G)^{**}_H$ and $(H)^G_{**} \in \mathrm{Sub}\,[G,H)$ implies $(H)^G_{**} \le \check{V}$. The next lemma shows that the reverse inequalities hold and in particular when (7) and (8) hold the joint value of the dual problems is $(G)^{**}_H$.

Lemma 5.4. *The concave biconjugate of G in the presence of the upper barrier H defined in* (57) *coincides with the value of one of the dual problems defined in* (12) *and the convex biconjugate of H in the presence of the lower barrier G defined in* (58) *coincides with the other dual problem, i.e.*

$$(G)^{**}_H = \hat{V}, \qquad (H)^G_{**} = \check{V}.$$

Proof. For the first statement it is sufficient to show that $\hat{V} \ge (G)^{**}_H$. Fix $x' \in I$ such that $(G)^{**}_H(x') < H(x')$ and suppose that $\hat{V}(x') < (G)^{**}_H(x') = (W^G)^{**}_H(F(x'))\psi(x')$. Define another set

$$\widetilde{\mathrm{S}}\mathrm{up}[G,H) := \left\{U : \mathbb{R}_+ \to [W^G, W_H] \,|\, U \text{ is continuous and concave on } \{U < W_H\}\right\}$$

Let $x = F(x')$ then by assumption there exists $f \in \widetilde{\mathrm{S}}\mathrm{up}[G,H)$ such that $f(x) = W^G(x) + \varepsilon$ for some $\varepsilon < \varepsilon^*(x)$ where ε^* is as defined in (38). Let

$$l_f(x) := \sup\{y \le x \,|\, f(y) = W^G(y)\}, \qquad r_f(x) := \inf\{y \ge x \,|\, f(y) = W^G(y)\}$$

so that the function f is concave on $[l_f(x), r_f(x)]$. A tangent to f at x can be expressed as

$$u(y;x) := f(x) + c'(y-x) \quad \text{for } c' \in \left[\frac{d^+}{dy}f(x), \frac{d^-}{dy}f(x)\right].$$

Consider

$$\begin{aligned} l'_\eta &:= l^x_G(c', f(x)+\eta) \vee l^x_H(c', f(x)+\eta), \\ r'_\eta &:= r^x_G(c', f(x)+\eta) \wedge r^x_H(c', f(x)+\eta). \end{aligned}$$

As $\partial^H f(x) \subseteq \partial^H_\varepsilon G(x) = \emptyset$ it follows that either $l'_0 = l^x_G(c', f(x))$ and/or $r'_0 = r^x_G(c', f(x))$. Consequently, for sufficiently small $\eta > 0$, either $l'_\eta = l^x_G(c', f(x)+\eta)$ and/or $r'_\eta = r^x_G(c', f(x)+\eta)$. Thus $f(y) = f(x) + \eta + c'(y-x)$ for some $y \in [l_f(x), r_f(x)]$ which contradicts that f is concave on $[l_f(x), r_f(x)]$. We may conclude that it is not possible to construct a function $f \in \widetilde{\mathrm{Sup}}(G,H]$ passing through $p \in [W^G(x), W^G(x) + \varepsilon^*(x))$ or equivalently, it is not possible to construct a function $\tilde{f} \in \mathrm{Sup}(G,H]$ passing though $\tilde{p} \in [G(x'), G(x') + \varepsilon^*(F(x')))$ for arbitrary $x' \in I$. Hence $(G)_H^{**} \geq \hat{V}$ and a symmetric argument can be used to show that assuming $(H)_{**}^G < \check{V}$ leads to a similar contradiction. $\square$

The next theorem is the main result in this section and shows when (7) and (8) hold, $\hat{V} = \check{V}$ implies the existence of a Nash equilibrium in the optimal stopping game defined in (9)–(10). As such it is a purely analytical version of one direction of Theorem 2.1 in [18]. Take

$$\mathcal{D}^+ := \left\{ x \in I \,|\, \partial^H G\left(F(x)\right) \neq \emptyset \right\}, \qquad \mathcal{D}^- := \left\{ x \in I \,|\, \partial^G H\left(F(x)\right) \neq \emptyset \right\} \quad (65)$$

as the candidate stopping regions.

Theorem 5.5. *Let*

$$\tau^* = \inf\{t \geq 0 \,|\, X_t \in \mathcal{D}^+\}, \qquad \sigma^* = \inf\{t \geq 0 \,|\, X_t \in \mathcal{D}^-\} \quad (66)$$

and suppose that assumptions (7) *and* (8) *hold. The optimal stopping game* (9)–(10) *has a Nash equilibrium and* (τ^*, σ^*) *is a saddle point, i.e. for any* τ, σ

$$R_x\left(\tau, \sigma^*\right) \leq R_x\left(\tau^*, \sigma^*\right) \leq R_x\left(\tau^*, \sigma\right). \quad (67)$$

for all $x \in I$.

Proof. When (7) and (8) hold, it was shown in Theorem 5.1 that the optimal stopping game has a value and $V = (G)_H^{**} = (H)_{**}^G$. The candidate stopping regions (65) suggest the following candidates for the optimal stopping times

$$\tau^* = \inf\{t \geq 0 \,|\, (G)_H^{**}(X_t) = G(X_t)\}, \qquad \sigma^* = \inf\{t \geq 0 \,|\, (G)_H^{**}(X_t) = H(X_t)\}.$$

It follows from Lemmata 5.3 and 5.4 that $V = \check{V} = \hat{V}$ is r-harmonic on $I \setminus (\mathcal{D}^+ \cup \mathcal{D}^-)$ and $(G)_H^{**}(x) = R_x(\tau^*, \sigma^*)$. Let

$$x_+ = \inf\{z \geq x \,|\, \partial^H G(F(z)) \neq \emptyset\}, \qquad x_- = \sup\{y \leq x \,|\, \partial^H G(F(y)) \neq \emptyset\} \quad (68)$$

and

$$y_+ = \inf\{z \geq x \,|\, \partial^G H(F(z)) \neq \emptyset\}, \qquad y_- = \sup\{y \leq x \,|\, \partial^G H(F(y)) \neq \emptyset\}. \quad (69)$$

so that $\tau^* = T_{x_+} \wedge T_{x_-}$ and $\sigma^* = T_{y_+} \wedge T_{y_-}$. Take any $a \leq z_+ \leq x \leq z_- \leq b$, set $\sigma = T_{z+} \wedge T_{z_-}$ and consider the set $A_x := [x_-, x_+] \cap [z_-, z_+]$, the following four cases are illustrated in Figure 5.2. First suppose that $A_x = [x_-, x_+]$ then

$$h(y) = \left(\frac{G}{\psi}\right)(x_-) + \frac{\left(\frac{G}{\psi}\right)(x_+) - \left(\frac{G}{\psi}\right)(x_-)}{F(x_+) - F(x_-)}(y - F(x_-)) \geq \frac{(G)_H^{**}(y)}{\psi(y)}$$

for all $y \in A_x$, moreover, h is the largest F-convex function with $h(x_+) = (G/\psi)(x_+)$ and $h(x_-) = (G/\psi)(x_-)$ so $R_x(\tau^*, \sigma) = h(F(x))\psi(x) \geq R_x(\tau^*, \sigma^*)$. Secondly suppose that $A_x = [z_-, z_+]$ and let

$$g(y) = \left(\frac{H}{\psi}\right)(z_-) + \frac{\left(\frac{H}{\psi}\right)(z_+) - \left(\frac{H}{\psi}\right)(z_-)}{F(z_+) - F(z_-)}(y - F(z_-))$$

for $y \in A_x$. Define $\widetilde{H}(y) := H(y)\mathbb{I}_{(x_-,x_+)} + G(y)\mathbb{I}_{[y=x_-]\cup[y=x_+]}$ for $y \in [x_+, x_-]$, then the convex biconjugate of $\widetilde{W_H}(y) := (\widetilde{H}/\psi)\circ F^{-1}(y)$ satisfies $g(y) \geq \widetilde{W_H^{**}}(y)$ for all $y \in A_x$. Moreover, Lemma 5.4 implies that $\widetilde{W_H^{**}}(y) = (W^G)^{**}_H(y)$ so $R_x(\tau^*, \sigma) = g(F(x))\psi(x) \geq R_x(\tau^*, \sigma^*)$. Thirdly, let $A_x = [x_-, z_+]$ then

$$f(y) = \left(\frac{G}{\psi}\right)(x_-) + \frac{\left(\frac{H}{\psi}\right)(z_+) - \left(\frac{G}{\psi}\right)(x_-)}{F(z_+) - F(x_-)}(y - F(x_-))$$

and as in the previous case $f(y) \geq W_H^{**}(y)$ and $R_x(\tau^*, \sigma) = f(F(x))\psi(x) \geq (G)^{**}_H(x) = R_x(\tau^*, \sigma^*)$. The final case that $A_x = [z_-, x_+]$ follows by a symmetric argument. Thus we have shown that the second inequality in (67) holds. The first inequality in (67) can be shown using a similar argument and thus the stopping times (66) are a saddle point. □

The previous theorem shows that when assumptions (7) and (8) hold, the value function V derived in Theorem 5.1 can be used to characterise a Nash equilibrium of the optimal stopping game. On the other hand, if we assume a Nash equilbrium exists, Theorem 2.1 in [18] shows that the semiharmonic characterisation holds. Consequently, it follows from Lemma 5.4 that the equality between the two representations of the generalised Legendre transform in Theorem 4.7 holds (which is the argument presented in [19]). However, in Section 4 it has been shown that the two representations of the extension of the Legendre transform in Theorem 4.7 coincide without the need to appeal to the result in [18].

The next corollary relaxes the assumptions (7) and (8) and is a purely analytical version of [12] Theorems 4.4 and 4.5.

Corollary 5.6. *Define $U(X_\tau(\omega)) = 0$ on the event $\{\tau = +\infty\}$ for any Borel measurable function U. Take $G \leq H$ such that (7) holds, and consider the optimal stopping game (9)–(10). Let*

$$l_a := \lim_{x\downarrow a} \frac{G(x) \vee 0}{\psi(x)}, \qquad l_b := \lim_{x\uparrow b} \frac{G(x) \vee 0}{\varphi(x)}. \tag{70}$$

and assume that $l_a \in [W^G(0+), W_H(0+)]$ and $l_b = 0$.

(*i*) *Extend W^G and W_H onto $\mathbb{R}_+$ by setting $W^G(0) = [W^G(0+), l_a]$ and $W_H(0) = [l_a, W_H(0+)]$. The value of the optimal stopping game (9)–(10) is*

$$V(x) = (G)^{**}_H(x) = (W^G)^{**}_H(F(x))\psi(x)$$

for all $x \in I$ where $(W^G)^{**}_H$ is as defined in Corollary 4.9 with $w_0 = l_a$.

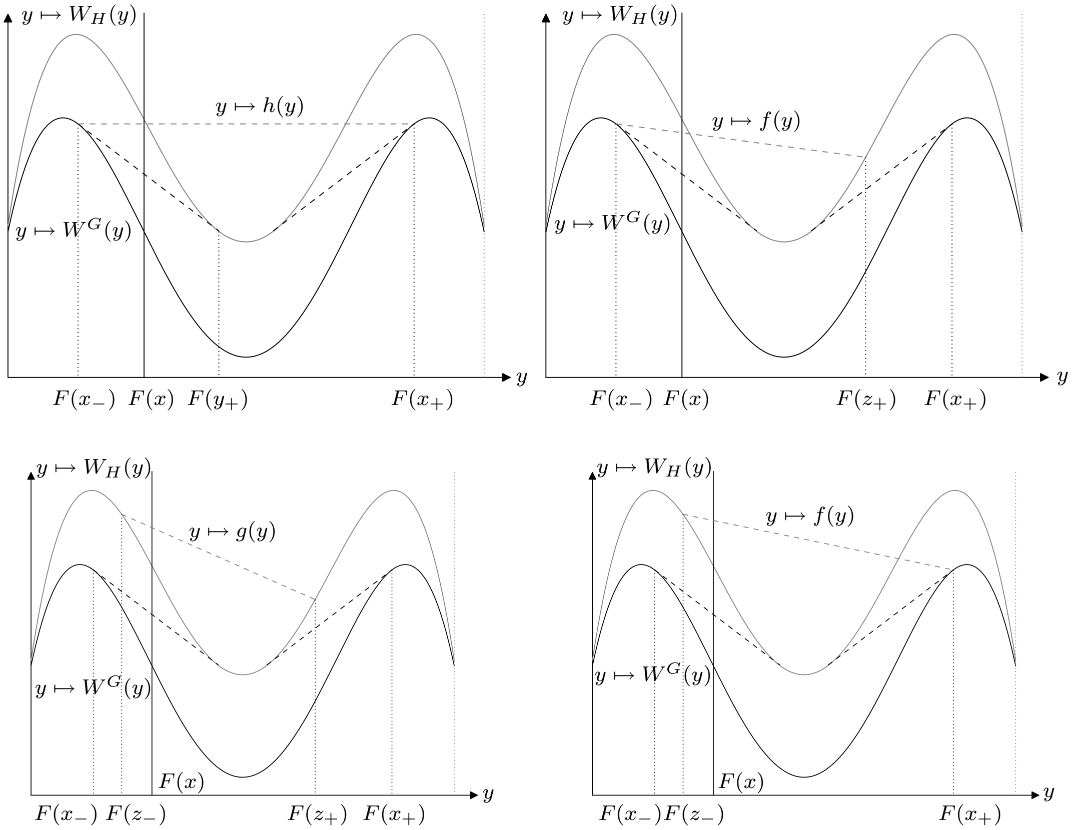

Figure 5.2: For particular choices of $[z_-, z_+]$, these four figures illustrate that the line f, g or h dominates the black dotted line $y \mapsto (W^G)^{**}_H(y)$.

(ii) *When $l_a = 0$ the optimal stopping game* (9)–(10) *has a Nash equilibrium and (τ^*, σ^*) is a saddle point.*

(iii) *When $l_a > 0$ and there exists $l > a$ such that $(G)^{**}_H(x) > G(x)$ for all $x \in (a, l)$ then the optimal stopping game* (9)–(10) *does not have a Nash equilibrium.*

(iv) *When $l_a > 0$ and there exists $l > a$ such that $(G)^{**}_H(x) = G(x)$ and/or $(G)^{**}_H(x) = H(x)$ for all $x \in (a, l)$ then the optimal stopping game* (9)–(10) *has a Nash equilibrium and (τ^*, σ^*) is a saddle point.*

Proof. (i) In the case that $l_a = W^G(0+)$ the first part was shown in Theorem 5.1. When $l_a > W^G(0+)$, $l_a = 0$ and $G(a+) = -\infty$. In this case the functions W^G and W_H are extended onto $\mathbb{R}_+$ by setting $W^G(0) = [W^G(0+), l_a]$ and $W_H(0) = [[l_a, W_H(0+)]$. This ensures that the function $(W^G)^{**}_H$ introduced in Corollary 4.9 has $(W^G)^{**}_H(0) = (W^G)^{**}_H(0+) = 0$. Furthermore, it can be shown using the same approach as in the proof of Theorem 5.1 that the value of this optimal stopping game is $V(x) = (W^G)^{**}_H(F(x))\psi(x)$ for all $x \in I$.

(ii) The boundaries of the diffusion X are natural so $P_x(T_a < \infty) = 0$ for all $x \in I$. Thus, to study whether or not a Nash equilibrium exists we extend V onto $[a, b)$ by setting $V(a) = V(X_{T_a}) = 0$. Let $W^V(x) := (W^G)^{**}_H(x)$ for

$x \in (0, \infty)$ and $W^V(0) := 0$. When $l_a = 0$, W^V and is continuous on $\mathbb{R}_+$, so the first case follows in exactly the same way as in Theorem 5.5.

(iii) In the case that $l_a > 0$ the function W^V is lower-semicontinuous on $\mathbb{R}_+$. In this case there exists a maximising sequence of stopping times but their limiting value need not be attained. To see this, let $x_\pm$, $y_\pm$ are defined as in (68)–(69) and suppose that $x_- = y_- = a$. Recall that $R_x(\tau, \sigma)$ is defined in (11). Take $\tau_n = T_{a+1/n} \wedge T_{x_+ \wedge y_+}$ then by assumption $\lim_{n\to\infty} \tau_n = \tau^*$ and the approach used in the proof of Theorem 5.5 shows that $R_x(\tau_n, \sigma^*) \leq (G)^{**}_H(x)$ for all $n \geq 1$ and $x \in (a, x_+ \wedge y_+)$ and by construction $\lim_{n\to\infty} R_x(\tau_n, \sigma^*) = (G)^{**}_H(x)$. However, using the convention at the boundary,

$$
\begin{aligned}
&R_x(\tau^*, \sigma^*) \\
&= E_x\left[e^{-rT_{x_+ \wedge y_+}}(G(X_{T_{x_+}})\mathbb{I}_{[x_+ \leq y_+]} + H(X_{T_{y_+}})\mathbb{I}_{[y_+ < x_+]})\mathbb{I}_{[T_{x_+ \wedge y_+} \leq T_a]}\right] \\
&= \left(\frac{G(x_+)}{\psi(x_+)} \frac{F(x) - F(a)}{F(x_+) - F(a)} \mathbb{I}_{[x_+ \leq y_+]} + \frac{H(y_+)}{\psi(y_+)} \frac{F(x) - F(a)}{F(y_+) - F(a)} \mathbb{I}_{[y_+ < x_+]}\right) \psi(x) \\
&= G(x_+) \frac{\varphi(x)}{\varphi(x_+)} \mathbb{I}_{[x_+ \leq y_+]} + H(y_+) \frac{\varphi(x)}{\varphi(y_+)} \mathbb{I}_{[y_+ < x_+]}.
\end{aligned}
$$

for all $x \in (a, x_+ \wedge y_+)$. Moreover,

$$
R_x(\tau_n, \sigma^*) = G(a + 1/n) \frac{\psi(x)}{\psi(a + 1/n)} \frac{F(x_+ \wedge y_+) - F(x)}{F(x_+ \wedge y_+) - F(a + 1/n)} + R_x(\tau^*, \sigma^*).
$$

Since $F(a+) = 0$ it follows that

$$
\lim_{n\to\infty} R_x(\tau_n, \sigma^*) = l_a \left(1 - \frac{F(x)}{F(x_+ \wedge y_+)}\right) \psi(x) + R_x(\tau^*, \sigma^*) > R_x(\tau^*, \sigma^*)
$$

and the first term is strictly positive as the function F is strictly increasing. Thus a saddle point does not exist as the limit of the maximising sequence is not attained.

(iv) The situation described above can not occur under the assumptions of part *(iii)* of the corollary because in this case it is not possible to take $x \in I$ such that $x_- \vee y_- = a$ where $x_\pm$ and $y_\pm$ are as defined in (68)–(69). □

A similar approach can be taken to study the case that $l_b > 0$ but requires Theorem 5.1 and Theorem 5.5 to be formulated in terms of the other ratio of the fundamental solutions using Corollary 4.8.

Remark 5.7. The previous result is heavily dependent on the assumption that $U(X_\tau(\omega)) = 0$ on the event $\{\tau = +\infty\}$. This assumption extends any continuous function $U : I \to \mathbb{R}$ onto $[a, b]$ by setting $U(a) = U(b) = 0$. Consequently, the associated function $W^U(y) := (U/\psi) \circ F^{-1}(y)$ is extended onto $\mathbb{R}_+$ by setting $W^U(0) = 0$. Hence, the function U (resp. W^U) need not be upper-semicontinuous on $[a, b)$ (resp. $\mathbb{R}_+$) which allows for the possibility that the value associated with the optimising sequence of stopping times need not be attained. If instead we extend G and H onto the closure of I using the approach discussed in Remark 4.4 the optimal stopping game (9)–(10) has a saddle point in all the cases discussed in Corollary 5.6.

We conclude by examining the Israeli-δ put-option introduced in [17].

Example 5.8 (Israeli-δ Put). Suppose that X is a geometric Brownian motion. That is X solves the SDE

$$dX_t = rX_t\,dt + \sigma X_t\,dW_t$$

with initial point $X_0 = x \in (0, \infty)$ where $r > 0$ is the discount rate and $\sigma > 0$ is a volatility parameter. In this case

$$\mathbb{L}_X u(x) = rx\frac{du}{dx}(x) + \frac{1}{2}\sigma^2\frac{d^2u}{dx^2}(x)$$

and the two fundamental solutions to $\mathbb{L}_X u = ru$ are $\varphi(x) = x$ and $\psi(x) = x^{-2r/\sigma^2}$. Moreover,

$$-\widetilde{F}(x) = \left(\frac{\psi}{\varphi}\right)(x) = x^{-(1+2r/\sigma^2)}, \qquad F(x) = \left(\frac{\varphi}{\psi}\right)(x) = x^{1+2r/\sigma^2}.$$

and $(-\widetilde{F})^{-1}(y) = y^{-\alpha}$, $F^{-1}(y) = y^{\alpha}$ where $\alpha = 1/(1 + 2r/\sigma^2) \in [0, 1]$. For some $K > 0$ and $\delta > 0$, let

$$G(x) = (K - x)^+, \qquad H(x) = (K - x)^+ + \delta.$$

The maximising agent has bought a perpetual American put option with strike $K > 0$ from the minimising agent but the minimising player retains the right to cancel the option by paying a fixed penalty of $\delta > 0$. The gains functions are rescaled by taking

$$\widetilde{W}^G(y) = \left(\frac{G}{\varphi}\right) \circ (-\widetilde{F})^{-1}(y) = (Ky^{\alpha} - 1)^+,$$
$$\widetilde{W}_H(y) = \left(\frac{H}{\varphi}\right) \circ (-\widetilde{F})^{-1}(y) = (Ky^{\alpha} - 1)^+ + \delta y^{\alpha}.$$

for $y \in (0, \infty)$. The functions G, H are illustrated in the left panel of Figure 5.3 and the functions $\widetilde{W}^G$ and $\widetilde{W}_H$ are illustrated in the right panel of Figure 5.3. The functions $\widetilde{W}^G$ and $\widetilde{W}_H$ are concave on the interval $[K^{-1/\alpha}, +\infty)$ and $\widetilde{W}_H$ is concave on the interval $[0, K^{-1/\alpha}]$.

The problem is degenerate in the sense that the inf-player will choose $\sigma^* = +\infty$ and the game has the same value as the perpetual put option examined in Example 3.6 if the concave biconjugate of $\widetilde{W}^G$ minorises $\widetilde{W}_H$, i.e. $\widetilde{W}^G_{**}(y) \leq \widetilde{W}_H(y)$ for all $y \in \mathbb{R}_+$ which is equivalent to

$$\delta \geq \widetilde{W}^G_{**}(K^{-1/\alpha})\varphi(K) = \frac{\sigma^2}{2r}\frac{K}{\left(1 + \frac{\sigma^2}{2r}\right)^{1+2r/\sigma^2}} =: \delta^*.$$

To avoid this case assume that $\delta \in (0, \delta^*]$. On $[0, K^{-1/\alpha}]$ the function $\widetilde{W}_H$ is concave so

$$(\widetilde{W}^G)^{**}_H(y) \leq \left(\frac{\widetilde{W}_H(K^{-1/\alpha})}{K^{-1/\alpha}}\right) y = \frac{\delta}{K} K^{1/\alpha} y \quad \forall y \in [0, K^{-1/\alpha}].$$

In fact this inequality holds with equality as for all $c < \widetilde{W}_H(K^{-1/\alpha})/K^{-1/\alpha}$

$$cy \vee \widetilde{W}^G(y) < \widetilde{W}_H(y) \quad \forall\, y \in (0, \infty).$$

On $[K^{-1/\alpha}, \infty)$, we can use an approach similar to that used in Example 3.6. There exists y^* such that

$$\widetilde{W}_H(K^{-1/\alpha}) = \widetilde{W}^G(y^*) - \frac{d}{dy}\widetilde{W}^G(y^*)(y - K^{-1/\alpha}).$$

This y^* solves the polynomial

$$\frac{\delta}{K} + 1 = K y^\alpha (1 - \alpha) + \alpha y^{\alpha - 1} K^{(\alpha-1)/\alpha}.$$

Let x^* be defined as $y^* = (x^*)^{-1/\alpha}$, then the constant x^* satisfies

$$(1 + 2r/\sigma^2)\left(1 + \frac{\delta}{K}\right)\left(\frac{x}{K}\right) = 2r/\sigma^2 + \left(\frac{x}{K}\right)^{(1+2r/\sigma^2)}$$

which is the same condition as in [17]. Moreover, $(\widetilde{W}^G)^{**}_H(y) = \widetilde{W}^G(y)$ for $y \geq y^*$ and on $[K^{-1/\alpha}, y^*]$ we have

$$\begin{aligned}
(W^G)^{**}_H(y) &= \widetilde{W}_H(K^{-1/\alpha}) + \frac{\widetilde{W}^G(y^*) - \widetilde{W}_H(K^{-1/\alpha})}{y^* - K^{-1/\alpha}}(y - K^{-1/\alpha}) \\
&= \frac{\delta}{K} + \left(K(y^*)^\alpha - 1 - \frac{\delta}{K}\right)\frac{y - K^{-1/\alpha}}{y^* - K^{-1/\alpha}} \\
&= \frac{\delta}{K}\frac{(x^*)^{-1/\alpha} - y}{(x^*)^{-1/\alpha} - K^{-1/\alpha}} + \frac{1}{x^*}(K - x^*)\frac{y - K^{-1/\alpha}}{(x^*)^{-1/\alpha} - K^{-1/\alpha}}.
\end{aligned}$$

The function $y \mapsto (\widetilde{W}^G)^{**}_H(y)$ is the black line in the lower panel of Figure 5.3.

According to Theorem 5.1 the value of this game option satisfies $V(x) = (\widetilde{W}^G)^{**}_H(-\widetilde{F}(x))\varphi(x)$ so

$$V(x) = \begin{cases} (K - x) & \text{for } x < x^* \\ \delta\left(\frac{x}{K}\right)\frac{(x^*)^{-1/\alpha} - x^{-1/\alpha}}{(x^*)^{-1/\alpha} - K^{-1/\alpha}} + (K - x^*)\left(\frac{x}{x^*}\right)\frac{x^{-1/\alpha} - K^{-1/\alpha}}{(x^*)^{-1/\alpha} - K^{-1/\alpha}} & \text{for } x \in [x^*, K] \\ \delta\left(\frac{x}{K}\right)^{(\alpha-1)/\alpha} & \text{for } x > K \end{cases}$$

which can be rearranged into the form derived in [17]. In this example assumptions (7) and (8) hold so Theorem 5.5 tells us that a saddle point of the game option is

$$\tau^* = \inf\{t \geq 0 \,|\, X_t \leq x^*\}, \qquad \sigma^* = \inf\{t \geq 0 \,|\, X_t = K\}.$$

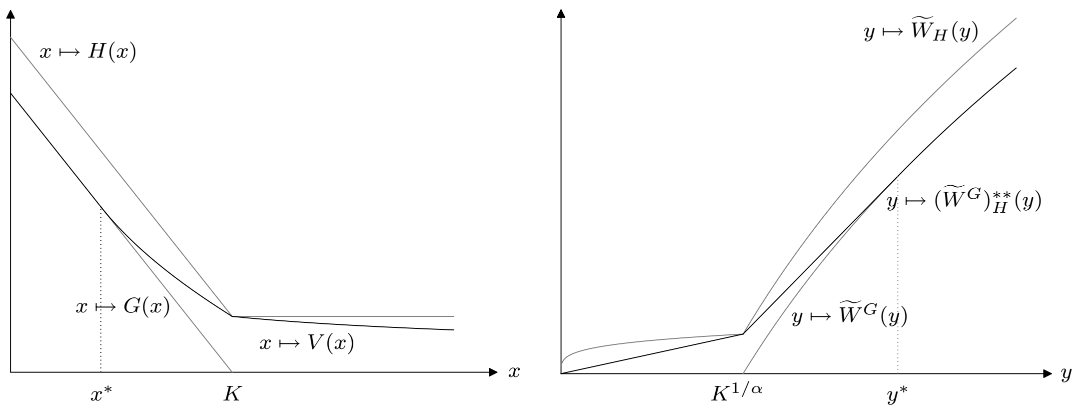

Figure 5.3: The top figure illustrates the payoff functions of the game. The black line is the value function of the game. The lower panel illustrates the transformed payoffs $\widetilde{W}_H$ and $\widetilde{W}^G$ the black line is the function $(\widetilde{W}^G)^{**}_H$.

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