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# The Flow of a Bingham Fluid under a Pseudo-Monotone Perturbation*

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We show existence of weak solutions for a three dimensional Bingham flow with a pseudo–monotone perturbation. For the two dimensional case and if the main operator is linear, we show more regularity of the solution and also derive the uniqueness result for the solution.

## Contents



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## 1. Introduction

A Bingham fluid is a rigid visco plastic fluid which is a kind of non Newtonian fluid. The behavior of this material is considered as rigid when a certain function of the stress does not reach the yield limit. Some substances are well approximated as Bingham fluid such as drilling muds, cement slurries and toothpastes.

The difficulty in a Bingham fluid problem is that we do not know how to separate the fluid zone and the rigid zone due to the relation between strains and stresses which becomes very different depending on the state of stresses. To overcome that difficulty, in [3], the authors formulated an initial boundary value problems for a Bingham fluid as a variational inequality, that is

$$\left\langle \frac{\partial u}{\partial t}, v-u \right\rangle + \mathscr{a}(u, v-u) + \mathscr{b}(u,u,v) + j(v) - j(u) \geq \langle f, v-u \rangle \,, \tag{1a}$$

in $(0,T)$ for each test function $v$ such that $\nabla \cdot v = 0$ in $\Omega$ and $v = 0$ on $\partial\Omega$. Furthermore $u$ satisfies

$$\nabla \cdot u = 0 \quad \text{in } \Omega \times (0,T), \tag{1b}$$

$$u = 0 \quad \text{on } \partial\Omega \times [0,T], \tag{1c}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega. \tag{1d}$$

Duvaut and Lions in [3] proved the existence of weak solutions of (1a)–(1d). The weak solutions according to the definition in their work belong to the same function class as the Leray–Hopf weak solutions of the Navier–Stokes equations and they also obtained more regularity of weak solutions in a two dimensional domain and the uniqueness of weak solutions was also achieved. However, the uniqueness of weak solutions is still an open problem in the three dimensional case as well as for the usual Navier-Stokes equations.

After the fundamental work of Duvaut and Lions, Bingham fluids became a very interesting topic and many papers have been devoted to it see e.g. the following articles and the references therein. Neumann and Wulst in [10] obtained the existence of the same kind of regular solutions in three dimensional domain through a different method under the assumption that the initial data belong to a special class and the assumption of average nonlinear viscosity.

In this article, we study weak solutions in dimension three of the Bingham fluid under a pseudo–monotone perturbation which extends the result in [8] from the case of monotone perturbation. In the article [8], the author used exactly the same approach as that in Lions and Duvaut, while we use here a different approach namely finite differences. We also deal with the non vanishing initial data as compared to the vanishing initial data in [2] using semi–discretization in time by finite differences.

Our work is organized as follows: we introduce the Bingham problem and its variational setting in Section 2 and state our main result and present its constructive proof in Section 3.

## 2. The Bingham fluid problem and the variational form of a Bingham fluid under a Pseudo–monotone perturbation

### 2.1. Constitutive law for a Bingham fluid

Throughout this article, we use the Einstein summation convention. We first recall the following constitutive relations from e.g. [3] that apply to the setting for our Bingham fluid problem:

$$\sigma_{ij} = -p\delta_{ij} + gD_{ij}/(D_{II})^{\frac{1}{2}} + 2\mu D_{ij}. \tag{2}$$

an expression which makes sense only when $D_{II} \neq 0$. If $D_{II} = 0$, we say the stress tensor is indeterminate. Here

- $p$ denotes the (undetermined) pressure,
- $D = D_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the stretching tensor,
- $D_{II}(u) = \frac{1}{2}D_{ij}(u)D_{ij}(u)$ is the second invariant of $D$,
- $g > 0$ is the plasticity coefficient,
- $\mu > 0$ is the viscosity of the Bingham fluid.

In order to complete the formulation of the constitutive laws, let us first note that (2) implies

$$\sigma^D_{II} = (g + 2\mu D^{\frac{1}{2}}_{II})^2, \tag{3}$$

where

$$\sigma^D_{II} = \frac{1}{2}\sigma^D_{ij}\sigma^D_{ij}, \tag{4}$$

denotes the second invariant of the deviatoric stress tensor $\sigma^D = \{\sigma^D_{ij}\}$, $\sigma^D_{ij} = \sigma_{ij} + p\delta_{ij}$.

The equation (3) implies $\sigma^{\frac{1}{2}}_{II} \geq g$ so that (2) is inverted to give:

$$D_{ij} = \frac{1}{2\mu}(1 - g/\sigma^{\frac{1}{2}}_{II})\sigma^D_{ij}. \tag{5}$$

Returning to the case $D_{II} = 0$, we should add that, while the stress tensor is then indeterminate, it is still so that $\sigma^{\frac{1}{2}}_{II} \leq g$ because, in the opposite case, (5) furnishes the tensor $D$ with $D_{II} > 0$.

To sum up, the constitutive laws for the Bingham fluid can be written as follows:

$$\sigma^{\frac{1}{2}}_{II} = g \;\text{ if and only if } D_{ij} = 0,$$
$$\sigma^{\frac{1}{2}}_{II} > g \;\text{ if and only if } D_{ij} = \frac{1}{2\mu}(1 - g/\sigma^{\frac{1}{2}}_{II})\sigma^D_{ij}.$$

### 2.2. Space setting

We introduce the spaces needed for our problem and their related properties as in [3] as follows:

$$\begin{aligned} \mathbb{V} &= \{\phi : \ \phi \in [\mathcal{D}(\Omega)]^d, \operatorname{div} \phi = 0\}, \\ H &= \text{closure of } \mathbb{V} \text{ in } [L^2(\Omega)]^d, \\ V &= \text{closure of } \mathbb{V} \text{ in } [H_0^1(\Omega)]^d, \\ V_s &= \text{closure of } \mathbb{V} \text{ in } [H^s(\Omega)]^d, \end{aligned}$$

where $d = 3$ in our problem; $H, V, V_s$ are Hilbert spaces with respect to the scalar products

$$\begin{aligned} \langle u, v \rangle &= \int_\Omega u_i v_i \, dx, \\ \langle u, v \rangle_V &= \int_\Omega D_{ij}(u) D_{ij}(v) \, dx, \\ \langle u, v \rangle_{V_s} &= \langle u_i, v_i \rangle_{H^s(\Omega)} . \end{aligned}$$

The later symmetric bilinear form on $V \times V$ becomes a scalar product on $V$ thanks to the Korn inequality, the generalized Poincaré inequality and the vanishing traces of functions in $V$, see e.g. [12, Remark 1.3]. The norms on $H, V$ hence are respectively

$$\begin{aligned} |u| &= \langle u, u \rangle^{\frac{1}{2}} = \left( \int_\Omega u_i u_i \, dx \right)^{\frac{1}{2}}, \\ \|u\| &= \langle u, u \rangle_V^{\frac{1}{2}} = \left( \int_\Omega D_{II}(u) \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

We finally set $\mathcal{V} = L^p(0,T;V)$, $\mathcal{H} = L^2(0,T;H)$ and $\mathcal{V}' = L^q(0,T;V')$ with $\frac{1}{p} + \frac{1}{q} = 1$ for convenience of use throughout this work.

### 2.3. The Bingham problem with a pseudo–monotone perturbation as a variational inequality

Let $P : V \to V'$ be a pseudo–monotone operator, see e.g. [1], that is $P$ satisfies:

i. $P$ is bounded,
ii. if $u_j \rightharpoonup u$ weakly in $V$ and $\limsup_{j\to\infty} \langle P(u_j), u_j - u \rangle \leq 0$ then

$$\liminf_{j\to\infty} \langle P(u_j), u_j - v \rangle \geq \langle P(u), u - v \rangle , \quad \forall v \in V.$$

We further assume that

$$\begin{aligned} \langle Pu, u \rangle &\geq \alpha_0 \|u\|^p - \alpha_1, \\ \|Pu\|_{V'} &\leq \beta_0 \|u\|^{p/p'} + \beta_1, \end{aligned} \tag{6}$$

for $\alpha_i, \beta_i > 0, i = 0, 1$.

Our variational inequality for a Bingham problem with a pseudo–monotone perturbation reads: To find $u = u(t)$ mapping $[0, T]$ into $V$ such that

$$\int_0^T \langle v', v-u\rangle + \mu a(u, v-u) - b(u, v, u) \tag{7a}$$

$$+ \langle Pu, v-u\rangle + j(v) - j(u) - \langle f, v-u\rangle \geq 0, \tag{7b}$$

for each test function $v \in \mathcal{V}$ such that $v' = \frac{dv}{dt} \in \mathcal{V}'$, and $v(0) = u_0$. The bilinear and trilinear forms $a(\cdot,\cdot), b(\cdot,\cdot,\cdot)$, and the functionals $j(\cdot)$ are defined by:

$$a(v, w) = 2\int_\Omega D_{ij}(v) D_{ij}(w)\, dx, \tag{7c}$$

$$b(u, v, w) = \int_\Omega u_i v_{i,j} w_j\, dx, \tag{7d}$$

$$j(u) = 2g \int (D_{II}(u))^{1/2}\ dx, \tag{7e}$$

where $g$ is the yield limit. Furthermore $u = u(t)$ satisfies the following incompressibility, boundary and initial conditions

$$\operatorname{div} u(t) = 0, \quad \forall t \in (0, T), \tag{7f}$$

$$u = 0 \quad \text{on } \partial\Omega \times [0, T], \tag{7g}$$

$$u(x, 0) = u_0(x) \quad \text{in} \in \Omega, \tag{7h}$$

for $u_0 \in H$. It is easy to check that

$$\begin{aligned} |a(u, v)| &\leq 2\|u\|\,\|v\|, \\ a(u, u) &\geq 2\|u\|^2. \end{aligned} \tag{8}$$

We have for all $u, v, w \in V$

$$\begin{aligned} |b(u, v, w)| &\leq C\|u\|\,\|v\|\,\|w\|, \\ b(u, v, w) + b(u, w, v) &= 0. \end{aligned} \tag{9}$$

For $u, v \in V$, we denote by $B(u, v)$ the element of $V'$ defined by

$$\langle B(u, v), w\rangle = b(u, v, w), \quad \forall w \in V. \tag{10}$$

For each $u \in V$, we also denote by $Au$ the element in $V'$ such that

$$\langle Au, v\rangle = a(u, v), \quad \forall v \in V. \tag{11}$$

In this article, we are looking for a weak solution of (7a) as stated in the coming section.

## 3. The main result

We have the following existence result for the nonlinear variational inequality for Bingham fluid:

**Theorem 3.1.** *Let there be given* $f \in \mathcal{V}' = L^{p'}(0,T;V')$ *and* $u_0 \in H$. *Then, there exists a function* $u \in L^p(0,T;V) \cap L^\infty(0,T;H)$ *that satisfies* (7a) *for all* $v \in \mathcal{V} = L^p(0,T;V)$ *with* $v' \in \mathcal{V}'$ *and* $v(0) = u_0$.

**Proof.** The proof of existence is conducted in a constructive way, using finite differences in time. Let $N$ be a positive integer; we consider the time step $k = \Delta t = T/N$ and let $f^n$ be defined by

$$f^n = \frac{1}{k}\int_{(n-1)k}^{nk} f(t)dt, \tag{12}$$

for $n = 1, 2, \ldots, N$.
For an $\epsilon > 0$ fixed, we regularize $j$ and consider $j_\epsilon$ defined by

$$j_\epsilon(u) = 2g\int_\Omega (\epsilon + D_{II}(u))^{\frac{1}{2}} dx. \tag{13}$$

We easily find the derivative of $j_\epsilon$ at $u$ in the direction $v$, see Lemma A.3, to be

$$\langle j'_\epsilon(u), v\rangle = \frac{d}{d\lambda} j_\epsilon(u+\lambda v)|_{\lambda=0} = g\int_\Omega (\epsilon + D_{II}(u))^{-\frac{1}{2}} D_{ij}(u)D_{ij}(v)\, dx. \tag{14}$$

Also from Lemma A.3, $j_\epsilon$ is convex which implies

$$j_\epsilon(v) - j_\epsilon(u) \geq \langle j'_\epsilon(u), v-u\rangle\,, \quad \text{for all } u, v \in V. \tag{15}$$

Our agenda is as follows:

- Step 1: build approximate solutions $u_k$ of $u$.
- Step 2: derivation of the inequality for $u_k$ and preparation to pass to the limit in the inequality in which we prove a priori estimates on the $u_k$: this includes two sub–steps
  - 2a: to derive weak convergence subsequences,
  - 2b: to derive strong convergence subsequences.
- Step 3: pass to the limit $k \to 0$ ($N \to \infty$) to show the existence of a solution $u$ to (7a).

### 3.1. Approximate solutions $u_k$

We set $u^0 = u_0$ and for each $n = 1, 2\ldots, N$, we recursively define $u^n \in V$ as a solution of the following implicit Euler discretization in time of (7a):

$$\begin{aligned}\left\langle \frac{u^n - u^{n-1}}{k}, v - u^n \right\rangle &+ \mu\mathscr{a}(u^n, v-u^n) + \mathscr{b}(u^{n-1}, u^n, v-u^n) \\ &+ \langle Pu^n, v-u^n\rangle + \langle j'_\epsilon(u^n), v-u^n\rangle \geq \langle f^n, v-u^n\rangle\,,\end{aligned} \tag{16}$$

for all $v \in V$.
In order to show the existence of such a $u^n$ in $V$, we rewrite (16) as

$$\frac{1}{k}\langle u^n, v-u^n\rangle + \mu a(u^n, v-u^n) + b(u^{n-1}, u^n, v-u^n) \\ + \langle Pu^n, v-u^n\rangle + \langle j'_\epsilon(u^n), v-u^n\rangle \geq \left\langle f^n + \frac{u^{n-1}}{k}, v-u^n\right\rangle, \tag{17}$$

for all $v \in V$.
Since $u^{n-1}$ and $f^n$ are known at time step $n$, (17) is seen as an elliptic variational inequality. We hence use Theorem A.8 in the Appendix to show the existence of $u^n$.

For a fixed time step $k$ and a known $u^{n-1}$, we define an operator $\mathcal{A}^k : V \to V'$ by

$$\left\langle \mathcal{A}^k u, v\right\rangle = \frac{1}{k}\langle u, v\rangle + \mu a(u,v) + \langle Pu, v-u\rangle + b(u^{n-1}, u, v) + \langle j'_\epsilon(u), v\rangle, \tag{18}$$

for $u$, $v \in V$.
We will show that $\mathcal{A}^k$ is pseudo–monotone and coercive to prove the existence of $u^n$. In order to show that $\mathcal{A}^k$ is pseudo–monotone, we use the fact that the sum of a pseudo–monotone and a monotone hemicontinuous operator is pseudo–monotone, see e.g. [11, Lemma A.5]. Since $P$ is pseudo–monotone, $\frac{I}{k}$ and $j'_\epsilon$ are monotone and $a(\cdot,\cdot)$ is coercive, we are left with showing that $A^k_b$ defined by $\left\langle A^k_b u, v\right\rangle = b(u^{n-1}, u, v)$ is monotone, that is

$$\left\langle A^k_b u - A^k_b v, u-v\right\rangle = b(u^{n-1}, u, u-v) - b(u^{n-1}, v, u-v) \geq 0,$$

for all $u, v \in V$. However, since

$$b(u^{n-1}, u, u-v) - b(u^{n-1}, v, u-v) = b(u^{n-1}, u-v, u-v) = 0,$$

for $u^{n-1}$ fixed, the result follows.
We now show that $\mathcal{A}^k$ is coercive. We have for any $v_0 \in V$:

$$\begin{aligned}
&\left\langle \mathcal{A}^k u, u-v_0\right\rangle \\
&\geq \frac{1}{k}\langle u,u\rangle - \frac{1}{k}\langle u, v_0\rangle + \mu a(u,u) - \mu a(u,v_0) - b(u^{n-1}, u, v_0) \\
&\qquad + \langle Pu, u\rangle - \langle Pu, v_0\rangle + \langle j'_\epsilon(u), u\rangle - \langle j'_\epsilon(u), v_0\rangle \\
&\geq \text{(by (8), (9), (41), (6) and (42))} \\
&\geq \frac{1}{k}\|u\|^2 - \frac{1}{k}\|u\|\,\|v_0\| + 2\mu\|u\|^2 - 2\mu\|u\|\,\|v_0\| - C\left\|u^{n-1}\right\|\|u\|\,\|v_0\| \qquad (19) \\
&\qquad + \alpha_0\|u\|^p - \alpha_1 - (\beta_0\|u\|^{p/p'} + \beta_1)\|v_0\| - \frac{g}{2}\|v_0\| \\
&= \alpha_0\|u\|^p + \left(\frac{1}{k} + 2\mu\right)\|u\|^2 - \left(\frac{1}{k} + 2\mu + C\left\|u^{n-1}\right\|\right)\|v_0\|\,\|u\| \\
&\qquad - \left(\beta_0\|u\|^{p/p'} + \beta_1 + \frac{g}{2}\right)\|v_0\| - \alpha_1.
\end{aligned}$$

For $p \geq 2$ (hence $q = p' \leq 2$), we easily see that $\lim_{\|u\|\to\infty} \frac{\langle \mathcal{A}^k u, u-v_0 \rangle}{\|u\|} = \infty$ which means that $\mathcal{A}^k$ is coercive. By (41), we find that $j'_\epsilon$ is bounded in $\mathcal{L}(V, V')$ hence $\mathcal{A}^k$ is bounded. Furthermore, by Lemma A.1, $j'_\epsilon$ is hemicontinuous and $\mathcal{A}^k$ is consequently hemicontinuous. We then conclude that $\mathcal{A}^k$ is pseudo–monotone. The existence of $u^n$ then follows by Theorem A.8.

We finally define the approximate function $u_k$ to be a step function on $(0, T)$ with value $u^n$ in $V$ on each time interval $((n-1)k, nk]$. We next derive a priori estimates for $u_k$ in the next subsection.

### 3.2. A priori estimates on the approximate functions $u_k$ and derivation of the variational inequality for $u_k$

In this section, we derive a priori estimates for the $u^n$ and hence for $u_k$. We then derive a variational inequality for the $u_k$ in preparation for the passage to the limit.

First, we show that the $u^n$ are bounded independently of $n$ as follows:

**Lemma 3.2.** *The constructed functions $u^n$ satisfy the following bounds:*

$$|u^r|^2 + \sum_{n=1}^{r} \left|u^n - u^{n-1}\right| + k\alpha_0 \sum_{n=1}^{r} \|u^n\|^p + 4k\mu \sum_{n=1}^{r} \|u^n\|^2 + 2k \sum_{n=1}^{r} j_\epsilon(u^n) \leq \kappa, \tag{20}$$

*where* $\kappa = \kappa(f, \alpha_0, u_0, p) = \frac{2^{p'/p+1}}{p'(\alpha_0 p')^{p'/p}} \|f\|_{\mathcal{V}'}^{p'} + |u|_0^2 + 2Tg\,|\Omega|$.

**Proof.** We take $v = 0$ in (16) and deduce from (16) and (15) that

$$\left\langle \frac{u^n - u^{n-1}}{k}, u^n \right\rangle + \mu a(u^n, u^n) + b(u^{n-1}, u^n, u^n) + \langle P u^n, u^n \rangle + j_\epsilon(u^n) - j_\epsilon(0) \leq \langle f^n, u^n \rangle .$$

We multiply both sides of this relation by $2k$ and use classically

$$2\left\langle u^n - u^{n-1}, u^n \right\rangle = |u^n|^2 - \left|u^{n-1}\right|^2 + \left|u^n - u^{n-1}\right|^2 ,$$

to obtain

$$\begin{aligned} &|u^n|^2 - \left|u^{n-1}\right|^2 + \left|u^n - u^{n-1}\right|^2 + 2k\mu a(u^n, u^n) + 2k b(u^{n-1}, u^n, u^n) \\ &\quad + 2k \langle P u^n, u^n \rangle + 2k j_\epsilon(u^n) - 2k j_\epsilon(0) \leq 2k \langle f^n, u^n \rangle . \end{aligned} \tag{21}$$

Thanks to $(9)_2$, $(8)_2$, $(6)_1$, (41) and (42), we arrive at

$$\begin{aligned} &|u^n|^2 - \left|u^{n-1}\right|^2 + \left|u^n - u^{n-1}\right|^2 + 4k\mu \|u^n\|^2 \\ &\quad + 2k\alpha_0 \|u^n\|^p + 2k j_\epsilon(u^n) - 2k j_\epsilon(0) \leq 2k \langle f^n, u^n \rangle . \end{aligned}$$

Applying the Young inequality on the right hand side, we find

$$\begin{aligned}&|u^n|^2 - \left|u^{n-1}\right|^2 + \left|u^n - u^{n-1}\right|^2 + 4k\mu \left\|u^n\right\|^2 + 2k\alpha_0 \left\|u^n\right\|^p \\ &\quad + 2kj_\epsilon(u^n) - 2kj_\epsilon(0) \leq 2k\left(\frac{\alpha_0}{2}\left\|w^n\right\|^p + \frac{2^{p'/p}}{p'(\alpha_0 p')^{p'/p}}\left\|f^n\right\|^{p'}\right).\end{aligned}$$

This is the same as

$$\begin{aligned}&|u^n|^2 - \left|u^{n-1}\right|^2 + \left|u^n - u^{n-1}\right|^2 + 4k\mu \left\|u^n\right\|^2 + k\alpha_0 \left\|u^n\right\|^p \\ &\quad + 2kj_\epsilon(u^n) - 2kj_\epsilon(0) \leq 2k\frac{2^{p'/p}}{p'(\alpha_0 p')^{p'/p}}\left\|f^n\right\|^{p'}.\end{aligned}$$

Summing the above inequalities for $n = 1, 2 \ldots, r$, we obtain

$$\begin{aligned}&|u^r|^2 + \sum_{n=1}^{r}\left|u^n - u^{n-1}\right| + k\alpha_0\sum_{n=1}^{r}\left\|u^n\right\|^p + 4k\mu\sum_{n=1}^{r}\left\|w^n\right\|^2 + 2k\sum_{n=1}^{r}j_\epsilon(u^n) \\ &\leq 2k\frac{2^{p'/p}}{p'(\alpha_0 p')^{p'/p}}\left\|f^n\right\|^{p'} + \left|u^0\right|^2 + 2k\sum_{n=1}^{r}\left|j_\epsilon(0)\right|.\end{aligned}$$

We have $\|f^n\|_{V'}^{p'} = \frac{1}{k^{p'}}\left\|\int_{(n-1)k}^{nk} f(t)\,dt\right\|_{V'}^{p'} \leq \frac{k^{p'/p}}{k^{p'}}\int_{(n-1)k}^{nk}\|f(t)\|_{V'}^{p'}\,dt$ by Hölder's inequality. Therefore

$$\begin{aligned}k\sum_{n=1}^{N}\|f^n\|_{V'}^{p'} &\leq k^{\frac{p'}{p}-p'+1}\sum_{n=1}^{N}\int_{(n-1)k}^{nk}\|f(t)\|_{\mathcal{V}'}^{p'}\,dt \\ &\leq \left(\text{since } \frac{1}{p} + \frac{1}{p'} = 1\right) \leq \int_0^T \|f(t)\|_{\mathcal{V}'}^{p'}\,dt.\end{aligned}$$

Hence (20) holds true for $\kappa = 2\frac{2^{p'/p}}{p'(\alpha_0 p')^{p'/p}}\|f\|_{\mathcal{V}'}^{p'} + \left|u^0\right|^2 + 2g\,|\Omega|$ using that $j_\epsilon(0) \leq g\,|\Omega|$.
This concludes the proof of the lemma. □

For each fixed $k$ we recall that we associate to the elements $u^1, u^2, \ldots, u^N$ the approximate function $u_k : [0, T] \to V$:

$$u_k(t) = u^m, \quad t \in ((m-1)k, mk].$$

We also define $\tilde{u}_k : [0, T] \to H$ as the continuous linear function on each time interval $((m-1)k, mk]$ and equal to $u^m$, $m = 0, 1, \ldots, N$ at $t = mk$. From (20), we conclude that $u_k$, $\tilde{u}_k$ are bounded in $L^\infty(0, T; H)$ and in $L^2(0, T, V)$.
Another property of $\tilde{u}_k$ represented in Lemma A.9 in the Appendix will be used later to show a compactness result for the functions $u_k$, $\tilde{u}_k$.

From (20), we can extract a subsequence still denoted by $k$ such that

$$u_k \rightharpoonup u \text{ weakly in } \mathcal{V} \text{ and weak - star in } L^\infty(0,T;H). \tag{22}$$

In order to claim that $u$ is a solution of the variational inequality (7a) with $f$ replaced by $f_k$ and $j$ by $j_\epsilon$. We first derive the variational inequalities for $u_k$ and $\tilde{u}_k$ to prepare for the passing to the limit step. For this task, we consider a test function $v \in \mathcal{C}^2([0,T];V)$ such that $v(0) = u_0$. We then define two approximate functions $v_k$ and $\tilde{v}_k$ of $v$ as follows: $v_k$ is the step function on $[0,T]$ with values $v(nk)$ at $t = nk$ and $\tilde{v}_k$ is the piecewise linear function interpolating those values. For $f \in L^q(0,T;V')$, we also denote by $f_k$ the step function with values $f^n = \frac{1}{k}\int_{I_n} f(t)\,dt$ for $t \in I_n = ((n-1)k, nk]$.

We now claim that $u$ is a solution of the variational inequality (7a) with $f$ replaced by $f_k$ and $j$ by $j_\epsilon$. We first take $v = v^n$ in (16) and multiply by $k$ to find:

$$\begin{aligned}\left\langle u^n - u^{n-1}, v^n - u^n\right\rangle &+ \mu\, a(u^n, v^n - u^n)k + b(u^{n-1}, u^n, v^n - u^n)k \\ &+ \langle Pu^n, v^n - u^n\rangle k + \langle j'_\epsilon(u^n), v^n - u^n\rangle k \geq \langle f^n, v^n - u^n\rangle k.\end{aligned}$$

Adding the term $\langle w^n - w^{n-1}, w^n\rangle$ to both sides where $w^n = v^n - u^n$ and summing for $n = 1, \ldots, m$ $(m \leq N)$, we then obtain

$$\begin{aligned}\sum_{n=1}^m \left\langle v^n - v^{n-1}, v^n - u^n\right\rangle &+ \mu \sum_{n=1}^m a(u^n, v^n - u^n)k + \sum_{n=1}^m b(u^{n-1}, u^n, v^n - u^n)k \\ &+ \sum_{n=1}^m \langle Pu^n, v^n - u^n\rangle k + \sum_{n=1}^m \langle j'_\epsilon(u^n), v^n - u^n\rangle k \geq \sum_{n=1}^m \langle f^n, v^n - u^n\rangle k,\end{aligned}$$

where we have used $\sum_{n=1}^m \langle w^n - w^{n-1}, w^n\rangle \geq 0$ which is shown in Lemma A.6. The above inequality is hence equivalent to

$$\begin{aligned}\int_0^t \langle \tilde{v}'_k, v_k - u_k\rangle\,dt &+ \mu \int_0^t a(u_k, v_k - u_k)\,dt + \int_0^t b(u_{k\circ\tau_h}, u_k, v_k - u_k)\,dt \\ &+ \int_0^t \langle P(u_k), v_k - u_k\rangle\,dt + \int_0^t \langle j'_\epsilon(u_k), v_k - u_k\rangle\,dt \geq \int_0^t \langle f_k, v_k - u_k\rangle\,dt,\end{aligned} \tag{23}$$

where $u_{k\circ\tau_h} = u^{n-1}$ on $I_n$, $n = 2, \ldots, N$, and $u_{k\circ\tau_h} = u_0$ on $I_1$. In particular when $t = T$, we have

$$\begin{aligned}\int_0^T \langle \tilde{v}'_k, v_k - u_k\rangle\,dt &+ \mu \int_0^T a(u_k, v_k - u_k)\,dt + \int_0^T b(u_{k\circ\tau_h}, u_k, v_k - u_k)\,dt \\ &+ \int_0^T \langle P(u_k), v_k - u_k\rangle\,dt + \int_0^T \langle j'_\epsilon(u_k), v_k - u_k\rangle\,dt \geq \int_0^T \langle f_k, v_k - u_k\rangle\,dt.\end{aligned}$$

This is the same as

$$\begin{aligned}\int_0^T \langle \tilde{v}_k', v_k - u_k\rangle \, dt + \mu \int_0^T a(u_k, v_k)\, dt + \int_0^T b(u_{k\circ\tau_h}, u_k, v_k - u_k)\, dt \\ + \int_0^T \langle P(u_k), v_k\rangle \, dt + \int_0^T \langle j_\epsilon'(u_k), v_k - u_k\rangle \, dt \\ \geq \int_0^T \langle f_k, v_k - u_k\rangle \, dt + \mu \int_0^T a(u_k, u_k)\, dt + \int_0^T \langle P(u_k), u_k\rangle \, dt.\end{aligned}$$

We make use of the identity $\int_0^T b(u_{k\circ\tau_h}, u_k, u_k) = 0$ and obtain

$$\begin{aligned}\int_0^T \langle \tilde{v}_k', v_k - u_k\rangle \, dt + \mu \int_0^T a(u_k, v_k)\, dt + \int_0^T b(u_{k\circ\tau_h}, u_k, v_k)\, dt \\ + \int_0^T \langle P(u_k), v_k\rangle \, dt + \int_0^T \langle j_\epsilon'(u_k), v_k - u_k\rangle \, dt \\ \geq \int_0^T \langle f_k, v_k - u_k\rangle \, dt + \mu \int_0^T a(u_k, u_k)\, dt + \int_0^T \langle P(u_k), u_k\rangle \, dt.\end{aligned} \tag{24}$$

We next show a compactness result which will be used to show the existence of a subsequence of $u_k$ converging strongly in $L^2(0,T;H)$ which is needed to pass to the limit in the nonlinear convection term in (24).

### 3.3. A compactness result

**Lemma 3.3.** *The functions $u_k$ and $\tilde{u}_k$ converge strongly to $u$ in $L^q(0,T;H)$, for any $q$, $1 \leq q < \infty$, as $k \to \infty$.*

**Proof.** From Lemma A.9, we have

$$|\langle \tilde{u}_k(t+r) - \tilde{u}_k(t), w\rangle| \leq M r^\ell \|w\|,$$

where $M > 0$.
Setting $w = \tilde{u}_k(t+r) - \tilde{u}_k(t)$ in (50), we find

$$\begin{aligned}\int_0^{T-r} \|\tilde{u}_k(t+r) - \tilde{u}_k(t)\|^2 \, dt &\leq M r^\ell \int_0^{T-r} \|\tilde{u}_k(t+r) - \tilde{u}_k(t)\| \, dt \\ &\leq M r^\ell T^{\frac{1}{2}} \left( \int_0^{T-r} \|\tilde{u}_k(t+r) - \tilde{u}_k(t)\|^2 \, dt \right)^{\frac{1}{2}} \\ &\leq (\text{since } \tilde{u}_k \text{ is bounded in } L^2(0,T;V)) \leq M_3 r^\ell.\end{aligned}$$

We consider the family of functions $\tilde{u}_k$ which is bounded in $L^2(0,T;V)$ and $L^\infty(0,T;H)$. By [13, Theorem 1.3.3], we conclude that $\tilde{u}_k$ is relatively compact in $L^q(0,T;H)$, $1 \leq q < 2$. Since $\tilde{u}_k$ converges weakly in $L^q(0,T;H)$ to $u$, this means that $\tilde{u}_k$ converges strongly to $u$ in $L^q(0,T;H)$, $1 \leq q < 2$.

Because the sequence $\tilde{u}_k$ is bounded in $L^\infty(0,T;H)$, by the Lebesgue dominated convergence theorem, we conclude that

$$\tilde{u}_k \to u \text{ strongly in } L^q(0,T;H) \text{ for all } 1 \le q < \infty. \tag{25}$$

By direct computation, we have

$$|u_k - \tilde{u}_k|_{L^2(0,T;H)} = \sqrt{\frac{k}{3}} \left( \left| u^m - u^{m-1} \right|^2 \right)^{\frac{1}{2}} \le (\text{by Lemma 3.2}) \le \frac{\kappa}{3} k,$$

where $\kappa$ is as in Lemma 3.2.
This fact together with (25) shows that, as $k \to 0$

$$u_k \to u \text{ strongly in } L^q(0,T;H),\ 1 \le q < \infty.$$

Lemma A.4 now gives the conclusion of this lemma. □

To pass to the limit as $k \to 0$ in the nonlinear term of $j'_\epsilon$, we derive a bound which implies weak convergence result for this term as follows: thanks to Lemma A.3 and the Cauchy–Schwarz inequality, we see that for any $v \in \mathcal{V}$

$$\begin{aligned} |\langle j'_\epsilon(u_k), v\rangle| &= \left| g \int_\Omega (\epsilon + D_{II}(u_k))^{-\frac{1}{2}} D_{ij}(u_k) D_{ij}(v)\, dx \right| \\ &\le \frac{1}{2} \left( \int_\Omega \frac{D_{II}(u_k)}{\epsilon + D_{II}(u_k)\, dx} \right)^{\frac{1}{2}} \left( \int_\Omega D_{II}(v)\, dx \right)^{\frac{1}{2}} = \frac{1}{2} g |\Omega|^{\frac{1}{2}} \|v\|. \end{aligned}$$

This implies that

$$j'_\epsilon(u_k) \text{ stays in a bounded set of } L^\infty(0,T;V') \text{ thus of } L^{p'}(0,T;V'). \tag{26}$$

### 3.4. Passage to the limit

We consider a test function $v \in \mathcal{C}^2([0,T];V)$ and its approximate functions $v_k$ and $\tilde{v}_k$ defined as above.

We now claim that $u$ is a solution of the variational inequality (7a). For this purpose, we recall the following classical facts, see e.g. [14, 11], when we pass to the limit as $k \to 0$ in (24):

- $u_k \to u$ strongly in $L^q(0,T;H)$ for all $1 \le q < \infty$. (27a)
- $\tilde{v}'_k \to v'$ strongly in $L^{p'}(0,T;V')$. (27b)
- $f_k \to f$ strongly in $L^{p'}(0,T;V')$. (27c)
- $Pu_k \to \chi$ weakly in $L^{p'}(0,T;V')$. (27d)
- $v_k \to v$ strongly in $L^p(0,T;V)$. (27e)

Also from (26):

- $j'_\epsilon(u_k) \to \xi_\epsilon$ weakly in $L^{p'}(0,T;V')$. (27f)

Passing to the limit $k \to 0$ in the above inequality (24) with the help of Lemma A.4, we obtain

$$\begin{aligned}
&\limsup_{k\to 0} \mu \int_0^T a(u_k, u_k)\, dt + \limsup_{k\to 0} \int_0^T \langle P(u_k), u_k\rangle\ dt + \int_0^T \langle f, v-u\rangle\ dt \\
\le\ & \int_0^T \langle v', v-u\rangle\ dt + \mu \int_0^T a(u,v)\, dt + \int_0^T b(u,u,v)\, dt \\
&\quad + \int_0^T \langle \chi, v\rangle\ dt + \int_0^T \langle \xi_\epsilon, v-u\rangle\ dt.
\end{aligned}$$

Now from $(8)_2$, we know that $a(u_k - u, u_k - u) \ge 0$. Since $u_k \rightharpoonup u$ weakly in $L^2(0,T;V)$, we find

$$\limsup_{k\to 0} \int_0^T a(u_k, u_k)\, dt \ge \liminf_{k\to 0} \int_0^T a(u_k,u_k)\, dt \ge \int_0^T a(u,u)\, dt.$$

Thus

$$\begin{aligned}
&\limsup_{k\to 0} \int_0^T \langle P(u_k), u_k\rangle\ dt \\
\le\ & -\int_0^T \langle f, v-u\rangle\ dt + \int_0^T \langle v', v-u\rangle\ dt + \mu \int_0^T a(u, v-u)\, dt \\
&\quad + \int_0^T b(u,u,v)\, dt + \int_0^T \langle \chi, v\rangle\ dt + \int_0^T \langle \xi_\epsilon, v-u\rangle\ dt. \qquad (28)
\end{aligned}$$

We now show that $P$ satisfies the assumption of pseudo–monotonicity, that is we need to verify that

$$\limsup_{k\to 0} \int_0^T \langle P(u_k), u_k - u\rangle\ dt \le 0. \qquad (29)$$

We have

$$\begin{aligned}
&\limsup_{k\to 0} \int_0^T \langle P(u_k), u_k - u\rangle\ dt \\
&\le \limsup_{k\to 0} \int_0^T \langle P(u_k), u_k\rangle\ dt - \liminf_{k\to 0} \int_0^T \langle P(u_k), u\rangle\ dt \\
\text{(by (28))} &\le -\int_0^T \langle f, v-u\rangle\ dt + \int_0^T \langle v', v-u\rangle\ dt + \mu \int_0^T a(u, v-u)\, dt \qquad (30) \\
&\quad + \int_0^T b(u,u,v)\, dt + \int_0^T \langle \chi, v\rangle\ dt + \int_0^T \langle \xi_\epsilon, v-u\rangle\ dt.
\end{aligned}$$

We now let $\tilde{u}_\tau$ be a sequence approximating $u$ defined by:

$$\begin{cases} \frac{1}{\tau}\tilde{u}_\tau' + \tilde{u}_\tau = u, \\ \tilde{u}_\tau(0) = u_0. \end{cases}$$

From the proof in [11, Lemma A.3], we know that $\tilde{u}_\tau, \tilde{u}_\tau' \in L^p(0,T;V)$ and $\tilde{u}_\tau \to u$ in $L^p(0,T;V)$ as $\tau \to \infty$. We also proved in [11] the followings:

* $\displaystyle\int_0^T \langle \tilde{u}_\tau', \tilde{u}_\tau - u\rangle\, dt = -\tau \int_0^T \|\tilde{u}_\tau - u\|_H^2\, dt \leq 0,$
* $\displaystyle\int_0^T \langle f, \tilde{u}_\tau - u\rangle\, dt \leq \theta_1(\tau) \overset{\tau\to\infty}{\longrightarrow} 0,$
* $\displaystyle\int_0^T \mathscr{b}(u, u, \tilde{u}_\tau - u) dt \leq \theta_2(\tau) \overset{\tau\to\infty}{\longrightarrow} 0.$

As $\tau$ goes to infinity, equation (30) with $v = \tilde{u}_\tau$ gives

$$\limsup_{k\to 0} \int_0^T \langle P(u_k), u_k - u\rangle\; dt \leq 0.$$

Hence, by the pseudo–monotonicity of $P$, we find

$$\liminf_{k\to 0} \int_0^T \langle P(u_k), u_k - w\rangle\; dt \geq \int_0^T \langle P(u), u - w\rangle\; dt, \tag{31}$$

for all $w \in \mathcal{V}$.
Letting $w = 0$ in the above inequality and combining with (28), we find

$$\begin{aligned} &\int_0^T \langle Pu, u\rangle\; dt \\ \leq\; &-\int_0^T \langle f, v - u\rangle\; dt + \int_0^T \langle v', v - u\rangle\; dt + \mu \int_0^T \mathscr{a}(u, v - u)\, dt \\ &+ \int_0^T \mathscr{b}(u, u, v)\, dt + \int_0^T \langle \chi, v\rangle\; dt + \int_0^T \langle \xi_\epsilon, v - u\rangle\; dt. \end{aligned} \tag{32}$$

It is left to show that $\chi = Pu$. From the inequality (31), we find for any $w \in \mathcal{V}$:

$$\begin{aligned} &\int_0^T \langle Pu, u - w\rangle\, dt \\ \leq\; &\limsup_{k\to 0} \int_0^T \langle P(u_k), u_k - w\rangle\, dt \\ \leq\; &\limsup_{k\to 0} \int_0^T \langle P(u_k), u_k - u\rangle\, dt + \limsup_{k\to 0} \int_0^T \langle P(u_k), u - w\rangle\, dt \\ \leq\; &(\text{by } (29)) \leq \limsup_{k\to 0} \int_0^T \langle P(u_k), u - w\rangle\, dt = \int_0^T \langle \chi, u - w\rangle\, dt. \end{aligned}$$

This means

$$\int_0^T \langle Pu, u - w\rangle \, dt \le \int_0^T \langle \chi, u - w\rangle \, dt,$$

for all $w \in \mathcal{V}$ and this implies $\chi = Pu$.
With this new information we will pass to the limit as $k \to 0$ in (24) again with $\epsilon$ being fixed so far. Before that, we rewrite (24) by noticing that $j_\epsilon(v_k) - j_\epsilon(u_k) \ge \langle j'_\epsilon(u_k), v_k - u_k\rangle$ which comes from (15):

$$\begin{aligned} \int_0^T \langle \tilde{v}'_k, v_k - u_k\rangle \, dt + \mu \int_0^T a(u_k, v_k) \, dt + \int_0^T b(u_{k \circ \tau_h}, u_k, v_k) \, dt \\ + \int_0^T \langle P(u_k), v_k\rangle \, dt + \int_0^T j_\epsilon(v_k) \, dt - \int_0^T j_\epsilon(u_k) \, dt \\ \ge \int_0^T \langle f_k, v_k - u_k\rangle \, dt + \mu \int_0^T a(u_k, u_k) \, dt + \int_0^T \langle P(u_k), u_k\rangle \, dt. \end{aligned} \tag{33}$$

We use the same procedure to pass to the limit as $k \to 0$ in (33) with the the knowledge that $Pu_k \rightharpoonup \chi = Pu$. Since $v \mapsto \int_0^T j_\epsilon(v) \, dt$ is continuous and convex on $L^2(0, T; V)$, it is lower semi–continuous in the weak topology of $L^2(0, T; V)$. Therefore

$$\liminf_{k \to 0} \int_0^T j_\epsilon(u_k) \, dt \ge \int_0^T j_\epsilon(u) \, dt.$$

With the aid of Lemma A.5, passing to the limit $k \to 0$ in (33), we arrive at

$$\begin{aligned} \int_0^T \langle v', v - u\rangle \, dt + \mu \int_0^T a(u, v - u) \, dt + \int_0^T b(u, u, v) \, dt \\ + \int_0^T \langle P(u), v - u\rangle \, dt + \int_0^T j_\epsilon(v) \, dt - \int_0^T j_\epsilon(u) \, dt - \int_0^T \langle f, v - u\rangle \, dt \ge 0, \end{aligned} \tag{34}$$

for all $v \in \mathcal{C}^2([0, T], V)$ such that $v(0) = u_0$.
Notice that, $u = u_\epsilon$ depends on $\epsilon$ in (34). From Lemma 3.2, $u_\epsilon$ is bounded independently of $\epsilon$ in $L^p(0, T; V) \cap L^\infty(0, T; H)$. Therefore, we can extract a subsequence still denoted by $\epsilon$ such that $u_\epsilon \rightharpoonup u$ weakly in $L^p(0, T; V)$ and weak star in $L^\infty(0, T; H)$ and the other convergences as in (27a)–(27f) hold.
Finally, letting $\epsilon \to 0$ in (34) as we did when passing to the limit $k \to 0$ in (33), we conclude that (7a) holds true for all $v \in \mathcal{C}^2([0, T], V)$ with $v(0) = u_0$. The proof of Theorem 3.1 is now complete thanks to Lemma A.2 which shows that (7a) holds for all $v \in \mathcal{V}$ with $v' \in \mathcal{V}'$ and $v(0) = u_0$. $\square$

The following shows a regularity result for the derivative $du/dt$ of the limit solution:

**Remark 3.4.** The result in Theorem 3.1 is still valid in the case of dimension $d > 3$ if the estimate (9) is replaced by the following estimate (see [14, Lemma 4.1]):

$$|\mathscr{b}(u, v, w)| \leq c\,|u|\,\|v\|\,\|w\|_{V_s},$$

for $s \geq d/2$. Then the proof of Theorem 3.1 will be still valid since we only use this bound in (19), but they are not the dominant terms in that estimate leading to the coercivity.

**Lemma 3.5.** *If $s \geq d/2$ then the sum $k\sum_{n=1}^{N}\left\|\frac{u^n-u^{n-1}}{k}\right\|_{V_s'}^{p'}$ is bounded independently of $k$.*

**Proof.** From (49), we obtain

$$\frac{u^n - u^{n-1}}{k} + \mu A u^n + B(u^{n-1}, u^n) + j_\epsilon'(u^n) + P(u^n) = f^n. \tag{35}$$

Taking the norm in the above equation (35), we obtain

$$\begin{aligned}
&\left\|\frac{u^n - u^{n-1}}{k}\right\|_{V_s'} \\
&\leq \|f^n\|_{V_s'} + \mu\,\|Au^n\|_{V_s'} + \|j_\epsilon'(u^n)\|_{V_s'} + \|P(u^n)\|_{V_s'} \\
&\leq (\text{using } V' \subset V_s',\ (41) \text{ and } (6)_1) \\
&\leq c_1(\|f^n\|_{V'} + \|Au^n\|_{V'} + \|u^n\|^{p/p'} + 1) + \left\|B(u^{n-1}, u^n)\right\|_{V_s'} + \frac{g}{2}\,\|u^n\|_V.
\end{aligned}$$

This and the fact that $\|Au^n\|_{V'} \leq \|A\|_{\mathcal{L}(V,V')}\,\|u^n\|$ imply that

$$\begin{aligned}
&\left\|\frac{u^n - u^{n-1}}{k}\right\|_{V_s'}^{p'} \\
&\leq c_2(\|f^n\|_{V'}^{p'} + \|u^n\|^{p'} + \|u^n\|^{p} + 1) + \left\|B(u^{n-1}, u^n)\right\|_{V_s'}^{p'} + \|u^n\|_{V}^{p'}).
\end{aligned} \tag{36}$$

For $s \geq d/2$, by [14, Lemma 4.1] we have $|b(u^{n-1}, u^n, w)| \leq c_2|u^{n-1}|\,\|u^n\|\,\|w\|_{V_s}$ for all $w \in V_s'$ and thus

$$\left\|B(u^{n-1}, u^n)\right\|_{V_s'}^{p'} \leq c_3|u^{n-1}|^{p'}\,\|u^n\|^{p'} \leq (\text{by } (20)) \leq c_4\,\|u^n\|^{p'}. \tag{37}$$

Multiplying (36) by $k$, combing (36) and (37) and summing for $n = 1, \ldots, N$, we finally have

$$k\sum_{n=1}^{N}\left\|\frac{u^n - u^{n-1}}{k}\right\|_{V_s'}^{p'} \leq c_5 k \sum_{n=1}^{N}\left(\|f^n\|_{V'}^{p'} + \|u^n\|_{V}^{p'} + \|u^n\|_{V}^{p} + 1\right).$$

From (20) and $f \in L^{p'}(0, T; V')$, the lemma follows. $\square$

**Remark 3.6 (Solution and uniqueness in dimension two).** Thanks to Lemma 3.5, for $d = 2$, we find the followings:

i. Regularity of the weak solution $u$: since $s = d/2 = 1$, we have $V' = V'_s$. With $P$ as above, we conclude that $u \in L^p(0,T;V)$ and $\frac{du}{dt} \in L^{p'}(0,T;V'_s) = L^{p'}(0,T;V')$. Hence $u \in \mathcal{C}([0,T],H)$.

ii. Uniqueness: We assume further that $P$ is monotone, that is

$$\langle P(u_1) - P(u_2), u_1 - u_2 \rangle \geq 0, \tag{38}$$

for all $u_1, u_2 \in V$.

Suppose that $u_1$ and $u_2$ are two solutions of (7a). Then

$$\int_0^T \langle v', v - u_i \rangle + \mu a(u_i, v - u_i) + b(u_i, u_i, v) \\ + j(v) - j(u_i) + \langle P(u_i), v - u_i \rangle - \langle f, v - u_i \rangle \geq 0,$$

for $i = 1, 2$. We notice that from (23) and following exactly the same step as in the passage to the limit, we actually have the inequality for $t \leq T$, that is

$$\int_0^t \langle v', v - u_i \rangle + \mu a(u_i, v - u_i) + b(u_i, u_i, v) \\ + j(v) - j(u_i) + \langle P(u_i), v - u_i \rangle - \langle f, v - u_i \rangle \geq 0,$$

Since $\frac{du_i}{dt} \in L^2(0,T;V')$ and $u_i(0) = u_0$, we can take $v = u_i$ in the inequality for $i = 1, 2$. Taking the sum of the two inequalities, we find the equation for the difference $U = u_1 - u_2$:

$$\int_0^t \left( - \langle U', U \rangle - \mu a(U,U) + b(u_2, u_2, U) \right. \\ \left. - b(u_1, u_1, U) + \langle P(u_1) - P(u_2), u_2 - u_1 \rangle \right) \geq 0.$$

Hence

$$\int_0^t \left( \langle U', U \rangle + \mu a(U,U) + \langle P(u_1) - P(u_2), u_1 - u_2 \rangle \right) \\ \leq \int_0^t \left( b(u_1, u_1, U) - b(u_2, u_2, U) \right).$$

Using $(8)_2$, this implies

$$\frac{1}{2} \|U(t)\|^2 + 2\mu \int_0^t \|U\|^2 + \int_0^t \langle P(u_1) - P(u_2), u_1 - u_2 \rangle \\ \leq \int_0^t \left( b(u_1 - U, u_1 - U, U) - b(u_1, u_1, U) \right) = - \int_0^t b(U, u_1, U).$$

From (9), we find

$$|\mathscr{B}(U, u_1, U)| \le c \|U\|^2 |u_1| \le \text{(Cauchy Schwarz)} \le 2\mu \|U\|^2 + \frac{1}{4\mu} \|u_1\|^2 \|U\|^2$$

This and (38) yield

$$\|U(t)\|^2 + 4 \int_0^t \mu \|U\|^2 \le \int_0^t \left( 4\mu \|U\|^2 + \frac{1}{2\mu} \|u_1\|^2 \|U\|^2 \right).$$

Finally, we find

$$\|U(t)\|^2 \le \int_0^t \frac{1}{2\mu} \|u_1\|^2 \|U\|^2 .$$

Since $\|u_1\|^2 \in L^1(0, T)$ and $U(0) = 0$, by Gronwall's inequality, we obtain $\|U(t)\| \le 0$ for all $t > 0$ which says $U = 0$.

## A. Appendix

We recall a classical result from e.g. [4]:

**Lemma A.1.** *If $v \mapsto J(v)$ is Gateaux differentiable and convex, then $u \mapsto J'(u)$ is monotone and hemicontinuous.*

**Proof.** Since J is convex, we have

$$J((1-\theta)u + \theta v) \le (1-\theta)J(u) + \theta J(v),$$

and hence

$$\frac{1}{\theta} J((1-\theta)u + \theta v) - J(u) \le J(v) - J(u).$$

Letting $\theta \to 0$, we find

$$\langle J'(u), u - v \rangle \le J(u) - J(v).$$

We add this inequality to the similar inequality where $u$ and $v$ have been exchanged and we find

$$\langle J'(u) - J'(v), v - u \rangle \ge 0. \qquad \square$$

**Lemma A.2.** *We assume that $u_0 \in V$ and we set*

$$\mathcal{K}_{u_0} := \{v \in L^p(0, T; V) \text{ such that } v' \in L^{p'}(0, T; V') \text{ and } v(0) = u_0\},$$
$$\mathcal{K}^1_{u_0} := \{v \in \mathcal{C}^2([0, T]; V) : v(0) = u_0\}.$$

*Then $\mathcal{K}^1_{u_0}$ is dense in $\mathcal{K}_{u_0}$ in the sense that for each $v \in \mathcal{K}_{u_0}$ there exists a sequence $v_j \in \mathcal{K}^1_{u_0}$ such that $v_j \to v$ strongly in $\mathcal{V}$ and $v'_j \to v'$ strongly in $\mathcal{V}'$.*

**Proof.** Let $v$ be given in $\mathcal{K}_{u_0}$, let $\tilde{v}$ be its extension by $u_0$ for $t < 0$. Let $\rho$ be a positive mollifier such that $\operatorname{supp} \rho \subset (0, \infty)$.
Then $\rho_\epsilon * \tilde{v}$ is defined on $(-\infty, T)$; it belongs to $\mathcal{C}^\infty((-\infty, T); V)$ and $\rho_\epsilon * \tilde{v} \to \tilde{v}$ in $L^p(-\infty, T; V)$ as $\epsilon \to 0$. Thus $\rho_\epsilon * \tilde{v}|_{[0,T]} \to v$ in $L^p(0, T; V)$ as $\epsilon \to 0$. Note that by construction $v_\epsilon(0) = \rho_\epsilon * \tilde{v}(0) = u_0$.
Similarly, $\tilde{v}'$ is the function equal to 0 for $t < 0$ or to $v'$ for $t \in (0, T)$ so that we obtain, for $\epsilon \to 0$: $\rho_\epsilon * \tilde{v} \to \tilde{v}'$ in $L^{p'}(-\infty, T; V')$ which implies $v'_\epsilon \to v'$ in $L^{p'}(0, T; V')$. □

**Lemma A.3.** *The first and second Gateaux derivatives of the function $j_\epsilon$ defined by* (13) *are given by:*

$$\langle j'_\epsilon(u), v\rangle = \frac{d}{d\lambda} j_\epsilon(u + \lambda v)|_{\lambda=0} = g \int_\Omega (\epsilon + D_{II}(u))^{-\frac{1}{2}} D_{ij}(u) D_{ij}(v)\, dx \tag{39}$$

$$\langle j''_\epsilon(u)v, v\rangle = \frac{d^2}{d^2\lambda} j_\epsilon(u + \lambda v)|_{\lambda=0} \tag{40}$$

$$= 2g \int_\Omega (\epsilon + D_{II}(u))^{-\frac{1}{2}} D_{II}(v)\, dx - \frac{g}{2} \int_\Omega (\epsilon + D_{II}(u))^{-\frac{3}{2}} \left(D_{ij}(u) D_{ij}(v)\right)^2\, dx.$$

*They enjoy the following properties*

$$|\langle j'_\epsilon(u), v\rangle| \le \frac{g}{2} \|v\|, \tag{41}$$

$$\langle j'_\epsilon(u), u\rangle \ge 0, \tag{42}$$

$$\langle j''_\epsilon(u)v, v\rangle \ge 0. \tag{43}$$

**Proof.** We have

$$\begin{aligned}\frac{d}{d\lambda} j_\epsilon(u + \lambda v) &= \frac{d}{d\lambda} 2g \int_\Omega (\epsilon + D_{II}(u + \lambda v))^{\frac{1}{2}}\, dx \\ &= g \int_\Omega (\epsilon + D_{II}(u + \lambda v))^{-\frac{1}{2}} \frac{d}{d\lambda} D_{II}(u + \lambda v)\, dx.\end{aligned}$$

For the last term, we write

$$\frac{d}{d\lambda} D_{II}(u + \lambda v) = \frac{d}{d\lambda} \frac{1}{2} \sum_{i,j=1}^{3} D_{ij}(u + \lambda v)^2 = D_{ij}(u + \lambda v) D_{ij}(v).$$

Thus

$$\frac{d}{d\lambda} j_\epsilon(u + \lambda v) = g \int_\Omega (\epsilon + D_{II}(u + \lambda v))^{-\frac{1}{2}} D_{ij}(u + \lambda v) D_{ij}(v)\, dx. \tag{44}$$

Since $u, v \in V$, using the Lebesgue dominated convergence theorem, we pass to the limit $\lambda \to 0$ in (44) and we find (39). We immediately find (41) and (42) as direct consequences of (39).

Differentiating again the RHS of (44) in $\lambda$, we obtain

$$\frac{d^2}{d^2\lambda} j_\epsilon(u+\lambda v) = -\frac{g}{2}\int_\Omega (\epsilon + D_{II}(u+\lambda v))^{-\frac{3}{2}} \left(D_{ij}(u+\lambda v)D_{ij}(v)\right)^2 \, dx \\ + g\int_\Omega (\epsilon + D_{II}(u+\lambda v))^{-\frac{1}{2}} \frac{d}{d\lambda}\left(D_{ij}(u+\lambda v)D_{ij}(v)\right) \, dx.$$

It is clear that

$$\frac{d}{d\lambda}\left(D_{ij}(u+\lambda v)D_{ij}(v)\right) = D_{ij}(v)D_{ij}(v) = 2D_{II}(v).$$

Thus

$$\frac{d^2}{d^2\lambda} j_\epsilon(u+\lambda v) = -\frac{g}{2}\int_\Omega (\epsilon + D_{II}(u+\lambda v))^{-\frac{3}{2}} \left(D_{ij}(u+\lambda v)D_{ij}(v)\right)^2 \, dx \\ + 2g\int_\Omega (\epsilon + D_{II}(u+\lambda v))^{-\frac{1}{2}} D_{II}(v) \, dx. \tag{45}$$

Passing to the limit $\lambda \to 0$ in (45) we find (40).

We then obtain (43) from (40) and $D_{ij}(u)D_{ij}(v) \leq \frac{1}{2}D_{II}(u)^{\frac{1}{2}}D_{II}(v)^{\frac{1}{2}}$ which is true by the Schwarz inequality applied to the sum in $i, j$. □

**Lemma A.4.** *If $u_k$ converges to $u$ in $L^2(0,T;L^p)$ weakly and in $L^2(0,T;H)$ strongly then for any vector function $v_k$ with components in $\mathcal{C}^1([0,T];L^r)$, for $r = \frac{2p}{p-2}$ we have:*

$$\lim_{k\to 0}\int_0^T \mathscr{b}(u_{k\circ\tau_h}, u_k, v_k)\, dt = \int_0^T \mathscr{b}(u,u,v)\, dt. \tag{46}$$

**Proof.** We write

$$\int_0^T \mathscr{b}(u_{k\circ\tau_h}, u_k, v_k)\, dt = -\int_0^T \mathscr{b}(u_{k\circ\tau_h}, v_k, u_k)\, dt \\ = -\sum_{i,j=1}^{3}\int_0^T\int_\Omega (u_{k\circ\tau_h})_i (D_i v_k)_j (u_k)_j \, dxdt.$$

Thus

$$\left|\int_0^T \mathscr{b}(u_{k\circ\tau_h}, u_k, v_k) - \mathscr{b}(u,u,v)\, dt\right|$$
$$\leq \sum_{i,j=1}^{3}\int_0^T\int_\Omega |(D_i v_k)_j|\,|(u_{k\circ\tau_h})_i(u_k)_j - (u)_i(u)_j| + |(D_i v_k)_j - D_i v|\,|(u)_i(u)_j|\, dxdt$$
$$\leq \sum_{i,j=1}^{3}\int_0^T\int_\Omega |(D_i v_k)_j|\,|(u_{k\circ\tau_h})_i(u_k)_j - (u)_i(u)_j| + |(D_i v_k)_j - D_i v|\,|(u)_i(u)_j|\, dxdt.$$

We consider the first term:

$$
\begin{aligned}
T_1 = &\int_0^T \int_\Omega |(D_i v_k)_j| \left|(u_{k\circ\tau_h})_i (u_k)_j - (u)_i (u)_j\right| \, dxdt \le \text{(Triangle inequality)} \\
\le &\int_0^T \int_\Omega |(D_i v_k)_j| \left( |(u_{k\circ\tau_h})_i ((u_k)_j - (u)_j)| + |(u)_j ((u_k)_i - (u)_i)| \right) dxdt \\
\le &\ \text{(by Hölder inequality in } x) \\
= &\int_0^T \|(D_i v_k)_j\|_{L^r} \big( \|(u_{k\circ\tau_h})_i\|_{L^p} \|(u_k)_j - (u)_j\|_H \\
&\qquad + \|(u)_j\|_{L^p} \|(u_k)_i - (u)_i\|_H \big) \, dt \\
\le &\ \|(D_i v_k)_j\|_{\mathcal{C}([0,T];L^r)} \int_0^T \big( \|(u_{k\circ\tau_h})_i\|_{L^p} \|(u_k)_j - (u)_j\|_H \\
&\qquad + \|(u)_j\|_{L^p} \|(u_k)_i - (u)_i\|_H \big) \, dt \\
\le &\ \text{(by Cauchy Schwarz inequality in } t) \\
\le &\ \|(D_i v_k)_j\|_{\mathcal{C}([0,T];L^r)} \Big( \|(u_{k\circ\tau_h})_i\|_{L^2(0,T;L^p)} \|(u_k)_j - (u)_j\|_{L^2(0,T;H)} \\
&\qquad + \|(u)_j\|_{L^2(0,T;L^p)} \|(u_k)_i - (u)_i\|_{L^2(0,T;H)} \Big).
\end{aligned}
$$

Since $u_{k\circ\tau_h}, u_k$ converge strongly in $L^2(0,T;H)$ to $u$ and are bounded in $L^2(0,T;L^p)$, and $D_i v_k$ is bounded in $\mathcal{C}([0,T];L^r)$, we easily see that $T_1$ goes to zero as $k$ tends to zero.
We now treat the second term:

$$
\begin{aligned}
T_2 = &\int_0^T \int_\Omega |(D_i v_k)_j - (D_i v)_j| \, |(u)_i (u)_j| \, dxdt \le \text{(by Hölder inequality in } x) \\
\le &\int_0^T \|(D_i v_k)_j - (D_i v)_j\|_{L^r} \|(u)_i\|_{L^p} \|(u)_j\|_H \, dt \\
\le &\ \|(D_i v_k)_j - (D_i v)_j\|_{\mathcal{C}([0,T];L^r)} \int_0^T \|(u)_i\|_{L^p} \|(u)_j\|_H \, dt.
\end{aligned}
$$

By Cauchy Schwarz inequality in $t$, we find

$$
T2 \le \|(D_i v_k)_j - (D_i v)_j\|_{\mathcal{C}([0,T];L^r)} \|(u)_i\|_{L^2(0,T;L^p)} \|(u)_j\|_{L^2(0,T;H)} .
$$

Since $u$ belongs to $L^2(0,T;L^p)$, and $D_i v_k$ converges to $D_i v_k$ in $\mathcal{C}([0,T];L^r)$, $T_2$ also goes to zero as $k$ tends to zero. The proof of the lemma is hence done. □

**Lemma A.5.** *If $v_k$ converges to $v$ in $L^2(0,T;V)$ strongly then there exists a subsequence still denoted by $k$ such that*

$$
j_\epsilon(v_k) \to j_\epsilon(v) \quad \textit{strongly in } L^2(0,T). \tag{47}
$$

**Proof.** We can extract a subsequence $k$ such that $v_k(t) \to v(t)$ in $V$ for a.e. $t \in (0,T)$. Since $v_k(t)$ converges to $v(t)$ in $V$ it is easy to see that $D_{II}(v_k(t))^{\frac{1}{2}}$ converges to $D_{II}(v(t))^{\frac{1}{2}}$ in $H$ hence in $L^1(\Omega)$. This means that $j_\epsilon(v_k)(t)$ converges to $j_\epsilon(v)(t)$ for a.e. $t \in (0,T)$.
Furthermore, we have for a.e. $t \in (0,T)$

$$j_\epsilon(v_k)(t) = \int_\Omega \sqrt{\epsilon + D_{II}(v_k(t))}\, dx \le \int_\Omega \sqrt{1 + D_{II}(v(t))}\, dx,$$

where the dominating term is square integrable in $t$. We then conclude the lemma using the Lebesgue Dominated Convergence Theorem. $\square$

**Lemma A.6.** *If $w^0 = 0$, we have for all $2 \le m \le N$*

$$\sum_{n=1}^{m} \left\langle w^n - w^{n-1}, w^n \right\rangle \ge 0.$$

**Proof.** We use $2\langle w^n - w^{n-1}, w^n\rangle = \|w^n\|^2 - \|w^{n-1}\|^2 + \|w^n - w^{n-1}\|^2$ and take the sum in $n$ from 1 to $m$ to obtain

$$\begin{aligned} 2\sum_{n=1}^{m} \left\langle w^n - w^{n-1}, w^n \right\rangle &= \|w^m\|^2 - \left\|w^0\right\|^2 + \sum_{n=1}^{m} \left\|w^n - w^{n-1}\right\|^2 \\ &\ge \|w^m\|^2 - \left\|w^0\right\|^2. \end{aligned}$$

Since $w^0 = 0$, the lemma follows. $\square$

**Lemma A.7.** *Suppose that $v \in C^2([0,T];V')$. Then we have $D_t\tilde{v}_k \to D_t v = v'$ strongly in $\mathcal{V}' = L^{p'}(0,T;V')$ where*

$$\begin{cases} v_k = & \textit{the step function with value } v^n = v(nk), \\ \tilde{v}_k = & \textit{the linear function interpolating the values } v\left((n-1)k\right) \textit{ and } v(nk). \end{cases}$$

**Proof.** See [11, Lemma A.4]. $\square$

The following theorem is a special case when $K = V$ in [7, Theorem 8.2]:

**Theorem A.8 (Theorem 8.2 in [7]).** *Let $A$ be a Pseudo–monotone operator from $V$ to $V'$ which is coercive in the following sense: there exists $v_0$ in $V$ such that*

$$\frac{\langle Av, v - v_0\rangle}{\|v\|} \longrightarrow \infty \quad \textit{if } \|v\| \to \infty \textit{ for } v \in V.$$

*Then for $f \in V'$, there exists $u$ in $V$ such that*

$$Au = f.$$

**Lemma A.9.** *For $r > 0$ and $t \in (0, T)$, the approximate solution $\tilde{u}_k$ satisfies*

$$\langle \tilde{u}_k(t+r) - \tilde{u}_k(t), w\rangle \leq M r^{\ell} \|w\|, \tag{48}$$

*where $M = M(\kappa, \mu, g) > 0$ and $\ell = \min\left\{\frac{1}{p}, \frac{2p-3}{2p}\right\}$.*

**Proof.** Letting $v = u^n \pm w$ for $w \in V$ in (16), we find

$$\left\langle \frac{u^n - u^{n-1}}{k}, w\right\rangle + \mu a(u^n, w) + b(u^{n-1}, u^n, w) + \langle j_\epsilon(u^n)', w\rangle + \langle Pu^n, w\rangle = \langle f^n, w\rangle. \tag{49}$$

The equation (49) is then interpreted in terms of the functions $u_k, \tilde{u}_k$ as:

$$\left\langle \frac{d\tilde{u}_k}{dt}, w\right\rangle + \mu a(u_k, w) + b(u_k(\tau_k), u_k, w) + \langle j_\epsilon(u_k)', w\rangle + \langle Pu_k, w\rangle = \langle f_k, w\rangle, \tag{50}$$

where $\tau_k(t) = t - k$, $u_{k\circ\tau_h} = u^{m-1}$ and $f_k(t) = f^m$ for $t \in [(m-1)k, mk]$. We rewrite the equation (50) as

$$\left\langle \frac{d\tilde{u}_k}{dt}, w\right\rangle = -\mu a(u_k, w) - b(u_k \circ \tau_k, u_k, w) - \langle j_\epsilon(u_k)', w\rangle + \langle f_k, w\rangle - \langle Pu_k, w\rangle,$$

and integrate from $t$ to $t + r$ to obtain:

$$\begin{aligned}
&\langle \tilde{u}_k(t+r) - \tilde{u}_k(t), w\rangle \\
&= -\mu \int_t^{t+r} a(u_k, w) - \int_t^{t+r} b(u_k \circ \tau_k, u_k, w) \\
&\quad - \int_t^{t+r} \langle j_\epsilon(u_k)', w\rangle + \int_t^{t+r} \langle f_k, w\rangle - \int_t^{t+r} \langle Pu_k, w\rangle.
\end{aligned} \tag{51}$$

We then majorize each term of the right hand side of (51) as follows:

$$\begin{aligned}
&\left|\mu \int_t^{t+r} a(u_k, w)\, ds\right| \\
&\leq (\text{by } (8)_1) \leq \mu \int_t^{t+r} 2\|u_k\|\,\|w\|\; ds = 2\mu\|w\| \int_t^{t+r} \|u_k\|\; ds \\
&\leq 2\mu\|w\|\, r^{\frac{1}{p'}} \left(\int_t^{t+r} \|u_k\|^p\; ds\right)^{\frac{1}{p}} \leq M_1 \|w\|\, r^{\frac{1}{p'}},
\end{aligned}$$

where $M_1 = 2\mu\frac{\kappa^{1/p}}{\alpha_0}$, thanks to (20).

$$\begin{aligned}
\left|\int_t^{t+r} \langle j_\epsilon(u_k)', w\rangle\; ds\right| &\leq \int_t^{t+r} |\langle j_\epsilon(u_k)', w\rangle|\; ds \leq \int_t^{t+r} |\langle j_\epsilon(u_k)', w\rangle|\; ds \\
&\leq (\text{ by } (41)) \leq \frac{g}{2}\|w\|\, r \leq (\text{since } r < 1) \leq \frac{g}{2}\|w\|\, \sqrt[p]{r}.
\end{aligned}$$

We next use [13, Lemma 2.1] with $m_1 = \frac{1}{2}, m_2 = m_3 = 1$ to bound the second term on the right hand side

$$\left| \int_t^{t+r} \mathscr{b}(u_k \circ \tau_k, u_k, w)\, ds \right| \le C_1 \int_t^{t+r} |u_k(s-k)|_{\frac{1}{2}} \|u_k(s)\| \, \|w\| \; ds,$$

where $|\cdot|_{\frac{1}{2}}$ is the norm of $H^{\frac{1}{2}}$ which satisfies $|\cdot|_{\frac{1}{2}} \le C\,|\cdot|^{\frac{1}{2}} \|\cdot\|^{\frac{1}{2}}$. Hence

$$\begin{aligned}
&\left| \int_t^{t+r} \mathscr{b}(u_k \circ \tau_k, u_k, w)\, ds \right| \\
&\le C_2 \|w\| \int_t^{t+r} |u_k(s-k)|^{\frac{1}{2}} \|u_k(s)\|^{\frac{3}{2}} \; ds \\
&\le (\text{ by Hölder's inequality}) \\
&\le C_2 \|w\| \, |u_k(s-k)|^{\frac{1}{2}}_{L^\infty(0,T;H)} \, r^{\frac{2p-3}{2p}} \left( \int_t^{t+r} \|u_k(s)\|^p \; ds \right)^{\frac{3}{2p}} \\
&\le M_2 r^{\frac{2p-3}{2p}} \|w\|,
\end{aligned}$$

with $M_2 = C_2 \kappa^{\frac{1}{2} + \frac{3}{2p}}$ thanks to (20).
We finally bound the nonlinear pseudo–monotone term

$$\begin{aligned}
&\left| \int_t^{t+r} \langle P u_k(s), w \rangle \; ds \right| \\
\le\; & \int_t^{t+r} |\langle P u_k(s), w\rangle| \; ds \le \|w\| \int_t^{t+r} \left( \beta_0 \|u_k\|^{p/p'} + \beta_1 \right) \, ds \\
\le\; & \|w\| \left[ \beta_0 \left( \int_t^{t+r} \left( \|u_k\|^{p/p'} \right)^{p'} \; ds \right)^{\frac{1}{p'}} + \beta_1 r \right] \\
\le\; & (\text{Hölder's inequality and } r < 1) \le \|w\| \left( \beta_0 \|u_k\|_{\mathcal{V}} \, r^{\frac{1}{p}} + \beta_1 r^{\frac{1}{p}} \right) \\
\le\; & M_3 \|w\| \, r^{\frac{1}{p}},
\end{aligned}$$

where $M_3 = \beta_0 \kappa^{\frac{1}{p'}} + \beta_1$. Consider
Now (51) yields

$$|\langle \tilde{u}_k(t+r) - \tilde{u}_k(t), w \rangle| \le M r^\ell \|w\|,$$

where $M = M_1 + M_2 + M_3$ and $\ell = \min \left\{ \frac{1}{p}, \frac{2p-3}{2p} \right\}$. Lemma A.9 is proven. □

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