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On Extreme Operators Whose Adjoints Preserve Extreme Points*

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In this paper we study Banach spaces X, which we call nice, such that any extreme operator T from a Banach space Y to X is a nice operator, that is, T^*, the adjoint of T, preserves extreme points. We get several necessary conditions for being nice. The main result in the paper is the characterization of nice finite-dimensional Banach spaces.

Keywords: Banach space, extreme operator, nice operator

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1. Introduction

Throughout this paper, all Banach spaces are assumed to be real. Given a Banach space X, B_X, S_X and E_X will stand for the closed unit ball of X, the unit sphere of X and the set of extreme points of B_X, respectively.

The space of all bounded linear operators from a Banach space X to a Banach space Y will be denoted by $\mathcal{L}(X,Y)$ and, as usual $\mathcal{L}(X,\mathbb{R})$ will be written X^*. For any T in $\mathcal{L}(X,Y)$, T^* will denote its adjoint. Elements in $E_{\mathcal{L}(X,Y)}$ will be called extreme operators and we will write $E(X,Y)$ instead of $E_{\mathcal{L}(X,Y)}$.

Now we introduce a class of operators which is closely related to the class of extreme operators and which plays a central role in this paper.

Definition 1.1. Let X and Y be Banach spaces. An operator T in $\mathcal{L}(X,Y)$ is said to be *nice* if $T^*(E_{Y^*}) \subseteq E_{X^*}$.

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The set of nice operators in $\mathcal{L}(X,Y)$ is denoted by $N(X,Y)$.

It is easy to prove that every nice operator is extreme. Nice operators have been widely studied since Blumenthal, Lindenstrauss and Phelps introduced then in [2] (see, for example [1, 3, 5, 6, 11, 12, 14]). In [2] the authors posed the problem about the coincidence of nice and extreme operators defined in spaces of continuous functions (i.e., spaces $\mathcal{C}(K)$, where K is a compact Hausdorff space). Several positive results have been obtained since then (see [2, 4, 13, 15, 16, 19]) but Sharir proved in [17, 18] the existence of non-nice extreme operators in $\mathcal{C}(K)$-type spaces.

The main goal in this paper is to study the Banach spaces appearing in the following definition.

Definition 1.2. A Banach space X is said to be "*nice*" if $N(Y,X) = E(Y,X)$ for every Banach space Y.

One can ask for the "dual" definition of nice Banach spaces, that is, Banach spaces X such that $N(X,Y) = E(X,Y)$ for any Banach space Y, but the following result shows that this kind of spaces cannot exist.

Proposition 1.3. *Let X be a Banach space, then there exists a Banach space Y such that $N(X,Y) \neq E(X,Y)$.*

Proof. Given a Banach space Y, we can consider the Banach space $X \oplus_2 Y$ consisting of the vector space $X \oplus Y$ endowed with the norm $\|(x,y)\| = \sqrt{\|x\|^2 + \|y\|^2}$ and we can also consider the operator $T : X \to X \oplus_2 Y$ defined by $T(x) = (x,0)$ for all x in X. Note that

$$\|T(x)\| = \|x\| \le \|(x,y)\| \text{ for all } x \in X \text{ and } y \in Y,$$

so for x in S_X $\|T(x)\| = 1$ and $\|T\| = 1$.

Let us see that T is an extreme operator.

Let S be in $\mathcal{L}(X, X \oplus_2 Y)$ such that $\|T \pm S\| \le 1$. Then $S(x) = (S_1(x), S_2(x))$ for all x in X, where $S_1 \in \mathcal{L}(X)$ and $S_2 \in \mathcal{L}(X,Y)$. Therefore, for each x in S_X we have

$$\begin{aligned}\|x \pm S_1(x)\|^2 &\le \|x \pm S_1(x)\|^2 + \|S_2(x)\|^2 \\ &= \|(x,0) \pm (S_1(x), S_2(x))\|^2 = \|T(x) \pm S(x)\|^2 \le 1.\end{aligned}$$

Since Id_X (the identity map on X) is an extreme operator, we have $S_1 = 0$. Hence, for each x in S_X we have

$$1 + \|S_2(x)\|^2 = \|x\|^2 + \|S_2(x)\|^2 \le 1$$

and consequently $S_2 = 0$. So $S = 0$, and T is an extreme operator in $\mathcal{L}(X, X \oplus_2 Y)$.

Let us see that T is a non-nice operator.

Via the identification $(X \oplus_2 Y)^* = X^* \oplus_2 Y^*$, the operator $T^* : X^* \oplus_2 Y^* \to X^*$ is given by

$$T^*(x^*, y^*) = x^* \text{ for all } x^* \in X^* \text{ and } y^* \in Y^*.$$

Let y^* be in E_{Y^*} then $(0, y^*) \in E_{X^* \oplus_2 Y^*}$, therefore

$$T^*(0, y^*) = 0 \notin E_{X^*}.$$

So T is a non-nice operator. □

One of the main results in this paper is the characterization of nice finite-dimensional Banach spaces (Theorem 2.12). For infinite-dimensional spaces we give necessary conditions for being nice. As a consequence we characterize the spaces of continuous functions on a locally compact Hausdorff space, vanishing at infinity which are nice (Corollary 2.5).

2. The results

We begin this section by showing that the class of nice Banach spaces is non-empty.

Proposition 2.1. *Let I be a nonempty set, then $c_0(I)$ is nice.*

Proof. We only have to prove that $E(X, c_o(I)) \subseteq N(X, c_0(I))$ for every Banach space X. For all $i \in I$, we denote by $e_i^* : c_0(I) \to \mathbb{R}$ the functional defined by $e_i^*(x) = x(i)$. It is easy to see that $E_{c_0(I)^*} = \{\pm e_i^* : i \in I\}$. Let T be an extreme operator in $\mathcal{L}(X, c_0(I))$ and $x_i^* = T^*(e_i^*)$. We are going to prove that $x_i^* \in E_{X^*}$ for every $i \in I$ and so T is nice. If there exists $i_0 \in I$ such that $x_{i_0}^* \notin E_{X^*}$, then there exist $y^*, z^* \in B_{X^*}$ with $y^* \neq z^*$ such that $x_{i_0}^* = \frac{1}{2}(y^* + z^*)$. Let $S_1, S_2 : X \to c_0(I)$ defined by

$$S_1(x)(i) = \begin{cases} y^*(x) & \text{if } i = i_0 \\ T(x)(i) & \text{if } i \neq i_0 \end{cases}$$

$$S_2(x)(i) = \begin{cases} z^*(x) & \text{if } i = i_0 \\ T(x)(i) & \text{if } i \neq i_0 \end{cases}$$

so S_1, $S_2 \in \mathcal{L}(X, c_0(I))$, $S_1 \neq S_2$, $\|S_1\| \leq 1$, $\|S_2\| \leq 1$ and $T = \frac{1}{2}(S_1 + S_2)$, which contradicts that T is an extreme operator. Thus $T^*(e_i^*) \in E_{X^*}$ for all $i \in I$ and $T \in N(X, c_0(I))$. □

It is clear that if I is finite with $\text{card}(I) = n$, then $c_0(I) = l_\infty^n$, so this Banach space is nice.

We are going to give a sufficient condition on a Banach space for being non-nice.

Theorem 2.2. *Let X be a Banach space such that there exists $e_0^* \in E_{X^*}$ which satisfies:*

i) $X^* = \overline{\mathrm{lin}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*}$.

ii) *For each $e^* \in E_{X^*} \setminus \{\pm e_0^*\}$, there exists $x \in B_X$ such that $e_0^*(x) = 0$ and $e^*(x) = 1$.*

Then X is non-nice.

Proof. We define a new norm $\|\cdot\|_0$ on X as follows

$$\|x\|_0 = \max\{\|x\|, |\, 2e_0^*(x)\, |\} \qquad (x \in X)$$

The norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent in X. So it is easy to see that

$$B_{(X,\|\cdot\|_0)^*} = \mathrm{co}(B_{(X,\|\cdot\|)^*} \cup \{\pm 2e_0^*\}).$$

Now, we consider the operator $T : (X, \|\cdot\|_0) \to (X, \|\cdot\|)$ defined by $T(x) = x$ and we are going to show that T is a non-nice extreme operator. The operator $T^* : (X^*, \|\cdot\|^*) \to (X^*, \|\cdot\|_0^*)$ is given by $T^*(x^*) = x^*$ for all x^* in X^*, so $T^*(e_0^*) = e_0^* \notin E_{(X,\|\cdot\|_0)^*}$ and T is non-nice. We claim that $E_{X^*} \setminus \{\pm e_0^*\} \subseteq E_{(X,\|\cdot\|_0)^*}$. Let e^* be in $E_{X^*} \setminus \{\pm e_0^*\}$. First let's suppose that

$$e^* = \alpha x^* + 2(\lambda - \mu)e_0^*$$

where $\alpha, \lambda, \mu \in \mathbb{R}_0^+$ such that $\alpha + \lambda + \mu = 1$ and $x^* \in B_{X^*}$. For *ii*), there exists $x \in B_X$ such that $e_0^*(x) = 0$ and $e^*(x) = 1$. So we have that

$$1 = e^*(x) = \alpha x^*(x) \;\text{ and from here }\; 1 = \alpha \mid x^*(x) \mid \leq \alpha \|x^*\| \leq \alpha$$

therefore $\alpha = 1$ and then $\lambda = \mu = 0$ and $e^* = x^*$.

Now assume that $e^* = \frac{1}{2}(y^* + z^*)$ with $y^*, z^* \in B_{(X,\|\cdot\|_0)^*}$. Then there exist $x_1^*, x_2^* \in B_{X^*}$ and $\alpha_1, \alpha_2, \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}_0^+$ such that

$$y^* = \alpha_1 x_1^* + 2(\lambda_1 - \mu_1)e_0^*, \qquad \alpha_1 + \lambda_1 + \mu_1 = 1,$$

$$z^* = \alpha_2 x_2^* + 2(\lambda_2 - \mu_2)e_0^*, \qquad \alpha_2 + \lambda_2 + \mu_2 = 1.$$

If $\alpha_1 + \alpha_2 = 0$, then $e^* = (\lambda_1 + \lambda_2 - \mu_1 - \mu_2)e_0^*$, so $e^* = \pm e_0^*$, a contradiction. If $\alpha_1 + \alpha_2 \neq 0$, we write

$$e^* = \frac{\alpha_1 + \alpha_2}{2} \left(\frac{\alpha_1 x_1^* + \alpha_2 x_2^*}{\alpha_1 + \alpha_2} \right) + 2 \left[\frac{\lambda_1 + \lambda_2}{2} - \frac{\mu_1 + \mu_2}{2} \right] e_0^*.$$

Since $\frac{\alpha_1 x_1^* + \alpha_2 x_2^*}{\alpha_1 + \alpha_2} \in B_{X^*}$, then $\frac{\alpha_1 + \alpha_2}{2} = 1$, that is $\alpha_1 + \alpha_2 = 2$, and thus $\alpha_1 = \alpha_2 = 1$, which implies that $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$. Therefore $e^* = \frac{1}{2}(x_1^* + x_2^*)$, but $x_1^*, x_2^* \in B_{X^*}$ and we conclude $e^* = x_1^* = x_2^*$. That is $e^* \in E_{(X,\|\cdot\|_0)^*}$.

Now that the claim is proved, we are going to show that T is an extreme operator. Let S_1, S_2 be in $\mathcal{L}((X, \|\cdot\|_0), (X, \|\cdot\|))$ be with $\|S_1\| \leq 1$, $\|S_2\| \leq 1$ such that $T = \frac{1}{2}(S_1 + S_2)$. Then $T^* = \frac{1}{2}(S_1^* + S_2^*)$. It follows from the claim that $T^*(e^*) \in E_{(X,\|\cdot\|_0)^*}$ for all e^* in $E_{X^*} \setminus \{\pm e_0^*\}$ and hence

$$T^*(e^*) = S_1^*(e^*) = S_2^*(e^*), \quad \text{for all } e^* \in E_{X^*} \setminus \{\pm e_0^*\}.$$

Thus, by *i*), $S_1^* = S_2^*$, and therefore $S_1 = S_2$. So T is a non-nice extreme operator, and as a result X is non-nice. □

Remark 2.3. If we have that $e_0^* \in \overline{E_{X^*} \setminus \{e_0^*\}}^{w^*}$, then *i*) is verified. Indeed, if $e_0^* \in \overline{E_{X^*} \setminus \{e_0^*\}}^{w^*}$, then $e_0^* \in \overline{E_{X^*} \setminus \{\pm e_0^*\}}^{w^*}$, so $-e_0^* \in \overline{E_{X^*} \setminus \{\pm e_0^*\}}^{w^*}$ and therefore $B_{X^*} = \overline{\text{co}}^{w^*}(E_{X^*} \setminus \{\pm e_0^*\})$. So

$$X^* = \overline{\text{lin}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*}$$

and *i*) is verified.

We are going to obtain several important consequences of this theorem.

Corollary 2.4. *Let L be a locally compact Hausdorff space, and let L' denote the set of all cluster points of L. If L' is nonempty, then $\mathcal{C}_0(L)$ is non-nice.*

Proof. Let t be in L' and $e_0^* = \delta_t$. Let us see that we have the hypotheses of the above theorem. For each s in $L \setminus \{t\}$, there exists $f_s \in B_{\mathcal{C}_0(L)}$ such that $\delta_t(f_s) = f_s(t) = 0$ and $\delta_s(f_s) = f_s(s) = 1$ and then *ii*) is verified. Now we are going to prove that $\delta_t \in \overline{E_{\mathcal{C}_0(L)^*} \setminus \{\delta_t\}}^{w^*}$. Since $t \in L'$ then there exists a net $\{s_i\}_{i\in I}$ in $L \setminus \{t\}$ such that $\{s_i\}_{i\in I} \to t$, so $\{\delta_{s_i}\}_{i\in I} \xrightarrow{w^*} \delta_t$. Taking into account the previous remark, we have that $X^* = \overline{\text{lin}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*}$. Now we can apply Theorem 2.2 and conclude that $\mathcal{C}_0(L)$ is non-nice. □

As a result of the above corollary and Proposition 2.1 we obtain:

Corollary 2.5. *Let L be a locally compact Hausdorff space, then $\mathcal{C}_0(L)$ is nice if and only if L is discrete. In particular, if K is a compact Hausdorff space, then $\mathcal{C}(K)$ is nice if and only if K is finite.*

Another consequence of Theorem 2.2 is about spaces of type $L_1(\mu)$.

Corollary 2.6. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space such that μ is σ-finite, and set $L_1(\mu) := L_1(\Omega, \mathcal{A}, \mu)$. If $\dim(L_1(\mu)) > 2$, then $L_1(\mu)$ is non-nice.*

Proof. If $L_1(\mu)$ is finite-dimensional, then it is isometrically isomorphic to l_1^n with $n = \dim(L_1(\mu)) > 2$. We can apply Theorem 2.2, by considering

$e_0^* = (1,\dots,1)$. Now $l_\infty^n = \operatorname{lin}(E_{l_\infty^n} \setminus \{\pm e_0^*\})$ because it is easy to check that $\{e_1^*, e_2^*, \dots, e_n^*\}$ is a basis of $\mathbb{R}^n$, where

$$\begin{aligned} e_1^* &= (-1,1,1,\dots,1) \\ e_2^* &= (1,-1,1,\dots,1) \\ &\cdots \\ e_i^* &= (1,1,\dots,1,\overset{i}{-1},1,\dots,1) \\ &\cdots \\ e_n^* &= (1,1,\dots,1,-1) \end{aligned}$$

such that $e_i^* \in E_{l_\infty^n}$, $(i = 1,2,\dots,n)$ and so condition $i)$ in Theorem 2.2 is satisfied. For $ii)$, let $e^* \in E_{l_\infty^n} \setminus \{\pm e_0^*\}$, we know that at least there exist $i,j \in \{1,\dots,n\}$ such that $e^*(i) = 1$ and $e^*(j) = -1$ then we take $x \in B_{l_1^n}$ as

$$x = \left(0,\dots,\overset{i}{\frac{1}{2}},\dots,\overset{j}{\frac{-1}{2}},\dots,0\right).$$

Therefore, $e_0^*(x) = 0$ and $e^*(x) = 1$. So l_1^n is non-nice.

If $L_1(\mu)$ is infinite-dimensional, we take $e_0^* \equiv 1 \in L_1(\mu)^* = L_\infty(\mu)$ and in this case, $e^* \in E_{L_\infty(\mu)}$ if and only if $\mid e^*(t) \mid = 1$ a.e.

First we are going to prove that $e_0^* \in \overline{E_{L_\infty(\mu)} \setminus \{e_0^*\}}^{w^*}$ and then Remark 2.3 tells us that condition $i)$ in Theorem 2.2 is fulfilled.

Let U be a basic w^*-neigborhood of 0 in $L_\infty(\mu)$. There exist $f_1, f_2, \dots, f_n$ in $L_1(\mu)$ and $\varepsilon > 0$ such that

$$U = \{\varphi \in L_\infty(\mu) : \left|\int_\Omega f_i(t)\varphi(t)d\mu\right| < \varepsilon \ \text{ for all } \ i = 1,\dots,n\}.$$

Since $\dim(L_1(\mu)) = \infty$, there exist $\{A_n\}_{n\in\mathbb{N}}$ measurable sets such that $A_n \cap A_m = \emptyset$ for all $n,m \in \mathbb{N}$ with $n \neq m$, and $0 < \mu(A_n) < +\infty$ for all $n \in \mathbb{N}$. Let $A = \bigcup_{n=1}^\infty A_n$. Then, for f in $L_1(\mu)$ we have

$$+\infty > \int_A f d\mu = \sum_{n=1}^\infty \int_{A_n} f d\mu \ \text{ so } \ \left\{\int_{A_n} f d\mu\right\}_{n\in\mathbb{N}} \to 0.$$

Therefore, for each $\varepsilon > 0$ there exists $N(f,\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(f,\varepsilon)$

$$\left|\int_{A_n} f d\mu\right| < \varepsilon.$$

Thus, given $f_1, f_2, \dots, f_n \in L_1(\mu)$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mid \int_{A_N} f_i d\mu \mid < \frac{\varepsilon}{2}$, for all $i = 1,\dots,n$ with $0 < \mu(A_N) < +\infty$. If we consider

$$e^*(t) = \begin{cases} 1 & \text{if } t \in \Omega \setminus A_N \\ -1 & \text{if } t \in A_N, \end{cases}$$

then we have that $e^* \neq e_0^*$ and $\mid e^*(t) \mid = 1$ for all $t \in \Omega$, so $e^* \in E_{L_\infty(\mu)}$. Therefore

$$\begin{aligned}\left|\int_\Omega f_i(t)(e^*(t) - e_0^*(t))d\mu(t)\right| &= \left|\int_{A_N} f_i(t)(e^*(t) - e_0^*(t))d\mu(t)\right| \\ &= \left|2\int_{A_N} f_i(t)d\mu(t)\right| < \varepsilon, \quad \text{for all } i = 1, \dots, n.\end{aligned}$$

Therefore, $e^* \in e_0^* + U$. So $e_0^* \in \overline{E_{L_\infty(\mu)} \setminus \{e_0^*\}}^{w^*}$.

Now, we are going to prove that *ii*) is verified. Let e^* be in $E_{L_\infty(\mu)} \setminus \{\pm e_0^*\}$. We consider the following sets

$$A = \{t \in \Omega : e^*(t) = 1\} \quad \text{and} \quad B = \{t \in \Omega : e^*(t) = -1\}$$

and note that $\mu(A) > 0$ and $\mu(B) > 0$. Since μ is a σ-finite measure, $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$ with $\mu(\Omega_n) < +\infty$ for all $n \in \mathbb{N}$. Then $A = \bigcup_{n\in\mathbb{N}}(A \cap \Omega_n)$ and $\mu(A) \leq \sum_{n=1}^\infty \mu(A \cap \Omega_n)$. Thus there exists $n \in \mathbb{N}$ such that $A_0 := A \cap \Omega_n$ is a measurable set with $0 < \mu(A_0) < +\infty$ and such that $e^*(t) = 1$ for all t in A_0. With the same arguments for B we obtain a measurable set B_0 such that for all t in B_0, $e^*(t) = -1$ with $0 < \mu(B_0) < +\infty$. Now we define

$$f = \frac{1}{2\mu(A_0)}\chi_{A_0} - \frac{1}{2\mu(B_0)}\chi_{B_0},$$

where χ_{A_0} and χ_{B_0} denote the characteristic functions of A_0 and B_0 , respectively. So

$$\|f\| = \int_\Omega \mid f \mid d\mu = \int_\Omega \left(\frac{1}{2\mu(A_0)}\chi_{A_0} + \frac{1}{2\mu(B_0)}\chi_{B_0}\right) d\mu = \frac{\mu(A_0)}{2\mu(A_0)} + \frac{\mu(B_0)}{2\mu(B_0)} = 1,$$

that is $f \in B_{L_1(\mu)}$. Let us see that $e_0^*(f) = 0$ and $e^*(f) = 1$:

$$\begin{aligned}e_0^*(f) &= \int_\Omega f d\mu = \int_\Omega \left(\frac{1}{2\mu(A_0)}\chi_{A_0} - \frac{1}{2\mu(B_0)}\chi_{B_0}\right) d\mu \\ &= \frac{\mu(A_0)}{2\mu(A_0)} - \frac{\mu(B_0)}{2\mu(B_0)} = 0\end{aligned}$$

and

$$\begin{aligned}e^*(f) &= \int_\Omega e^*(t)f(t)d\mu = \int_\Omega e^*(t)\left(\frac{1}{2\mu(A_0)}\chi_{A_0} - \frac{1}{2\mu(B_0)}\chi_{B_0}\right) d\mu \\ &= \int_\Omega (\chi_A - \chi_B)\frac{1}{2\mu(A_0)}\chi_{A_0} d\mu - \int_\Omega (\chi_A - \chi_B)\frac{1}{2\mu(B_0)}\chi_{B_0} d\mu \\ &= \int_\Omega \frac{1}{2\mu(A_0)}\chi_{A_0} d\mu + \int_\Omega \frac{1}{2\mu(B_0)}\chi_{B_0} d\mu \\ &= \frac{\mu(A_0)}{2\mu(A_0)} + \frac{\mu(B_0)}{2\mu(B_0)} = 1.\end{aligned}$$

Then *ii*) is verified and we can apply Theorem 2.2 for getting that $L_1(\mu)$ is non-nice. □

If $\dim(L_1(\mu)) = 1$ or 2, we have that $L_1(\mu)$ is isometrically isomorphic, respectively, to $\mathbb{R}$ and to l^2_∞, so the space is nice.

In [2] it is proved a stronger result for spaces $\mathcal{C}(K)$ where K is a compact Hausdorff space. Now we prove the same result for spaces $\mathcal{C}_0(L)$.

Theorem 2.7. *Let L be a locally compact Hausdorff space such that $L' \neq \emptyset$. Then there exist a Banach space X and an extreme operator T in $\mathcal{L}(X, \mathcal{C}_0(L))$ such that $T^*(\delta_t) \notin E_{X^*}$ for all $t \in L'$.*

Proof. Let $M = L'$ and Y be a vectorial subspace of $l_\infty(M)$ with norm $\|\cdot\|_Y$ such that $(Y, \|\cdot\|_Y)$ is a Banach space for which:

i) $e_t \in B_Y$ for every t in M and $\overline{\text{lin}\{e_t : t \in M\}}^{\|\cdot\|_Y} = Y$.
ii) $\{e_t^* : t \in M\}$ is a bounded subset of $(Y, \|\cdot\|_Y)^*$.

(For example, $Y = l_p(M)$ $(1 \leq p < \infty)$ or $Y = c_0(M)$). We consider $X = \mathcal{C}_0(L) \times Y$ with the norm given by

$$\|(f, y)\| = \max\{\|f\|_\infty, \|y\|_Y, \sup_{t \in M} \mid f(t) \pm y(t) \mid\},$$

$\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent in $\mathcal{C}_0(L) \times Y$. Let's note that

$$\|(f, 0)\| = \|f\|_\infty \quad \text{for all } f \in \mathcal{C}_0(L)$$

$$\|(0, y)\|_\infty = \|y\|_Y \quad \text{for all } y \in Y.$$

We define the operator $T : \mathcal{C}_0(L) \times Y \to \mathcal{C}_0(L)$ by $T(f, y) = f$. It is clear that T is a norm-one bounded linear operator. Moreover

$$T^* : \mathcal{C}_0(L)^* \to \mathcal{C}_0(L)^* \times Y^*$$

is given by $T^*(x^*) = (x^*, 0)$ for all $x^* \in \mathcal{C}_0(L)^*$, and $T^*(\delta_s) = (\delta_s, 0)$ for all s in L. Let's see that if t is in M then $(\delta_t, 0) \notin E_{(\mathcal{C}_0(L) \times Y)^*}$. We can write $(\delta_t, 0)$ as

$$(\delta_t, 0) = \frac{1}{2}((\delta_t, e_t^*) + (\delta_t, -e_t^*)).$$

If $\|(f, y)\| \leq 1$, we have

$$\mid (\delta_t, \pm e_t^*)(f, y) \mid = \mid f(t) \pm y(t) \mid \leq \|(f, y)\| \leq 1,$$

therefore $\|(\delta_t, \pm e_t^*)\| \leq 1$. Thus, T is a non-nice operator.

Now, let's prove that T is an extreme operator.

Let $S : \mathcal{C}_0(L) \times Y \to \mathcal{C}_0(L)$ be a bounded linear operator such that $\|T \pm S\| \leq 1$. Then, for all (f, y) in $\mathcal{C}_0(L) \times Y$ with $\|(f, y)\| \leq 1$, we have

$$\|f \pm S(f, y)\|_\infty = \|T(f, y) \pm S(f, y)\|_\infty \leq 1$$

and in particular

$$\|f \pm S(f,0)\|_\infty \leq 1 \ \text{ for all } f \in \mathcal{C}_0(L) \text{ with } \|f\|_\infty \leq 1.$$

Since $\mathrm{Id}_{\mathcal{C}_0(L)}$ is an extreme operator in $\mathcal{L}(\mathcal{C}_0(L))$, we obtain

$$S(f,0) = 0 \ \text{ for all } f \in \mathcal{C}_0(L).$$

Thus, for all (f,y) in $\mathcal{C}_0(L) \times Y$ with $\|(f,y)\| \leq 1$,

$$\|f \pm S(0,y)\|_\infty \leq 1.$$

If t is in M, for all f in $\mathcal{C}_0(L)$ with $\|(f,e_t)\| \leq 1$, we have

$$\|f \pm S(0,e_t)\|_\infty \leq 1.$$

For all k in L, $k \neq t$, there exists f_k in $B_{\mathcal{C}_0(L)}$ such that $f_k(t) = 0$ and $f_k(k) = 1$, so $\|(f_k,e_t)\| = 1$. Then

$$| f_k(k) \pm S(0,e_t)(k) | = | 1 \pm S(0,e_t)(k) | \leq \|f_k \pm S(0,e_t)\| \leq 1,$$

therefore

$$S(0,e_t)(k) = 0 \ \text{ for all } k \in L, k \neq t.$$

Since $S(0,e_t) \in \mathcal{C}_0(L)$ and $t \in M = L'$, we get

$$S(0,e_t)(k) = 0 \ \text{ for all } k \in L,$$

that is

$$S(0,e_t) = 0 \ \text{ for all } t \in M,$$

so $S(0,y) = 0$ for all y in $\overline{\mathrm{lin}\{e_t : t \in M\}}^{\|\cdot\|_Y} = Y$.

Now $S(f,y) = S(f,0)+S(0,y) = 0$, that is, $S = 0$ and this finishes the proof. □

Now we give a new condition for being nice. This result allows us to prove that a lot of Banach spaces are non-nice and also it will play a key role for determining nice finite-dimensional spaces.

Proposition 2.8. *Let X be a Banach space that is nice and such that E_X is nonempty. Then*

$$| x^*(x) | = 1 \ \textit{ for all } x^* \in E_{X^*} \textit{ and } x \in E_X.$$

Proof. We have that $N(Y,X) = E(Y,X)$ for every Banach space Y. In particular, $N(\mathbb{R},X) = E(\mathbb{R},X)$. Let x be in E_X and let T be in $\mathcal{L}(\mathbb{R},X) \equiv X$ defined by $T(\alpha) = \alpha x$, for all $\alpha \in \mathbb{R}$. Then $T \in E(\mathbb{R},X) = N(\mathbb{R},X)$, that is

$$T^*(x^*) \in E_{\mathbb{R}^*} \equiv E_{\mathbb{R}} \ \text{ for all } x^* \in E_{X^*}.$$

Thus $T^*(x^*)(\alpha) = x^*(T(\alpha)) = x^*(\alpha x) = \alpha x^*(x)$ and we conclude that

$$| x^*(x) | = 1 \ \text{ for all } x^* \in E_{X^*} \text{ and } x \in E_X. \qquad \square$$

In view of Corollaries 2.5 and 2.6 the property appearing in the above proposition is not sufficient for being nice. This last result shows that being nice is a very strong condition. For example, it is easy to prove that $\mathbb{R}$ is the unique nice strictly convex space. Also, by using [10, Proposition 2], any infinite-dimensional reflexive Banach space cannot be nice.

For finite-dimensional spaces the above condition has been studied in connection with the class of CL-spaces, whose definition we recall now.

Definition 2.9. A Banach space X is a *CL-space* if for each maximal face F of B_X we have $B_X = \mathrm{co}(F \cup -F)$.

In [7, Corollary 3.7] it was shown that, if X is a finite-dimensional space, then X is a CL-space if and only if $| x^*(x) |= 1$ for all $x^* \in E_{X^*}$ and $x \in E_X$.

Our next goal is to describe all the nice finite-dimensional spaces. In order to do that we recall the main result in [9] which reads as follows.

Theorem 2.10 ([9, Theorem 1.1]). *The following three assertions, concerning a finite dimensional Banach space X, are equivalent.*

(*1*) $T \in E(X,X),\ x \in E_X \Rightarrow Tx \in E_X$.
(*2*) $T_1, T_2 \in E(X,X) \Rightarrow T_1 \circ T_2 \in E(X,X)$.
(*3*) $\{T_i\}_{i=1}^m \subseteq E(X,X) \Rightarrow \|T_1 \circ \cdots \circ T_m\| = 1$ *for all* m.

As a consequence of the above theorem we can state condition (*1*) by using nice operators as the following result shows.

Corollary 2.11. *If X is a finite-dimensional Banach space, then the following assertions are equivalent:*

i) $N(X,X) = E(X,X)$.
ii) $T \in E(X,X),\ x \in E_X \Rightarrow T(x) \in E_X$.

Now we have all the tools for proving the aforementioned characterization of nice finite-dimensional spaces.

Theorem 2.12. *Let X be a finite-dimensional Banach space, then X is nice if, and only if, $X = l_\infty^n$ for some $n \in \mathbb{N}$.*

Proof. The "if" part follows from Proposition 2.1. In order to prove the "only if" part, assume that X is nice. Then X is a CL-space and, by Corollary 2.11, X satisfies condition (*1*) in Theorem 2.10. Therefore, by [8, Theorem 6.5 and Theorem 4.1], we have that X is $l_1^m \oplus_1 l_\infty^3 \oplus_1 \overset{p}{\cdots} \oplus_1 l_\infty^3$ or $l_\infty^n \oplus_\infty l_1^3 \oplus_\infty \overset{k}{\cdots} \oplus_\infty l_1^3$ with m, n, p, $k \in \mathbb{N} \cup \{0\}$. We are going to see that the only possibilities are $m = 0$ and $p = 1$, or $m = 1, 2$ and $p = 0$ in the first case, and $k = 0$, in the second one.

Let's see first that X can not be of the form $l_1^m \oplus_1 l_\infty^3 \oplus_1 \overset{p}{\cdots} \oplus_1 l_\infty^3$ with $p \geq 2$.

We consider $Z = X^* = l_\infty^m \oplus_\infty l_1^3 \oplus_\infty \overset{p}{\cdots} \oplus_\infty l_1^3$ and the operator $I : l_\infty^4 \to Z$ defined by

$$I(x,y,z,t) = \left(x, \overset{m}{\ldots}, x, \frac{x+y}{2}, \frac{x-y}{2}, 0, \frac{z+t}{2}, \frac{z-t}{2}, 0, x, 0, 0, \ldots, x, 0, 0\right).$$

We have that $I(E_{l_\infty^4}) \subseteq E_Z$ and I is an isometry. If $Y = l_1^4$, then, by [8, Corollary 3.6.], there exist $U \in E(Z,Y)$ and $z \in E_Z$ such that $U(z) \notin E_Y$. From here we obtain that $T = U^* \in E(Y^*, X)$ and T is a non-nice operator, so X is non-nice.

If $p = 0$ then $X = l_1^m$ and Corollary 2.6 applies, so that X is non-nice for $m \geq 3$.

If $p = 1$ and $m \geq 2$, then we consider $Z = l_\infty^m \oplus_\infty l_1^3$ and we build the operator $I : l_\infty^4 \to Z$ defined by

$$I(x,y,z,t) = \left(x, y, x, x, \ldots, x, \frac{z+t}{2}, \frac{z-t}{2}, 0\right).$$

We have that $I(E_{l_\infty^4}) \subseteq E_Z$ and I is an isometry, and, by [8, Corollary 3.6], we obtain that $Z^* = X$ is non-nice.

If $p = 1$ and $m = 1$ we consider $Z = \mathbb{R} \oplus_\infty l_1^3$ and, by [8, Lemma 3.5], we have that $Z^* = X$ is non-nice.

Summarizing, if $X = l_1^m \oplus_1 l_\infty^3 \oplus_1 \overset{p}{\cdots} \oplus_1 l_\infty^3$ then $(m,p) = (1,0), (2,0)$ or $(0,1)$, that is $X = \mathbb{R}, l_\infty^2$ or l_∞^3.

Now, assume that $X = l_\infty^n \oplus_\infty l_1^3 \oplus_\infty \overset{k}{\cdots} \oplus_\infty l_1^3$. We know that l_1^3 satisfies the hypotheses in Theorem 2.2 so, as in the proof of this theorem, there exist a Banach space X_0 and a linear operator $T_0 : X_0 \to l_1^3$ which is a non-nice operator and such that $T_0^*(A_0) \subseteq E_{X_0^*}$, where $A_0 \subseteq E_{l_\infty^3}$ and $\mathrm{lin}(A_0) = l_\infty^3$.

If $k \geq 1$, then we consider

$$T : l_\infty^n \oplus_\infty X_0 \oplus_\infty \overset{k}{\cdots} \oplus_\infty l_1^3 \to l_\infty^n \oplus_\infty l_1^3 \oplus_\infty \overset{k}{\cdots} \oplus_\infty l_1^3$$

defined by

$$T(y, x_1, x_2, \ldots, x_k) = (y, T_0(x_1), x_2, \ldots, x_k),$$

and note that $T^* : l_1^n \oplus_1 l_\infty^3 \oplus_1 \overset{k}{\cdots} \oplus_1 l_\infty^3 \to l_1^n \oplus_1 X_0^* \oplus_1 \overset{k}{\cdots} \oplus_1 l_\infty^3$ is given by

$$T^*(y^*, x_1^*, x_2^*, \ldots, x_k^*) = (y^*, T_0^*(x_1^*), x_2^*, \ldots, x_k^*).$$

Since $T^*(x^*) \in E_{l_1^n \oplus_1 l_\infty^3 \oplus_1 \overset{k}{\cdots} \oplus_1 l_\infty^3}$, for all $x^* \in A$ with $A \subseteq E_{l_1^n \oplus_1 l_\infty^3 \oplus_1 \overset{k}{\cdots} \oplus_1 l_\infty^3}$ such that $\mathrm{lin}(A) = l_1^n \oplus_1 l_\infty^3 \oplus_1 \overset{k}{\cdots} \oplus_1 l_\infty^3$, T is a non-nice extreme operator. Therefore, $k = 0$, and hence $X = l_\infty^n$. $\square$

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