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Some Remarks on an Idempotent and Non-Associative Convex Structure

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$\mathbb{B}$-convexity was defined in [6] as a suitable Kuratowski-Painlevé upper limit of linear convexities over a finite dimensional Euclidean vector space. Except for the special case where convex sets are subsets of $\mathbb{R}^n_+$, $\mathbb{B}$-convexity was not defined with respect to a given explicit algebraic structure. This is done in that paper, which proposes an extension of $\mathbb{B}$-convexity to the whole Euclidean vector space. An unital idempotent and non-associative magma is defined over the real set and an extended n-ary operation is introduced. Along this line, the existence of the Kuratowski-Painlevé limit of the convex hull of two points over $\mathbb{R}^n$ is shown and an explicit extension of $\mathbb{B}$-convexity is proposed.

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1. Introduction

$\mathbb{B}$-convexity was defined in [6]. One can say, loosely speaking, that this $\mathbb{B}$-convexity is obtained from the usual linear convexity making the formal transformation $+ \to \max$. By definition, a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ is a connected upper semilattice. $\mathbb{B}$-convex functions were analyzed in [1]. Hahn-Banach like separation properties [8] as well as fixed point results [7] – see also [10] – have been established. The standard form of $\mathbb{B}$-convexity is defined on the nonegative Euclidean orthant $\mathbb{R}^n_+$ and is linked to Max-Plus algebra via a suitable homeomorphism. In finite dimensional space, Max-Plus convexity and $\mathbb{B}$-convexity are isomorphic topological Maslov's semi-modules [9] and, consequently, a proposition that is true in the framework of $\mathbb{B}$-convexity holds, with obvious lexical modifications, in Max-Plus convexity. Though $\mathbb{B}$-convexity was initially defined over $\mathbb{R}^n$ as a Kuratowski-Painlevé upper limit of linear convexities, it was not described in term of an explicit algebraic structure, excepted in the case where convex sets are subsets of $\mathbb{R}^n_+$. More recently, $\mathbb{B}^{-1}$-convex sets were introduced in [2] and [12].

This paper introduces a suitable algebraic structure extending $\mathbb{B}$-convexity to the whole Euclidean vector space. However, there do not exist non trivial algebraic structures being both idempotent, associative, and having inverse elements.

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Therefore, a special class of idempotent magma is considered in which associativity is relaxed to preserve symmetry and idempotence. This binary operation is based upon absolute value function. An n-ary extension of this algebraic structure is proposed and related to the pointwise limit of a generalized Hölder sum. Some algebraic properties are established and an extended definition of $\mathbb{B}$-convexity is then proposed, including as a special case that one proposed in [6]. It is shown that such a notion of convexity is equivalently characterized from the Kuratowski-Painlevé limit of the generalized convex hull of two points defined in [6].

The paper unfolds as follows. Section 2 focusses on a special class of symmetrical idempotent magmas. An n-ary extension of this operation is proposed and is related to the limit of a generalized Hölder sum. In Section 3, it is established that such an algebraic structure yields a very simple extension of $\mathbb{B}$-convexity over $\mathbb{R}^n$. In Section 4, a relation to the limit of linear convexity is established. Among other things, it is shown that, given two points in the whole Euclidean vector space, the Painlevé-Kuratowski limit of their generalized convex hull exists and is characterized from the algebraic structure defined in the paper.

In this paper, the following notations are used: $\mathbb{N}$ and $\mathbb{Z}$ denote respectively the sets of natural numbers and integers; for all natural numbers $n \geq 1$, $[n] = \{1, ..., n\}$; $\mathbb{R}^n_+ = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i \geq 0, i \in [n]\}$; $\mathbb{R}^n_{++} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i > 0, i \in [n]\}$; for all finite set A, let $\mathrm{Card}(A)$ denotes its cardinality.

2. Pointwise Limit of a Generalized Sum and Algebraic Structure

For all $p \in \mathbb{N}$, we consider a bijection $\varphi_p : \mathbb{R} \longrightarrow \mathbb{R}$ defined for all $x \in \mathbb{R}^n$ by: $\varphi_p : \lambda \longrightarrow \lambda^{2p+1}$ and $\Phi_p(x_1, ..., x_n) = (\varphi_p(x_1), ..., \varphi_p(x_n))$. We can induce a field structure on $\mathbb{R}$ for which φ_p becomes a field isomorphism. Given this change of notation via φ_p and Φ_p we can define a $\mathbb{R}$-vector space structure on $\mathbb{R}^n$ by: $\lambda \overset{\varphi_p}{\cdot} x = \Phi_p^{-1}(\varphi_p(\lambda).\Phi_p(x)) = \lambda.x$ and for all $x_1, x_2 \in \mathbb{R}$, $x_1 \overset{\varphi_p}{+} x_2 = \Phi_p^{-1}\left(\Phi_p(x_1) + \Phi_p(x_2)\right)$; we call these two operations the indexed scalar product and the indexed sum (indexed by φ of course). Suppose that $A = \{x_1, ..., x_m\}$. The φ_p-sum, denoted $\sum^{\varphi_p}$ of $(x_1, ..., x_m) \in \mathbb{R}^n$, is defined by $\sum^{\varphi_p}_{i\in[m]} x_i = \Phi_p^{-1}\left(\sum_{i\in[m]} \Phi_p(x_i)\right)$. In the remainder of the paper we denote for all $x, y \in \mathbb{R}^n$ $x \overset{p}{+} y = x \overset{\varphi_p}{+} y$.

2.1. On some Idempotent, Symmetrical and Non-associative Algebraic Structure

If $x, y \in \mathbb{R}_+$ then one has $\lim_{p\longrightarrow+\infty} x \overset{p}{+} y = \max\{|x|, |y|\}$. In the case where $x, y \in \mathbb{R}$, it is easy to establish the following property.

Lemma 2.1. *For all $x, y \in \mathbb{R}$ we have:*

$$\lim_{p \longrightarrow +\infty} x \overset{p}{+} y = \lim_{p \longrightarrow +\infty} \left(x^{2p+1} + y^{2p+1}\right)^{\frac{1}{2p+1}} = \begin{cases} x & \text{if } |x| > |y| \\ \frac{1}{2}(x+y) & \text{if } |x| = |y| \\ y & \text{if } |x| < |y|. \end{cases}$$

Proof. Suppose that $x = -y$. Then for all $p \in \mathbb{N}$ we have $(x^{2p+1} + y^{2p+1})^{\frac{1}{2p+1}} = (x^{2p+1} - x^{2p+1})^{\frac{1}{2p+1}} = 0$, which proves this case. If $x = y$, then $\lim_{p \longrightarrow +\infty}(x^{2p+1} + y^{2p+1})^{\frac{1}{2p+1}} = x \lim_{p \longrightarrow +\infty} 2^{\frac{1}{2p+1}} = x$. To end the proof, suppose, for example, that $|x| > |y|$. The map $t \mapsto \ln(1+t)$ is continuous at point 0. Thus, since $\left|\frac{y}{x}\right| < 1$, we have $\lim_{p \longrightarrow +\infty} \ln\left(\left(1 + \left(\frac{y}{x}\right)^{2p+1}\right)^{\frac{1}{2p+1}}\right) = 0$. Consequently,

$$\lim_{p \longrightarrow +\infty} \left(x^{2p+1} + y^{2p+1}\right)^{\frac{1}{2p+1}} = x \left[\lim_{p \longrightarrow +\infty} \left(1 + \left(\frac{y}{x}\right)^{2p+1}\right)^{\frac{1}{2p+1}}\right] = x. \qquad \square$$

Let $(M, \boxplus)$ be a magma or groupoid, that is a set M equipped with a single closed binary operation $M \times M \to M$ defined by $(x, y) \to x \boxplus y$. M is unital if it has a neutral element 0. This binary operation is idempotent if, for all $x \in M$, $x \boxplus x = x$. It is associative if for all $(x, y, z) \in M^3$, one has $(x \boxplus y) \boxplus z = x \boxplus (y \boxplus z) = x \boxplus y \boxplus z$. M has inverse elements if, for all $x \in M$, there is some $-x$ in M such that $x \boxplus (-x) = (-x) \boxplus x = 0$. We say that x and $-x$ are symmetrical if $-x$ is the inverse of x and conversely. It is a standard fact that a nontrivial group is not idempotent. In general idempotence is compatible with a semigroup structure that is an algebraic structure consisting of a set together with an associative binary operation. A semigroup generalizes a group to a type where every element did not have to have a symmetrical element. In the following a non-associative and idempotent magma is considered. The real set $\mathbb{R}$ is endowed with the binary operation $\boxplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$x \boxplus y = \lim_{p \longrightarrow +\infty} x \overset{p}{+} y = \lim_{p \longrightarrow +\infty} \left(x^{2p+1} + y^{2p+1}\right)^{\frac{1}{2p+1}} \tag{1}$$

The map $(x, y) \mapsto x \boxplus y$ is not continuous over $\mathbb{R}^2$. Moreover, this operation is not associative. For example, one has $(1 \boxplus 1) \boxplus (-1) = 1 \boxplus (-1) = 0$ and $1 \boxplus (1 \boxplus (-1)) = 1 \boxplus 0 = 1$ which contradicts associativity. Associativity is replaced with a weakened assumption which only requires that associativity works for any pair of non symmetrical elements. By definition, for all $x, y \in \mathbb{R}_+$ one has $x \boxplus y = \max\{x, y\}$. Moreover, if $x, y \in \mathbb{R}_-$ then $x \boxplus y = \min\{x, y\}$. It follows that the operation $\boxplus$ defines a total order on $\mathbb{R}_+$ and $\mathbb{R}_-$, but not on the whole real set. In the remainder $(\mathbb{R}, \boxplus)$ denotes the set $\mathbb{R}$ equipped with the operation $\boxplus$. The proof of the following proposition is left to the reader.

Proposition 2.2. *The set $\mathbb{R}$ equipped with the operation $\boxplus$ and the scalar multiplication satisfies the following properties:*

(a) *For all* $x \in \mathbb{R}$, $x \boxplus x = x$ (*idempotence*).
(b) *For all* $x \in \mathbb{R}$, $x \boxplus 0 = 0 \boxplus x = x$ (*neutral element*).
(c) *For all* $x \in \mathbb{R}$, *there exists a uniqueness symmetrical element* $-x \in \mathbb{R}$ *such that* $x \boxplus (-x) = (-x) \boxplus x = 0$ (*symmetrical element*).
(d) *For all* $x, y \in \mathbb{R}$, *we have* $x \boxplus y = y \boxplus x$ (*commutativity*).
(e) *For all* $(x, y, z) \in \mathbb{R}^3$, *if* x, y *and* z *are mutually non symmetrical then* $(x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$ (*weakened form of associativity*).
(f) *For all* $(x, y, z) \in \mathbb{R}^3$, *one has* $z(x \boxplus y) = (x \boxplus y)z = (zx) \boxplus (zy)$ (*distributivity*).

The properties above show that $(\mathbb{R}, \boxplus, \cdot)$ is endowed with some kind of "scalar field like" algebraic structure. It is not a scalar field because $(\mathbb{R}, \boxplus)$ is not a group. The next statement is an immediate consequence of Lemma 2.1.

Lemma 2.3. *For all* $x, y \in \mathbb{R}$, *the following inequalities are equivalent:* (a) $x \leq y$; (b) $0 \leq (-x) \boxplus y$; (c) $x \boxplus (-y) \leq 0$.

Proof. First, note that the distributivity of the scalar multiplication on the operation $\boxplus$ implies that (b) and (c) are equivalent. All we need to prove is that (a) and (b) are equivalent. Let us prove the first implication. If $x \leq y$, then for all natural number $p \geq 1$, we have $((-x)^{2p+1} + y^{2p+1})^{\frac{1}{2p+1}} \geq 0$. It follows that $\lim_{p \longrightarrow +\infty} ((-x)^{2p+1} + y^{2p+1})^{\frac{1}{2p+1}} = (-x) \boxplus y \geq 0$, which proves ($b$). Conversely, suppose that (b) holds. By hypothesis $0 \leq (-x) \boxplus y$. If $(-x) \boxplus y = 0$ then one has $x = y$ and (a) is immediate. Suppose now that $(-x) \boxplus y = y \geq 0$. This implies that one has $|y| \geq |-x| = |x|$. Since $y \geq 0$, we have $y \geq x$, and we deduce condition (a). Finally, if $(-x) \boxplus y = -x \geq 0$, from the distributivity of the scalar multiplication on the operation $\boxplus$, we have $x \boxplus (-y) = x \leq 0$. Since $|x| \geq |y|$, this implies that $x \leq y$, which ends the proof. $\square$

2.2. Construction of a n-ary Operation

In the following it is established that, though the operation $\boxplus$ does not satisfy associativity, it can be extended by constructing a non-associative algebraic structure which returns to a given n-tuple a real value. For all $x \in \mathbb{R}^n$ and all subset I of $[n]$, let us consider the map $\xi_I[x] : \mathbb{R} \longrightarrow \mathbb{Z}$ defined for all $\alpha \in \mathbb{R}$ by

$$\xi_I[x](\alpha) = \text{Card}\{i \in I : x_i = \alpha\} - \text{Card}\{i \in I : x_i = -\alpha\}. \tag{2}$$

This map measures the symmetry of the occurrences of a given value α in the components of a vector x. This map satisfies the following properties whose the proofs are obvious and left to the reader.

Lemma 2.4. *For all* $x \in \mathbb{R}^n$ *and for all subset* I *of* $[n]$ *the map* $\xi_I[x]$ *defined in* (2) *satisfies the following properties:*

(a) $\xi_I[x]$ *is an impair map, that is for all* $\alpha \in \mathbb{R}$, $\xi_I[x](-\alpha) = -\xi_I[x](\alpha)$.
(b) *For all* $\alpha \in \mathbb{R}$ *the map* $x \mapsto \xi_I[x](\alpha)$ *is impair.*

(c) *If* $\{\alpha, -\alpha\} \cap \{x_i : i \in I\} = \emptyset$ *then* $\xi_I(\alpha) = 0$.
(d) *For all* $i \in I$, $\xi_{I\setminus\{i\}}(x_i) = \xi_I(x_i) - 1$.
(e) *If* $\xi_I[x]\left(\max_{i\in I}|x_i|\right) > 0$ *then* $\max_{i\in I} x_i = \max_{i\in I}|x_i|$.
(f) *If* $\xi_I[x]\left(\max_{i\in I}|x_i|\right) < 0$ *then* $\min_{i\in I} x_i = -\max_{i\in I}|x_i|$.
(g) *For all subsets* I *and* J *of* $[n]$ *and all* $\alpha \in \mathbb{R}$, $\xi_{I\cup J}[x](\alpha) = \xi_I[x](\alpha) + \xi_J[x](\alpha) - \xi_{I\cap J}[x](\alpha)$.
(h) $\xi_\emptyset[x](\alpha) = 0$, *for all* $\alpha \in \mathbb{R}$.

For all $x \in \mathbb{R}^n$ let $\mathcal{J}_I(x)$ be a subset of I defined by

$$\mathcal{J}_I(x) = \{j \in I : \xi_I[x](x_j) \neq 0\} = I\setminus\left(\xi_I[x]^{-1}(0)\right). \tag{3}$$

$\mathcal{J}_I(x)$ is called **the residual index set** of x. It is obtained by dropping from I all the i's such that $\mathrm{Card}\{j \in I : x_j = x_i\} = \mathrm{Card}\{j \in I : x_j = -x_i\}$.

Definition 2.5. For all positive natural number n and for all subset I of $[n]$, let $\digamma_I : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the map defined for all $x \in \mathbb{R}^n$ by

$$\digamma_I(x) = \begin{cases} \max_{i\in\mathcal{J}_I(x)} x_i & \text{if } \xi_I[x]\left(\max_{i\in\mathcal{J}_I(x)}|x_i|\right) > 0 \\ \min_{i\in\mathcal{J}_I(x)} x_i & \text{if } \xi_I[x]\left(\max_{i\in\mathcal{J}_I(x)}|x_i|\right) < 0 \\ 0 & \text{if } \xi_I[x]\left(\max_{i\in\mathcal{J}_I(x)}|x_i|\right) = 0. \end{cases} \tag{4}$$

where $\xi_I[x]$ is the map defined in (2) and $\mathcal{J}_I(x)$ is the residual index set of x. The operation that takes an n-tuple $(x_1, ..., x_n)$ of $\mathbb{R}^n$ and returns a single real element $\digamma_I(x_1, ..., x_n)$ is called a n-ary extension of the binary operation $\boxplus$.

Notice that, $\mathcal{J}_I(x) = \emptyset$ if and only if $\xi_I[x](\max_{i\in\mathcal{J}_I(x)}|x_i|) = 0$. To see the key idea of the definition above let us define the generalized sum of n real numbers $x_1, ..., x_n$ as $S_p(x_1, ..., x_n) = \left(\sum_{i\in[n]} {x_i}^p\right)^{\frac{1}{p}}$, say a Hölder sum. When one consider the subsequence of pair natural numbers, this generalized sum has the limit: $\lim_{p\longrightarrow+\infty} S_{2p}(x_1, ..., x_n) = \lim_{p\longrightarrow+\infty}\left(\sum_{i\in[n]} {x_i}^{2p}\right)^{\frac{1}{2p}} = \max_{i\in[n]}|x_i|$. The case where the generalized sum is defined with respect to the impair natural numbers is analyzed in this section. It is shown below that $\digamma_I(x)$ is the limit of the generalized sum $S_{2p+1}(x_1, ..., x_n)$.

Proposition 2.6. *For all natural number* $n \geq 1$ *and all* $x \in \mathbb{R}^n$, *if* I *is a nonempty subset of* $[n]$ *then:*

$$\digamma_I(x) = \lim_{p\longrightarrow\infty}\left(\sum_{i\in I} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} = \lim_{p\longrightarrow\infty} \sum_{i\in I}^{\varphi_p} x_i.$$

Proof. Let $\mathcal{J}_I(x)$ be the residual index set of x. We have

$$\left(\sum_{i\in I} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} = \left(\sum_{i\in I\backslash\mathcal{J}_I(x)} {x_i}^{2p+1} + \sum_{i\in\mathcal{J}_I(x)} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}}.$$

By definition, there exists a partition of $I\backslash\mathcal{J}_I(x)$ whose any block contains two symmetric elements. Hence it follows that

$$\sum_{i\in I\backslash\mathcal{J}_I(x)} {x_i}^{2p+1} = -\sum_{i\in I\backslash\mathcal{J}_I(x)} {x_i}^{2p+1} = 0.$$

Hence, we deduce that

$$\left(\sum_{i\in I} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} = \left(\sum_{i\in\mathcal{J}_I(x)} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}}.$$

Suppose that $\mathcal{J}_I(x) = \emptyset$. In such a case

$$\left(\sum_{i\in I} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} = \left(\sum_{i\in\mathcal{J}_I(x)} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} = 0.$$

Consequently $\lim_{p\longrightarrow\infty}\left(\sum_{i\in I} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} = 0 = \digamma_I(x)$ which proves this case.

Suppose now that $\mathcal{J}_I(x) \neq \emptyset$. Let us denote

$$\mathcal{M}_I(x) = \{i \in \mathcal{J}_I(x) : |x_i| = \max_{i\in\mathcal{J}_I(x)} |x_i|\}.$$

Then, from the definition of map $\xi_I[x]$ in equation (2), we have

$$\sum_{i\in\mathcal{M}_I(x)} \left(\frac{x_i}{\max_{i\in\mathcal{J}_I(x)} |x_i|}\right)^{2p+1} = \xi_I[x]\left(\max_{i\in\mathcal{J}_I(x)} |x_i|\right).$$

It follows that for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\left(\sum_{i\in\mathcal{J}_I(x)} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} &= \max_{i\in\mathcal{J}_I(x)} |x_i| \left(\sum_{i\in\mathcal{J}_I(x)} \frac{{x_i}^{2p+1}}{\max_{i\in\mathcal{J}_I(x)} |x_i|^{2p+1}}\right)^{\frac{1}{2p+1}} \qquad (5)\\ &= \left(\max_{i\in\mathcal{J}_I(x)} |x_i|\right)\alpha_p(x).\end{aligned}$$

where

$$\alpha_p(x) = \left(\xi_I[x]\left(\max_{i\in\mathcal{J}_I(x)} |x_i|\right) + \sum_{i\notin\mathcal{M}_I(x)} \left(\frac{x_i}{\max_{i\in\mathcal{J}_I(x)} |x_i|}\right)^{2p+1}\right)^{\frac{1}{2p+1}}.$$

We need to compute the limit of $\alpha_p(x)$, when $p \longrightarrow \infty$. Clearly, for all $i \notin \mathcal{M}_I(x)$, we have

$$\Big|\frac{x_i}{\max_{i\in\mathcal{J}_I(x)}|x_i|}\Big| < 1. \tag{6}$$

Since $\mathcal{J}_I(x) \neq \emptyset$, $\xi_I[x]\left(\max_{i\in\mathcal{J}_I(x)}|x_i|\right) \neq 0$, hence we consider two cases:

(i) $\xi_I[x]\left(\max_{i\in\mathcal{J}_I(x)}|x_i|\right) > 0$.

For the sake of simplicity, define $a = \xi_I[x](\max_{i\in\mathcal{J}_I(x)}|x_i|)$ and $b_i = |\frac{x_i}{\max_{i\in\mathcal{J}_I(x)}|x_i|}|$ for each i. We then obtain $\alpha_p(x) = \left(a + \sum_{i\notin\mathcal{M}_I(|x|)} b_i^{2p+1}\right)^{\frac{1}{2p+1}}$. Moreover, from (6), we have $|b_i| < 1$ for all $i \notin \mathcal{M}_I(|x|)$. By hypothesis $a > 0$ and we deduce that $\lim_{p\longrightarrow+\infty}\ln(a + \sum_{i\notin\mathcal{M}_I(|x|)} b_i^{2p+1}) = \ln(a)$. Hence, we have

$$\lim_{p\longrightarrow+\infty}\ln\alpha_p(x) = \lim_{p\longrightarrow+\infty}\frac{\ln(a + \sum_{i\notin\mathcal{M}_I(|x|)} b_i^{2p+1})}{2p+1} = 0.$$

Thus, $\lim_{p\longrightarrow+\infty}\alpha_p(x) = 1$. Hence, from (5), we deduce that:

$$\lim_{p\longrightarrow\infty}\left(\sum_{i\in\mathcal{J}_I(x)} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} = \max_{i\in\mathcal{J}_I(x)}|x_i| = \digamma_I(x).$$

(ii) $\xi_I[x]\left(\max_{i\in\mathcal{J}_I(x)}|x_i|\right) < 0$.

Applying *(i)*, we then obtain

$$-\lim_{p\longrightarrow\infty}\left(\sum_{i\in\mathcal{J}_I(x)} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} = \lim_{p\longrightarrow\infty}\left(\sum_{i\in\mathcal{J}_I(x)} (-x_i)^{2p+1}\right)^{\frac{1}{2p+1}} = \max_{i\in\mathcal{J}_I(x)}|x_i|.$$

Thus

$$\lim_{p\longrightarrow\infty}\left(\sum_{i\in\mathcal{J}_I(x)} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} = -\max_{i\in\mathcal{J}_I(x)}|x_i| = \min_{i\in\mathcal{J}_I(x)} x_i = \digamma_I(x). \qquad \square$$

Let us introduce for all n-tuple $x = (x_1, ..., x_n)$ the operation defined by:

$$\mathop{\boxplus}_{i\in I} x_i = \lim_{p\longrightarrow\infty}\left(\sum_{i\in I} {x_i}^{2p+1}\right)^{\frac{1}{2p+1}} = \lim_{p\longrightarrow\infty}\sum_{i\in I}^{\varphi_p} x_i. \tag{7}$$

Clearly, this operation encompasses as a special case the binary operation defined in equation (1). From Lemma 2.1 and Definition 2.5 if $n = 2$ and $I = \{1, 2\}$, then, for all $(x_1, x_2) \in \mathbb{R}^2$

$$\mathop{\boxplus}_{i\in\{1,2\}} x_i = x_1 \boxplus x_2.$$

Example 2.7. Suppose that $x = (2, 3, -2, -3, 1, -3, 3, -\frac{1}{2})$. First, note that $[8] = \{1, \cdots, 8\}$ and $\{x_i : i \in [8]\} = \{1, -2, 2, -3, 3, -\frac{1}{2}\}$. We have $\{i : x_i = 3\} = \{2, 7\}$ and $\{i : x_i = -3\} = \{4, 6\}$. Therefore, $\mathrm{Card}\{i : x_i = 3\} = \mathrm{Card}\{i : x_i = -3\} = 2$, $\xi_{[8]}[x](x_i) = 0$ for $i \in \{2, 4, 6, 7\}$. Moreover: $\{i : x_i = 2\} = \{1\}$ and $\{i : x_i = -2\} = \{3\}$. Consequently, $\mathrm{Card}\{i : x_i = 2\} = \mathrm{Card}\{i : x_i = -2\} = 1$ and $\xi_{[8]}[x](x_i) = 0$ for $i \in \{1, 3\}$. Hence, we have

$$\mathcal{J}_{[8]}(x) = [8] \setminus (\{2, 4, 6, 7\} \cup \{1, 3\}) = \{5, 8\}.$$

Therefore $\mathcal{J}_{[8]}(x) = \{5, 8\}$. Hence

$$\boxplus_{i \in [8]} x_i = \boxplus_{i \in \{5,8\}} x_i = 1 \boxplus \left(-\frac{1}{2}\right) = x_5 = 1.$$

2.3. Some Algebraic Properties

A few immediate properties whose the proofs are obvious are established in the next Lemma.

Proposition 2.8. *For all $x \in \mathbb{R}^n$ and all nonempty subset I of $[n]$, we have:*

(a) *If $\mathcal{J}_I(x) \neq \emptyset$ then there is some $i_0 \in I$ such that $x_{i_0} = \boxplus_{i \in I} x_i$. Moreover, $\xi[x](x_{i_0}) > 0$.*

(b) *If $\alpha \in \mathbb{R}$ and $x_i = \alpha$ for all $i \in I$ then $\boxplus_{i \in I} x_i = \alpha$.*

(c) *Moreover, if all the elements of the family $\{x_i\}_{i \in I}$ are mutually non symmetrical, then:*

$$\boxplus_{i \in I} x_i = \arg\max_{\lambda} \{|\lambda| : \lambda \in \{x_i\}_{i \in I}\}.$$

(d) *For all $\alpha \in \mathbb{R}$, one has:*

$$\alpha \left(\boxplus_{i \in I} x_i \right) = \boxplus_{i \in I} (\alpha x_i).$$

(e) *Suppose that $x \in \epsilon \mathbb{R}^n_+$ where ϵ is $+1$ or -1. Then $\boxplus_{i \in I} x_i = \epsilon \max_{i \in I} \{\epsilon x_i\}$.*

(f) *We have $|\boxplus_{i \in I} x_i| \leq \boxplus_{i \in I} |x_i|$.*

(g) *For all permutation $\sigma : I \to I$, we have $\boxplus_{i \in I} x_i = \boxplus_{i \in I} x_{\sigma(i)}$.*

(h) *If there exists $j, k \in I$ with $j \neq k$ and $x_j + x_k = 0$, then $\boxplus_{i \in I \setminus \{j,k\}} x_i = \boxplus_{i \in I} x_i$. Moreover $\boxplus_{i \in I} x_i = \boxplus_{i \in \mathcal{J}_I(x)} x_i$.*

Proof. (a) By hypothesis, the subset $\mathcal{J}_I(x) = \{i \in I : \xi_I[x](x_i) \neq 0\}$ is nonempty. Therefore, there exists some $i_0 \in \mathcal{J}_I(x)$ such that $|x_{i_0}| \geq |x_i|$ for all $i \in \mathcal{J}_I(x)$. There are two possibilities. If $\xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) > 0$, then, from Lemma 2.4.($e$), $\max_{i \in \mathcal{J}_I(x)} |x_i| = \max_{i \in \mathcal{J}_I(x)} x_i = x_{i_0}$. Thus $\xi_I[x](x_{i_0}) >$

0. If $\xi_I[x](\max_{i\in\mathcal{J}_I(x)}|x_i|) < 0$, from Lemma 2.4.$(f)$, we have $\min_{i\in\mathcal{J}_I(x)} x_i = -\max_{i\in\mathcal{J}_I(x)}|x_i| = x_{i_0}$. Since $\xi_I[x]$ is an impair map, this also implies that $\xi_I[x](x_{i_0}) > 0$.

(b) is immediate setting $x_i = \alpha$ for all $i \in I$.

(c) If all the x_i's are mutually non symmetrical, then $\xi_I[x](x_i) \neq 0$ for all $i \in I$. Hence, $\xi_I[x](\max_{i\in I}|x_i|\,) \neq 0$ and there is some $\epsilon \in \{-1, 1\}$ such that $\digamma_I(x) = \epsilon \max_{j\in\mathcal{J}_I(x)}|x_i|$, we deduce (c).

(d) Since the scalar multiplication is distributive on addition, it is an immediate consequence of Proposition 2.6.

(e) If $x \in \mathbb{R}^n_+$ then $\boxplus_{i\in I} x_i = \max_{i\in I} x_i$. $x \in -\mathbb{R}^n_+$ implies that $\xi_I[x](\max_{i\in I}|x_i|\,) < 0$ and $\boxplus_{i\in I} x_i = \min_{i\in I} x_i = -\max_{i\in I}(-x_i)$.

(f) For all $x \in \mathbb{R}^n$ and all $I \subset [n]$, we have $\digamma_I(|x|) = \max_{i\in I}|x_i|$. Moreover, by definition there is some $\delta \in \{-1, 0, 1\}$ such that $\digamma_I(x) = \delta \max_{j\in\mathcal{J}_I(x)} \delta x_i$. Therefore $\digamma_I(|x|) \geq |\digamma_I(x)|$.

(g) Given a nonempty subset I of $[n]$, the Hölder sum is independent of any permutation of the index set I. Hence, from Proposition 2.6, we deduce (g).

(h) In such a case $\{i \in I\backslash\{j,k\} : \xi_I[x](x_i) \neq 0\} = \{i \in I : \xi_I[x](x_i) \neq 0\}$. Therefore $\mathcal{J}_{I\backslash\{j,k\}}(x) = \mathcal{J}_I(x)$, which proves the first part of the statement. Since $\mathcal{J}_{\mathcal{J}_I(x)}(x) = \mathcal{J}_I(x)$, the second part is immediate. $\square$

In the following we introduce the operation $\langle\cdot,\cdot\rangle_\infty : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined for all $x, y \in \mathbb{R}^n$ by $\langle x, y\rangle_\infty = \boxplus_{i\in[n]} x_i y_i$. Let $\|\cdot\|_\infty$ be the Tchebychev norm defined by $\|x\|_\infty = \max_{i\in[n]}|x_i|$. The next result is an immediate consequence of Proposition 2.8.

Proposition 2.9. *For all $x, y \in \mathbb{R}^n$, we have:*

(a) $\sqrt{\langle x, x\rangle_\infty} = \|x\|_\infty$.
(b) $|\langle x, y\rangle_\infty| \leq \|x\|_\infty\|y\|_\infty$.
(c) *For all $\alpha \in \mathbb{R}$, $\alpha\langle x, y\rangle_\infty = \langle \alpha x, y\rangle_\infty = \langle x, \alpha y\rangle_\infty$.*

Proof. (a) By definition $\langle x, x\rangle_\infty = \boxplus_{i\in[n]} {x_i}^2 = \max_{i\in[n]} {x_i}^2 = {\|x\|_\infty}^2$, which ends the proof.

(b) From Proposition 2.8.(f), $|\langle x, y\rangle_\infty| = |\boxplus_{i\in[n]} x_i y_i| \leq \boxplus_{i\in[n]} |x_i y_i| \leq (\max_{i\in[n]}|x_i|)(\max_{i\in[n]}|y_i|)$, which proves (b). (c) is immediate from Proposition 2.8.(d). $\square$

The next statement establishes a key property resulting from Proposition 2.10.

Proposition 2.10. *Suppose that $x = (x_1, ..., x_n) \in \mathbb{R}^n$. For all nonempty subset*

I of $[n]$ and all $i \in I$:

$$\left[x_i \boxplus \left(\boxplus_{j \in I \backslash \{i\}} x_j\right)\right] \in \left\{0, \boxplus_{j \in I} x_j\right\}$$

and

$$\boxplus_{i \in I} x_i = \boxplus_{i \in I} \left[x_i \boxplus \left(\boxplus_{j \in I \backslash \{i\}} x_j\right)\right].$$

Proof. If $\mathcal{J}_I(x) = \emptyset$, then this property is immediate. In such case, since I is nonempty, from Proposition 2.8.(a), there is some $i_0 \in I \backslash \{i\}$ such that $x_{i_0} = -x_i$. Therefore $x_i \boxplus \left(\boxplus_{j \in I \backslash \{i\}} x_j\right) = 0$. Suppose now that $\mathcal{J}_{I \backslash \{i\}}(x) \neq \emptyset$ and let us consider four cases:

(i) $|x_i| > |\boxplus_{j \in I} x_j|$. This implies that $i \in I \backslash \mathcal{J}_I(x)$. Thus $\xi_I[x](x_i) = 0$. Hence, $\xi_{I \backslash \{i\}}[x](x_i) < 0$ and from Proposition 2.8.(a) $\boxplus_{j \in I \backslash \{i\}} x_j \neq x_i$. Moreover $\xi_{I \backslash \{i\}}[x](-x_i) > 0$, and by hypothesis, $|x_i| \geq |x_j|$ for all $j \in \mathcal{J}_{I \backslash \{i\}}(x)$. Therefore $\boxplus_{j \in I \backslash \{i\}} x_j = -x_i$. It follows that

$$x_i \boxplus \left(\boxplus_{j \in I \backslash \{i\}} x_j\right) = x_i \boxplus (-x_i) = 0,$$

which proves this case.

(ii) $x_i = \boxplus_{j \in I} x_j$. By definition, this implies that $\xi_I[x](x_i) > 0$. Since $\xi_{I \backslash \{i\}}[x](x_i) = \xi_I[x](x_i) - 1$, one has $\xi_{I \backslash \{i\}}[x](x_i) \geq 0$ and, consequently, $\xi_{I \backslash \{i\}}[x](-x_i) \leq 0$. Thus, from Proposition 2.8.(a), $\boxplus_{j \in I \backslash \{i\}} x_j \neq -x_i$. Moreover, $|\boxplus_{j \in I \backslash \{i\}} x_j| \leq |x_i|$ and we have

$$x_i \boxplus \left(\boxplus_{j \in I \backslash \{i\}} x_j\right) = x_i = \boxplus_{j \in I} x_j.$$

(iii) $x_i = -\boxplus_{j \in I} x_j$. Equivalently, we have $-x_i = \boxplus_{j \in I} x_j$ and, from Proposition 2.8.(a), this implies that there is some $i_0 \in I$ such that $x_{i_0} = \boxplus_{j \in I} x_j = -x_i$ with $\xi_I[x](x_{i_0}) > 0$. Since $\mathcal{J}_I(x) \neq \emptyset$, $x_{i_0} \neq 0$ and $x_{i_0} \neq x_i$. It follows that $i_0 \neq i$. Thus $\xi_{I \backslash \{i\}}[x](x_{i_0}) > 0$. Therefore, $\boxplus_{j \in I \backslash \{i\}} x_j = x_{i_0} = -x_i$. Consequently:

$$x_i \boxplus \left(\boxplus_{j \in I \backslash \{i\}} x_j\right) = x_i \boxplus (-x_i) = 0.$$

(iv) $|x_i| < |\boxplus_{j \in I} x_j|$. Hence, from Proposition 2.8.(a) there is some $i_0 \in I$ such that $x_{i_0} = \boxplus_{j \in I} x_j$. This implies that $\xi_I[x](x_{i_0}) > 0$. Therefore, since

$|x_{i_0}| > |x_i|$, we have $\xi_{I\backslash\{i\}}[x](x_{i_0}) > 0$ and it follows that $|x_i| < |\boxplus_{j\in I\backslash\{i\}} x_i|$. Hence $x_i \boxplus \left(\boxplus_{j\in I\backslash\{i\}} x_j\right) = x_{i_0} = \boxplus_{j\in I} x_j$, which ends the proof of the first part of the statement.

To prove the second part of the statement, we need to establish that there exists some $i \in I$ such that $x_i \boxplus \left(\boxplus_{j\in I\backslash\{i\}} x_j\right) = x_i$. If $\mathcal{J}_I(x) = \emptyset$ then $\boxplus_{j\in I} x_j = 0$ and from the statement above $\left[x_i \boxplus \left(\boxplus_{j\in I\backslash\{i\}} x_j\right)\right] \in \{0\}$ for all i. In such a case, this property is obviously true. Suppose that $\mathcal{J}_I(x) \neq \emptyset$. Recall that from Proposition 2.8.(a) there is some $i \in I$ such that $x_i = \boxplus_{j\in I} x_j$. Then, using ($ii$), the second statement follows. □

For all $(u, v, w, t) \in \mathbb{R}^4$, let us denote $u \boxplus v \boxplus w = \digamma_{[3]}(u, v, w)$ and $u \boxplus v \boxplus w \boxplus t = \digamma_{[4]}(u, v, w, t)$. The result above implies that for all $x, y, z \in \mathbb{R}$ we have the identities: $x \boxplus y = x \boxplus y \boxplus x \boxplus y$ and $x \boxplus y \boxplus z = [x \boxplus (y \boxplus z)] \boxplus [y \boxplus (z \boxplus x)] \boxplus [z \boxplus (x \boxplus y)]$.

Let $\Lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the map defined by:

$$\Lambda(x_1, ..., x_n) = \left(x_1 \boxplus \left(\boxplus_{j\in[n]\backslash\{1\}} x_j\right), ..., x_n \boxplus \left(\boxplus_{j\in[n]\backslash\{n\}} x_j\right)\right). \tag{8}$$

Lemma 2.11. *Suppose that $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Let $\epsilon \in \{-1, 1\}$ such that $\boxplus_{i\in I} x_i \in \epsilon\mathbb{R}_+$. For all $i \in I$, let $\Lambda_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the map defined by $\Lambda_i(x) = x_i \boxplus \left(\boxplus_{j\in[n]\backslash\{i\}} x_j\right)$. Then:*

(a) $\boxplus_{i\in I} x_i = \epsilon \max_{i\in I}\{\epsilon\Lambda_i(x)\}$;

(b) *For all $i, j, k \in [n]$, we have: $\Lambda_i(x) \boxplus \Lambda_j(x) \boxplus \Lambda_k(x) = (\Lambda_i(x) \boxplus \Lambda_j(x)) \boxplus \Lambda_k(x) = \Lambda_i(x) \boxplus (\Lambda_j(x) \boxplus \Lambda_k(x))$.*

Proof. (a) From Proposition 2.10 we have $\Lambda_i(x) \in \{0, \boxplus_{i\in I} x_i\}$ for all $i \in I$. Hence $\Lambda_i(x) \in \epsilon\mathbb{R}_+$ for all i and from Proposition 2.8.(c), the result follows. (b) is an immediate consequence of (a). □

Example 2.12. Let $x = (4, -3, -4, 2, 3, 2, -2) \in \mathbb{R}^7$. We have $\mathcal{J}_{[7]}(x) = \{4, 6, 7\}$ and $[7]\backslash\mathcal{J}_{[7]}(x) = \{1, 2, 3, 5\}$. We have $4 \boxplus (-3) \boxplus (-4) \boxplus 2 \boxplus 3 \boxplus 2 \boxplus (-2) = 2$. Moreover, $\Lambda_1(x) = 4 \boxplus ((-3) \boxplus (-4) \boxplus 2 \boxplus 3 \boxplus 2 \boxplus (-2)) = 4 \boxplus (-4) = 0$; Similarly we obtain $\Lambda_2(x) = (-3) \boxplus 3 = 0$; $\Lambda_3(x) = (-4) \boxplus 4 = 0$; $\Lambda_4(x) = 2 \boxplus 0 = 2$; $\Lambda_5(x) = 3 \boxplus (-3) = 0$; $\Lambda_6(x) = 2 \boxplus 0 = 2$; $\Lambda_7(x) = (-2) \boxplus 2 = 0$. It follows that

$$\Lambda(x) = (0, 0, 0, 2, 0, 2, 0) = \left(0, 0, 0, \boxplus_{i\in[7]} x_i, 0, \boxplus_{i\in[7]} x_i, 0\right).$$

Lemma 2.13. *Let n be a positive natural number and I be a nonempty subset of $[n]$. Let $\mathfrak{P}(I) = \{I_j : j \in [m]\}$ be a partition of I with m nonempty subsets I_j. If for all $(j,k) \in [m] \times [m]$*

$$\left(\bigboxplus_{i\in I_j} x_i\right) + \left(\bigboxplus_{i\in I_k} x_i\right) \neq 0 \quad \text{then} \quad \bigboxplus_{j\in[m]} \left(\bigboxplus_{i\in I_j} x_i\right) = \bigboxplus_{i\in I} x_i.$$

Proof. For all $j \in [m]$, let us denote $y_j = \bigboxplus_{i\in I_j} x_i$. By hypothesis the y_j's are mutually non symmetrical, it follows that there exists some $j_0 \in [m]$ such that

$$y_{j_0} = \bigboxplus_{j\in[m]} y_j = \arg\max_{y_j}\{|y_j| : j \in [m]\}.$$

Therefore, for all $i \in I$ such that $|x_i| > |y_{j_0}|$, and all $j \in [m]$ $\xi_{I_j}[x](x_i) = 0$. However, by hypothesis $\mathfrak{P}(I)$ is a partition of I. Hence $I = \bigcup_{j\in[m]} I_j$ with $I_j \cap I_k = \emptyset$, for all $j \neq k$. Thus, for all $i \in I$ such that $|x_i| > |y_{j_0}|$ we have from Lemma 2.4.(g) $\xi_I[x](x_i) = \sum_{j\in[m]} \xi_{I_j}[x](x_i) = 0$. It follows that

$$\mathcal{J}_I(x) \subset \{i \in I : |x_i| \leq |y_{j_0}|\}.$$

Therefore, $|\bigboxplus_{j\in[m]} \left(\bigboxplus_{i\in I_j} x_i\right)| = |\bigboxplus_{j\in[m]} y_j| = |y_{j_0}| \geq |\bigboxplus_{i\in I} x_i|$. From Proposition 2.8.$(a)$, $y_{j_0} \neq 0$ implies that there is some $i_0 \in I_{j_0}$ and such that $x_{i_0} = \bigboxplus_{i\in I_{j_0}} x_i = y_{j_0}$. Since $|x_i| \leq |x_{i_0}|$ for all $i \in \mathcal{J}_I(x)$ and the y_j's are not symmetrical it follows that $\xi_I[x](x_{i_0}) > 0$ which ends the proof. $\square$

2.4. Euclidean Orthant, Absolute Value and Upper Semi-Lattice Structure

The algebraic structure $(\mathbb{R}, \boxplus, \cdot)$ can be extended to $\mathbb{R}^n$. Suppose that $x, y \in \mathbb{R}^n$, and let us denote

$$x \boxplus y = (x_1 \boxplus y_1, \cdots, x_n \boxplus y_n). \tag{9}$$

Moreover, let us consider m vectors $x_1, ..., x_m \in \mathbb{R}^n$, and define

$$\bigboxplus_{j\in[m]} x_j = \left(\bigboxplus_{j\in[m]} x_{j,1}, \cdots, \bigboxplus_{j\in[m]} x_{j,n}\right). \tag{10}$$

Let the triple $(\mathbb{R}^n, \boxplus, \cdot)$ denotes the n-dimensional Euclidean vector space equipped with the operation binary operation $(x, y) \mapsto x \boxplus y$ and the external scalar multiplication of vectors by real numbers $\cdot$.

For all $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ we have $x \boxplus y = x \vee y$. Moreover, for all $(x, y) \in \mathbb{R}^n_- \times \mathbb{R}^n_-$, $x \boxplus y = x \wedge y$ where $\vee$ and $\wedge$ respectively denote the maximum and the minimum with respect to partial order of $\mathbb{R}^n$ associated to the positive cone, that is, the coordinatewise supremum and infimum. For all x and y in $\mathbb{R}^n$, $x \leq y$ means $y - x \in \mathbb{R}^n_+$. It follows that given a subset $\{x_1, ..., x_m\}$ of $\mathbb{R}^m_+$, we have

$$\boxplus_{j \in [m]} x_j = \bigvee_{j \in [m]} x_j. \tag{11}$$

If $\{x_1, ..., x_m\}$ is a subset of $\mathbb{R}^m_-$ then

$$\boxplus_{j \in [m]} x_j = \bigwedge_{j \in [m]} x_j. \tag{12}$$

For all $x, y \in \mathbb{R}^n$, let us denote $x \boxdot y = (x_1 y_1, ..., x_n y_n)$. In the following we say that two vectors $x, y \in \mathbb{R}^n$ are **copositive** if

$$x \boxdot y \in \mathbb{R}^n_+. \tag{13}$$

We say that a subset K of $\mathbb{R}^n$ is copositive if for all $x, y \in K$ one has $x \boxdot y \geq 0$. For all subset L of $\mathbb{R}^n$, K is copositive and maximal in L if there does not exists a copositive subset $K' \subset L$ which contains K. A n-dimensional ortant in $\mathbb{R}^n$ is copositive and maximal in $\mathbb{R}^n$. Equivalently, a n-dimensional orthant in $\mathbb{R}^n$ is a subset defined by a system of inequalities: $\epsilon_i x_i \geq 0$ for any $i \in [n]$, where each ϵ_i is $+1$ or -1. A n-dimensional closed orthant K of $\mathbb{R}^n$ can be written $K = \prod_{i=1}^n (\epsilon_i \mathbb{R}_+)$. Let $\Psi_K : \mathbb{R}^n_+ \longrightarrow K$ be the map defined by $\Psi_K(x) = (\epsilon_1 x_1, ..., \epsilon_n x_n)$ with $|\epsilon_i| = 1$, for all $i \in [n]$. ψ_K is a linear homeomorphism such that $\psi_K(\mathbb{R}^n_+) = K$ and one has ${\psi_K}^{-1} = \psi_K$, which implies that $\psi_K(K) = \mathbb{R}^n_+$.

For all $x \in \mathbb{R}^n$, let us denote $|x| = (|x_1|, ..., |x_n|)$. Let K be a n-dimensional orthant and let us consider the binary relation defined by $x \leqslant y \iff |x| \leq |y|$. $\geqslant$ is a partial order over K. For all x, y, and z in K, we have $x \leqslant x$ (reflexivity); if $x \leqslant y$ and $y \leqslant x$ then $x = y$ (antisymmetry); if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$ (transitivity). A n-dimensional closed orthant K equipped with the partial order $\leqslant$ is a partially ordered set (or a poset). Then $\boxplus$ is a join on K, and the triple $(K, \boxplus, \geqslant)$ is an upper-semilattice.

If $\{x_1, ..., x_m\}$ is a subset of K then

$$\boxplus_{j \in [m]} x_j = \Psi_K \left(\bigvee_{j \in [m]} \Psi_K(x_j) \right). \tag{14}$$

3. On Some Idempotent Convex Structure

A subset C of a $\mathbb{R}^n_+$ is $\mathbb{B}$-convex if and only if for all $t \in [0, 1]$ and all $x, y \in C$, $x \vee ty \in C$. Equivalently, we say that a subset C of a n-dimensional orthant K is $\mathbb{B}$-convex if and only if for all $t \in [0, 1]$ and all $x, y \in C$, $x \boxplus ty \in C$. Such a definition is equivalent to that one proposed in [6]. It is also the definition proposed further in the paper to define $\mathbb{B}$-convex sets on the whole Euclidean vector space.

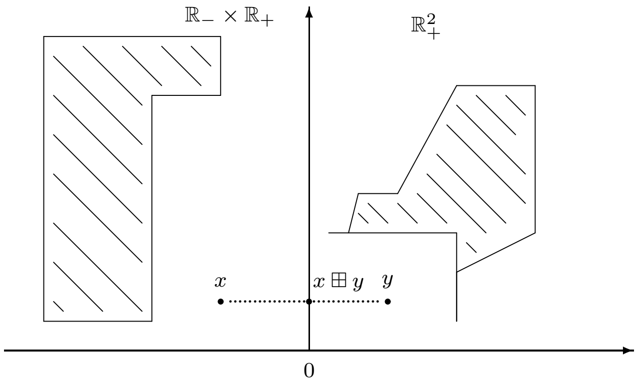

Figure 3.1: $\mathbb{B}$-convex sets and symmetrical points

Equivalently, a subset C of K is $\mathbb{B}$-convex if and only if $\Psi_K(C)$ is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$. For all finite subset $A = \{x_1, ..., x_m\}$ of K the smallest $\mathbb{B}$-convex set which contains it is $\mathbb{B}[A] = \Big\{ \boxplus_{i=1,\cdots,m} t_i x_i : t_i \in [0,1], \max_{i\in[m]} t_i = 1 \Big\}$. For the sake of simplicity, let $\mathbb{B}[x, y]$ denote the $\mathbb{B}$-convex hull of $\{x, y\}$ for all $x, y \in K$.

The binary operation $\boxplus$ yields a simple formulation of $\mathbb{B}$-convexity on each orthant. However, the problem to solve is much more complex over $\mathbb{R}^n$. Suppose for example that $x, y \in \mathbb{R}^n$, $|x| = |y|$ and $x \neq y$, then $\{tx \boxplus sy : \max\{t, s\} = 1, t, s \geq 0\} = \{x, x \boxplus y, y\}$ that is not a path-connected subset of $\mathbb{R}^n$ (see Figure 3.1).

3.1. An Extended Definition of $\mathbb{B}$-convexity

In [6] $\mathbb{B}$-convexity is introduced as a limit of linear convexities. More precisely, for all $p \in \mathbb{N}$ the Φ_p-convex hull of a finite set $A \subset \mathbb{R}^n$ is defined by:

$$\mathrm{Co}^{\Phi_p}(A) = \left\{ \overset{\varphi_p}{\sum_{i\in[m]}} t_i \overset{\varphi_p}{\cdot} x_i : \overset{\varphi_p}{\sum_{i\in[m]}} t_i = \varphi_p^{-1}(1), \varphi_p(t_i) \geq 0, i \in [m] \right\} \tag{15}$$

which can be rewritten:

$$\mathrm{Co}^{\Phi_p}(A) = \left\{ \Phi_p^{-1}\left(\sum_{i\in[m]} t_i^{2p+1}.\Phi_p(x_i) \right) : \left(\sum_{i\in[m]} t_i^{2p+1} \right)^{\frac{1}{2p+1}} = 1,\, t_i \geq 0, i \in [m] \right\}.$$

This is basically the approach of Ben-Tal [5] and Avriel [3].

Equivalently, one has $\mathrm{Co}^{\Phi_p}(A) = \Phi_p^{-1}(\mathrm{Co}(\Phi_p(A)))$. Recall that, for all $x, y \in \mathbb{R}^n$, $x \overset{p}{+} y = x \overset{\varphi_p}{+} y$. For simplicity, throughout the remainder of the paper we denote for all subset L of $\mathbb{R}^n$ $\mathrm{Co}^p(L) = \mathrm{Co}^{\Phi_p}(L)$.

From Briec and Horvath [6] a subset L of $\mathbb{R}^n$ is $\mathbb{B}$-convex if for all finite subset $A \subset L$ the $\mathbb{B}$-polytope $\mathrm{Co}^\infty(A) = \mathrm{Ls}_{p\longrightarrow\infty} \mathrm{Co}^p(L)$ is contained in L. In the following, we show that the Painlevé-Kuratowski limit of the Φ_p-convex hull of two points x, y exists in $\mathbb{R}^n$ and we give an algebraic characterization[1].

In this paper, a weaker definition is proposed in line with the algebraic structure above introduced.

Definition 3.1. A subset C of $\mathbb{R}^n$ is $\mathbb{B}^\sharp$-convex if and only if for all $t \in [0,1]$ and all $(x,y) \in C \times C$, $x \boxplus ty \in C$.

Notice that for all n-dimensional orthant K of $\mathbb{R}^n$, a $\mathbb{B}$-convex subset of K is $\mathbb{B}^\sharp$-convex. It is shown that the following definitions of $\mathbb{B}^\sharp$-convexity are equivalent.

Proposition 3.2. *For all subset C of $\mathbb{R}^n$, the following claims are equivalent:*

(a) *C is a $\mathbb{B}^\sharp$-convex subset of $\mathbb{R}^n$.*

(b) *For all $(x_1, ..., x_m) \in C^m$ we have:*

$$\left\{ \boxplus_{i\in[m]} t_i x_i : \max_{i\in[m]} t_i = 1, t \in [0,1]^m \right\} \subset C.$$

Proof. Let us prove that (a) implies (b). If C is $\mathbb{B}^\sharp$-convex, this property is true for $m = 2$. Suppose it is true at rank m and let us prove that it is true at rank $m+1$. In other words, assume that for all $(x_1, ..., x_m) \in C^m$ we have: $\left\{ \boxplus_{i\in[m]} t_i x_i : \max_{i\in[m]} t_i = 1, t \in [0,1]^m \right\} \subset C$, we need to prove that if $(x_1, ..., x_m, x_{m+1}) \in C^{m+1}$ then for all $t \in [0,1]^{m+1}$ such that $\max_{i\in[m+1]} t_i = 1$ we have $\boxplus_{i\in[m+1]} t_i x_i \in C$. To establish this property, we use Proposition 2.10 which implies that if $(x_1, ..., x_m, x_{m+1}) \in C^{m+1}$ then, for all $t \in [0,1]^{m+1}$ such that $\max_{i\in[m+1]} t_i = 1$, we have

$$\boxplus_{i\in[m+1]} t_i x_i = \boxplus_{i\in[m+1]} \left[t_i x_i \boxplus \left(\boxplus_{j\in[m+1]\setminus\{i\}} t_j x_j \right) \right].. \qquad (\star)$$

For all i set $t_i^\star = \max\{t_j : j \in [m+1]\setminus\{i\}\}$. It follows that

$$\boxplus_{i\in[m+1]} t_i x_i = \boxplus_{i\in[m+1]} \left[t_i x_i \boxplus t_i^\star \left(\boxplus_{j\in[m+1]\setminus\{i\}} (\frac{t_j}{t_i^\star})\, x_j \right) \right].$$

(i) By definition, if $t_i^\star = \max\{t_j : j \in [m+1]\setminus\{i\}\} < 1$ then, since $\max_{j\in[m+1]} t_j = 1$, we have $t_i = 1$. Moreover, $\max_{j\in[m+1]\setminus\{i\}}(\frac{t_j}{t_i^\star}) = 1$ and since, by hypothesis, the

[1]The Kuratowski-Painlevé lower limit of the sequence of sets $\{A_n\}_{n\in\mathbb{N}}$, denoted $\mathrm{Li}_{n\to\infty} A_n$, is the set of points p for which there exists a sequence $\{p_n\}$ of points such that $p_n \in A_n$ for all n and $p = \lim_{n\to\infty} p_n$; a sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^m$ is said to converge, in the Kuratowski-Painlevé sense, to a set A if $\mathrm{Ls}_{n\to\infty} A_n = A = \mathrm{Li}_{n\to\infty} A_n$, in which case we write $A = \mathrm{Lim}_{n\to\infty} A_n$.

property is assumed to be true at rank m, it follows that $\boxplus_{j\in[m+1]\backslash\{i\}}(\frac{t_j}{t_i^\star})\,x_j \in C$. Hence, we deduce that

$$x_i \boxplus t_i^\star \left(\bigboxplus_{j\in[m+1]\backslash\{i\}} \left(\frac{t_j}{t_i^\star}\right) x_j \right) \in C.$$

(ii) If $t_i^\star = 1$ then there is some $i_0 \in [m+1]\backslash\{i\}$ such that $t_{i_0} = 1$ and, by hypothesis, it follows that $\boxplus_{j\in[m+1]\backslash\{i\}} t_j x_j \in C$. Furthermore, since $t_i \in [0,1]$ we deduce from (a) that $t_i x_i \boxplus \left(\boxplus_{j\in[m+1]\backslash\{i\}} t_j x_j\right) \in C$. For all i, set $\Lambda_i = t_i x_i \boxplus \left(\boxplus_{j\in[m+1]\backslash\{i\}} t_j x_j\right)$. We have proven that, for each i, $\Lambda_i \in C$. Moreover, from Propositions 2.10 and Lemma 2.11, the Λ_i's belong to a n-dimensional orthant K and, then, can be composed associatively using the operation $\boxplus$. Thus, we deduce from $(\star)$ that $\boxplus_{i\in[m+1]} t_i x_i \in C$ which ends the proof of (b). The converse inclusion is immediate. □

Proposition 3.3.

(a) *The emptyset, $\mathbb{R}^n$, as well as all the singletons are $\mathbb{B}^\sharp$-convex.*

(b) *If $\{D_\delta : \delta \in \Delta\}$ is an arbitrary family of $\mathbb{B}^\sharp$-convex sets then $\bigcap_\delta D_\delta$ is $\mathbb{B}^\sharp$-convex.*

(c) *If $\{D_\delta : \delta \in \Delta\}$ is a family of $\mathbb{B}^\sharp$-convex sets such that $\forall \delta_1, \delta_2 \in \Delta$ $\exists \delta_3 \in \Delta$ such that $D_{\delta_1} \cup D_{\delta_2} \subset D_{\delta_3}$ then $\bigcup_\delta D_\delta$ is $\mathbb{B}^\sharp$-convex.*

(d) *If C a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ then it is $\mathbb{B}^\sharp$-convex.*

Given a set $S \subset \mathbb{R}^n$ there is, according to (a) above, a $\mathbb{B}^\sharp$-convex set which contains S; by (b) the intersection of all such $\mathbb{B}^\sharp$-convex sets is $\mathbb{B}$-convex; we call it the $\mathbb{B}^\sharp$-convex hull of S and we write $\mathbb{B}^\sharp[S]$ for the $\mathbb{B}^\sharp$-convex hull of S.

Proposition 3.4. *The following properties hold:*

(a) $\mathbb{B}^\sharp[\emptyset] = \emptyset$, $\mathbb{B}^\sharp[\mathbb{R}^n] = \mathbb{R}^n$, *for all* $x \in \mathbb{R}^n$, $\mathbb{B}^\sharp[\{x\}] = \{x\}$.

(b) *For all* $S \subset \mathbb{R}^n$, $S \subset \mathbb{B}^\sharp[S]$ *and* $\mathbb{B}^\sharp[\,[\mathbb{B}^\sharp[S]\,] = \mathbb{B}^\sharp[S]$.

(c) *For all* $S_1, S_2 \subset \mathbb{R}^n$, *if* $S_1 \subset S_2$ *then* $\mathbb{B}^\sharp[S_1] \subset \mathbb{B}^\sharp[S_2]$.

(d) *For all* $S \subset \mathbb{R}^n$, $\mathbb{B}^\sharp[S] = \bigcup\{\mathbb{B}^\sharp[A] : A$ *is a finite subset of* $S\}$.

(e) *A subset $L \subset \mathbb{R}^n$ is $\mathbb{B}^\sharp$-convex if and only if, for all finite subset A of L, $\mathbb{B}^\sharp[A] \subset L$.*

3.2. Intermediate Points and Copositivity

A set of points, we term the **intermediate points**, is introduced to characterize the $\mathbb{B}$-convex hull of two points on the whole Euclidean vector space. For all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, let us consider the map $\gamma : \mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty] \longrightarrow \mathbb{R}^n$ defined by:

$$\gamma(x, y, t) = (\max\{1, t\})^{-1}\,(x \boxplus ty)\,, \quad \text{for all } t \geq 0 \tag{16}$$

and by $\gamma(x,y,+\infty)=y$. For all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$, let $\mathcal{I}(x,y)$ be the subset defined by $\mathcal{I}(x,y)=\{i\in[n]:x_iy_i<0\}$ and let $n(x,y)$ be its cardinal. Remark that $\gamma(x,y,0)=x$.

For all $i\in\mathcal{I}(x,y)$ and all $t_i^\star\in\mathbb{R}_{++}$ a point $\gamma\in\mathbb{R}^n$ is called a **i-intermediate point** between x and y if there is some $t_i^\star\in]0,+\infty[$ such that

$$\gamma_i(x,y,t_i^\star):=(\gamma(x,y,t_i^\star))_i=0. \tag{17}$$

Lemma 3.5. *Let $\gamma:\mathbb{R}^n\times\mathbb{R}^n\times(\mathbb{R}_+\cup\{+\infty\})\longrightarrow\mathbb{R}^n$ be the map defined in* (16). *Suppose that $\mathcal{I}(x,y)\neq\emptyset$. We have the the following properties:*

(a) *For all $i\in\mathcal{I}(x,y)$, the map $t\mapsto\gamma_i(x,y,t)$ has a uniqueness zero $t_i^\star=-\frac{x_i}{y_i}=|\frac{x_i}{y_i}|>0$ and there is a uniqueness i-intermediate point*

$$\gamma(x,y,t_i^\star)=\left(\frac{|y_i|}{\max\{|x_i|,|y_i|\}}\,x\right)\boxplus\left(\frac{|x_i|}{\max\{|x_i|,|y_i|\}}\,y\right).$$

(b) *For all $i\in\mathcal{I}(x,y)$ and all $t\geq 0$*

$$\gamma_i(x,y,t)=\begin{cases}\max\{1,t\}^{-1}x_i & \textit{if } t<-\frac{x_i}{y_i}\\ 0 & \textit{if } t=-\frac{x_i}{y_i}\\ t\max\{1,t\}^{-1}y_i & \textit{if } t>-\frac{x_i}{y_i}\end{cases}$$

(c) *If $t>\max\{1,-\frac{x_i}{y_i}\}$ then $\gamma_i(x,y,t)=y_i$. Moreover, if $t<\min\{1,-\frac{x_i}{y_i}\}$ then $\gamma_i(x,y,t)=x_i$.*

(d) $\lim_{t\longrightarrow 0}\gamma(x,y,t)=x$ *and* $\lim_{t\longrightarrow+\infty}\gamma(x,y,t)=y$.

Proof. (a) For all $i\in\mathcal{I}(x,y)$, we have $x_iy_i<0$, which implies that $-\frac{x_i}{y_i}=|\frac{x_i}{y_i}|>0$. Moreover, $\gamma_i(x,y,t)=0$ if and only if $(\max\{1,t\}^{-1}x_i)\boxplus(\max\{1,t\}^{-1}ty_i)=0$. Since this is equivalent to $x_i\boxplus ty_i=0$, we deduce that $t_i^\star=-\frac{x_i}{y_i}$ is the uniqueness positive zero of the equation $\gamma_i(x,y,t)=0$. Moreover $\gamma_i(x,y,0)=x_i\neq 0$ and $\gamma_i(x,y,+\infty)=y_i\neq 0$, which ends the proof.

(b) The case $t=-\frac{x_i}{y_i}$ is an immediate consequence of (a). Assume that $t\neq-\frac{x_i}{y_i}$. In such a case, one has

$$\gamma_i(x,y,t)=\begin{cases}\max\{1,t\}^{-1}x_i & \text{if } |\max\{1,t\}^{-1}x_i|>|t\max\{1,t\}^{-1}y_i|\\ t\max\{1,t\}^{-1}y_i & \text{if } |\max\{1,t\}^{-1}x_i|<|t\max\{1,t\}^{-1}y_i|.\end{cases}$$

Since $|\frac{x_i}{y_i}|=-\frac{x_i}{y_i}$, it follows that

$$\gamma_i(x,y,t)=\begin{cases}\max\{1,t\}^{-1}x_i & \text{if } t<-\frac{x_i}{y_i}\\ t\max\{1,t\}^{-1}y_i & \text{if } t>-\frac{x_i}{y_i}.\end{cases}$$

(c) If $t>\max\{1,-\frac{x_i}{y_i}\}$, then, from ($b$) one has $\gamma_i(x,y,t)=t\max\{1,t\}^{-1}y_i$. Moreover $\max\{1,t\}=t$. Therefore $\gamma_i(x,y,t)=y_i$. If $t<\min\{1,-\frac{x_i}{y_i}\}$, then $\gamma_i(x,y,t)=\max\{1,t\}^{-1}x_i$. Moreover $\max\{1,t\}=1$. Therefore $\gamma_i(x,y,t)=x_i$.

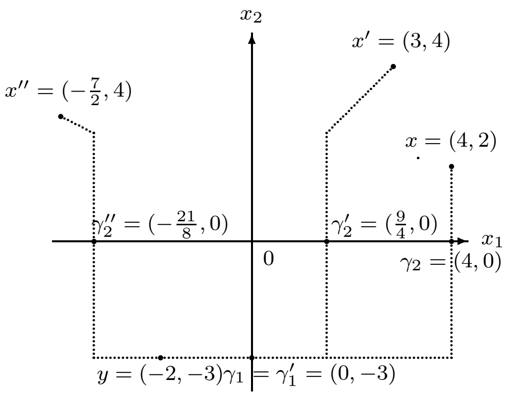

Figure 3.2.1: Intermediate points

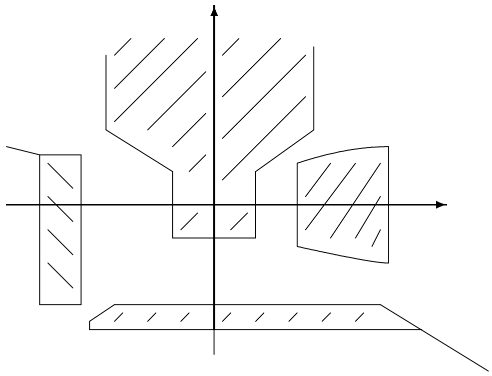

Figure 3.2.2: Examples of $\mathbb{B}^\sharp$-convex sets

(d) Suppose that $j \notin \mathcal{I}(x,y)$, then there is $\epsilon \in \{-1,1\}$ such that $\left(\max\{1,t\}^{-1}x\right)_j \boxplus \left(t\max\{1,t\}^{-1}y\right)_j \in \epsilon\mathbb{R}_+$. It follows that the map $t \mapsto \gamma_j(x,y,t)$ is continuous. Hence, we clearly have $\lim_{t\longrightarrow 0}\gamma_j(x,y,t) = x_j$ and $\lim_{t\longrightarrow +\infty}\gamma_j(x,y,t) = y_j$. Suppose now that $i \in \mathcal{I}(x,y)$. From (c) $\lim_{t\longrightarrow 0}\gamma_i(x,y,t) = x_i$ and $\lim_{t\longrightarrow +\infty}\gamma_i(x,y,t) = y_i$ which ends the proof. $\square$

Notice that it may happen that there are two indexes $i,k \in \mathcal{I}(x,y)$ such that $\gamma(x,y,t_i^\star) = \gamma(x,y,t_k^\star)$. Let $\Theta(x,y) = \{0,+\infty,-\frac{x_i}{y_i} : i \in \mathcal{I}(x,y)\}$. If $\mathcal{I}(x,y) = \emptyset$, then $\Theta(x,y) = \{x,y\}$.

Example 3.6. Let $x = (4,2)$, $x' = (3,4)$, $x'' = (-\frac{7}{2},3)$ and $y = (-2,-3)$ four points of $\mathbb{R}^2$. Clearly, we have $\mathcal{I}(x,y) = \mathcal{I}(x',y) = \{1,2\}$. Let us denote γ_i the intermediate points between x and y. We have $\gamma_1 = \max\{4,2\}^{-1}(2(4,2) \boxplus 4(-2,-3)) = (0,-3)$. The second intermediate point between x and y is $\gamma_2 = \max\{2,3\}^{-1}(3(4,2) \boxplus 2(-2,-3)) = (4,0)$. Let us denote γ_i' the intermediate points between x' and y. We have $\gamma_1' = \max\{3,2\}^{-1}(2(3,4) \boxplus 3(-2,-3)) = (0,-3)$. Following a similar procedure, we get $\gamma_2' = (\frac{9}{4},0)$. Finally, we have $\mathcal{I}(x'',y) = \{2\}$, and using similar notations we obtain $\gamma_2'' = (-\frac{21}{8},0)$.

The following figure depicts the example above. The doted lines between the points and their respective intermediate points represent the $\mathbb{B}$-convex hull defined on each orthant respectively.

Lemma 3.7. *For all $x,y \in \mathbb{R}^n$ such that $\mathcal{I}(x,y) \neq \emptyset$, let $\{i_m\}_{m\in[n(x,y)]}$ be an index sequence of $\mathcal{I}(x,y)$ such that*

$$i_m \in \arg\max_{i\in\mathcal{I}(x,y)}\left\{-\frac{x_i}{y_i} : i \geq m\right\} \quad \textit{for all } m \in [n(x,y)].$$

We have for all $m \in [n(x,y)-1]$:

$$\gamma\left(x,y,-\frac{x_{i_m}}{y_{i_m}}\right) \boxdot \gamma\left(x,y,-\frac{x_{i_{m+1}}}{y_{i_{m+1}}}\right) \geq 0.$$

Moreover $x \boxdot \gamma\left(x, y, -\frac{x_{i_1}}{y_{i_1}}\right) \geq 0$, $\gamma\left(x, y, -\frac{x_{i_{n(x,y)}}}{y_{i_{n(x,y)}}}\right) \boxdot y \geq 0$.

Proof. Suppose that $j \notin \mathcal{I}(x,y)$. In such a case $x_j y_j \geq 0$. It follows that there is some $\epsilon \in \{-1,1\}$ such that $x_j, y_j \in \epsilon\mathbb{R}_+$. Therefore $\gamma_j(x,y,t) = \left[\left(\max\{1,t\}^{-1} x\right) \boxplus \left(t \max\{1,t\}^{-1} y\right)\right]_j \in \epsilon\mathbb{R}_+$, for all $t \geq 0$. Hence if $\gamma(x,y,t^\star_{i_m})$ and $\gamma(x,y,t^\star_{i_{m+1}})$ are two intermediate points, then for all $j \in [n]\backslash\mathcal{I}(x,y)$ one has

$$\left(\gamma_j(x,y,t^\star_{i_m})\right)\left(\gamma_j(x,y,t^\star_{i_{m+1}})\right) \geq 0.$$

Suppose now that $i \in \mathcal{I}(x,y)$. Set $t^\star_i = -\frac{x_i}{y_i}$ for all $i \in \mathcal{I}(x,y)$. By construction $\{t^\star_{i_m}\}_{m\in[n(x,y)]}$ is a nondecreasing sequence of $\mathbb{R}_{++}$. Since $t^\star_{i_m}$ and $t^\star_{i_{m+1}}$ are two consecutive terms of this sequence, for all $i \in \mathcal{I}(x,y)$ we have $t^\star_i \notin]t^\star_{i_m}, t^\star_{i_{m+1}}[$. Thus one has either $t^\star_i \leq t^\star_{i_m} \leq t^\star_{i_{m+1}}$ or $t^\star_i \geq t^\star_{i_{m+1}} \geq t^\star_{i_m}$. From Lemma 3.5.$(b)$,

$$\begin{aligned}
&\gamma_i(x,y,t^\star_{i_m})\ \gamma_i(x,y,t^\star_{i_{m+1}}) \\
&= \begin{cases} \max\{1,t^\star_{i_m}\}^{-1}\max\{1,t^\star_{i_{m+1}}\}^{-1}(x_i)^2 \geq 0 & \text{if } t^\star_i > t^\star_{i_{m+1}} \geq t^\star_{i_m} \\ 0 & \text{if } t^\star_i \in \{t^\star_{i_m}, t^\star_{i_{m+1}}\} \\ t^\star_{i_m} t^\star_{i_{m+1}} \max\{1,t^\star_{i_m}\}^{-1}\max\{1,t^\star_{i_{m+1}}\}^{-1}(y_i)^2 \geq 0 & \text{if } t^\star_i < t^\star_{i_m} \leq t^\star_{i_{m+1}} \end{cases}
\end{aligned} \tag{18}$$

It follows that for all $m \in [n(x,y)-1]$, $\gamma(x,y,t_{i_m}) \boxdot \gamma(x,y,t_{i_{m+1}}) \geq 0$. Let us prove that $x \boxdot \gamma(x,y,t_{i_1}) \geq 0$ and $\gamma(x,y,t_{i_{n(x,y)+1}}) \boxdot y \geq 0$. Since for all $i \in \mathcal{I}(x,y)$ $t^\star_{i_1} \leq t^\star_i \leq t^\star_{i_{n(x,y)}}$ we have

$$x_i\ \gamma_i(x,y,t^\star_{i_1}) = \begin{cases} \max\{1,t^\star_{i_1}\}^{-1}(x_i)^2 \geq 0 & \text{if } t^\star_i > t^\star_{i_1} \\ 0 & \text{if } t^\star_i = t^\star_{i_1} \end{cases}$$

and

$$\gamma_i(x,y,t^\star_{i_{n(x,y)}})\ y_i = \begin{cases} 0 & \text{if } t^\star_i = t^\star_{i_{n(x,y)}} \\ t^\star_{i_{n(x,y)}} \max\{1,t^\star_{i_{n(x,y)}}\}^{-1}(y_i)^2 \geq 0 & \text{if } t^\star_i < t^\star_{i_{n(x,y)}} \end{cases}$$

Since $t^\star_i = -\frac{x_i}{y_i} > 0$ for all $i \in \mathcal{I}(x,y)$, this ends the proof. $\square$

Set $t^\star_{i_0} = 0$, $t^\star_{i_{n(x,y)+1}} = +\infty$ and $t^\star_{i_m} = -\frac{x_{i_m}}{y_{i_m}}$ for all $m \in [n(x,y)]$. A sequence $\{t^\star_{i_m}\}_{m=0}^{n(x,y)+1}$ of $\Theta(x,y)$ satisfying the conditions of Lemma 3.7 is called an **intermediate sequence**. Moreover, let $\Gamma(x,y) = \{\gamma(x,y,t^\star_i) : t^\star_i \in \Theta(x,y)\}$ be the set of the i-intermediate points together with x and y.

One can then establish the following inclusion, whose the proof is immediate.

Proposition 3.8. *Let C be a $\mathbb{B}^\sharp$-convex set of $\mathbb{R}^n$. Suppose that $x, y \in C$, and let $\{t^\star_{i_m}\}_{m=0}^{n(x,y)+1}$ be an intermediate sequence of $\Theta(x,y)$. Then, for all $m \in$*

$[n(x,y)]$, $\gamma(x,y,t^\star_{i_m}) \in C$. *Furthermore*

$$\bigcup_{m=0}^{n(x,y)} \mathbb{B}\left[\gamma(x,y,t^\star_{i_m}), \gamma(x,y,t^\star_{i_{m+1}})\right] \subset C.$$

Proof. For all $t \geq 0$, $\max\{\max\{1,t\}^{-1}, t\max\{1,t\}^{-1}\} = 1$. Consequently, since $x, y \in C$, all the intermediate points lie in C. Since for all $m \in \{0,1,...,n(x,y)\}$ $\gamma(x,y,t^\star_{i_m})$ and $\gamma(x,y,t^\star_{i_{m+1}})$ are copositive, it follows that $\mathbb{B}\left[\gamma(x,y,t^\star_{i_m}), \gamma(x,y,t^\star_{i_{m+1}})\right] \subset C$ for all m, which ends the proof. □

The next results will be useful in the remainder of the paper.

Lemma 3.9. *For all* $(a,b,c,d) \in \mathbb{R}^4$, *if* $(a \boxplus b)(c \boxplus d) \geq 0$ *then*

$$(a \boxplus b) \boxplus (c \boxplus d) = \digamma_{[4]}(a,b,c,d).$$

Proof. We first assume that $(a \boxplus b)(c \boxplus d) = 0$. In such a case, one has either $a \boxplus b = 0$ or $c \boxplus d = 0$. Suppose, for example, that $a \boxplus b = 0$. Then $(a \boxplus b) \boxplus (c \boxplus d) = 0 \boxplus (c \boxplus d) = c \boxplus d$. Moreover, $\digamma_{[4]}(a,b,c,d) = \digamma_{[4]}(a,-a,c,d) = c \boxplus d$ and the equality holds true. The proof is similar in the case where $c \boxplus d = 0$.

Suppose now that $(a \boxplus b)(c \boxplus d) > 0$. Then $a \boxplus b \neq 0$ and $c \boxplus d \neq 0$ and from Lemma 2.13, we deduce the result. □

For all $u, v, w, z \in \mathbb{R}^n$, let us denote

$$u \boxplus v \boxplus w \boxplus z = \left(\digamma_{[4]}(u_1,v_1,w_1,z_1), \cdots, \digamma_{[4]}(u_n,v_n,w_n,z_n)\right). \tag{19}$$

Proposition 3.10. *For all* $x, y \in \mathbb{R}^n$, *let* $\{t^\star_{i_m}\}_{m=0}^{n(x,y)+1}$ *be an intermediate sequence of* $\Theta(x,y)$. *Then*

$$\begin{aligned}&\bigcup_{m=0}^{n(x,y)} \mathbb{B}\left[\gamma(x,y,t^\star_{i_m}), \gamma(x,y,t^\star_{i_{m+1}})\right]\\ &\subset \{tx \boxplus rx \boxplus sy \boxplus wy : \max\{t,r,s,w\} = 1, t,r,s,w \geq 0\}.\end{aligned}$$

Proof. We have just to show that if $\gamma, \gamma' \in \Gamma(x,y)$ and $\gamma \boxdot \gamma' \geq 0$, then $\mathbb{B}[\gamma,\gamma'] \subset \{tx \boxplus rx \boxplus sy \boxplus wy : \max\{t,r,s,w\} = 1, t,r,s,w \geq 0\}$. Suppose that $z \in \mathbb{B}[\gamma,\gamma']$. Hence, by hypothesis, there are $\alpha, \alpha' \in \mathbb{R}_+$ such that $\max\{\alpha,\alpha'\} = 1$ and $z = \alpha\gamma \boxplus \alpha'\gamma'$. Since γ and γ' are two intermediate points there exists $s,t,s',t' \geq 0$ with $\max\{s,t\} = 1$ and $\max\{s',t'\} = 1$ and such that $\gamma = sx \boxplus ty$ and $\gamma' = s'x \boxplus t'y$. It follows that $z = [\alpha(sx \boxplus ty)] \boxplus [\alpha'(s'x \boxplus t'y)] = [(\alpha sx) \boxplus (\alpha ty)] \boxplus [(\alpha's'x) \boxplus (\alpha't'y)]$. Since γ and γ' are copositive, for all $i \in [n]$, $[(\alpha sx_i) \boxplus (\alpha ty_i)][(\alpha's'x_i) \boxplus (\alpha't'y_i)] \geq 0$. We deduce from Lemma 3.9 that

$$[(\alpha sx_i) \boxplus (\alpha ty_i)] \boxplus [(\alpha's'x_i) \boxplus (\alpha't'y_i)] = (\alpha sx_i) \boxplus (\alpha ty_i) \boxplus (\alpha's'x_i) \boxplus (\alpha't'y_i).$$

It follows that $z = (\alpha s)x \boxplus (\alpha t)y \boxplus (\alpha's')x \boxplus (\alpha't')y$. Moreover, one has $\max\{\alpha s, \alpha t, \alpha's', \alpha't'\} = 1$. Hence, $z \in \{tx \boxplus rx \boxplus sy \boxplus wy : \max\{t,r,s,w\} = 1, t,r,s,w \geq 0\}$, which ends the proof. □

3.3. Some Topological Properties

In the following we show that $\mathbb{B}^\sharp$-convex sets have a path-connected structure. This we do using the intermediate function and focusing on the copositive case.

Proposition 3.11. *Let $\gamma : \mathbb{R}^n \times \mathbb{R}^n \times [0,+\infty] \longrightarrow \mathbb{R}^n$ be the map defined in* (16)*. Suppose that x and y are copositive. Then:*

(a) *The map $t \mapsto \gamma(x,y,t)$ is continuous on $\mathbb{R}_+$.*

(b) *One has $\lim_{t\to 0}\gamma(x,y,t) = \gamma(x,y,0) = x$ and $\lim_{t\to\infty}\gamma(x,y,t) = \gamma(x,y,+\infty) = y$.*

(c) *For all copositive pairs (x,y), we have $\gamma(x,y,[0,+\infty]) = \mathbb{B}[x,y]$.*

Proof. (a) The maps $t \mapsto \max\{1,t\}^{-1}$ is continuous over $\mathbb{R}_+$. Since x and y are copositive, for all i there is some $\epsilon_i \in \{-1,1\}$ such that

$$\gamma_i(x,y,t) = \epsilon_i \max\left\{\epsilon_i \max\{1,t\}^{-1}x_i, \epsilon_i t \max\{1,t\}^{-1}y_i\right\}.$$

Consequently, each map $\gamma_i(x,y,\cdot)$ is continuous in t and the result follows.

(b) Clearly $\lim_{t\to 0}\max\{1,t\}^{-1} = 1$ and $\lim_{t\to 0} t\max\{1,t\}^{-1} = 0$. Hence, we have $\lim_{t\to 0}\gamma(x,y,t) = x$. Moreover, $\lim_{t\to\infty}\max\{1,t\}^{-1} = 0$ and $\lim_{t\to\infty} t\max\{1,t\}^{-1} = 1$. Consequently, $\lim_{t\to\infty}\gamma(x,y,t) = y$.

(c) Since x and y are copositive, we have $\mathbb{B}[x,y] = \{tx \boxplus sy : t,s \in [0,1], \max\{t,s\} = 1\}$. Since for all $t \geq 0$ $\max\left\{\max\{1,t\}^{-1}, t\max\{1,t\}^{-1}\right\} = 1$, we deduce that $\gamma(x,y,\mathbb{R}_+) \subset \mathbb{B}[x,y]$. However, from [6], $\mathbb{B}[x,y]$ is a closed subset of $\mathbb{R}^n$. From (a) and (b), we deduce that $\gamma(x,y,[0,+\infty]) \subset \mathbb{B}[x,y]$. Let us show the converse inclusion. By definition, we have

$$\mathbb{B}[x,y] = \{tx \boxplus y : t \in [0,1]\} \cup \{x \boxplus sy : s \in [0,1]\}.$$

Suppose that $0 < t \leq 1$, and set $t' = t^{-1}$. We have $tx \boxplus y = (\max\{1,t'\}^{-1})\, x \boxplus (t'\max\{1,t'\}^{-1})\, y \in \gamma(x,y,[0,+\infty])$. If $t = 0$ then $x = \gamma(x,y,0)$. Furthermore, if $0 \leq s \leq 1$, then $x \boxplus sy = (\max\{1,s\}^{-1}x) \boxplus (s\max\{1,s\}^{-1}y) \in \gamma(x,y,[0,+\infty])$, which proves the converse inclusion. $\square$

Corollary 3.12. *Let a and b two real numbers with $a < b$. Let $h : [a,b] \longrightarrow \bar{\mathbb{R}}_+$ be an homeomorphism such that $h(a) = 0$ and $h(b) = \infty$. Let $\xi_h : \mathbb{R}^n \times \mathbb{R}^n \times [a,b]$ be the map defined by $\xi_h(x,y,s) = \gamma(x,y,h(s))$. If x and y are copositive, then the map $s \mapsto \xi_h(x,y,s)$ is continuous. Moreover $\xi_h(x,y,a) = x$, $\xi_h(x,y,b) = y$ and $\xi_h(x,y,[a,b]) = \mathbb{B}[x,y]$.*

It is shown below that a $\mathbb{B}^\sharp$-convex set is path-connected.

Proposition 3.13. *A non empty $\mathbb{B}^\sharp$-convex of $\mathbb{R}^n$ is path-connected.*

Proof. We first establish that for all $x,y \in \mathbb{R}^n$, there exists a continuous map

$\xi(x, y, \cdot) : [0,1] \longrightarrow \mathbb{R}^n$ such that

$$\xi(x, y, [0,1]) = \bigcup_{m=0}^{n(x,y)} \mathbb{B}\left[\gamma(x, y, t^\star_{i_m}), \gamma(x, y, t^\star_{i_{m+1}})\right],$$

with $\xi(x, y, 0) = x$ and $\xi(x, y, 1) = y$. Let $\{t^\star_{i_m}\}_{m=0}^{n(x,y)+1}$ be an intermediate sequence of $\Theta(x, y)$ from x to y. Set $x = \gamma(x, y, 0)$ and $y = \gamma(x, y, \infty)$. From Corollary 3.12, for all $m \in \{0, ..., n(x, y)\}$ there is collection of continuous maps

$$\eta_m : [t^\star_{i_m}, t^\star_{i_{m+1}}] \longrightarrow \mathbb{B}\left[\gamma(x, y, t^\star_{i_m}), \gamma(x, y, t^\star_{i_{m+1}})\right]$$

such that $\eta_m([t^\star_{i_m}, t^\star_{i_{m+1}}]) = \mathbb{B}\left[\gamma(x, y, t^\star_{i_m}), \gamma(x, y, t^\star_{i_{m+1}})\right]$, $\eta(t_m) = \gamma(x, y, t^\star_{i_m})$ and $\eta_m(t_{m+1}) = \gamma(x, y, t_{i_{m+1}})$. Let $\xi : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R}^n$ be the map defined by $\xi(x, y, t) = \eta_m(t)$ for all $t \in [t_m, t_{m+1}]$. Clearly, the map $t \mapsto \xi(x, y, t)$ is continuous and we have $\xi(x, y, 0) = x$ and $\xi(x, y, 1) = y$. Moreover, $\xi(x, y, [0,1]) = \bigcup_{m=0}^{n(x,y)} \mathbb{B}\left[\gamma(x, y, t^\star_{i_m}), \gamma(x, y, t^\star_{i_{m+1}})\right]$. Suppose that L is a $\mathbb{B}^\sharp$-convex subset of $\mathbb{R}^n$. From Proposition 3.8 we have, for all $x, y \in L$, the inclusion $\bigcup_{m=0}^{n(x,y)} \mathbb{B}\left[\gamma(x, y, t^\star_{i_m}), \gamma(x, y, t^\star_{i_{m+1}})\right] \subset L$. Hence, we deduce that $\xi(L \times L \times [0,1]) \subset L$ and this ends the proof. □

3.4. Separation of Copositive $\mathbb{B}$-Convex Sets

We say that two subsets C_1 and C_2 of $\mathbb{R}^n$ are copositive if for all $(x_1, x_2) \in C_1 \times C_2$, $x_1 \boxdot x_2 \in \mathbb{R}^n_+$. In this subsection, it is shown that the inner product $(x, y) \mapsto \langle x, y\rangle_\infty = \boxplus_{i\in[n]} x_i y_i$ can be used to separate two copositive $\mathbb{B}$-convex sets. For all $u, v \in \mathbb{R}$, let us define the binary operation

$$u \smile v = \begin{cases} u & \text{if } |u| > |v| \\ \min\{u, v\} & \text{if } |u| = |v| \\ v & \text{if } |u| < |v|. \end{cases}$$

An elementary calculus shows that $u \boxplus v = \frac{1}{2}(u \smile v - [(-u) \smile (-v)])$. It has been established in [8] that the set $\mathbb{R}$ equipped with the semilattice operation $\smile$ and the usual multiplication $\cdot$ by positive real numbers is a semimodule over the semifield of positive real numbers $\mathbb{R}_+$. Furthermore, both $(\mathbb{R}_+, \smile, \cdot)$ and $(\mathbb{R}_-, \smile, \cdot)$ are sub-semimodules isomorphic to $(\mathbb{R}_+, \max, \cdot)$; the isomorphisms are, respectively, given by the inclusion, $u \mapsto u$, and the negative of the inclusion, $u \mapsto -u$.

Given m elements $u_1, ..., u_m$ of $\mathbb{R}$, not all of which are 0, let I_+, respectively I_-, be the set of indices for which $0 < u_i$, respectively $u_i < 0$. We can then write $u_1 \smile \cdots \smile u_m = (\smile_{i\in I_+} u_i) \smile (\smile_{i\in I_-} u_i) = (\max_{i\in I_+} u_i) \smile (\min_{i\in I_-} u_i)$ from which we have

$$u_1 \smile \cdots \smile u_m = \begin{cases} \max_{i\in I_+} u_i & \text{if } I_- = \emptyset \text{ or } \max_{i\in I_-} |u_i| < \max_{i\in I_+} u_i \\ \min_{i\in I_-} u_i & \text{if } I_- = \emptyset \text{ or } \max_{i\in I_+} u_i < \max_{i\in I_-} |u_i| \\ \min_{i\in I_-} u_i & \text{if } \max_{i\in I_-} |u_i| = \max_{i\in I_+} u_i. \end{cases} \quad (20)$$

We define a $\mathbb{B}$-form on $\mathbb{R}^n_+$ as a map $f : \mathbb{R}^n_+ \to \mathbb{R}$ such that, for all $u_1, ..., u_m$ in $\mathbb{R}^n_+$ and all $t_1, ..., t_m$ in $\mathbb{R}_+$ $f(t_1 u_1 \vee \cdots \vee t_m u_m) = t_1 f(u_1) \smile \cdots \smile t_m f(u_m)$. It has been shown in [8] that a map $f : \mathbb{R}^n_+ \to \mathbb{R}$ is a $\mathbb{B}$-form if and only if there exists $(a_1, ..., a_n) \in \mathbb{R}^n$, necessarily unique, such that, for all $(x_1, ..., x_n) \in \mathbb{R}^n_+$,

$$f(x_1, ..., x_n) = a_1 x_1 \smile \cdots \smile a_n x_n. \tag{21}$$

Moreover for all $\mathbb{B}$-forms $f : \mathbb{R}^n_+ \to \mathbb{R}$ and all real numbers c :

$$\text{if } 0 \leq c \text{ then } f(x) \leq c \text{ if and only if } \max_{i \in I_+}\{a_i x\} \leq \max_{i \in I_-}\{-a_i x, c\} \tag{22}$$

and

$$\text{if } c \leq 0 \text{ then } f(x) \leq c \text{ if and only if } \max_{i \in I_+}\{a_i x, -c\} \leq \max_{i \in I_-}\{-a_i x\}. \tag{23}$$

For all $c \in \mathbb{R}$ and all subset I of $[n]$ the map $y \mapsto \max_{i \in I}\{y_i, c\}$ is continuous over $\mathbb{R}^n$. Therefore for all $c \in \mathbb{R}$, $f^{-1}(\,]-\infty, c]) = \{x \in \mathbb{R}^n : f(x) \leq c\}$ is closed. It follows that a $\mathbb{B}$-form is lower semi-continuous.

The largest (smallest) lower (upper) semi-continuous minorant (majorant) of a map h is said to be the lower (upper) semi-continuous regularization of h.

Proposition 3.14. *Let f be a $\mathbb{B}$-form defined by $f(x_1, ..., x_n) = a_1 x_1 \smile \cdots \smile a_n x_n$, for some $a \in \mathbb{R}^n$. Then f is the lower semi-continuous regularization of the map $x \mapsto \langle a, x \rangle_\infty = \boxplus_{i \in [n]} a_i x_i$.*

Proof. Suppose that $\varphi_a : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the lower semi-continuous regularization of $\langle a, x \rangle_\infty$. First, remark that for all $x \in \mathbb{R}^n$:

$$f(x) \leq \langle a, x \rangle_\infty.$$

Therefore, all we need to prove is that $\varphi_a(x) = f(x)$. By definition, since f is lower semi-continuous, we have for all $x \in \mathbb{R}^n$

$$f(x) \leq \varphi_a(x) \leq \langle a, x \rangle_\infty.$$

Let $I^a_- = \{i \in [n] : a_i x_i < 0\}$. If $I^a_- = \emptyset$ then $\langle a, x \rangle_\infty = \max_{i \in [n]} |a_i x_i| = \max_{i \in [n]}\{a_i x_i\}$. Moreover by definition $f(x) = \max_{i \in [n]}\{a_i x_i\} = \langle a, x \rangle_\infty$. Consequently, since $f(x) \leq \varphi_a(x) \leq \langle a, x \rangle_\infty$, we deduce that $f(x) = \langle a, x \rangle_\infty = \varphi_a(x)$.

Suppose now that $I^a_- \neq \emptyset$ and pick some $i_0 \in I^a_-$. By hypothesis, we have $a_{i_0} \neq 0$. Now, let $\{x_k\}_{k \in \mathbb{N}}$ be the sequence defined as:

$$x_{k,i} = \begin{cases} x_i & \text{if } i \neq i_0 \\ x_{i_0} + \frac{1}{a_{i_0} k} & \text{if } i = i_0. \end{cases}$$

Hence, since $a_{i_0}x_{i_0} < 0$ and $i_0 \in I_-^a$ we have $\langle a, x_k\rangle_\infty = a_{i_0}x_{i_0} - \frac{1}{k} = -\max_{i\in[n]} |a_i x_i| - \frac{1}{k}$. Thus:

$$\lim_{k\longrightarrow\infty} \langle a, x_k\rangle_\infty = \lim_{k\longrightarrow\infty} \left(-\max_{i\in[n]} |a_i x_i| - \frac{1}{k}\right) = -\max_{i\in[n]} |a_i x_i|.$$

Moreover, since φ_a is lower semi-continuous and $\lim_{k\longrightarrow\infty} x_k = x$:

$$\liminf_{k\longrightarrow\infty} \varphi_a(x_k) \geq \varphi_a(x)$$

By hypothesis φ_a is the lower semi-continuous regularization of $\langle a, \cdot\rangle_\infty$, thus, by definition, $\langle a, x_k\rangle_\infty \geq \varphi_a(x_k)$. Therefore:

$$-\max_{i\in[n]} |a_i x_i| = \lim_{k\longrightarrow\infty} \langle a, x_k\rangle_\infty \geq \liminf_{k\longrightarrow\infty} \varphi_a(x_k) \geq \varphi_a(x)$$

Hence $\varphi_a(x) \leq -\max_{i\in[n]} |a_i x_i|$. However, since $I_-^a \neq \emptyset$, $f(x) = -\max_{i\in[n]} |a_i x_i|$, and we deduce that:

$$\varphi_a(x) \leq f(x).$$

But since φ_a is the lower semi-continuous regularization of the map $x \mapsto \langle a, x\rangle_\infty$ and $f(x) \leq \langle a, x\rangle_\infty$ $\forall x \in \mathbb{R}^n$, we also have:

$$\varphi_a(x) \geq f(x).$$

Consequently, $\varphi_a(x) = f(x)$ which ends the proof. $\square$

In [8] it was established that if C_1 and C_2 are nonproximate $\mathbb{B}$-convex subsets of $\mathbb{R}_+^n$ then there exists a $\mathbb{B}$-form $f : \mathbb{R}_+^n \to \mathbb{R}$ such that $\sup_{x\in C_1} f(x) < \inf_{x\in C_2} f(x)$. In the following, this result is extended to the inner product $(x, y) \mapsto \langle x, y\rangle_\infty = \boxplus_{i\in[n]} x_i y_i$.

Proposition 3.15. *If C_1 and C_2 are nonproximate copositive $\mathbb{B}$-convex subsets of $\mathbb{R}^n$ then there exists some $a \in \mathbb{R}^n$ such that*

$$\sup_{x\in C_1} \langle a, x\rangle_\infty < \inf_{x\in C_2} \langle a, x\rangle_\infty.$$

Proof. If C_1 and C_2 are copositive, then they belong to the same n-dimensional orthant K that is homeomorphic to $\mathbb{R}_+^n$ using a suitable linear homeomorphism. Therefore, for sake of simplicity, we shall assume that $K = \mathbb{R}_+^n$. From [8], there is some $a \in \mathbb{R}^n$ such that the map $x \mapsto a_1x_1 \smile \cdots \smile a_nx_n$ separates C_1 and C_2. This implies that $\inf_{x\in C_2} f(x) > \sup_{x\in C_1} f(x)$. Since f is the lower semi-continuous regularization of $\langle a, \cdot\rangle_\infty$, it follows that $\langle a, x\rangle_\infty \geq f(x)$ for all $x \in C_2$. Therefore

$$\inf_{x\in C_2} \langle a, x\rangle_\infty \geq \inf_{x\in C_2} f(x). \qquad (\star)$$

Let us consider the map $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined for all $x \in \mathbb{R}^n$ by $g(x) = -f(-x)$. Since $x \mapsto -x$ is continuous, g is upper semi-continuous. Moreover, for all

$x \in \mathbb{R}^n$, $f(-x) \leq \langle a, -x \rangle_\infty = -\langle a, x \rangle_\infty$ implies that $g(x) \geq \langle a, x \rangle_\infty$. From equations (22) and (23), for all real numbers c :

$$\text{if } c \leq 0 \text{ then } g(x) \geq c \text{ if and only if } \max_{i \in I_+}\{-a_i x\} \leq \max_{i \in I_-}\{a_i x, -c\}$$

and

$$\text{if } c \geq 0 \text{ then } g(x) \geq c \text{ if and only if } \max_{i \in I_+}\{-a_i x, c\} \leq \max_{i \in I_-}\{a_i x\}.$$

Hence $g(x) < c$ if and only if $f(x) < c$ and $\sup_{x \in C_1} g(x) < \inf_{x \in C_2} f(x) \leq \inf_{x \in C_2} \langle a, x \rangle_\infty$. Since $\langle a, x \rangle_\infty \leq g(x)$ for all $x \in C_1$, this ends the proof from $(\star)$. $\square$

4. Relation to a Limit of Linear Convexities

4.1. Intermediate Points of Order p

For all natural number p, let us consider the map $\gamma^{(p)} : \mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty] \longrightarrow \mathbb{R}$ defined by:

$$\gamma^{(p)}(x, y, t) = \left(\frac{1}{1 \overset{p}{+} t} \right) x \overset{p}{+} \left(\frac{t}{1 \overset{p}{+} t} \right) y, \quad \text{for all } t \geq 0 \tag{24}$$

and by $\gamma^{(p)}(x, y, +\infty) = y$.

Lemma 4.1. *For all $p \in \mathbb{N}$, the map defined in* (24) *satisfies the following properties. For all $x, y \in \mathbb{R}^n$:*

- (a) *The map $t \mapsto \gamma^{(p)}(x, y, t)$ is continuous over $\mathbb{R}_+$.*
- (b) *We have $\lim_{t \longrightarrow 0} \gamma^{(p)}(x, y, t) = \gamma^{(p)}(x, y, 0) = x$ and $\lim_{t \longrightarrow +\infty} \gamma^{(p)}(x, y, t) = \gamma^{(p)}(x, y, +\infty) = y$.*
- (c) *We have $\gamma^{(p)}(x, y, [0, +\infty]) = \mathrm{Co}^p(x, y)$.*
- (d) *For all $t \in [0, +\infty]$, we have $\lim_{p \longrightarrow +\infty} \gamma^{(p)}(x, y, t) = \gamma(x, y, t)$.*
- (e) *For all $t \in [0, +\infty]$, $\gamma^{(p)}(x, y, t)$ and $\gamma(x, y, t)$ are copositive.*

Proof. (a) The φ_p generalized sum is continuous. Moreover, for all $t \geq 0$ the map $t \mapsto (1 \overset{p}{+} t)^{-1}$ is continuous and positive.

(b) follows from the continuity and using the fact that $\lim_{t \longrightarrow +\infty} \frac{t}{1 \overset{p}{+} t} = 1$.

(c) Let us show that for all $t \geq 0$ $\gamma^{(p)}(x, y, t) \in \mathrm{Co}^p(x, y)$. It is easy to see that $\left(\frac{1}{1 \overset{p}{+} t} \right) \overset{p}{+} \left(\frac{t}{1 \overset{p}{+} t} \right) = 1$ and it follows that $\gamma^{(p)}(x, y, t) \in \mathrm{Co}^p(x, y)$ for all $t \geq 0$. Since $\gamma^{(p)}(x, y, \infty) = y$, we deduce that $\gamma^{(p)}(x, y, [0, +\infty]) \subset \mathrm{Co}^p(x, y)$. Conversely, suppose that $z \in \mathrm{Co}^p(x, y)$. By hypothesis there is some $\theta \in [0, 1]$ such that $z = \theta x \overset{p}{+} (1 \overset{p}{-} \theta) y$. If $\theta \in \{0, 1\}$ then either $z = x$ or $z = y$.

Suppose that $\theta \in]0,1[$, then setting $t = \frac{1 \overset{p}{-} \theta}{\theta}$, we obtain $\theta = \frac{1}{1 \overset{p}{+} t}$ and $1 \overset{p}{-} \theta = \frac{t}{1 \overset{p}{+} t}$. Consequently, $z \in \gamma(x, y, \mathbb{R}_{++})$. Therefore, the converse inclusion is true and we deduce that $\gamma^{(p)}(x, y, [0, +\infty]) = \mathrm{Co}^p(x, y)$.

(d) If either $t = 0$ or $t = \infty$ then this property obviously holds true. Suppose that $t \in \mathbb{R}_{++}$. For all $j \in [n]$, we have

$$\lim_{p \longrightarrow \infty} \gamma_j^{(p)}(x, y, t) = \lim_{p \longrightarrow \infty} \left(\frac{x_j}{1 \overset{p}{+} t} \right) \overset{p}{+} \left(\frac{t y_j}{1 \overset{p}{+} t} \right) = \lim_{p \longrightarrow \infty} \left(\frac{x_j^{2p+1} + (t y_j)^{2p+1}}{1 + t^{2p+1}} \right)^{\frac{1}{2p+1}}$$

$$= \lim_{p \longrightarrow \infty} \frac{(x_j^{2p+1} + (t y_j)^{2p+1})^{\frac{1}{2p+1}}}{(1 + t^{2p+1})^{\frac{1}{2p+1}}} = \frac{x_j \boxplus t y_j}{\max\{1, t\}} = \gamma_j(x, y, t).$$

(e) From Lemma 2.3, we have for all $j \in [n]$, $\left(x_j^{2p+1} + (t y_j)^{2p+1}\right)^{\frac{1}{2p+1}} \geq 0$ if and only if $x_j \boxplus t y_j \geq 0$. Using distributivity of scalar multiplication, we deduce (e). □

For all $i \in \mathcal{I}(x, y)$, we say that $\gamma^{(p)}(x, y, t_i^\star)$ is **a i-intermediate point of order** p between x and y if $\gamma_i^{(p)}(x, y, t_i^\star) = 0$.

Corollary 4.2. *Let $x, y \in \mathbb{R}^n$ and suppose that $\mathcal{I}(x, y) \neq \emptyset$. Then, for all $i \in \mathcal{I}(x, y)$ one has the following properties:*

(a) *For all $p \in \mathbb{N}$, there is a uniqueness i-intermediate point of order p*

$$\gamma^{(p)}(x, y, t_i^\star) = \left(\frac{|y_i|}{|x_i| \overset{p}{+} |y_i|} \right) x \overset{p}{+} \left(\frac{|x_i|}{|x_i| \overset{p}{+} |y_i|} \right) y,$$

with $t_i^\star = -\frac{x_i}{y_i} = |\frac{x_i}{y_i}|$.

(b) *For all $p \in \mathbb{N}$ and all $i \in \mathcal{I}(x, y)$, $\gamma^{(p)}\left(x, y, -\frac{x_i}{y_i}\right)$ is a i-intermediate point of order p if and only if $\gamma\left(x, y, -\frac{x_i}{y_i}\right)$ is a i-intermediate point.*

(c) *Let $\gamma(x, y, t_i^\star)$ be a i-intermediate point and let $\{\gamma^{(p)}(x, y, t_i^\star)\}_{p \in \mathbb{N}}$ be a sequence of i-intermediate points of order p. Then $\lim_{p \longrightarrow \infty} \gamma^{(p)}(x, y, t_i^\star) = \gamma(x, y, t_i^\star)$.*

(d) *If $\{t_{i_m}^\star\}_{m=0}^{n(x,y)+1}$ is an intermediate sequence of $\Theta(x, y)$ satisfying the conditions of Lemma 3.7 with $t_{i_0}^\star = 0$, $t_{i_{n(x,y)+1}}^\star = +\infty$ and $t_{i_m}^\star = -\frac{x_{i_m}}{y_{i_m}}$ for all $m \in [n(x, y)]$, then:*

$$\mathrm{Co}^p(x, y) = \bigcup_{m=0}^{n(x,y)} \mathrm{Co}^p\left(\gamma^{(p)}\left(x, y, t_{i_m}^\star\right), \gamma^{(p)}\left(x, y, t_{i_{m+1}}^\star\right)\right).$$

Proof. (a) $\gamma_i(x, y, t_i) = 0$ if and only if $\left(\frac{1}{1 \overset{p}{+} t}\right) x_i \overset{p}{+} \left(\frac{t}{1 \overset{p}{+} t}\right) y_i = 0$ which is equivalent to $t_i = -\frac{x_i}{y_i} = |\frac{x_i}{y_i}|$.

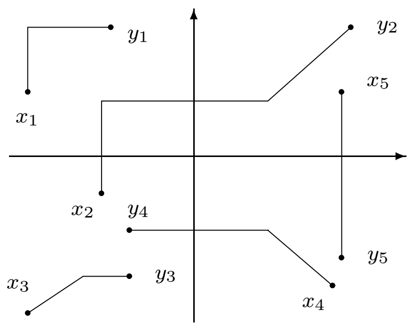

Figure 4.1.1: Examples of sets $\mathrm{Co}^\infty(x, y)$

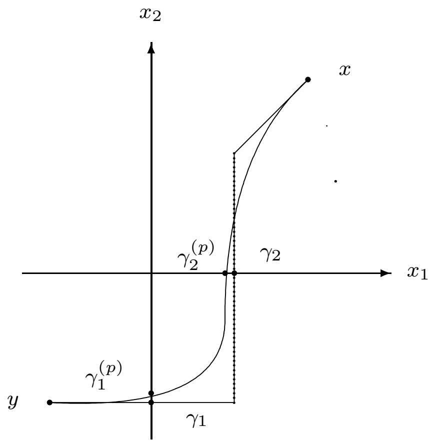

Figure 4.1.2: Limit of $\mathrm{Co}^p(x, y)$

(b) and (c) are two immediate consequences of (a) and Lemma 4.1.(d).

(d) By definition, for all m $\gamma^{(p)}\left(x, y, t^\star_{i_m}\right), \gamma^{(p)} \in \mathrm{Co}^p(x, y)$. Therefore $\mathrm{Co}^p(x, y) \supset \bigcup_{m=0}^{n(x,y)} \mathrm{Co}^p\left(\gamma^{(p)}\left(x, y, t^\star_{i_m}\right), \gamma^{(p)}\left(x, y, t^\star_{i_{m+1}}\right)\right)$. Moreover, since $x = \gamma^{(p)}\left(x, y, t^\star_{i_0}\right)$ and $y = \gamma^{(p)}\left(x, y, t^\star_{i_{n(x,y)+1}}\right)$, the converse inclusion holds. □

4.2. Painlevé-Kuratowski Limit

In the following, it is proven that $\mathrm{Co}^\infty(\{x, y\}) = \mathrm{Lim}_{p \longrightarrow +\infty} \mathrm{Co}^p(\{x, y\})$. This means that, given two points in the whole Euclidean vector space, the Painlevé-Kuratowski limit of their generalized convex hull exists. Moreover it is established that it has an algebraic description. For the sake of simplicity let $\mathrm{Co}^\infty(x, y)$ and $\mathrm{Co}^p(x, y)$ denote these convex hulls for all $p \in \mathbb{N}$.

From Lemma 3.7 the Φ_p-convex hull of x and y is the finite union of the Φ_p-convex hull of two consecutive intermediate points of order p. The sequence which these intermediate points of order p are arranged is identical to the copositive sequence of the intermediate points.

For future reference, we gather in the lemma below some elementary facts, which are a slight extension of a result established in [6].

Lemma 4.3. *Let K be a n-dimensional orthant of $\mathbb{R}^n$. Let $A = \{x_1, ..., x_m\}$ be a finite subset of K. For all natural number p let $A^{(p)} = \{x_1^{(p)}, ..., x_m^{(p)}\}$ be a finite collection of m vectors in K.*

(a) *If there exists an an increasing sequence of natural numbers $\{p_k\}_{k \in \mathbb{N}}$ such that for $i = 1, ..., m$ $\lim_{k \longrightarrow \infty} x_i^{(p_k)} = x_i$, then:*

$$\lim_{k \longrightarrow \infty} \sum_{i \in [m]}^{\varphi_{p_k}} x_i^{(p_k)} = \boxplus_{i \in [m]} x_i.$$

(b) *If for* $i = 1, ..., m$ $\lim_{p\longrightarrow\infty} x_i^{(p)} = x_i$, *then*

$$\operatorname*{Lim}_{p\to\infty} \mathrm{Co}^p(A^{(p)}) = \left\{ \boxplus_{i\in[m]} t_i x_i : \max_i t_i = 1, t_i \geq 0 \right\} = \mathbb{B}[A].$$

Proof. (a) Let $\Psi_K : K \longrightarrow \mathbb{R}^n_+$ be the function characterizing the n-dimensional orthant K. By definition for all $x \in \mathbb{R}^n$ one has $\Psi_K(x) = (\epsilon_1 x_1, ..., \epsilon_n x_n)$ where $\epsilon_j \in \{-1, 1\}$ for all $j \in [n]$. By definition one has for all j:

$$\sum_{i\in[m]}^{\varphi_{p_k}} x_{i,j}^{(p_k)} = \epsilon_j \sum_{i\in[m]}^{\varphi_{p_k}} |x_{i,j}^{(p_k)}|.$$

From the Lemma 2.0.1.(b) established in [6], we have $\lim_{k\longrightarrow+\infty} \sum_{i\in[n]}^{\varphi_{p_k}} |x_{i,j}^{(p_k)}| = \max_i |x_{i,j}|$. Consequently, for all $j \in [n]$

$$\lim_{k\longrightarrow+\infty} \sum_{i\in[m]}^{\varphi_{p_k}} x_{i,j}^{(p_k)} = \epsilon_j \max_i |x_{i,j}| = \boxplus_{i\in[m]} x_{i,j},$$

which ends the proof.

(b) We first establish that $\mathbb{B}[A] = \left\{ \boxplus_{i=1,\cdots,m} t_i x_i : t_i \in [0,1], \max_{i\in[m]} t_i = 1 \right\} \subset \mathrm{Li}_{p\longrightarrow\infty} \mathrm{Co}^p(A)$. Let $y = t_1 x_1 \boxplus \cdots \boxplus t_m x_m$ with $t_1, \cdots, t_m \in [0,1]$ and $\max_{i\in[m]} t_i = 1$. Define $y_p \in \mathrm{Co}^p\,(A)$ by

$$y^{(p)} = \frac{1}{t_1 \overset{p}{+} \cdots \overset{p}{+} t_m} \left(t_1 \overset{p}{\cdot} x_1^{(p)} \overset{p}{+} \cdots \overset{p}{+} t_m \overset{p}{\cdot} x_m^{(p)} \right)$$

Since $x_1^{(p)}, \cdots, x_m^{(p)} \in K$ and $\lim_{p\longrightarrow\infty} \left(t_1 \overset{p}{+} \cdots \overset{p}{+} t_m \right) = \max_{i\in[m]} t_i = 1$ we deduce from (a) that

$$\lim_{p\to\infty} y^{(p)} = \lim_{p\longrightarrow\infty} \left(t_1 x_1^{(p)} \overset{p}{+} \cdots \overset{p}{+} t_m x_m^{(p)} \right) = t_1 x_1 \boxplus \cdots \boxplus t_m x_m = y.$$

This completes the first part of the proof.

Next, we establish that $\mathrm{Ls}_{p\longrightarrow\infty} \mathrm{Co}^p(A) \subset \mathbb{B}[A]$. Take $z \in \mathrm{Ls}_{p\longrightarrow\infty} \mathrm{Co}^p(A)$; there is an increasing sequence $\{p_k\}_{k\in\mathbb{N}}$ and a sequence of points $\{z_k\}_{k\in\mathbb{N}}$ such that $z_k \in \mathrm{Co}^{p_k}(A^{(p_k)})$ and $\lim_{k\longrightarrow\infty} z_k = z$. Each z_k being in $\mathrm{Co}^{p_k}(A^{(p_k)})$, we can write

$$z_k = t_{k,1} x_1^{(p_k)} \overset{p_k}{+} \cdots \overset{p_k}{+} t_{k,m} x_m^{(p_k)}.$$

Since $t_k = (t_{k,1}, ..., t_{k,m}) \in [0,1]^m$ one can extract a subsequence $(t_{k_l})_{l\in\mathbb{N}}$ converges to a point $t^* = (t_1^*, ..., t_m^*) \in [0,1]^m$. It follows that for all $i \in [m]$

$\lim_{l\longrightarrow\infty} t_{k_l} x_i^{(p_{k_l})} = x_i$. Furthermore, from (a), $\lim_{l\to\infty}\left(\sum_{i=1}^m t_{k,i}^{2p_{k_l}+1}\right)^{1/(2p_{k_l}+1)} = \max_i t_i^* = 1$. It follows that for all $i \in [m]$ $\lim_{l\longrightarrow\infty} t_{k_l} x_i^{(p_{k_l})} = t_i^* x_i$. From (a) we deduce that $x = \boxplus_{i=1}^m t_i^* x_i$ with $\max_{i\in[m]}\{t_i^*\} = 1$. The first and the second part of the proof show that

$$\operatorname*{Ls}_{p\longrightarrow\infty} \mathrm{Co}^p(A^{(p)}) \subset \mathbb{B}[A] \subset \operatorname*{Li}_{p\longrightarrow\infty} \mathrm{Co}^p(A^{(p)})$$

and this completes the proof since we always have the inclusion $\mathrm{Li}_{p\longrightarrow\infty}\mathrm{Co}^p(A^{(p)}) \subset \mathrm{Ls}_{p\longrightarrow\infty}\mathrm{Co}^p(A^{(p)})$ □

Let us consider ℓ sequences of subsets of $\mathbb{R}^n$ $\{A_m^{(p)}\}_{p\in\mathbb{N}}$, $m \in [\ell]$. If there exists a subset A_m of $\mathbb{R}^n$ such that $\mathrm{Lim}_{p\longrightarrow\infty} A_m^{(p)} = A_m$ for all $m \in [\ell]$, then it is easy to show that:

$$\operatorname*{Lim}_{p\longrightarrow\infty}\left(\bigcup_{m\in[\ell]} A_m^{(p)}\right) = \bigcup_{m\in[\ell]} A_m. \tag{25}$$

Proposition 4.4. *For all $x, y \in \mathbb{R}^n$, let $\{t_{i_m}^\star\}_{m=0}^{n(x,y)+1}$ be an intermediate sequence of $\Theta(x,y)$. Then*

$$\mathrm{Co}^\infty(x,y) = \operatorname*{Lim}_{p\longrightarrow+\infty}\mathrm{Co}^p(x,y) = \bigcup_{m=0}^{n(x,y)} \mathbb{B}\left[\gamma(x,y,t_{i_m}^\star), \gamma(x,y,t_{i_{m+1}}^\star)\right].$$

Proof. From Lemma 4.1.(d), we have for all $i \in \mathcal{I}(x,y)$

$$\lim_{p\longrightarrow+\infty}\gamma^{(p)}\left(x,y,-\frac{x_i}{y_i}\right) = \gamma\left(x,y,-\frac{x_i}{y_i}\right).$$

Moreover, from Lemma 4.1.(e), for all $p \in \mathbb{N}$ and all $i \in \mathcal{I}(x,y)$, $\gamma^{(p)}\left(x,y,-\frac{x_i}{y_i}\right)$ and $\gamma\left(x,y,-\frac{x_i}{y_i}\right)$ are copositive. Recall that two vectors are copositive if their components have the same sign. From Lemma 3.7, for all $m \in [n(x,y)]$ $\gamma(x,y,t_{i_m}^\star)$ and $\gamma(x,y,t_{i_{m+1}}^\star)$ are copositive. Hence it follows that for all m, $\gamma^{(p)}(x,y,t_{i_m}^\star)$ and $\gamma^{(p)}(x,y,t_{i_{m+1}}^\star)$ are copositive.

From Proposition 4.3, we have for all m

$$\operatorname*{Lim}_{p\longrightarrow+\infty}\mathrm{Co}^p\left(\gamma^{(p)}(x,y,t_{i_m}^\star),\gamma^{(p)}(x,y,t_{i_{m+1}}^\star)\right) = \mathbb{B}\left[\gamma(x,y,t_{i_m}^\star),\gamma(x,y,t_{i_{m+1}}^\star)\right].$$

Moreover, we have from Corollary 4.2.(d)

$$\mathrm{Co}^p(x,y) = \bigcup_{m=0}^{n(x,y)} \mathrm{Co}^p\left(\gamma^{(p)}(x,y,t_{i_m}^\star),\gamma^{(p)}(x,y,t_{i_{m+1}}^\star)\right).$$

Hence, from equation (25), and Corollary 4.2.(c), the result follows. □

This property has an immediate consequence.

Proposition 4.5. *A subset C of $\mathbb{R}^n$ is $\mathbb{B}^\sharp$-convex if and only if for all $x, y \in \mathbb{R}^n$ $\mathrm{Co}^\infty(x,y) \subset C$.*

Proof. This is an immediate consequence of Propositions 3.8 and 3.10. □

Notice that it is not clear from the definition of $\mathbb{B}^\sharp$-convex sets that, for an arbitrary couple (x,y) of $\mathbb{R}^n \times \mathbb{R}^n$, $\mathrm{Co}^\infty(x,y)$ is $\mathbb{B}^\sharp$-convex.

Corollary 4.6. *A $\mathbb{B}$-convex subset of $\mathbb{R}^n$ is $\mathbb{B}^\sharp$-convex.*

In [6] it was established that a $\mathbb{B}$-convex subset of $\mathbb{R}^n$ is connected. A stronger property is established below. It is shown that $\mathbb{B}$-convex sets are path-connected.

Proposition 4.7. *A non empty $\mathbb{B}$-convex of $\mathbb{R}^n$ is path-connected.*

Proof. By definition a subset L of $\mathbb{R}^n$ is $\mathbb{B}$-convex if for all finite subset $A \subset L$ we have $\mathrm{Co}^\infty(x,y) \subset L$. This implies that $\mathrm{Co}^\infty(x,y) \subset L$ for all $x, y \in L$, which yields the result from Proposition 3.13. □

In the next statement, an algebraic characterization of $\mathrm{Co}^\infty(x,y)$ is provided. This we do by using the fact that the convex hull is not modified whenever one consider several occurrences of a given point. For example, for all $p \in \mathbb{N}$, one can equivalently write:

$$\begin{aligned}\mathrm{Co}^p(x,y) &= \Big\{\alpha x \overset{p}{+} \beta y : \alpha \overset{p}{+} \beta = 1, \alpha, \beta \geq 0\Big\} \qquad (26)\\ &= \Big\{\alpha_1 x \overset{p}{+} \alpha_2 x \overset{p}{+} \beta_1 y \overset{p}{+} \beta_2 y : \alpha_1 \overset{p}{+} \alpha_2 \overset{p}{+} \beta_1 \overset{p}{+} \beta_2 = 1, \alpha_i, \beta_i \geq 0\Big\}.\end{aligned}$$

Lemma 4.8. *For all $x, y \in \mathbb{R}^n$,*

$$\mathrm{Co}^\infty(x,y) = \{tx \boxplus rx \boxplus sy \boxplus wy : \max\{t,r,s,w\} = 1, t,r,s,w \geq 0\}.$$

Proof. Suppose that $z \in \{tx \boxplus rx \boxplus sy \boxplus wy : \max\{t,r,s,w\} = 1, t,r,s,w \geq 0\}$. By definition there exists $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ with $\max\{\alpha_1, \alpha_2, \beta_1, \beta_2\} = 1$ and such that

$$z = \alpha_1 x \boxplus \alpha_2 x \boxplus \beta_1 y \boxplus \beta_2 y.$$

Define

$$z^{(p)} = \frac{1}{\alpha_1 \overset{p}{+} \alpha_2 \overset{p}{+} \beta_1 \overset{p}{+} \beta_2}\Big(\alpha_1 x \overset{p}{+} \alpha_2 x \overset{p}{+} \beta_1 y \overset{p}{+} \beta_2 y\Big).$$

By construction $z^{(p)} \in \mathrm{Co}^p(x,y)$. Taking the limit on both sides yields from Proposition 2.6:

$$\begin{aligned}\lim_{p \longrightarrow \infty} z^{(p)} &= \frac{1}{\max\{\alpha_1, \alpha_2, \beta_1, \beta_2\}}(\alpha_1 x \boxplus \alpha_2 x \boxplus \beta_1 y \boxplus \beta_2 y)\\ &= \alpha_1 x \boxplus \alpha_2 x \boxplus \beta_1 y \boxplus \beta_2 y = z.\end{aligned}$$

Consequently, $z \in \mathrm{Li}_{p \longrightarrow +\infty} \mathrm{Co}^p(x,y)$. □

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