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Partial Hölder Continuity of Minimizers of Functionals Satisfying a General Asymptotic Relatedness Condition

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We consider the partial Hölder continuity of minimizers of functionals of the form

$$v \mapsto \int_\Omega f(x, v, Dv)\ dx,$$

where $\Omega \subseteq \mathbb{R}^n$ is open and bounded. In our setting the integrand $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is not necessarily continuous in any of its three arguments. In particular, due to the use of a suitable asymptotic relatedness condition, f possesses continuity and convexity only as the norm of its third argument tends to infinity. Since, in particular, v is possibly vector-valued, this provides a generalization of certain existing regularity results in the literature and helps to further build a low-order regularity theory.

Keywords: Partial regularity, Morrey regularity, Hölder continuity

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1. Introduction

In this paper we consider the partial Hölder continuity of minimizers of the functional

$$v \mapsto \int_\Omega f(x, v, Dv)\ dx, \tag{1}$$

where the integrand $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, with $\Omega \subseteq \mathbb{R}^n$ open and bounded, is in possession of very weak structure conditions; note that we thus consider

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the case in which v is allowed to be vector-valued. Indeed, as will be elucidated in the sequel, all of the continuity and convexity properties of f are essentially captured at infinity by means of a suitable asymptotic relatedness condition. Succinctly, the idea is to require the existence of functions $a : \Omega \times \mathbb{R}^N \to \mathbb{R}$ and $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ such that for each $\varepsilon > 0$ there exists a function $\sigma_\varepsilon : [0,+\infty) \to [0,+\infty)$, where $\sigma_\varepsilon := \sigma(\varepsilon)$ is understood, such that

$$|f(x,u,\xi) - a(x,u)F(\xi)| < \varepsilon|\xi|^p, \tag{2}$$

for each $(x,u) \in \Omega \times \mathbb{R}^N$ whenever $\xi \in \mathbb{R}^{N\times n}$ satisfies $|\xi| > \sigma_\varepsilon$; here and throughout we shall assume that $p \in [2,+\infty)$. As will be described in Section 2, the functions a and F will be required to satisfy certain structure conditions. In particular, the function a is assumed to be continuous, whereas F is a function enjoying p-Uhlenbeck structure – see Section 2 for additional details. Consequently, this offloads any structural hypotheses, save but a standard growth condition, that would be placed on f itself and instead places such hypotheses on a and F. These considerations, therefore, increase the generality of the problems to which our results may be applied. It is worth noting that in this context the function a, though required to be continuous on $\Omega \times \mathbb{R}^N$, will not be required to possess any degree of Hölder continuity. In addition, as shall also be explicated in the sequel, we may, in fact, assume that σ_ε belongs only to the Morrey space $L^{\gamma,\beta}(\Omega)$, for some $\gamma > p$ and some $\beta \in (n-p,n)$. Due to the fact that σ_ε may then possess singularities, it follows that in this case condition (2) may not even hold at some $x \in \Omega$. Furthermore, we do not invoke any sort of dimension restriction – i.e., our results are not restricted to the low-dimensional setting.

In particular, the discussion in the preceding paragraph implies that the integrand of the functional in (1) may be discontinuous when the norm of its third argument is small. Moreover, due to the fact that f is asymptotically related to $a(x,u)F(\xi)$, it follows that f may possess a fairly general form – see the example at the end of Section 3 for a specific illustration of this fact. In any case, our main result, which is stated precisely in Section 3, shows that under these very general conditions we find that a minimizer u of the functional (1) possesses low-order partial Hölder continuity; specifically, it holds that $u \in \mathcal{C}_{\mathrm{loc}}^{0,1-\frac{n-\alpha}{p}}(\Omega_0)$ for each $\Omega_0 \Subset \Omega$, where $|\Omega \setminus \Omega_0| = 0$, and for each $\alpha \in (n-p,\beta)$, where β is the Morrey exponent for σ_ε. Thus, the more regularity imposed on σ_ε, the stronger partial Hölder continuity that we may expect a minimizer u to satisfy.

Let us comment briefly on the strategy used to obtain our results. The general strategy we use is well-known in the regularity literature. In particular, we choose an arbitrary but fixed Lebesgue point, say $x_0 \in \Omega$, of the gradient of the minimizer of (1). Using a variety of estimates, we then show that the quantity

$$\frac{1}{\rho^\alpha}\int_{\mathcal{B}_\rho(x_0)} |Du|^p \, dx \tag{3}$$

is controllable as $\rho \to 0^+$ for each $\alpha \in (n-p, \beta)$. By a standard continuity argument we obtain from (3) that the quantity

$$\frac{1}{\rho^\alpha} \int_{\mathcal{B}_\rho(x)} |Du|^p \, dx \tag{4}$$

remains finite at each x in some appropriately small ball centered at x_0 as $\rho \to 0^+$. From (4) we then deduce the Morrey regularity of the gradient of u – namely, $Du \in L^{p,\alpha}(\mathcal{B})$, where $\mathcal{B}$ is again some suitably small ball centered at x_0. This result and a standard compactness argument together with the properties of Campanato and Sobolev-Morrey spaces imply that there is an open set $\Omega_0 \subseteq \Omega$ of full measure such that for each $\Omega' \Subset \Omega$ it holds that $u \in \mathcal{C}^{0,\alpha}(\Omega')$, from which the desired result follows at once. As part of this argument a central strategy is to compare the minimizer u of the complicated functional (1) to a minimizer, which we label w, of the simpler functional

$$w \mapsto \int_{\mathcal{B}_R(x_0)} F(Dw) \, dx, \tag{5}$$

for some $R > 0$ and x_0 a Lebesgue point for Du as before. Because F is a function with p-Uhlenbeck structure, it follows, say by a result of Hamburger [25] (see also [21]), who gave a generalization of Uhlenbeck's results [37], that w inherits good estimates of which we make use as part of estimates of integrals involving u. Essentially this program has been used in a variety of contexts and is similar, for example, to certain of the arguments provided in [15, Chapter 6] as well as a well-known paper of Giaquinta and Giusti [17]. Nonetheless, to the best of our knowledge the application of this program to problem (1)–(2) is new, and given the weak conditions imposed on f we believe it to be mathematically interesting.

We next attempt to contextualize properly our work. In particular, the problem of regularity of minimizers in the vectorial setting is quite complicated and subtle. Indeed, since the important work of De Giorgi [7] in the late 1960s, it has been known that vectorial minimizers to variational problems need not be everywhere continuous. Rather, one typically shows that there is a set, say $\Omega_0 \subseteq \Omega$, of full measure on which the minimizer possesses some degree of regularity and outside of which – that is on the set $\Omega \setminus \Omega_0$ – the minimizer may possibly be singular. There has been considerable work conducted over the past 40 years not only with regard to the problem of the regularity of minimizers but also with regard to the measure, especially the Hausdorff dimension, of the so-called singular set. Papers by Giaquinta and Modica [19, 20, 21] provided some fundamental results along these lines, whereas some more recent results may be found in [1, 3, 4, 6, 11, 26] and certain of the references therein. Furthermore, one may consult both the excellent survey papers by Mingione [28, 29] as well as the monograph by Giusti [22] for a comprehensive review of these efforts.

On the other hand, the use of asymptotic relatedness properties has proved to be a very useful tool in partial regularity theory over the past 25 years; this concept has even proved useful in the study of boundary value problems for ordinary differential eqautions – see, for example, [23, 24]. In particular, Chipot and Evans [5] were the first to employ such conditions in the study of the partial regularity problem. Later Raymond [34] studied the regularity properties of minimizers of functions of the form

$$v \mapsto \int_\Omega f(Dv) + g(x,v)\ dx, \tag{6}$$

where f is asymptotically related to the p-norm and g is a Carathéodory function. In the case of Raymond's contribution we note that the superquadratic setting was treated, whereas later Pasarelli di Napoli and Verde [30] treated the analogous problem in the subquadratic setting. More recent contributions include Fey and Foss [10] and Foss [12] who use various asymptotic relatedness condition in the context of proving global Lipschitz regularity of minimizers to certain functionals. In addition, Scheven and Schmidt [35, 36] have recently produced a variety of results for problems involving asymptotic relatedness conditions. In particular, they have shown both higher integrability – see [36] – in the case of relatively general functionals and partial Lipschitz regularity – see [35] – in the case of more restricted functionals (i.e., those whose integrands depend solely on the gradient of the minimizer). As a consequence of higher integrability, Scheven and Schmidt also demonstrate in [36, Theorem 8.1] everywhere Hölder continuity, although they assume a strict dimension assumption, namely that $n \leq p+2$. Finally, Foss and Goodrich [13] have recently succeeded in utilizing asymptotic relatedness conditions to prove partial Hölder continuity as well as associated Caccioppoli and reverse Hölder inequalities for minimizers of functionals mapping between Riemannian manifolds. The reader may consult the well-known paper by Luckhaus [27] and the references therein for some older results on partial regularity in the case of minimizing maps between Riemannian manifolds.

It should also be mentioned that obtaining partial regularity in spite of weak structure conditions imposed on the integrand of various functionals has been studied in a few different contexts to date. Some first results in this direction are due Giaquinta and Giusti [16]. In that work the authors demonstrate the Hölder continuity of minimizers of the functional

$$v \mapsto \int_\Omega f(x,v,Dv)\ dx, \tag{7}$$

with f a Carathéodory function satisfying standard growth; they provide various results in both the scalar and vectorial settings. More recently Bögelein, et al. [2] proved partial Hölder continuity for minimizers of functionals of the form provided by functional (7) where f is assumed to be continuous in v, to be VMO in x, and such that the partial map $\xi \mapsto f(\cdot,\cdot,\xi)$ is twice-differentiable. Obviously, the VMO condition in x is quite general, whereas the condition on the partial map $\xi \mapsto f(\cdot,\cdot,\xi)$ is much stronger than what we present herein.

In addition, no asymptotic relatedness conditions are utilized in [2]. Similarly, a series of articles by Ragusa and Tachikawa [31, 32, 33] also address partial regularity for minimizers under weak structure conditions. More specifically, this problem is tackled in the quadratic functionals case [32], whereas in [31, 33] the authors address the problem for the somewhat more general case of functions of the form (7). In each case, the authors assume a VMO-type condition on the partial map $x \mapsto f(x, \cdot, \cdot)$ with continuity (not necessarily Hölder) and $\mathcal{C}^2$ assumptions on, respectively, the partial maps $v \mapsto f(\cdot, v, \cdot)$ and $\xi \mapsto f(\cdot, \cdot, \xi)$. But, as in [2], these conditions are imposed uniformly over the domain of f, and, in particular, no asymptotic conditions are utilized.

A final point worth mentioning is the recent paper by Foss and Mingione [14]. Insofar as the relevance of [14] is to our work here, we mention that the authors, in part, consider the question of partial Hölder continuity for variational integrals of the form given in (7). While the authors there suppose rather more stringent structure conditions on the integrand f than we do here, it is of relevance to note that the partial map $(x, u) \mapsto f(x, u, \cdot)$ is merely continuous. In particular, it is not necessarily Hölder continuous. As we have already mentioned the function a that we study in (2) is similarly not necessarily Hölder continuous.

In light of the preceding, our results here provide a complement in the asymptotically convex setting to the results of [2, 14, 31, 32, 33, 36]. Let us first mention the sharpness of the results we provide here since we believe this to be one of the principal contributions of this work. One may first observe that for $p < n$ the partial regularity we obtain cannot be strengthened to everywhere regularity on Ω due to the u-dependence present in the integrand; this holds even in spite of the assumption that f is asymptotically related to the splitting form $a(x, u)F(\xi)$. In addition, the counterexample provided by Scheven and Schmidt [35] shows that for $p < n - 2$ it is not possible to drop the symmetry assumption on F – even in case there is neither x- nor u-dependence. Finally, counterexamples provided by Dolzmann, et al. [8] demonstrate that one cannot reach the limiting exponent $\alpha = n$ (where α, recall, is the degree of Morrey regularity held by the minimizer u) – once again, even in case there is neither x- nor u-dependence, unless there is an extra assumption coupling the behavior of $\frac{\partial^2}{\partial \xi^2} f$ and $\frac{\partial^2}{\partial \xi^2} F$. In light of these observations, the results that we present in this paper are quite sharp.

On the other hand, our results also provide complement to existing works, such as [2, 14, 31, 32, 33, 36], insofar as the degree of regularity we obtain is concerned and the structural conditions under which we are able to achieve this regularity. A principle difference between the structural assumptions in [2, 14, 31, 32, 33, 36] and the ones we make here is our asymptotic relatedness assumption, which allows for relatively general integrands. On the other hand, in [2, 14, 31, 32, 33, 36] no rotational symmetry is supposed, whereas the function F, referenced in condition (2), is required to possess this symmetry. For example, in [2] while f is VMO with respect to x, it is uniformly strictly quasiconvex and $\mathcal{C}^2$ with respect to its third argument; we make these sorts of assumptions only "at infinity" in the form of an asymptotic relatedness condition. Furthermore, in [36] the authors establish higher integrability as well as partial Hölder continuity for minimizers of functions of the form in (7), wherein f is also very general – possessing continuity and convexity only at infinity. But their everywhere Hölder continuity result [36, Theorem 8.1] relies on a strict dimensional hypotheses – namely that $n \leq p+2$. Thus, their result gives only low-dimensional Hölder continuity. In contrast to this, our results give only partial Hölder continuity but without a restriction on the dimension of the domain, and so, in the case of partial regularity at least, we complement this result from [36]. We also complement the results of [31, 32, 33], which as explained previously treat the partial Hölder continuity of minimizers of functions with relatively weak structure conditions in both the quadratic functional and more general functional setting. Finally, we remark that our results are also novel due to the fact that the asymptotic relatedness condition as suggested by (2) is very general, primarily due to the fact that a may depend on u in addition to x. A comparison of this to the asymptotic relatedness conditions employed, say, by Scheven and Schmdit [36] reveals that they replace the function $a(x,u)F(\xi)$ by $g(x,\xi)$. Consequently, although their results yield everywhere regularity, their results cannot handle the function $a(x,u)|\xi|^p$, whereas our results can, again, at least in the partial regularity setting – cf., Example 1 at the end of Section 3.

To conclude this section let us finish by mentioning the basic outline of the remainder of this paper. In Section 2 we present some preliminary definitions and notations that are used throughout the paper. We also argue some preliminary lemmas as well as provide a precise description of the structure assumptions on the functions a, f, and F mentioned above. Finally, in Section 3 we provide the statement and proof of our regularity result. We then conclude with an illustrative example and two final remarks.

2. Preliminaries

In this section we introduce certain notations and results to which we will refer routinely in the sequel. We begin by stating the function spaces in which we work.

Definition 2.1. For each $p \in [1,+\infty)$ and $0 \leq \gamma \leq n$, we define the **Morrey space**

$$L^{p,\gamma}\left(E;\mathbb{R}^N\right) := \left\{ u \in L^p\left(E;\mathbb{R}^N\right) \ : \ \sup_{\substack{y\in E\\ \rho>0}} \frac{1}{\rho^\gamma}\int_{E\cap\mathcal{B}_\rho(y)} |u|^p \, dx < +\infty \right\}, \tag{8}$$

where $E \subseteq \mathbb{R}^n$ in (8) is some measurable set. We write $u \in L^{p,\gamma}_{\text{loc}}\left(E;\mathbb{R}^N\right)$ if $u \in L^{p,\gamma}\left(E';\mathbb{R}^N\right)$ for each $E' \Subset E$. Note that $\mathcal{B}_\rho(y) := \{z \in \mathbb{R}^n \ : \ |z-y| < \rho\}$.

Definition 2.2. For each $p \in [1, +\infty)$ and $0 \leq \kappa \leq n$ we say that a mapping $u \in W^{1,p}\left(\Omega, \mathbb{R}^N\right)$ belongs to the **Sobolev-Morrey space** $W^{1,(p,\kappa)}\left(\Omega; \mathbb{R}^N\right)$ provided that $u \in L^{p,\kappa}\left(\Omega; \mathbb{R}^N\right)$ and $Du \in L^{p,\kappa}\left(\Omega; \mathbb{R}^N\right)$. We write $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}\left(\Omega; \mathbb{R}^N\right)$ provided that $u \in W^{1,(p,\kappa)}\left(\Omega'; \mathbb{R}^N\right)$ for each $\Omega' \Subset \Omega$.

Definition 2.3. Given $[\Lambda_*, \Lambda^*] \subseteq (0, +\infty)$ and $p \in (1, +\infty)$ we say that $F \in \mathcal{C}^2\left(\mathbb{R}^{N\times n}\right)$ has a ***p*-Uhlenbeck structure** and belongs to $\mathcal{U}^p_{\Lambda_*,\Lambda^*}$ if there is a function $\widetilde{F} \in \mathcal{C}^2([0, +\infty))$ such that for each ξ, $\eta \in \mathbb{R}^{N\times n}$ it holds that:

(1) $F(\xi) = \widetilde{F}\left(|\xi|^2\right)$;

(2) $\Lambda_* \left(1 + |\xi|^2\right)^{\frac{p}{2}} \leq F(\xi) \leq \Lambda^* \left(1 + |\xi|^2\right)^{\frac{p}{2}}$;

(3) $\left|\frac{\partial}{\partial\xi}F(\xi)\right| \leq \Lambda^* \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\xi|$;

(4) $\left|\frac{\partial^2}{\partial\xi^2}F(\xi)\right| \leq \Lambda^* \left(1 + |\xi|^2\right)^{\frac{p-2}{2}}$; and

(5) $\frac{\partial^2}{\partial^2\xi}F(\xi) :: [\eta \otimes \eta] \geq \Lambda_* \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2$.

We next state the structural assumptions that we impose on the integrand, $f(x, u, \xi)$, of the functional which we study in this paper. As intimated in Section 1 our principal assumption is that f is asymptotically related to a function of the form $a(x, u)F(\xi)$, where $F \in \mathcal{U}^p_{\Lambda_*,\Lambda^*}$, in the sense that

$$\left|f(x, u, \xi) - a(x, u)F(\xi)\right| < \varepsilon|\xi|^p \tag{9}$$

whenever $|\xi|$ is sufficiently large. In particular, this allows us to capture all of the necessary continuity and convexity conditions in a and F and, in particular, avoid making such assumptions about f itself. Indeed, f need only satisfy these assumptions "at infinity."

(A1) It holds that $a \in \mathcal{C}\left(\Omega \times \mathbb{R}^N\right)$. Furthermore, there are λ, $L \in \mathbb{R}$, with $\lambda > 0$, such that $\lambda \leq a(x, u) \leq L$, for each $(x, u) \in \Omega \times \mathbb{R}^N$.

(A2) There is $\omega : \Omega \to [0, 1]$, where ω is assumed to be concave, continuous, and increasing, both such that $\omega(0) = 0$ and such that the function $a(x, u)$ satisfies $|a(x, u) - a(y, v)| \leq L\omega(|x - y| + |u - v|)$, for all (x, u), $(y, v) \in \Omega \times \mathbb{R}^N$.

(G1) The function $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ possesses p-Uhlenbeck structure in the sense of Definition 3 above.

(G2) The measurable function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the growth condition $|f(x, u, \xi)| \leq M\left(1 + |\xi|^p\right)$, for each $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ and some positive constant M.

(G3) There exists both a number $\beta > 0$ satisfying $\beta \in (n - p, n)$ and a number $\gamma > p$ such that for each $\varepsilon > 0$ given, there is $\sigma_\varepsilon \in L^{\gamma,\beta}(\Omega; [0, +\infty))$ such that $\left|f(x, u, \xi) - a(x, u)F(\xi)\right| < \varepsilon|\xi|^p$, for each $(x, u) \in \Omega \times \mathbb{R}^N$, whenever $|\xi| > \sigma_\varepsilon(x)$. Here the functions $(x, u) \mapsto a(x, u)$ and $\xi \mapsto F(\xi)$ satisfy the structural assumptions provided in assumptions (A1)–(A2), (G1) above.

We mention a remark regarding hypotheses (A1)–(A2) and (G1)–(G3).

Remark 2.4. Observe that condition (A2) implies that the function $a(x,u)$ is continuous by virtue of the assumptions on the function ω, which acts as a modulus of continuity with respect to the function a. On the other hand, condition (A2) also indicates that we make no assumption whatsoever regarding the decay of $\omega(s)$, say as $s \to 0^+$. More specifically, it may be the case that there does *not* exist an $\alpha > 0$ such that $\omega(s) \leq |s|^\alpha$, for $s > 0$ small.

Next we recall the following reverse Hölder-type inequality – see [22, Theorem 6.5]. Note that the inequality we provide in Lemma 2.5 is slightly different from that provided in [22, (6.34)–(6.35)].

Lemma 2.5. *Assume that $u \in W^{1,p}\left(\Omega; R^N\right)$ is a local minimizer for the functional*

$$\mathcal{G}(u,\Omega) := \int_\Omega g(x,u(x),Du(x))\ dx$$

with g satisfying both $|g(x,u,z)| \leq C_1\left(|z| + \vartheta(x)\right)^p$ and $g(x,u,z) \geq \widetilde{F}(z) - \vartheta(x)^p$, for a constant C_1, and where $\widetilde{F}$ is strictly quasiconvex at 0, $1 < p \leq n$, and $\vartheta \in L^p$. Then there exists $R_0 > 0$, depending only on u, such that for $R < R_0$ and for each ball $\mathcal{B}_R \Subset \Omega$ it holds that

$$\fint_{\mathcal{B}_R} |Du|^p\ dx \leq C\left\{\left(\fint_{\mathcal{B}_{2R}} |Du|^{pm}\ dx\right)^{\frac{1}{m}} + \fint_{\mathcal{B}_{2R}} (\vartheta(x))^p\ dx\right\},$$

where $m := \frac{n}{p+n} < 1$.

We will also require a boundary version of the Caccioppoli inequality. For a discussion about this version see, for example, [22, §6.5].

Lemma 2.6. *Suppose that the growth assumptions of Lemma 2.5 are in force. Furthermore, suppose that $w \in W^{1,p}\left(\mathcal{B}_R\right)$ is a minimizer of the functional*

$$v \mapsto \int_{\mathcal{B}_R} |Dv|^p\ dx, \tag{10}$$

where $\mathcal{B}_{2R} \Subset \Omega$. If $u \in W^{1,p}(\Omega)$ is given such that $w - u \in W_0^{1,p}\left(\mathcal{B}_R\right)$, then for some $m \in (0,1)$ and for each $0 < r < R$ and $x \in \mathcal{B}_R$ such that $\mathcal{B}_r(x) \Subset \mathcal{B}_R$ it holds that

$$\fint_{\mathcal{B}_r} |D(w-u)|^p\ dx \leq C\left(\fint_{\mathcal{B}_{2r}} |D(w-u)|^{pm}\ dx\right)^{\frac{1}{m}} + C\fint_{\mathcal{B}_{2r}} 1 + |Du|^p\ dx.$$

Our next lemma proves that f appearing in (1) satisfies an appropriate coercivity condition. That this holds is essential to invoking an appropriate version of the reverse Hölder inequality of Lemma 2.5.

Lemma 2.7. *Suppose that conditions* (A1)–(A2) *and* (G1)–(G3) *hold. Then the function f satisfies the coercivity condition*

$$f(x,u,\xi) \geq \frac{1}{2}\lambda\Lambda^*|\xi|^p - \vartheta(x),$$

where $\xi \mapsto \frac{1}{2}\lambda\Lambda^*|\xi|^p$ is a map quasiconvex at 0 *and the function ϑ is defined by*

$$\vartheta(x) := \left(M + 2^{p-2}L\Lambda^*\right)\left(1 + \left(\sigma_{\frac{1}{2}\lambda\Lambda_*}(x)\right)^p\right)$$

satisfies $\vartheta \in L^{s,\beta}(\Omega)$, for some $s > 1$.

Proof. Let $x \in \Omega$ be fixed but arbitrary. First note by the asymptotic relatedness condition (G3) that if $|\xi| \geq \sigma_{\frac{1}{2}\lambda\Lambda_*}(x)$, then it holds that

$$|f(x,u,\xi) - a(x,u)F(\xi)| \leq \frac{1}{2}\lambda\Lambda_*|\xi|^p. \tag{11}$$

We now consider two cases. In case $|\xi| \leq \sigma_{\frac{1}{2}\Lambda_*\lambda}(x)$, using the fact that for A, $B \geq 0$ the inequality $-|A - B| \geq -A - B$ holds, we estimate

$$\begin{aligned}
f(x,u,\xi) &\geq a(x,u)F(\xi) - |f(x,u,\xi) - a(x,u)F(\xi)| \\
&\geq \lambda\Lambda_*\left(|\xi|^2\right)^{\frac{p}{2}} - f(x,u,\xi) - a(x,u)F(\xi) \\
&\geq \lambda\Lambda_*|\xi|^p - M\left(1 + |\xi|^p\right) - L\Lambda^*\left(1 + |\xi|^2\right)^{\frac{p}{2}} \\
&\geq \lambda\Lambda_*|\xi|^p - M\left(1 + \left(\sigma_{\frac{1}{2}\lambda\Lambda_*}(x)\right)^p\right) - L\Lambda^*\left(1 + \left(\sigma_{\frac{1}{2}\lambda\Lambda_*}(x)\right)^2\right)^{\frac{p}{2}} \\
&\geq \lambda\Lambda_*|\xi|^p - M\left(1 + \left(\sigma_{\frac{1}{2}\lambda\Lambda_*}(x)\right)^p\right) - 2^{\frac{p}{2}-1}L\Lambda^*\left(1 + \left(\sigma_{\frac{1}{2}\lambda\Lambda_*}(x)\right)^p\right) \\
&\geq \lambda\Lambda_*|\xi|^p - \left(M + 2^{p-2}L\Lambda^*\right)\left(1 + \left(\sigma_{\frac{1}{2}\lambda\Lambda_*}(x)\right)^p\right).
\end{aligned} \tag{12}$$

Conversely, in case $|\xi| > \sigma_{\frac{1}{2}\Lambda_*\lambda}(x)$, by invoking the asymptotic condition of (11) we deduce that

$$\begin{aligned}
f(x,u,\xi) &\geq a(x,u)F(\xi) - |f(x,u,\xi) - a(x,u)F(\xi)| \\
&\geq \lambda\Lambda_*\left(1 + |\xi|^2\right)^{\frac{p}{2}} - \frac{1}{2}\lambda\Lambda_*|\xi|^p \\
&\geq \frac{1}{2}\lambda\Lambda_*|\xi|^p.
\end{aligned} \tag{13}$$

Now, consider the map

$$\xi \mapsto \frac{1}{2}\lambda\Lambda^*|\xi|^p \tag{14}$$

and the function $\vartheta : \Omega \to [0, +\infty)$ by

$$\vartheta(x) := \left(M + 2^{p-2} L\Lambda^*\right)\left(1 + \left(\sigma_{\frac{1}{2}\lambda\Lambda_*}(x)\right)^p\right). \tag{15}$$

Clearly, the map $\frac{1}{2}\lambda\Lambda^*|\xi|^p$ is trivially quasiconvex at 0. In addition, the function ϑ satisfies $\vartheta \in L^{s,\beta}(\Omega)$, for some $s > 1$ due to the fact that $\sigma_\varepsilon \in L^{\gamma,\beta}(\Omega)$ for each $\varepsilon > 0$. In fact, due to the term $\left(\sigma_{\frac{1}{2}\lambda\Lambda^*}(x)\right)^p$ and the fact that $\sigma_\varepsilon \in L^{\gamma,\beta}(\Omega)$, it is seen that $s = \frac{\gamma}{p}$. In any case, in light of estimates (12)–(13), we deduce that $f(x, u, \xi) \geq \widetilde{F}(\xi) - \vartheta(x)$, and so, the coercivity of f holds, as claimed. And this completes the proof. □

Now, due to the fact that the function ϑ appears in the coercivity condition of Lemma 2.7 but not the growth condition (G2), we shall need to use a slightly modified form of Lemma 2.5 in the sequel. In particular, an examination of the proof of Lemma 2.5 as given in Giusti [22] reveals that under our assumptions, the Caccioppoli inequality that is obtained from Lemma 2.5 is instead

$$\fint_{\mathcal{B}_R} |Du|^p \, dx \leq C\left(\fint_{\mathcal{B}_{2R}} |Du|^{pm} \, dx\right)^{\frac{1}{m}} + \fint_{\mathcal{B}_{2R}} (\vartheta(x) + 1) \, dx, \tag{16}$$

for some $0 < m < 1$. We omit the proof of this fact since it involves a fairly straightforward reworking of the proof given in Giusti. The modified Caccioppoli inequality (16) then gives rise to the following reverse Hölder inequality; it is this reverse Hölder inequality (17) that we shall utilize in Section 3. We omit the proof of this lemma.

Lemma 2.8. *Let $u \in W^{1,p}(\Omega)$ be a minimizer of the functional* (1) *and let ϑ be defined as in the statement of Lemma 2.7. Then there exists a number $r_0 > 1$ such that for each $r \in (1, r_0)$ it holds that*

$$\fint_{\mathcal{B}_R} |Du|^q \, dx \leq C\left(\fint_{\mathcal{B}_{2R}} |Du|^p \, dx\right)^{\frac{q}{p}} + C\fint_{\mathcal{B}_{2R}} (1 + \vartheta(x))^{\frac{q}{p}} \, dx, \tag{17}$$

where $q := pr$ and $\mathcal{B}_R \Subset \Omega$.

Since we also require a boundary version of the reverse Hölder inequality, we state this as our next lemma; once again, we omit the proof.

Lemma 2.9. *Let $w \in W^{1,p}(\mathcal{B}_R(x_0))$ for some ball $\mathcal{B}_R(x_0) \Subset \Omega$, where w is a minimizer of the functional*

$$v \mapsto \int_{\mathcal{B}_R(x_0)} F(Dv) \, dx. \tag{18}$$

Suppose that $u - w \in W_0^{1,p}(\mathcal{B}_R)$, *where* $u \in W^{1,t}(\mathcal{B}_{2R})$ *for some* $t > p$. *Then there exists a number* $r_0 > 1$ *such that for each* $r \in (1, r_0)$ *it holds that*

$$\fint_{\mathcal{B}_R} |Dw - Du|^q \, dx \leq C \left(\fint_{\mathcal{B}_R} |Dw - Du|^p \, dx \right)^{\frac{q}{p}} + C \fint_{\mathcal{B}_{2R}} (1 + |Du|^p)^{\frac{q}{p}} \, dx, \quad (19)$$

where $q := pr$.

We next recall a basic property of Morrey spaces – see [22, §2.3] – together with an associated corollary. This property, especially Corollary 2.11, will be used in the proof given in Section 3.

Lemma 2.10. *Provided that* $s \geq p$ *and* $\frac{n-\lambda}{p} \geq \frac{n-\mu}{s}$, *it holds that*

$$L^{s,\mu}(\Omega) \hookrightarrow L^{p,\lambda}(\Omega).$$

Corollary 2.11. *Provided that* $s \geq p$, *it holds that*

$$L^{s,\alpha}(\Omega) \subseteq L^{p,\alpha}(\Omega),$$

for any $0 \leq \alpha < n$.

Finally, we end with a word about the notation utilized in Section 3.

Notation 2.12. Throughout Section 3 we adopt the convention that C shall denote a generic constant, satisfying always $C > 1$, that may change from line to line and whose specific dependency generally shall not be noted, save but for the fact that C does *not* depend on the functions, say u and w, involved in the arguments. Finally, we note that we utilize the notation

$$\fint_E f(x) \, dx := \frac{1}{|E|} \int_E f(x) \, dx,$$

where E is a Lebesgue measurable set and $|E|$ denotes Lebesgue measure of the set E.

3. Main Result

We now state and prove our main theorem. In particular, this result asserts that a $W^{1,p}(\Omega)$ minimizer of the functional (1) enjoys, in fact, $W_{\text{loc}}^{1,(p,\alpha)}(\Omega_0)$ Sobolev-Morrey regularity, for each $n - p < \alpha < \beta < n$ and some open measurable set $\Omega_0 \subseteq \Omega$ satisfying $|\Omega \setminus \Omega_0| = 0$. Using the fact that the embedding

$$W_{\text{loc}}^{1,(p,\alpha)}(\Omega_0) \hookrightarrow \mathcal{C}_{\text{loc}}^{0,1-\frac{n-\alpha}{p}}(\Omega_0)$$

holds, we thus argue that a minimizer of the functional (1) possesses low-order local Hölder continuity.

Theorem 3.1. *Let $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ satisfying $u \in W^{1,p}(\Omega)$ be a minimizer for the functional* (1) *with $\Omega \subseteq \mathbb{R}^n$ open and bounded. Assume that conditions* (A1)–(A2) *and* (G1)–(G3) *hold. Finally, assume that $p \in [2, +\infty)$. Then there is an open set $\Omega_0 \subseteq \Omega$, satisfying $|\Omega \setminus \Omega_0| = 0$, such that for each $\alpha \in \mathbb{R}$ satisfying $n - p < \alpha < \beta < n$ it holds that $u \in W^{1,(p,\alpha)}(\Omega_0)$ and, in fact, that*

$$u \in \mathcal{C}^{0,1-\frac{n-\alpha}{p}}(\Omega')$$

for each $\Omega' \Subset \Omega_0$ – that is,

$$u \in \mathcal{C}^{0,1-\frac{n-\alpha}{p}}_{loc}(\Omega_0).$$

Proof. We assume in the following that $u : \Omega \to \mathbb{R}^N$ is a minimizer of the functional (1). As in the statement of the theorem, we assume that u belongs to the Sobolev class $W^{1,p}(\Omega)$. Moreover, let $x_0 \in \Omega$ be an arbitrary but fixed Lebesgue point for Du and let $R > 0$ be chosen so that

$$0 < R < \frac{1}{4}\mathsf{dist}(x_0, \partial\Omega);$$

later we shall refine the choice of R – see (56) and following. Furthermore, let $w \in W^{1,p}(\mathcal{B}_R)$ be a minimizer of the functional

$$v \mapsto \int_{\mathcal{B}_R(x_0)} F(Dv)\,dx,$$

satisfying $w = u$, a.e. $x \in \partial\mathcal{B}_R(x_0)$, whence $u - w \in W^{1,p}_0(\mathcal{B}_R(x_0))$; henceforth, we shall employ the notation

$$\mathcal{B}_R := \{x \in \mathbb{R}^n : |x - x_0| < R\}.$$

So, with these preliminaries and for $0 < \rho < R$, it is possible to obtain an estimate similar to that obtained in the first part of the proof given in Raymond [34, Proposition 5.4]. In particular, using both the Euler-Lagrange equations and the fact that $F \in \mathcal{U}^p_{\Lambda_*,\Lambda^*}$ we estimate

$$\begin{aligned}
&\int_{\mathcal{B}_\rho} |Du|^p\,dx\\
&\le C\int_{\mathcal{B}_\rho} |Dw|^p\,dx + C\int_{\mathcal{B}_R} |Du - Dw|^p\,dx\\
&\le C\int_{\mathcal{B}_\rho} |Dw|^p\,dx + C\int_{\mathcal{B}_R} \left(1 + |Du|^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Du - Dw|^2\,dx\\
&\le C\int_{\mathcal{B}_\rho} |Dw|^p\,dx + C\int_{\mathcal{B}_R}\int_0^1 (1-s)\frac{\partial^2}{\partial\xi^2}F(Dw + s[Du - Dw])\\
&\qquad :: [Du - Dw]\otimes[Du - Dw]\,ds\,dx\\
&\le C\int_{\mathcal{B}_\rho} |Dw|^p\,dx + C\int_{\mathcal{B}_R} F(Du) - F(Dw)\,dx.
\end{aligned}\tag{20}$$

Now, recalling both the growth conditions imposed on the p-Uhlenbeck function F as well as a result of Hamburger [25, Theorem 4.1] we note that

$$\begin{aligned}\int_{\mathcal{B}_\rho}\left(1+|Dw|^2\right)^{\frac{p}{2}}\,dx &\le C\left(\frac{\rho}{R}\right)^n\int_{\mathcal{B}_R}\left(1+|Dw|^2\right)^{\frac{p}{2}}\,dx\\ &\le C\left(\frac{\rho}{R}\right)^n\int_{\mathcal{B}_R}\frac{1}{\Lambda_*}F(Dw)\,dx\\ &\le C\left(\frac{\rho}{R}\right)^n\int_{\mathcal{B}_R}\frac{1}{\Lambda_*}F(Du)\,dx\\ &\le C\left(\frac{\rho}{R}\right)^n\int_{\mathcal{B}_R}\frac{\Lambda^*}{\Lambda_*}\left(1+|Du|^2\right)^{\frac{p}{2}}\,dx\\ &\le C\rho^n+C\left(\frac{\rho}{R}\right)^n\int_{\mathcal{B}_R}|Du|^p\,dx, \end{aligned} \tag{21}$$

where we use the minimality of w in (21). So, putting (20)–(21) together it follows that

$$\begin{aligned}\int_{\mathcal{B}_\rho}|Du|^p\,dx &\le C\int_{\mathcal{B}_\rho}|Dw|^p\,dx+C\int_{\mathcal{B}_R}F(Du)-F(Dw)\,dx \qquad (22)\\ &\le C\int_{\mathcal{B}_\rho}\left(1+|Dw|^2\right)^{\frac{p}{2}}\,dx+C\int_{\mathcal{B}_R}F(Du)-F(Dw)\,dx\\ &\le C\left[C\rho^n+C\left(\frac{\rho}{R}\right)^n\int_{\mathcal{B}_R}|Du|^p\,dx\right]+C\int_{\mathcal{B}_R}F(Du)-F(Dw)\,dx\\ &\le C\rho^n+C\left(\frac{\rho}{R}\right)^n\int_{\mathcal{B}_R}|Du|^p\,dx+C\int_{\mathcal{B}_R}F(Du)-F(Dw)\,dx. \end{aligned}$$

So, we must now estimate from above the quantity

$$C\int_{\mathcal{B}_R}F(Du)-F(Dw)\,dx \tag{23}$$

appearing in (22).

To begin the program of estimating (23) let us first put $z:=(u)_R\in\mathbb{R}^N$. Second of all, putting $y:=x_0$ so that $y\in\mathcal{B}_R(x_0)$, observe that $|x-y|=|x-x_0|<R$ so that together with the fact that $\zeta\mapsto\omega(\zeta)$ is a monotone increasing map we estimate

$$\omega(|x-y|+|w-z|)\le\omega(R+|w-z|). \tag{24}$$

Now, note that we may estimate

$$
\begin{aligned}
&\int_{\mathcal{B}_R} F(Du) - F(Dw)\ dx \\
\leq &\int_{\mathcal{B}_R} \left| \frac{a(y,z) - a(x,u)}{a(y,z)} \right| F(Du)\ dx \\
&\qquad + \frac{1}{a(y,z)} \int_{\mathcal{B}_R} a(x,u) F(Du)\ dx - \int_{\mathcal{B}_R} F(Dw)\ dx \\
\leq C &\int_{\mathcal{B}_R} \Lambda^* \omega(|x-y| + |u-z|) \left(1 + |Du|^p\right)\ dx \\
&\qquad + \frac{1}{a(y,z)} \int_{\mathcal{B}_R} \{-f(x,u,Du) + a(x,u)F(Du) + f(x,u,Du)\}\ dx \\
&\qquad - \int_{\mathcal{B}_R} F(Dw)\ dx \\
\leq C &\int_{\mathcal{B}_R} \omega\left(R + |u - (u)_R|\right)\left(1 + |Du|^p\right)\ dx \\
&\qquad + C \int_{\mathcal{B}_R} \omega\left(R + |w - (u)_R|\right)\left(1 + |Dw|^p\right)\ dx \\
&\qquad + \frac{1}{a(y,z)} \int_{\mathcal{B}_R} -f(x,u,Du) + a(x,u)F(Du)\ dx \\
&\qquad + \frac{1}{a(y,z)} \int_{\mathcal{B}_R} f(x,u,Du) - a(x,w)F(Dw)\ dx.
\end{aligned} \tag{25}
$$

Observe that in (25) we have used the estimate

$$
\begin{aligned}
&-\int_{\mathcal{B}_R} F(Dw)\ dx \\
= &-\int_{\mathcal{B}_R} \frac{a(y,z) - a(x,w) + a(x,w)}{a(y,z)} F(Dw)\ dx \\
\leq &\int_{\mathcal{B}_R} \frac{|a(y,z) - a(x,w)|}{a(y,z)} F(Dw)\ dx - \int_{\mathcal{B}_R} \frac{a(x,w)}{a(y,z)} F(Dw)\ dx \\
\leq C &\int_{\mathcal{B}_R} \omega(R + |w - (u)_R|)\left(1 + |Dw|^p\right)\ dx - \int_{\mathcal{B}_R} \frac{a(x,w)}{a(y,z)} F(Dw)\ dx.
\end{aligned} \tag{26}
$$

We now consider the final term appearing in (25). In particular, letting $\varepsilon > 0$

be a fixed number to be specified later, we estimate

$$\begin{aligned}
&\frac{1}{a(y,z)}\int_{\mathcal{B}_R} f(x,u,Du)-a(x,w)F(Dw)\;dx\\
&\le \frac{1}{\lambda}\int_{\mathcal{B}_R\cap\{x\,:\,|Dw|<\sigma_\varepsilon(x)\}} f(x,w,Dw)-a(x,w)F(Dw)\;dx\\
&\qquad+\frac{1}{\lambda}\int_{\mathcal{B}_R\cap\{x\,:\,|Dw|\ge\sigma_\varepsilon(x)\}} |f(x,w,Dw)-a(x,w)F(Dw)|\;dx\\
&\le \frac{1}{\lambda}\int_{\mathcal{B}_R\cap\{x\,:\,|Dw|<\sigma_\varepsilon(x)\}} |f(x,w,Dw)|+|a(x,w)F(Dw)|\;dx+\frac{\varepsilon}{\lambda}\int_{\mathcal{B}_R}|Dw|^p\;dx\\
&\le \frac{1}{\lambda}CR^n+\frac{1}{\lambda}C\int_{\mathcal{B}_R}|\sigma_\varepsilon(x)|^p\;dx+\frac{\varepsilon}{\lambda}\int_{\mathcal{B}_R}|Dw|^p\;dx, \qquad (27)
\end{aligned}$$

where we have used the minimality of u in order to obtain the first inequality in (27) and the asymptotic relatedness condition to obtain the second-to-last inequality in (27). Next using the minimality for w as well as the upper growth estimate for $\xi\mapsto F(\xi)$ we estimate

$$\frac{\varepsilon}{\lambda}\int_{\mathcal{B}_R}|Dw|^p\;dx\le\frac{\varepsilon}{\lambda}\int_{\mathcal{B}_R}\frac{1}{\Lambda_*}F(Dw)\;dx\le\frac{\varepsilon}{\lambda\Lambda_*}\int_{\mathcal{B}_R}F(Du)\;dx. \qquad (28)$$

Consequently, we estimate

$$\begin{aligned}
&\int_{\mathcal{B}_R} F(Du)-F(Dw)\;dx\\
&\le C\int_{\mathcal{B}_R}\omega\left(R+|u-(u)_R|\right)\left(1+|Du|^p\right)\;dx\\
&\qquad+C\int_{\mathcal{B}_R}\omega\left(R+|w-(u)_R|\right)\left(1+|Dw|^p\right)\;dx\\
&\qquad+\frac{1}{a(y,z)}\int_{\mathcal{B}_R} a(x,u)F(Du)-f(x,u,Du)\;dx\\
&\qquad+\frac{1}{\lambda}CR^n+C\int_{\mathcal{B}_R}|\sigma_\varepsilon(x)|^p\;dx+\frac{\varepsilon}{\lambda\Lambda_*}\int_{\mathcal{B}_R}F(Du)\;dx. \qquad (29)
\end{aligned}$$

Finally, we focus on the term

$$\frac{1}{a(y,z)}\int_{\mathcal{B}_R} a(x,u)F(Du)-f(x,u,Du)\;dx \qquad (30)$$

appearing in (29). But similar to (27), we observe that

$$\begin{aligned}
&\frac{1}{a(y,z)}\int_{\mathcal{B}_R} a(x,u)F(Du)-f(x,u,Du)\;dx\\
&\le\frac{1}{\lambda}CR^n+C\int_{\mathcal{B}_R}|\sigma_\varepsilon(x)|^p\;dx+\frac{\varepsilon}{\lambda\Lambda_*}\int_{\mathcal{B}_R}F(Du)\;dx. \qquad (31)
\end{aligned}$$

Thus, in summary, we estimate

$$\begin{aligned}
&\int_{\mathcal{B}_R} F(Du) - F(Dw)\ dx \\
&\leq C \int_{\mathcal{B}_R} \omega\left(R + |u - (u)_R|\right)\left(1 + |Du|^p\right)\ dx \\
&\qquad + C \int_{\mathcal{B}_R} \omega\left(R + |w - (u)_R|\right)\left(1 + |Dw|^p\right)\ dx \\
&\qquad + \frac{C}{\lambda} R^n + \frac{2\varepsilon}{\lambda\Lambda_*} \int_{\mathcal{B}_R} F(Du)\ dx + C \int_{\mathcal{B}_R} |\sigma_\varepsilon(x)|^p\ dx.
\end{aligned} \tag{32}$$

We next estimate from above the terms

$$C \int_{\mathcal{B}_R} \omega(R + |u - (u)_R|)\left(1 + |Du|^p\right)\ dx \tag{33}$$

and

$$C \int_{\mathcal{B}_R} \omega(R + |w - (u)_R|)\left(1 + |Dw|^p\right)\ dx, \tag{34}$$

which occur in (32). We shall focus our attention first on (33). First of all, using the fact that u is a minimizer, due to Lemma 2.8 we may invoke a reverse Hölder inequality to estimate

$$⨍_{\mathcal{B}_R} |Du|^q\ dx \leq C \left(⨍_{\mathcal{B}_{2R}} |Du|^p\ dx \right)^{\frac{q}{p}} + C ⨍_{\mathcal{B}_{2R}} (1 + \vartheta(x))^{\frac{q}{p}}\ dx \tag{35}$$

where $q \in (p, p + \eta)$, for some $\eta > 0$ generated by the reverse Hölder inequality in (35). With this value of q fixed and in observance of the fact that $\frac{q}{q-p}$ and $\frac{q}{p}$ are Hölder conjugate, by Hölder's inequality we may provide the estimate

$$\begin{aligned}
&\int_{\mathcal{B}_R} \omega(R + |u - (u)_R|)\left(1 + |Du|^p\right)\ dx \\
&\leq CR^n \left(⨍_{\mathcal{B}_R} \left(\omega\left(R + |u - (u)_R|\right)\right)^{\frac{q}{q-p}}\ dx \right)^{\frac{q-p}{q}} \left(⨍_{\mathcal{B}_R} \left(1 + |Du|^p\right)^{\frac{q}{p}}\ dx \right)^{\frac{p}{q}}.
\end{aligned} \tag{36}$$

Now, observe that since the map $t \mapsto t^{\frac{q}{p}}$ is convex (since $q > p$), we may further estimate

$$⨍_{\mathcal{B}_R} \left(1 + |Du|^p\right)^{\frac{q}{p}}\ dx \leq C \left(1 + ⨍_{\mathcal{B}_R} |Du|^q\ dx \right). \tag{37}$$

Since it can be shown that $(x + y)^\gamma \leq x^\gamma + y^\gamma$, for $0 < \gamma < 1$ and $x,\ y \geq 0$, putting the preceding estimates (35) and (37) together, we deduce that

$$\begin{aligned}
&\left(\fint_{\mathcal{B}_R}(1+|Du|^p)^{\frac{q}{p}}\ dx\right)^{\frac{p}{q}}\\
&\le C\left(1+\fint_{\mathcal{B}_{2R}}|Du|^p\ dx\right)+C\left(\fint_{\mathcal{B}_{2R}}(1+\vartheta(x))^{\frac{q}{p}}\ dx\right)^{\frac{p}{q}}\\
&\le C+C\fint_{\mathcal{B}_{2R}}|Du|^p\ dx+C\left(1+\fint_{\mathcal{B}_{2R}}(\vartheta(x))^{\frac{q}{p}}\ dx\right)^{\frac{p}{q}}\\
&\le C+C\fint_{\mathcal{B}_{2R}}|Du|^p\ dx+CR^{-n}\int_{\mathcal{B}_{2R}}(\vartheta(x))^{\frac{q}{p}}\ dx. \qquad (38)
\end{aligned}$$

Hence, combining inequality (38) with inequality (36) we deduce that

$$\begin{aligned}
&\int_{\mathcal{B}_R}\omega(R+|u-(u)_R|)\,(1+|Du|^p)\ dx\\
&\le CR^n\left(\fint_{\mathcal{B}_R}\omega\left(R+|u-(u)_R|\right)\ dx\right)^{\frac{q-p}{q}}\\
&\qquad\left(1+\fint_{\mathcal{B}_{2R}}|Du|^p\ dx+R^{-n}\int_{\mathcal{B}_{2R}}(\vartheta(x))^{\frac{q}{p}}\ dx\right), \qquad (39)
\end{aligned}$$

where to obtain the inequality in (39) above we have used the fact that $\omega(\mathbb{R})\subseteq[0,1]$ so that

$$\left(\omega\left(R+|u-(u)_R|\right)\right)^{\frac{q}{q-p}}\le\omega\left(R+|u-(u)_R|\right) \qquad (40)$$

since $\frac{q}{q-p}>1$. Using Jensen's inequality we observe that

$$\begin{aligned}
&CR^n\left(\fint_{\mathcal{B}_R}\omega\left(R+|u-(u)_R|\right)\ dx\right)^{\frac{q-p}{q}}\\
&\quad\left(1+\fint_{\mathcal{B}_{2R}}|Du|^p\ dx+R^{-n}\int_{\mathcal{B}_{2R}}(\vartheta(x))^{\frac{q}{p}}\ dx\right)\\
&\le CR^n+CR^n\left(\omega\left(R+\fint_{\mathcal{B}_R}|u-(u)_R|\ dx\right)\right)^{\frac{q-p}{q}}\fint_{\mathcal{B}_{2R}}|Du|^p\ dx\\
&\quad+C\int_{\mathcal{B}_{2R}}(\vartheta(x))^{\frac{q}{p}}\ dx\\
&\le CR^n+CR^n\left(\omega\left(R+\fint_{\mathcal{B}_R}|u-(u)_R|\ dx\right)\right)^{\frac{q-p}{q}}\fint_{\mathcal{B}_{2R}}|Du|^p\ dx+CR^{\beta}. \qquad (41)
\end{aligned}$$

Note that to obtain the final step of inequality (41) we use the fact that since $\frac{q}{p}$ may be assumed without loss to be as close to unity as desired, it follows that we may assume that $1<\frac{q}{p}<s$, where s is the number from Lemma 2.7. In

particular, this implies by an application of Corollary 2.11 that $\vartheta \in L^{\frac{q}{p},\beta}(\Omega)$. Critically, then, we may estimate

$$\int_{\mathcal{B}_{2R}} (\vartheta(x))^{\frac{q}{p}}\ dx \le CR^{\beta}.$$

Finally, by invoking Poincaré's inequality, the monotonicity of $\omega(\cdot)$, and Hölder's inequality once more, we may use estimate (41) to arrive at the estimate

$$\begin{aligned}
&\int_{\mathcal{B}_\rho} |Du|^p\ dx \qquad (42)\\
&\le C\left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}_R} |Du|^p\ dx + \frac{C}{\lambda}R^n + \frac{2\varepsilon}{\lambda\Lambda_*}\int_{\mathcal{B}_R} F(Du)\ dx + C\rho^n\\
&\qquad + CR^n + CR^n\left(\omega\left(R + C\left(R^{p-n}\int_{\mathcal{B}_R} |Du|^p\ dx\right)^{\frac{1}{p}}\right)\right)^{\frac{q-p}{q}} ⨍_{\mathcal{B}_{2R}} |Du|^p\ dx\\
&\qquad + \int_{\mathcal{B}_R} \omega(R + |w-(u)_R|)\left(1+|Dw|^p\right)\ dx + C\int_{\mathcal{B}_R} |\sigma_\varepsilon(x)|^p\ dx + CR^{\beta}.
\end{aligned}$$

We now work on estimating from above the term

$$\int_{\mathcal{B}_R} \omega(R + |w-(u)_R|)\left(1+|Dw|^p\right)\ dx, \qquad (43)$$

which occurs in (42). To this end, we first estimate

$$\begin{aligned}
&\int_{\mathcal{B}_R} \omega(R + |w-(u)_R|)\left(1+|Dw|^p\right)\ dx \qquad (44)\\
&\le CR^n + CR^n\left(⨍_{\mathcal{B}_R} \left[\omega\left(R + |w-u| + |u-(u)_R|\right)\right]^{\frac{q}{q-p}}\ dx\right)^{\frac{q-p}{q}}\left(⨍_{\mathcal{B}_R} |Dw|^q\ dx\right)^{\frac{p}{q}}.
\end{aligned}$$

Noting by the minimality of w that

$$⨍_{\mathcal{B}_R} |Dw|^p\ dx \le \frac{CR^{-n}\Lambda^*}{\Lambda_*}\int_{\mathcal{B}_R}\left(1+|Du|^2\right)^{\frac{p}{2}}\ dx \le C + C⨍_{\mathcal{B}_R} |Du|^p\ dx, \qquad (45)$$

we may again appeal to Lemmas 2.8 and 2.9 to estimate

$$
\begin{aligned}
&\fint_{\mathcal{B}_R} |Dw|^q \, dx \\
&\le C\fint_{\mathcal{B}_R} |Dw - Du|^q \, dx + C\fint_{\mathcal{B}_R} |Du|^q \, dx \\
&\le C\left[\left(\fint_{\mathcal{B}_{2R}} |Dw - Du|^p \, dx\right)^{\frac{q}{p}} + \fint_{\mathcal{B}_{2R}} (1 + |Du|^p)^{\frac{q}{p}} \, dx\right] \\
&\qquad + C\left[\left(\fint_{\mathcal{B}_{2R}} |Du|^p \, dx\right)^{\frac{q}{p}} + \fint_{\mathcal{B}_{2R}} 1 + (\vartheta(x))^{\frac{q}{p}} \, dx\right] \\
&\le C\left[1 + \left(\fint_{\mathcal{B}_R} |Dw|^p \, dx\right)^{\frac{q}{p}} + \fint_{\mathcal{B}_{2R}} |Du|^q \, dx\right. \\
&\qquad \left. + \left(\fint_{\mathcal{B}_{2R}} |Du|^p \, dx\right)^{\frac{q}{p}} + \fint_{\mathcal{B}_{2R}} (\vartheta(x))^{\frac{q}{p}} \, dx\right] \\
&\le C\left[1 + \left(\fint_{\mathcal{B}_R} |Du|^p \, dx\right)^{\frac{q}{p}} + \fint_{\mathcal{B}_{2R}} |Du|^q \, dx + \left(\fint_{\mathcal{B}_{2R}} |Du|^p \, dx\right)^{\frac{q}{p}}\right. \\
&\qquad \left. + \fint_{\mathcal{B}_{2R}} (\vartheta(x))^{\frac{q}{p}} \, dx\right] \\
&\le C\left[1 + \left(\fint_{\mathcal{B}_{2R}} |Du|^p \, dx\right)^{\frac{q}{p}} + \fint_{\mathcal{B}_{2R}} |Du|^q \, dx + \fint_{\mathcal{B}_{2R}} (\vartheta(x))^{\frac{q}{p}} \, dx\right] \\
&\le C\left[1 + \left(\fint_{\mathcal{B}_{4R}} |Du|^p \, dx\right)^{\frac{q}{p}} + \left(\fint_{\mathcal{B}_{4R}} |Du|^p \, dx\right)^{\frac{q}{p}} + \fint_{\mathcal{B}_{2R}} (\vartheta(x))^{\frac{q}{p}} \, dx\right]. \qquad (46)
\end{aligned}
$$

Consequently, we estimate

$$
\begin{aligned}
&CR^n + CR^n\left(\fint_{\mathcal{B}_R} \left[\omega\left(R + |w - u| + |u - (u)_R|\right)\right]^{\frac{q}{q-p}} \, dx\right)^{\frac{q-p}{q}} \left(\fint_{\mathcal{B}_R} |Dw|^q \, dx\right)^{\frac{p}{q}} \\
&\le CR^n + CR^n\left(\omega\left(\fint_{\mathcal{B}_R} R + |w - u| + |u - (u)_R| \, dx\right)\right)^{\frac{q-p}{q}} \left(\fint_{\mathcal{B}_{4R}} |Du|^p \, dx\right) \\
&\qquad + CR^\beta, \qquad (47)
\end{aligned}
$$

where we have also invoked Jensen's inequality as well as the fact that ϑ is an element of the Morrey space $L^{s,\beta}(\Omega)$.

We next estimate from above the argument of ω appearing on the right-hand side of inequality (47) above. With the help of (45) we may argue that

$$\begin{aligned}
& R + \fint_{\mathcal{B}_R} |w-u| \; dx + \fint_{\mathcal{B}_R} |u-(u)_R| \; dx \\
&\leq R + CR\fint_{\mathcal{B}_R} |Dw - Du| \; dx + C\left(R^{p-n}\int_{\mathcal{B}_R} |Du|^p \; dx\right)^{\frac{1}{p}} \\
&\leq R + CR\left[\fint_{\mathcal{B}_R} |Dw| \; dx + \fint_{\mathcal{B}_R} |Du| \; dx\right] + C\left(R^{p-n}\int_{\mathcal{B}_R} |Du|^p \; dx\right)^{\frac{1}{p}} \\
&\leq R + C\left[R\left(\fint_{\mathcal{B}_R} |Dw|^p \; dx\right)^{\frac{1}{p}} + R\left(\fint_{\mathcal{B}_R} |Du|^p \; dx\right)^{\frac{1}{p}}\right] \\
&\qquad + C\left(R^{p-n}\int_{\mathcal{B}_R} |Du|^p \; dx\right)^{\frac{1}{p}} \\
&\leq R + C\left(R^p + R^{p-n}\int_{\mathcal{B}_R} |Du|^p \; dx\right)^{\frac{1}{p}} + C\left(R^{p-n}\int_{\mathcal{B}_R} |Du|^p \; dx\right)^{\frac{1}{p}}. \qquad (48)
\end{aligned}$$

Observe that in (48) we have used the fact that as $u(x) = w(x)$, a.e. $x \in \partial\mathcal{B}_R$, it follows that $u - w \in W_0^{1,p}(\mathcal{B}_R)$, and so, in observance of [9, Theorem 3, p. 265] we estimate

$$\fint_{\mathcal{B}_R} |w-u| \; dx \leq CR\fint_{\mathcal{B}_R} |Dw - Du| \; dx. \qquad (49)$$

Next notice that since $\sigma_\varepsilon \in L^{\gamma,\beta}(\Omega)$, it follows by another application Corollary 2.11 that $\sigma_\varepsilon \in L^{p,\beta}(\Omega)$. Thus, it can be shown that

$$C\int_{\mathcal{B}_R} |\sigma_\varepsilon(x)|^p \; dx \leq C_\varepsilon R^\beta,$$

where by C_ε we denote that this constant depends on the value of ε. Finally, noting that

$$\begin{aligned}
\frac{2\varepsilon}{\lambda\Lambda_*}\int_{\mathcal{B}_{4R}} F(Du) \; dx &\leq \frac{2\varepsilon\Lambda^*}{\lambda\Lambda_*}\int_{\mathcal{B}_{4R}} \left(1+|Du|^2\right)^{\frac{p}{2}} \; dx \\
&\leq \frac{2CR^n\varepsilon\Lambda^*}{\lambda\Lambda_*} + \frac{2\varepsilon\Lambda^* C}{\lambda\Lambda_*}\int_{\mathcal{B}_{4R}} |Du|^p \; dx \qquad (50)
\end{aligned}$$

and combining the preceding inequalities we arrive at the final estimate

$$
\begin{aligned}
&\int_{\mathcal{B}_\rho} |Du|^p\, dx \qquad (51)\\
&\le C\left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}_{4R}} |Du|^p\, dx + CR^n + \varepsilon CR^n + \frac{2\varepsilon\Lambda^* C}{\lambda\Lambda_*}\int_{\mathcal{B}_{4R}} |Du|^p\, dx + C_\varepsilon R^\beta\\
&\qquad + CR^n \left(\fint_{\mathcal{B}_{4R}} |Du|^p\, dx\right) \times \left(\omega\left(CR + C\left(R^p + R^{p-n}\int_{\mathcal{B}_{4R}} |Du|^p\, dx\right)^{\frac{1}{p}}\right.\right.\\
&\qquad\qquad \left.\left. + C\left(R^{p-n}\int_{\mathcal{B}_{4R}} |Du|^p\, dx\right)^{\frac{1}{p}}\right)\right)^{\frac{q-p}{q}}\\
&\le C\left(\frac{\rho}{4R}\right)^n \int_{\mathcal{B}_{4R}} |Du|^p\, dx + C\varepsilon \int_{\mathcal{B}_{4R}} |Du|^p\, dx + C_\varepsilon (4R)^\beta\\
&\qquad + C\left(\int_{\mathcal{B}_{4R}} |Du|^p\, dx\right)\left(\omega\left(C(4R) + C\left((4R)^{p-n}\int_{\mathcal{B}_{4R}} |Du|^p\, dx\right)^{\frac{1}{p}}\right)\right)^{\frac{q-p}{q}}
\end{aligned}
$$

Note that in (51) we use the fact that $C\rho^n \le CR^n$ since $\rho < R$ as well as the fact that $R^n < R^\beta$ since $n > \beta$, assuming without loss that $R < 1$.

We now provide an iteration similar to though not entirely the same as a well-known iteration lemma presented by Giaquinta [15, Lemma 2.1]. This will then provide the final step in the proof. To this end, let $\varphi : [0, +\infty) \to [0, +\infty]$ be the monotone increasing function defined by

$$
\varphi(R) := \int_{\mathcal{B}_R} |Du|^p\, dx. \qquad (52)
$$

Let C_0 henceforth represent the now fixed value of the constant C appearing on the right-hand side of inequality (51). Then using the notation introduced in (52) we may recast (51) in the form

$$
\begin{aligned}
&\varphi(\rho) \qquad (53)\\
&\le C_0\left[\left(\frac{\rho}{4R}\right)^n + \varepsilon + \left[\omega\left(C_0\left[4R + (4R)^{1-\frac{n}{p}}\left(\varphi(4R)\right)^{\frac{1}{p}}\right]\right)\right]^{\frac{q-p}{q}}\right]\varphi(4R) + C_\varepsilon (4R)^\beta.
\end{aligned}
$$

Now let $\alpha > 0$ satisfy the following:

$$
n - p < \alpha < \beta < n. \qquad (54)
$$

Then choose a $\tau \in \left(0, \frac{1}{4}\right)$ and $1 > \varepsilon,\, \varepsilon_1 > 0$ sufficiently small such that

$$
C_0\left[\tau^n + \varepsilon + \left[\omega\left(\varepsilon_1\right)\right]^{\frac{q-p}{q}}\right] < \tau^\alpha \qquad (55)
$$

holds. Observe that (55) fixes the values of ε, ε_1, and τ. Henceforth suppose that $R_0 > 0$ is a constant that is selected sufficiently small such that each of the inequalities

$$R_0 < \min\left\{\left(\frac{\varepsilon_1^p\left(1-\tau^{\beta-\alpha}\right)\tau^\alpha}{C_\varepsilon 2^{p+1}C_0^{p+1}}\right)^{\frac{1}{\beta+p-n}}, \frac{\varepsilon_1}{2C_0}, \operatorname{dist}\left(x_0, \partial\Omega\right)\right\} \tag{56}$$

and

$$R_0^{p-n}\varphi\left(R_0\right) < \frac{\varepsilon_1^p}{2^{p+1}C_0^p}. \tag{57}$$

is satisfied. By setting $R = \frac{1}{4}R_0$ and $\rho = \tau R_0$ we obtain from (53) the initial estimate

$$\varphi(\tau R_0) \le \tau^\alpha \varphi\left(R_0\right) + C_\varepsilon R_0^\beta. \tag{58}$$

We claim that we can now inductively iterate estimate (58). Let $k \in \mathbb{N}$ be fixed but otherwise arbitrary and suppose that the inequality

$$\varphi\left(\tau^k R_0\right) \le \tau^{k\alpha}\varphi\left(R_0\right) + C_\varepsilon R_0^\beta \sum_{j=0}^{k-1} \tau^{j\alpha+(k-j-1)\beta} \tag{59}$$

holds. Provided that the induction hypothesis (59) holds, it follows that the inequality

$$C_0\left(\tau^k R_0 + \left(\tau^k R_0\right)^{1-\frac{n}{p}}\left(\varphi\left(\tau^k R_0\right)\right)^{\frac{1}{p}}\right) < \varepsilon_1 \tag{60}$$

holds. Indeed, we simply note that

$$
\begin{aligned}
& C_0\left(\tau^k R_0+\left(\tau^k R_0\right)^{1-\frac{n}{p}}\left(\varphi\left(\tau^k R_0\right)\right)^{\frac{1}{p}}\right) \\
& \leq C_0 R_0+C_0\left[\left(\tau^k R_0\right)^{p-n}\left(\tau^{k\alpha} \varphi\left(R_0\right)+C_\varepsilon R_0^\beta \sum_{j=0}^{k-1} \tau^{j\alpha+(k-j-1)\beta}\right)\right]^{\frac{1}{p}} \\
& \leq C_0 R_0+C_0\left[\left(\tau^k R_0\right)^{p-n}\left(\tau^{k\alpha} \varphi\left(R_0\right)+C_\varepsilon R_0^\beta \tau^{k\alpha} \tau^{-\alpha} \sum_{j=0}^{\infty} \tau^{j(\beta-\alpha)}\right)\right]^{\frac{1}{p}} \\
& \leq C_0 R_0+C_0\left[\left(\tau^k R_0\right)^{p-n}\left(\tau^{k\alpha} \varphi\left(R_0\right)+\frac{C_\varepsilon}{1-\tau^{\beta-\alpha}} \tau^{k\alpha} \tau^{-\alpha} R_0^\beta\right)\right]^{\frac{1}{p}} \\
& =C_0 R_0+C_0\left[\tau^{k(\alpha+p-n)} R_0^{p-n} \varphi\left(R_0\right)+\frac{C_\varepsilon}{1-\tau^{\beta-\alpha}} \tau^{k(\alpha+p-n)} \tau^{-\alpha} R_0^{\beta+p-n}\right]^{\frac{1}{p}} \\
& \leq C_0 R_0+C_0\left[R_0^{p-n} \varphi\left(R_0\right)+\frac{C_\varepsilon}{1-\tau^{\beta-\alpha}} \tau^{-\alpha} R_0^{\beta+p-n}\right]^{\frac{1}{p}} \\
& \leq \frac{\varepsilon_1}{2}+C_0\left[\frac{\varepsilon_1^p}{2^{p+1} C_0^p}+\frac{\varepsilon_1^p}{2^{p+1} C_0^p}\right]^{\frac{1}{p}} \\
& =\varepsilon_1 .
\end{aligned} \tag{61}
$$

This means that the argument of the function ω remains sufficiently small in order to reapply (53) now with $\rho=\tau^{k+1}R_0$ and $R=\frac{1}{4}\tau^k R_0$. In fact, applying inequality (53) with $\rho=\tau^{k+1}R_0$ and $R=\frac{1}{4}\tau^k R_0$ yields

$$
\begin{aligned}
& \varphi\left(\tau^{k+1} R_0\right) \\
& \leq C_0\left[\left(\frac{\rho}{4R}\right)^n+\varepsilon+\left[\omega\left(C_0\left[4R+(4R)^{1-\frac{n}{p}}\left(\varphi(4R)\right)^{\frac{1}{p}}\right]\right)\right]^{\frac{q-p}{q}}\right] \varphi(4R)+C_\varepsilon(4R)^\beta \\
& =C_0\left[\tau^n+\varepsilon+\left[\omega\left(\varepsilon_1\right)\right]^{\frac{q-p}{q}}\right] \varphi\left(\tau^k R_0\right)+C_\varepsilon\left(\tau^k R_0\right)^\beta \\
& \leq \tau^\alpha \varphi\left(\tau^k R_0\right)+C_\varepsilon\left(\tau^k R_0\right)^\beta \\
& \leq \tau^\alpha\left[\tau^{k\alpha} \varphi\left(R_0\right)+C_\varepsilon R_0^\beta \sum_{j=0}^{k-1} \tau^{j\alpha+(k-j-1)\beta}\right]+C_\varepsilon\left(\tau^k R_0\right)^\beta \\
& =\tau^{(k+1)\alpha} \varphi\left(R_0\right)+C_\varepsilon R_0^\beta\left[\tau^\alpha \sum_{j=0}^{k-1}\left(\tau^{j\alpha+(k-j-1)\beta}\right)+\tau^{k\beta}\right] \\
& =\tau^{(k+1)\alpha} \varphi\left(R_0\right)+C_\varepsilon R_0^\beta \sum_{j=0}^{k} \tau^{j\alpha+(k-j)\beta},
\end{aligned} \tag{62}
$$

which completes the induction step; note that we have used the fact that (61) holds so that (55) can again be invoked. We conclude that (58) may inductively iterated to obtain inequality (59) for each $k \in \mathbb{N}$. Moreover, for any fixed but otherwise arbitrary $k \in \mathbb{N}$ we may further estimate from (62) that

$$\begin{aligned}
&\varphi\left(\tau^{k+1}R_0\right)\\
&\le C_0\left[\left(\frac{\rho}{4R}\right)^n+\varepsilon+\left[\omega\left(C_0\left[4R+(4R)^{1-\frac{n}{p}}\left(\varphi(4R)\right)^{\frac{1}{p}}\right]\right)\right]^{\frac{q-p}{q}}\right]\varphi(4R)+C_\varepsilon(4R)^\beta\\
&\le \tau^{(k+1)\alpha}\varphi\left(R_0\right)+C_\varepsilon R_0^\beta\sum_{j=0}^{k}\tau^{j\alpha+(k-j)\beta}\\
&\le \tau^{(k+1)\alpha}\varphi\left(R_0\right)+C_\varepsilon R_0^\beta\tau^{k\alpha}\sum_{j=0}^{\infty}\tau^{k(\beta-\alpha)}\\
&= \tau^{(k+1)\alpha}\varphi\left(R_0\right)+\frac{C_\varepsilon R_0^{\beta-\alpha}\tau^{-\alpha}\left(\tau^{k+1}R_0\right)^\alpha}{1-\tau^{\beta-\alpha}},
\end{aligned}$$

seeing as $\beta-\alpha>0$.

Crucially, note that in (61) we tacitly require that $p+\alpha-n>0$ so that $\tau^{k(p+\alpha-n)}<1$, for each k. But since $\alpha>n-p$ this inequality is satisfied for each n, and so, we do not require a dimension restriction at this point. Furthermore, note that $R_0^{\beta+p-n}$ remains uniformly bounded as $R_0\to 0^+$ due to the fact that $\beta+p-n>(n-p)+p-n=0$. In summary, recalling that τ and ε_1 were earlier fixed, we observe that if (57) holds, then the estimate in (58) may be iterated to obtain that (59) holds.

Suppose now that $0<\rho<\frac{1}{4}R_0$ is given. Then there is a $k\in\mathbb{N}$ such that

$$\tau^{k+1}R_0\le\rho<\tau^kR_0. \tag{63}$$

Then it holds that

$$\int_{\mathcal{B}_\rho}|Du|^p\,dx\le\left(\frac{\rho}{R_0}\right)^\alpha\int_{\mathcal{B}_{R_0}}|Du|^p\,dx+\left(\frac{\rho}{R_0}\right)^\alpha\frac{C_\varepsilon R_0^\beta\tau^{-\alpha}}{1-\tau^{\beta-\alpha}}. \tag{64}$$

Hence, we conclude that

$$\begin{aligned}
\sup_{\frac{1}{4}R_0>\rho>0}\frac{1}{\rho^\alpha}\int_{\mathcal{B}_\rho}|Du|^p\,dx&\le\frac{1}{R_0^\alpha}\int_{\mathcal{B}_{R_0}}|Du|^p\,dx+\frac{C_\varepsilon R_0^\beta\tau^{-\alpha}}{1-\tau^{\beta-\alpha}}\\
&<+\infty.
\end{aligned}\tag{65}$$

Estimate (65) holds only for balls centered at $x_0\in\Omega$, as originally deduced at the beginning of this proof. In order to complete the argument define the set $\Omega_0\subseteq\Omega$ by

$$\Omega_0:=\left\{x\in\Omega\ :\ \liminf_{\substack{0<R<\operatorname{dist}(x,\partial\Omega)\\ R\to 0}}R^p\fint_{\mathcal{B}_R(x)}|Du|^p\,dy<\frac{\varepsilon^p}{2^{p+1}C_0^p}\right\}. \tag{66}$$

Note that (57) and hence (65) holds at each $x_0 \in \Omega_0$. Furthermore, $\Omega_0 \supseteq \mathcal{L}_{Du}$, with $\mathcal{L}_{Du}$ the set of Lebesgue points for Du, and so, it follows that

$$|\Omega \setminus \Omega_0| = 0. \tag{67}$$

We can then use the absolute continuity of the map

$$x \mapsto \int_{\mathcal{B}_{R_0}(x)} |Du|^p \, dy \tag{68}$$

in the standard way to argue that Ω_0 is an open set. This together with a standard covering argument allows us to conclude, as in [15, pp. 171 ff.], that $Du \in L^{p,\alpha}(\Omega')$, for each $\Omega' \Subset \Omega_0$, as desired. If by $\mathcal{S}$ we denote the singular set (i.e., the set on which u may be singular), then it also holds that $\mathcal{S} \subseteq \Omega \setminus \Omega_0$ and, hence, $|\mathcal{S}| = 0$.

Finally, it remains to argue that u enjoys Sobolev-Morrey $W^{1,(p,\alpha)}(\Omega')$ regularity on each $\Omega' \Subset \Omega_0$. But this claim follows from a standard argument involving the Poincaré inequality, the isomorphism between Campanato and Morrey spaces, and the embedding $W^{1,(p,\alpha)}_{\text{loc}}(\Omega_0) \hookrightarrow \mathcal{C}^{0,1-\frac{n-\alpha}{p}}_{\text{loc}}(\Omega_0)$. Thus, we conclude that u is locally Hölder continuous on Ω_0 with Hölder exponent $1-\frac{n-\alpha}{p}$ with $n-p < \alpha < \beta < n$. As this is the desired conclusion, the proof is thus complete. □

We conclude this section and the paper with an example and three remarks.

Example 3.2. Consider the function

$$f(x,u,\xi) := a(x,u)|\xi|^p + b(x,u)|\xi|^r \tag{69}$$

with $r < p$. Supposing that b is bounded in u and that a satisfies conditions (A1)–(A2), then this function is admissible in our theory. To see this note in particular that $f(x,u,\xi)$ is asymptotically related to $a(x,u)\left(1+|\xi|^2\right)^{\frac{p}{2}}$. Since a is bounded, for $|\xi| \gg 1$ it follows that $1+|\xi|^2 \approx |\xi|^2$, and so, we find that $a(x,u)\left(1+|\xi|^2\right)^{\frac{p}{2}}$ is asymptotically related to $a(x,u)|\xi|^p$. Since the property of "asymptotic relatedness" generates an equivalence class, we conclude that f is asymptotically related to $a(x,u)|\xi|^p$, which is itself asymptotically related to $(x,u,\xi) \mapsto a(x,u)\left(1+|\xi|^2\right)^{\frac{p}{2}}$. Since this latter map is admissible, we obtain that (69) is admissible in our theory.

Furthermore, we observe that the coefficient of $|\xi|^p$ is allowed to depend on u as well as x. As mentioned in Section 1 this is one distinguishing aspect of our results; so far as we are aware this level of generality is new for this class of problem. We also point out that due to our weak structure conditions, it turns out that the function b in (69) need only satisfy

$$\sup_{\substack{0<r<\operatorname{dist}(x,\partial\Omega)\\ x\in\Omega}} \frac{1}{r^\beta} \int_{\mathcal{B}_r(x)} \left(\sup_{u\in\mathbb{R}^N} |b(y,u)|^{\frac{\gamma}{p-r}} \right) dy < +\infty, \tag{70}$$

where $\gamma > p$ and $\beta \in (n-p, n)$. Then condition (G3) is satisfied with σ_ε defined by

$$\sigma_\varepsilon(x) := \sup_{u \in \mathbb{R}^N} \left(\frac{1}{\varepsilon} |b(x,u)| \right)^{\frac{1}{p-r}}. \tag{71}$$

In particular, the function b is not required to be bounded but only to belong to $L^{\frac{\gamma}{p-r},\beta}(\Omega)$ uniformly with respect to u.

Remark 3.3. It is worth noting in Example 3.2 that the number b can take large negative values so that, in particular, the function f can be non-convex in ξ.

Remark 3.4. If we could allow the function σ_ε to depend on u, then it would be possible to further extend the preceding example to the case in which b could also be unbounded with respect to u. While this type of extension (i.e., allowing σ_ε to depend on u) has been successfully carried out in certain other settings (see [12], for example), for the sake of brevity and simplicity of exposition we have elected not to pursue this potential generalization in this paper, and we leave this for future work.

Remark 3.5. Finally, observe that if we strengthen our hypotheses in Theorem 3.1 so that σ_ε has no dependence on x, then we obtain that the conclusion of Theorem 3.1 holds for each $\alpha \in (n-p, n)$. In particular, the constant β of hypothesis (G3) no longer serves as an upper bound on the degree of partial Hölder continuity we can expect u to satisfy.

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