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Weak Convexity of Sets and Functions in a Banach Space

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We consider weakly convex sets with respect to (w.r.t.) a quasiball M (quasiball is a closed convex proper subset of a Banach space E with 0 being its interior point). We investigate the properties of M which are sufficient for equivalence of the weak convexity of a closed set A, single-valuedness and continuity of M-projection onto A from the M-tube around A, and Fréchet differentiability of the M-distance function on the M-tube around A. We show that a function f is weakly convex w.r.t. a convex function γ with $\gamma(0) < 0$ iff the epigraph of f is weakly convex w.r.t. the epigraph of γ. The weak convexity of f w.r.t. a uniformly convex coercive function γ is characterized in terms of well posedness of the infimal convolution problem for f and γ.

Keywords: Weak convexity, Minkowski functional, infimal convolution, quasiball

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1. Introduction

Let E be a real normed vector space. For a set $A \subset E$ by $\operatorname{int} A$, $\overline{A}$, and ∂A we denote the interior, the closure, and the boundary of A, respectively. We use $\langle p, x\rangle$ to denote the value of the functional $p \in E^*$ at the vector $x \in E$. For $R > 0$ and $c \in E$ we denote by $\mathfrak{B}_R(c)$ the closed ball with center c and radius R, in particular, we put $\mathfrak{B}_R = \mathfrak{B}_R(0)$. The *distance* from a point $x \in E$ to a set $A \subset E$ is defined as

$$\varrho(x, A) = \inf_{a\in A} \|x - a\|. \tag{1}$$

The *metric projection* of a point $x \in E$ onto a set $A \subset E$ is

$$P(x, A) = A \cap \mathfrak{B}_{\varrho(x,A)}(x).$$

The *proximal normal cone* to a set $A \subset E$ at a point $a \in A$ is defined by

$$N^P(a, A) = \{z \in E \mid \exists t > 0 :\ a \in P(a + tz, A)\}. \tag{2}$$

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Federer [11] for a set $A \subset \mathbb{R}^n$ defined

$$\operatorname{reach}(A) = \sup\{r > 0 \,|\, P(x, A) \text{ is a singleton } \forall x \in U^r(A)\},$$

where $U^r(A) = \{x \in \mathbb{R}^n \mid 0 < \varrho(x, A) < r\}$ is a *tube* around A.

Federer considered sets with positive reach, i.e., sets $A \subset \mathbb{R}^n$ such that $\operatorname{reach}(A) > 0$. In particular, he proved that the distance function $\varrho(\cdot, A)$ is continuously differentiable on the set $\{x \in \mathbb{R}^n \mid 0 < \varrho(x, A) < \operatorname{reach}(A)\}$.

Clarke, Stern and Wolenski [7] introduced and studied the *proximally smooth sets* in a Hilbert space H. A set $A \subset H$ is said to be proximally smooth if there exists $r > 0$ such that the distance function $\varrho(\cdot, A)$ is continuously differentiable on $U^r(A)$. Properties of proximally smooth sets in a Banach space and relations between such sets and akin classes of sets, including uniformly prox-regular sets, were investigated in [1]–[5], [9]. Reformulating Definition 2.5 due to Bernard, Thibault and Zlateva [5], we say that a closed set $A \subset E$ is uniformly r-prox-regular if

$$P(a + rz, A) = \{a\}, \quad \forall a \in A, \ \forall z \in N^P(a, A) : \ \|z\| < 1.$$

Poliquin, Rockafellar and Thibault [20] showed that in a Hilbert space proximally smooth sets are exactly *uniformly prox-regular sets*.

In [4], [5] it was assumed that the moduli of uniform convexity and uniform smoothness of E are of power type. The *modulus of convexity* of a normed space E is

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x + z\|}{2} \;\middle|\; x, z \in \partial\mathfrak{B}_1(0), \ \|x - z\| \geq \varepsilon\right\}, \quad \forall \varepsilon \in (0, 2].$$

The modulus of convexity is of *power type* q if for some $C > 0$ one has $\delta_E(\varepsilon) \geq C\varepsilon^q$ for all $\varepsilon \in (0, 2]$.

The *modulus of smoothness* of a normed space E is

$$\varrho_E(\tau) = \sup\left\{\frac{\|x + \tau z\|}{2} + \frac{\|x - \tau z\|}{2} - 1 \;\middle|\; x, z \in \partial\mathfrak{B}_1(0)\right\}, \quad \tau \geq 0.$$

The modulus of smoothness is of *power type* s with $s > 1$ if for some $c > 0$ one has $\varrho_E(\tau) \leq c\tau^s$ for all $\tau \geq 0$.

In particular, in [4] it was proved the following result (see also Theorem 2.6 in [5]).

Proposition 1.1. *Assume that the moduli of uniform convexity and uniform smoothness of a Banach space E are of power type. Then for a closed set $A \subset E$ the following statements are equivalent:*

(*i*) *A is uniformly r-prox-regular;*

(*ii*) $P(\cdot, A)$ *is single-valued and continuous on* $U^r(A)$*;*
(*iii*) $\varrho(\cdot, A)$ *is continuously differentiable on* $U^r(A)$*.*

Note that the results of the present paper imply that statements (*i*) and (*ii*) of Proposition 1.1 are equivalent provided that E is a uniformly convex Banach space (and even slightly less) without any assumption about smoothness of E. The assumption of Proposition 1.1 about power type of the moduli may be omitted. Our notion of weakly convex sets w.r.t. a quasiball (see Section 2) generalizes the notion of uniformly prox-regularity by substituting the norm by a nonsymmetric seminorm (Minkowski functional). This approach reveals the close relation between weakly convex sets and weakly convex functions.

Janin [16] considered PC2 functions which locally decompose into the sum of a convex function and a positive definite quadratic function. Rockafellar [22] introduced and studied akin class of lower-C^2 functions, favorable in subgradient optimization. Bougeard [6] considered the same decomposition as Janin but globally. Vial [23] called the functions of the same class as *weakly convex.* He said that a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is weakly convex if for some $r > 0$ the function $x \mapsto f(x) + r\|x\|^2$ is convex. Note that if we try to use this definition for functions defined on a Banach (not Hilbert) space, then most of the favorable properties of such functions will be lost.

Bernard, Thibault and Zlateva [4], [5] considered the class of *J-primal lower regular* (J-plr) functions defined on a Banach space. The notion of J-plr functions involves the following definitions.

Recall that the *epigraph* of function $f : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by

$$\operatorname{epi} f = \{(x, y) \in E \times \mathbb{R} \mid x \in E,\ y \geq f(x)\};$$

the *effective domain* f is

$$\operatorname{dom} f = \{x \in E \mid f(x) \in \mathbb{R}\}.$$

The *proximal subdifferential* of a function $f : E \to \mathbb{R} \cup \{+\infty\}$ at a point $x \in \operatorname{dom} f$ is defined by

$$\partial^P f(x) = \{u \in E \mid (u, -1) \in N^P((x, f(x)), \operatorname{epi} f)\}, \tag{3}$$

where the norm of the space $E \times \mathbb{R}$ is

$$\|(x, y)\| = \sqrt{\|x\|^2 + y^2}, \quad \forall x \in E,\ \forall y \in \mathbb{R}. \tag{4}$$

The mapping $J : E \to E^*$ defined by

$$J(x) = \{x^* \in E^* \mid \langle x^*, x\rangle = \|x^*\| \cdot \|x\|,\ \|x^*\| = \|x\|\}$$

is called *normalized duality mapping.*

According to [5], a function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be *uniformly* J-plr if there exist positive constants Θ, r such that for any $t \geq \Theta$, any $x \in E$, $u \in \partial^P f(x)$ with $\|u\| \leq rt$,

$$f(x+v) - f(x) \geq \langle p, v \rangle, \quad \forall v \in E, \ \forall p \in J(u - tv). \tag{5}$$

The following Proposition (Proposition 4.4 in [5]) shows the relation between J-primal lower regularity of a function and prox-regularity of its epigraph.

Proposition 1.2. *Assume that the moduli of uniform convexity and uniform smoothness of a Banach space E are of power type. If $f : E \to \mathbb{R} \cup \{+\infty\}$ is uniformly* J-plr *with parameters r, Θ, then* epi f *is uniformly r'-prox-regular for some $r' \geq \frac{1}{2} \min\left\{\frac{1}{\Theta}, r\right\}$.*

In Section 3 we introduce the notion of weakly convex functions defined on a Banach space with respect to some function γ. According to Theorem 3.5, this notion is equivalent to the weak convexity of the set $\operatorname{epi} f$ w.r.t. the quasiball $\operatorname{epi} \gamma$. Theorem 3.3 shows that a function $f : E \to \mathbb{R} \cup \{+\infty\}$ is uniformly J-plr provided that f is weakly convex w.r.t. the function $\gamma(x) = \sigma\|x\|^2$ for some $\sigma > 0$ and $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$ (symbol $\boxplus$ means infimal convolution, see (6)). Theorems 3.6 and 3.7 imply that the weak convexity of a function f with respect to a uniformly convex coercive function γ is the necessary and sufficient condition for well posedness of the infimal convolution problem for f and γ.

The *infimal convolution* [12], [21] of the functions $f : E \to \mathbb{R} \cup \{+\infty\}$ and $g : E \to \mathbb{R} \cup \{+\infty\}$ is the function $f \boxplus g$, defined by

$$(f \boxplus g)(x) = \inf_{u \in E} \left(f(x-u) + g(u)\right), \quad x \in E. \tag{6}$$

The problem to find a minimizer in (6) is quite common in optimization. The infimal convolution may be interpreted as a generalization of Moreau–Yosida regularization (or Moreau envelope, or proximal envelope) [19], [24].

Note that the infimal convolution of functions is related to the Minkowski sum of their epigraphs. Recall that the *Minkowski sum* of sets $A, B \subset E$ is

$$A + B = \{a + b \mid a \in A, \ b \in B\}.$$

For any functions $f, g : E \to \mathbb{R} \cup \{+\infty\}$ we have

$$\operatorname{epi} f + \operatorname{epi} g \subset \operatorname{epi}(f \boxplus g) \subset \overline{\operatorname{epi} f + \operatorname{epi} g}. \tag{7}$$

The rest of the paper is organized as follows. In Sections 2 and 3 the main results on weakly convex sets and function are formulated. In Section 4 we consider some families of subsets, most of them are various modifications of convex sets. In Section 5 we show that the weak convexity implies continuity of the M-projection. In Section 6 we prove the inverse implication. In Section 7 the differentiability of the M-distance function is considered. Section 8 is the proof of the main results for weakly convex sets. In Sections 9 and 10 we prove the results for weakly convex functions.

2. Weakly convex sets

We say that the subset M of E is a *quasiball* if M is closed convex, $0 \in \operatorname{int} M$ and $M \neq E$.

Note that a quasiball M is the unit ball for some norm equivalent to the original one iff M is bounded in the sense of the original norm and symmetric (i.e. $-M = M$).

Let M be a quasiball. Recall that the *Minkowski functional* of M is defined by

$$\mu_M(x) = \inf \{t > 0 \mid x \in tM\}, \quad x \in E.$$

Remark 2.1. The Minkowski functional of a quasiball $M \subset E$ is Lipschitz continuous on E with the constant $\frac{1}{\varrho(0,\partial M)}$.

A functional $\mu : E \to [0, +\infty)$ is called a *nonsymmetric seminorm* if it is *sublinear*, i.e., *positively homogeneous*:

$$\mu(tx) = t\mu(x), \quad \forall t \geq 0, \ \forall x \in E$$

and *subadditive*:

$$\mu(x + y) \leq \mu(x) + \mu(y), \quad \forall x, y \in E.$$

Note that a functional $\mu : E \to [0, +\infty)$ is a continuous at zero nonsymmetric seminorm iff it is the Minkowski functional of a quasiball.

For a quasiball $M \subset E$, a closed set $A \subset E$, and a point $x_0 \in E$ we consider the problem to approximate x_0 by a point $a \in A$ best in the sense of the nonsymmetric seminorm $\mu_M(\cdot)$:

$$\min_{a \in A} \mu_M(x_0 - a). \tag{8}$$

Through $\varrho_M(x_0, A)$ we denote the M-*distance* from the point x_0 to the set A:

$$\varrho_M(x_0, A) = \inf_{a \in A} \mu_M(x_0 - a). \tag{9}$$

Remark 2.2. For any $x \in E$ and $A \subset E$ we have

$$\varrho_M(x, A) = \inf \{t > 0 \mid x \in A + tM\} = \inf \{t > 0 \mid (x - tM) \cap A \neq \emptyset\}.$$

Given a function $F : A \to \mathbb{R} \cup \{+\infty\}$ a point $a_0 \in A$ is called a *minimizer* of the problem

$$\min_{a \in A} F(a) \tag{10}$$

if $F(a_0) = \min_{a \in A} F(a)$. A sequence $\{a_k\} \subset A$ is said to be a *minimizing sequence* of the problem (10) if

$$\lim_{k \to \infty} F(a_k) = \inf_{a \in A} F(a).$$

The problem (10) is said to be *well posed* if every minimizing sequence of this problem converges to the minimizer.

For a set $A \subset E$ and a point $x_0 \in E$ the *M-projection* of x_0 onto A is the set of all solutions of the problem (8):

$$P_M(x_0, A) = \left\{ a \in A \middle| \ \mu_M(x_0 - a) \leq \varrho_M(x_0, A) \right\}. \tag{11}$$

Remark 2.3. For any quasiball $M \subset E$, point $x_0 \in E$, and a set $A \subset E$ such that $\varrho_M(x_0, A) > 0$ we have

$$P_M(x_0, A) = A \bigcap (x_0 - \varrho_M(x_0, A)M).$$

Remark 2.4. If E is a Banach space, a set $A \subset E$ is closed, $x_0 \in E$, and the problem (8) is well posed, then $P_M(x_0, A)$ is a singleton.

The set of *unit M-normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N_M^1(a, A) = \{z \in \partial M \mid \exists t > 0 : \ a \in P_M(a + tz, A)\}. \tag{12}$$

A set $A \subset E$ is called *weakly convex* w.r.t. the quasiball $M \subset E$ if

$$a \in P_M(a + z, A), \quad \forall a \in A, \ \forall z \in N_M^1(a, A). \tag{13}$$

In the case $M = \mathfrak{B}_r(0)$, $r > 0$ the family of weakly convex sets is exactly the family of r-proximally smooth sets [4] and is the same as the family of sets satisfying the support condition of weak convexity with constant r (see [1]). In [9], [13], [14] the weakly convex sets were considered w.r.t. a nonsymmetric bounded quasiball.

Note that any convex set $A \subset E$ is weakly convex w.r.t. any quasiball $M \subset E$. If a set $A \subset E$ is weakly convex w.r.t. the quasiball $M \subset E$, then A is weakly convex w.r.t. the quasiball λM for all $\lambda \in (0, 1)$.

Through $U_M(A)$ we denote the *M-tube* around a set $A \subset E$:

$$U_M(A) = \{x \in E \mid 0 < \varrho_M(x, A) < 1\}. \tag{14}$$

A set $M \subset E$ is called *boundedly uniformly convex* if M is convex and for any bounded sequences $\{z_k\} \subset M$ and $\{z'_k\} \subset M$ the statement $\lim_{k\to\infty} \varrho\left(\frac{z_k + z'_k}{2}, \partial M\right)$ $= 0$ implies that $\lim_{k\to\infty} \|z_k - z'_k\| = 0$.

A set $M \subset E$ is said to be *parabolic* if it is closed convex and for every $b \in E$ the set $M \setminus (2M - b)$ is bounded. The parabolic sets were introduced in [15]. Obviously, any bounded set is parabolic. The term "parabolic" is due to the observation that the epigraph of the parabola $y = x^2$ is parabolic while the epigraph of the hyperbola $y = \frac{1}{x}$, $x > 0$ is not parabolic.

A functional $p \in E^*$ is called the *Fréchet derivative* of a function $f : E \to \mathbb{R}$ at $x_0 \in E$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} = 0. \tag{15}$$

The *Fréchet subdifferential* of a function $f : E \to \mathbb{R}$ at $x_0 \in E$ is defined by

$$\partial_F f(x_0) = \left\{ p \in E^* \;\middle|\; \liminf_{x \to x_0} \frac{f(x) - f(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \right\}. \tag{16}$$

Theorem 2.5. *Let E be a Banach space, $M \subset E$ be a boundedly uniformly convex and parabolic quasiball, $A \subset E$ be a closed set, $U_M(A) \neq \emptyset$. Then the assertions (i)–(iii) are equivalent:*

(*i*) *A is weakly convex w.r.t. M;*

(*ii*) *for any $x_0 \in U_M(A)$ the problem (8) is well posed;*

(*iii*) *the M-projection mapping $x \mapsto P_M(x, A)$ is single-valued and continuous on $U_M(A)$.*

If additionally the Minkowski functional of M is Fréchet differentiable on $E \setminus \{0\}$, then each statement (i)–(iii) is equivalent to each statement (iv)–(v):

(*iv*) *$A + \operatorname{int} M \neq E$ and the function $\varrho_M(\cdot, A)$ is Fréchet differentiable on $U_M(A)$;*

(*v*) *$A + \operatorname{int} M \neq E$ and the Fréchet subdifferential of the function $\varrho_M(\cdot, A)$ is not empty at any point of $U_M(A)$.*

If additionally the Minkowski functional of M is continuously differentiable on $E \setminus \{0\}$, then each assertion (i)–(v) is equivalent to

(*vi*) *$A + \operatorname{int} M \neq E$ and the function $\varrho_M(\cdot, A)$ is continuously differentiable on $U_M(A)$.*

The lemmas of Sections 5–7 show that some implications of Theorem 2.5 remain valid under weaker assumptions than the bounded uniform convexity of M.

3. Weakly convex functions

Given a function $\gamma : E \to \mathbb{R} \cup \{+\infty\}$ and a number $r > 0$ we consider the function

$$\gamma_r(x) = r \cdot \gamma\left(\frac{x}{r}\right), \quad \forall x \in E. \tag{17}$$

Note that $\operatorname{epi} \gamma_r = r \cdot \operatorname{epi} \gamma$.

The *γ-predifferential* of a function $f : E \to \mathbb{R} \cup \{+\infty\}$ at a point $x \in \operatorname{dom} f$ is defined by

$$\pi_\gamma f(x) = \{u \in \operatorname{dom} \gamma \mid \exists r > 0 : \ (f \boxplus \gamma_r)(x + ru) = f(x) + \gamma_r(ru)\}. \tag{18}$$

The following two lemmas show the relations between the γ-predifferential, the Fréchet subdifferential and the proximal subdifferential for functions defined on a normed space.

Lemma 3.1. *Assume that for functions $f, \gamma : E \to \mathbb{R} \cup \{+\infty\}$ we have $x \in \operatorname{dom} f$, $u \in \pi_\gamma f(x)$ and $\gamma'(u)$ is a Fréchet derivative of γ at u. Then $\gamma'(u) \in \partial_F f(x)$.*

Lemma 3.2. *Let $\gamma : E \to \mathbb{R}$ be defined by $\gamma(x) = \sigma\|x\|^2$, $x \in E$ for some $\sigma > 0$. Assume a function $f : E \to \mathbb{R} \cup \{+\infty\}$ to be such that* $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$ *and $x_0 \in \operatorname{dom} f$. Then*

$$\partial^P f(x_0) = 2\sigma\pi_\gamma f(x_0).$$

A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be *weakly convex* with respect to $\gamma : E \to \mathbb{R} \cup \{+\infty\}$ if

$$(f \boxplus \gamma)(x + u) = f(x) + \gamma(u), \quad \forall x \in \operatorname{dom} f, \ \forall u \in \pi_\gamma f(x). \tag{19}$$

The following two theorems show relations between the class of weakly convex functions and other functional classes.

Theorem 3.3. *Let $\gamma : E \to \mathbb{R}$ be defined by $\gamma(x) = \sigma\|x\|^2$, $x \in E$ for some $\sigma > 0$. Assume that a function $f : E \to \mathbb{R} \cup \{+\infty\}$ is weakly convex w.r.t. the function γ and* $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$. *Then f is uniformly J-*plr *(see* (5)*); moreover,*

$$f(x + v) \geq f(x) + \langle p, v\rangle,$$
$$\forall x \in \operatorname{dom} f, \ \forall u \in \partial^P f(x), \ \forall t \geq 2\sigma, \ \forall v \in E, \ \forall p \in J(u - tv).$$

Theorem 3.4. *Let $E = H$ be a Hilbert space. Assume that a function $\gamma : H \to \mathbb{R}$ be defined by $\gamma(x) = \sigma\|x\|^2$ for some $\sigma > 0$. Assume that a function $f : H \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and* $\operatorname{dom} f \neq \emptyset$. *Then the following statements are equivalent:*

(*i*) $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$ *and the function f is weakly convex w.r.t. γ;*
(*ii*) $f(x) - f(x_0) \geq \langle u, x - x_0\rangle - \sigma\|x - x_0\|^2$, $\forall x_0 \in \operatorname{dom} f$, $\forall x \in H$, $\forall u \in \partial^P f(x_0)$;
(*iii*) *the function $x \mapsto f(x) + \gamma(x)$ is convex.*

Theorem 3.4 implies that in a Hilbert space the weak convexity w.r.t. the function $\gamma(x) = \sigma\|x\|^2$ ($\sigma > 0$) is equivalent to weak convexity by the terminology of Vial [23] and lower-C^2 property due to Rockafellar [22].

Theorem 3.5. *Let $\gamma : E \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function, $\gamma(0) < 0$, and $0 \in \operatorname{int} \operatorname{dom} \gamma$. Then for any function $f : E \to \mathbb{R} \cup \{+\infty\}$ the following statements are equivalent:*

(*i*) *the function f is weakly convex w.r.t. the function γ;*
(*ii*) *the set* $\operatorname{epi} f$ *is weakly convex w.r.t. the quasiball* $\operatorname{epi} \gamma$.

A function $\gamma : E \to \mathbb{R}$ is called *uniformly convex* on a convex set $X \subset E$ if for any sequences $\{x_k\} \subset X$ and $\{x'_k\} \subset X$ the statement $\limsup_{k\to\infty} \big(\gamma(x_k) + \gamma(x'_k) - 2\gamma\big(\frac{x_k + x'_k}{2}\big)\big) \leq 0$ implies that $\lim_{k\to\infty} \|x_k - x'_k\| = 0$.

A function $\gamma : E \to \mathbb{R} \cup \{+\infty\}$ is *coercive* if $\lim_{\|x\|\to\infty} \frac{\gamma(x)}{\|x\|} = +\infty$.

Theorem 3.6. *Let E be a Banach space and let $\gamma : E \to \mathbb{R}$ be a coercive function, bounded on any bounded set, and uniformly convex on any convex bounded set. Suppose that a function $f : E \to \mathbb{R}\cup\{+\infty\}$ is lower semicontinuous and weakly convex w.r.t. γ, $\mathrm{dom}(f \boxplus \gamma) \neq \emptyset$, $r \in (0, 1)$. Then*

(*i*) *function $f \boxplus \gamma_r$ is real-valued and continuous on E;*
(*ii*) *the infimal convolution problem*

$$\min_{u\in E} (f(u) + \gamma_r(x_0 - u)) \tag{20}$$

is well posed for any $x_0 \in E$;

(*iii*) *for any $x_0 \in E$ the problem (20) has a unique solution $u_{\min}(x_0)$ and the function $u_{\min} : E \to E$ is continuous.*

The following theorem shows that the assumption about the weak convexity of f is necessary in Theorem 3.6.

Theorem 3.7. *Let E be a Banach space and let $\gamma : E \to \mathbb{R}$ be a coercive function, bounded on any bounded set, and uniformly convex on any convex bounded set. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function,* $\mathrm{dom}(f \boxplus \gamma) \neq \emptyset$. *Suppose that for any $r \in (0, 1)$ and $x_0 \in E$ the problem* (20) *is well posed. Then f is weakly convex w.r.t. γ.*

4. Some families of sets

A convex set $M \subset E$ is said to be

- *locally uniformly convex* if for any point $z_0 \in \overline{M}$ and any sequence $\{z_k\} \subset M$ the statement $\lim_{k\to\infty} \varrho\left(\frac{z_0+z_k}{2}, \partial M\right) = 0$ implies that $\lim_{k\to\infty} \|z_0 - z_k\| = 0$.
- *fully 2-convex* if for any bounded sequence $\{z_k\} \subset M$ the statement $\lim_{n,k\to\infty} \varrho\left(\frac{z_n+z_k}{2}, \partial M\right) = 0$ implies that $\{z_k\}$ is a Cauchy sequence.
- *Kadec* if for any bounded sequence $\{z_k\} \subset \partial M$ weak convergence $z_k \to z_0 \in \partial M$ implies strong convergence of $\{z_k\}$.

We shall use the following notations:

Notation	The family of
$\mathcal{LUC}$	locally uniformly convex sets;
$\mathcal{F}2\mathcal{C}$	fully 2-convex sets;
$\mathcal{K}$	Kadec sets;
$\mathcal{SC}$	strictly convex sets;
$\mathcal{BUC}$	boundedly uniformly convex sets;
$\mathcal{P}$	parabolic sets;
$\mathcal{Q}$	quasiballs;
$\mathcal{WCS}(M)$	weakly convex sets w.r.t. the quasiball M;
$\mathcal{WCF}(\gamma)$	weakly convex functions w.r.t. the function γ.

Recall that the definitions of boundedly uniformly convex sets, parabolic sets, quasiballs, and weakly convex sets are given in Section 2. The weakly convex functions are defined in Section 3.

If the unit ball $\mathfrak{B}_1(0)$ of a normed vector space E is strictly convex, locally uniformly convex, fully 2-convex or boundedly uniformly convex, then E and its norm are said to be strictly convex, locally uniformly convex, fully 2-convex or uniformly convex correspondingly.

Uniformly convex spaces were introduced by Clarkson [8]. Locally uniformly convex spaces were first considered by Lovaglia [18]. Fully 2-convex (and fully k-convex) spaces were introduced and studied by Fan and Glicksberg [10].

One can easily see that the families of sets mentioned above are related by the inclusions

$$\mathcal{BUC} \subset \mathcal{LUC} \cap \mathcal{F2C}, \qquad \mathcal{LUC} \cup \mathcal{F2C} \subset \mathcal{SC} \cap \mathcal{K}. \tag{21}$$

Lemma 5.1 in [15] implies the following proposition.

Proposition 4.1. *Let $M \subset E$ be a convex parabolic set, $0 < \lambda_1 < \lambda_2$, $x_1, x_2 \in E$. Then the set $(\lambda_1 M + x_1) \setminus (\lambda_2 \operatorname{int} M + x_2)$ is bounded.*

Lemma 4.2. *If the set $M \neq E$ is convex and parabolic, then the set $M \cap (-M)$ is bounded.*

Proof. Since $M \neq E$, there exists $b \in E \setminus \left(\frac{3}{2}M\right)$. The convexity of M implies that $\frac{3}{2}M = M + \frac{1}{2}M$. Consequently, $b - \frac{1}{2}M \subset E \setminus M$. Therefore,

$$b + \frac{1}{2}\left(M \cap (-M)\right) = \left(b + \frac{1}{2}M\right) \cap \left(b - \frac{1}{2}M\right) \subset \left(b + \frac{1}{2}M\right) \setminus M.$$

Using the parabolicity of M we deduce that the set $M \cap (-M)$ is bounded. □

Lemma 4.3. *Let $\{z_k\}$ be a sequence in the quasiball M. Then*

$$\lim_{k\to\infty} \varrho(z_k, \partial M) = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \mu_M(z_k) = 1.$$

If the sequence $\{z_k\}$ is bounded, then the inverse implication holds.

Proof. Assume that $\lim_{k\to\infty} \varrho(z_k, \partial M) = 0$. Then, there exists a sequence $\{z'_k\} \subset \partial M$ such that $\lim_{k\to\infty} \|z_k - z'_k\| = 0$. Using Remark 2.1, we have $\lim_{k\to\infty} \mu_M(z'_k - z_k) = 0$. The subadditivity of the Minkowski functional and the inclusions $z_k \in M$, $z'_k \in \partial M$ imply that $1 - \mu_M(z'_k - z_k) = \mu_M(z'_k) - \mu_M(z'_k - z_k) \leq \mu_M(z_k) \leq 1$. Consequently, $\lim_{k\to\infty} \mu_M(z_k) = 1$.

Now assume that the sequence $\{z_k\}$ is bounded. Prove the inverse implication. Fix any $z \in M$ such that $\mu_M(z) > 0$. Since $\frac{z}{\mu_M(z)} \in \partial M$, it follows that

$$\varrho(z, \partial M) \leq \left\| z - \frac{z}{\mu_M(z)} \right\| = \frac{\|z\|}{\mu_M(z)}(1 - \mu_M(z)). \tag{22}$$

Suppose that $\lim_{k\to\infty}\mu_M(z_k)=1$. Using (22), we get $\lim_{k\to\infty}\varrho(z_k,\partial M)=0$. $\square$

The next lemma follows directly from the definitions and Lemma 4.3.

Lemma 4.4. *For any $M\in\mathcal{Q}$ the following assertions are equivalent:*

(i) *$M\in\mathcal{F}2\mathcal{C}$;*

(ii) *for any bounded sequence $\{z_k\}\subset M$ the equation $\lim_{n,k\to\infty}\mu_M(z_n+z_k)=2$ implies that $\{z_k\}$ is a Cauchy sequence.*

Lemma 4.5. *If in a Banach space E there exists $M\in\mathcal{Q}\cap\mathcal{P}\cap\mathcal{F}2\mathcal{C}$, then*

(i) *E admits an equivalent fully 2-convex norm;*

(ii) *E is reflexive.*

Proof. Assume that $M\in\mathcal{Q}\cap\mathcal{P}\cap\mathcal{F}2\mathcal{C}$. We consider the sublinear function $f(z)=\mu_M(z)+\mu_M(-z)$ and the closed set $B=\{z\in E\mid f(z)\le 1\}$. Since $M\cap(-M)\subset 2B$ and $0\in\operatorname{int}M$, it follows that $0\in\operatorname{int}B$. As $B\subset M\cap(-M)$, by Lemma 4.2 the set B is bounded. Consequently, $f(z)=\mu_B(z)$ is a norm, equivalent to the original one.

We claim that $B\in\mathcal{F}2\mathcal{C}$. Indeed, let us consider a sequence $\{z_k\}\subset B$ such that $\lim_{n,k\to\infty}\mu_B(z_n+z_k)=2$, that is

$$\lim_{n,k\to\infty}\left(\mu_M(z_n+z_k)+\mu_M(-z_n-z_k)\right)=2. \tag{23}$$

Taking the proper subsequence, we assume that $\lim_{k\to\infty}\mu_M(z_k)=\mu^+\ge 0$ and $\lim_{k\to\infty}\mu_M(-z_k)=\mu^-\ge 0$. Since $z_k\in B$, we have $\mu_M(z_k)+\mu_M(-z_k)=f(z_k)\le 1$. Thus,

$$\mu^+ + \mu^- \le 1. \tag{24}$$

The subadditivity of the Minkowski functional and the relation (23) imply that $\liminf_{n,k\to\infty}(\mu_M(z_n)+\mu_M(z_k)+\mu_M(-z_n)+\mu_M(-z_k))\ge 2$. Using (24), we get $\mu^+ + \mu^- = 1$. For the sake of certainty we suppose that $\mu^+>0$ (otherwise $\mu^->0$ and we change the notations $\mu^+\leftrightarrow\mu^-$, $z_k\leftrightarrow -z_k$). By the subadditivity of the Minkowski functional we have $\limsup_{n,k\to\infty}\mu_M(-z_n-z_k)\le 2\mu^-$, $\limsup_{n,k\to\infty}\mu_M(z_n+z_k)\le 2\mu^+$. Hence, using (23), we deduce that

$$2\mu^+ = 2-2\mu^- \le 2-\limsup_{n,k\to\infty}\mu_M(-z_n-z_k)\le \limsup_{n,k\to\infty}\mu_M(z_n+z_k)\le 2\mu^+.$$

Thus,

$$\lim_{n,k\to\infty}\mu_M(z_n+z_k)=2\mu^+. \tag{25}$$

Since the sequence $\{z_k\}$ is bounded and $\lim_{k\to\infty}\mu_M(z_k)=\mu^+>0$, (25) implies that for the sequence of $w_k=\frac{z_k}{\mu_M(z_k)}$ we get $\lim_{n,k\to\infty}\mu_M(w_n+w_k)=2$. Recalling that $M\in\mathcal{Q}\cap\mathcal{F}2\mathcal{C}$, by Lemma 4.4 we deduce that $\{w_k\}$ is a Cauchy sequence. Therefore, $\{z_k\}$ is a Cauchy sequence too. Using once more Lemma 4.4, we get $B\in\mathcal{F}2\mathcal{C}$. The assertion (i) is proved. Combining (i) and Theorem 3 in [10], we deduce the assertion (ii). $\square$

Lemma 4.6. *Let $M \in \mathcal{Q}$, $z, w \in M$ and $\lambda \in (0,1)$. Then*

$$2 - \mu_M(w+z) \le \frac{1 - \mu_M(\lambda w + (1-\lambda)z)}{\min\{\lambda, 1-\lambda\}}.$$

Proof. Consider the case $\lambda \in \left(0, \frac{1}{2}\right]$. Since $\lambda w + (1-\lambda)z = \lambda(w+z) + (1-2\lambda)z$, it follows that $\mu_M(\lambda w + (1-\lambda)z) \le \lambda\mu_M(w+z) + (1-2\lambda)\mu_M(z) \le \lambda\mu_M(w+z) + 1 - 2\lambda$. Consequently, in this case the proof is completed. The case $\lambda \in \left(\frac{1}{2}, 1\right)$ can be reduced to the previous one by replacing $\lambda \leftrightarrow 1-\lambda$, $w \leftrightarrow z$. □

Lemma 4.7. *(i) Assume that $M \in \mathcal{Q} \cap \mathcal{F}2\mathcal{C}$, a bounded sequence $\{z_k\} \subset M$, and a sequence $\{\lambda_k\} \subset \mathbb{R}$ are such that*

$$\liminf_{n,k\to\infty} \mu_M(\lambda_n z_n + (1-\lambda_k)z_k) \ge 1, \qquad \lim_{k\to\infty} \lambda_k = \lambda_0 \in (0,1).$$

Then $\{z_k\}$ is a Cauchy sequence.

(ii) Assume that $M \in \mathcal{Q} \cap \mathcal{LUC}$, $z_0 \in M$, a bounded sequence $\{z_k\} \subset M$, and a sequence $\{\lambda_k\} \subset \mathbb{R}$ are such that

$$\liminf_{k\to\infty} \mu_M(\lambda_0 z_0 + (1-\lambda_k)z_k) \ge 1, \qquad \lim_{k\to\infty} \lambda_k = \lambda_0 \in (0,1).$$

Then $\lim_{k\to\infty} \|z_k - z_0\| = 0$.

Proof. (i). The boundedness of $\{z_k\}$ and Remark 2.1 imply that

$$\liminf_{n,k\to\infty} \mu_M(\lambda_0 z_n + (1-\lambda_0)z_k) = \liminf_{n,k\to\infty} \mu_M(\lambda_n z_n + (1-\lambda_k)z_k) \ge 1.$$

Hence, using Lemma 4.6, we get $\liminf_{n,k\to\infty} \mu_M\left(\frac{z_n+z_k}{2}\right) \ge 1$. Since $\mu_M\left(\frac{z_n+z_k}{2}\right) \le \frac{\mu_M(z_n)+\mu_M(z_k)}{2} \le 1$, it follows that $\lim_{n,k\to\infty} \mu_M\left(\frac{z_n+z_k}{2}\right) = 1$. Using Lemma 4.4, we deduce the assertion (i). A similar reasoning and Lemma 4.3 imply the assertion (ii). □

5. The weak convexity implies well posedness

Lemma 5.1. *Let $M \in \mathcal{Q}$, $A \subset E$.*

(i) *If $x \in E$, $\varrho_M(x,A) > 0$, then $x \notin A + \varrho_M(x,A)\operatorname{int} M$.*

(ii) *If $x \in E$, $\varrho_M(x,A) > 0$, $A \in \mathcal{WCS}(M)$, $a \in P_M(x,A)$, $z = \frac{x-a}{\varrho_M(x,A)}$, then $a + z \notin A + \operatorname{int} M$.*

Proof. In order to deduce (i), assume the contrary: there exists $a \in (x - \varrho_M(x,A)\operatorname{int} M) \cap A$. Hence, $\frac{x-a}{\varrho_M(x,A)} \in \operatorname{int} M$. Therefore, one can find $t \in (0, \varrho_M(x,A))$ such that $\frac{x-a}{t} \in M$. Consequently, $x \in A + tM$ and $\varrho_M(x,A) \le t < \varrho_M(x,A)$. The contradiction completes the proof of (i). Now we suppose that the assumptions of (ii) hold. According to (12), we have $z \in N_M^1(a,A)$. Hence, (13) implies that $a \in P_M(a+z,A)$. Using (11), we get $\varrho_M(a+z,A) \ge \mu_M(z) = 1$. Thus, (i) implies (ii). □

For any set $A \subset E$ and any quasiball $M \subset E$ we denote

$$S_M(A) = \left\{ x \in E \mid 0 < \varrho_M(x, A) < \sup_{x \in E} \varrho_M(x, A) \right\}. \tag{26}$$

We use $T_M(A)$ to denote the set of all $x_0 \in E$ such that the problem (8) is well posed.

Theorem 2.3 in [15] implies the following proposition.

Proposition 5.2. *Let E be a reflexive Banach space, $M \in \mathcal{Q} \cap \mathcal{P} \cap \mathcal{SC} \cap \mathcal{K}$. Let $A \subset E$ be a closed set. Then the set $T_M(A) \cap S_M(A)$ is a dense subset of $S_M(A)$.*

Lemma 5.3. *Let $M \in \mathcal{Q} \cap \mathcal{LUC}$, $A \in \mathcal{WCS}(M)$, $x_0 \in E$, $\varrho_M(x_0, A) \in (0, 1)$, $P_M(x_0, A) \neq \emptyset$. Then $x_0 \in T_M(A)$.*

Proof. Pick $a_0 \in P_M(x_0, A)$. Let $\{a_k\}$ be a minimizing sequence of the problem (8). It suffices to show that $\lim_{k\to\infty} \|a_k - a_0\| = 0$. Note that $\mu_M(x_0 - a_k) \geq \varrho_M(x_0, A) > 0$ for all $k \in \mathbb{N} \cup \{0\}$. For every $k \in \mathbb{N} \cup \{0\}$ we denote

$$\varrho_k = \mu_M(x_0 - a_k), \qquad z_k = \frac{x_0 - a_k}{\varrho_k}.$$

By Lemma 5.1 we have $a_0 + z_0 \notin A + \operatorname{int} M$. Consequently, $(1 - \varrho_0)z_0 + \varrho_k z_k = a_0 + z_0 - a_k \notin \operatorname{int} M$ and $\mu_M\left((1 - \varrho_0)z_0 + \varrho_k z_k\right) \geq 1$ for all $k \in \mathbb{N}$. Taking into account that $z_k \in M \setminus \left(\frac{1}{\varrho_k} \operatorname{int} M - \frac{1-\varrho_0}{\varrho_k} z_0\right)$ for all $k \in \mathbb{N}$ and $\varrho_k \to \varrho_0 < 1$ as $k \to \infty$, by Proposition 4.1 we deduce that the sequence $\{z_k\}$ is bounded. Using Lemma 4.7(ii), we get $\lim_{k\to\infty} \|z_k - z_0\| = 0$. Therefore, $\lim_{k\to\infty} \|a_k - a_0\| = 0$. □

Lemma 5.4. *Let E be a reflexive Banach space, $M \in \mathcal{Q} \cap \mathcal{P} \cap \mathcal{SC} \cap \mathcal{K}$. Assume that for a closed set $A \in \mathcal{WCS}(M)$ there exists $x_0 \in E$ such that $\varrho_M(x_0, A) > 0$. Then $A + \operatorname{int} M \neq E$.*

Proof. If $\varrho_M(x_0, A) \geq 1$, it suffices to use Lemma 5.1(i). Therefore, we assume that $\varrho_M(x_0, A) \in (0, 1)$. Fix $a_0 \in A$. Using continuity of the function $\varrho_M(\cdot, A)$, we pick $x_1 \in [x_0, a_0]$ such that $0 = \varrho_M(a_0, A) < \varrho_M(x_1, A) < \varrho_M(x_0, A)$. Therefore, $x_1 \in S_M(A)$. According to Proposition 5.2, there exists $x \in T_M(A) \setminus A$ close to x_1. Taking into account Remark 2.4, one can find $a \in P_M(x, A)$. Denoting $z = \frac{a-x}{\varrho_M(a,A)}$, by Lemma 5.1$(ii)$ we have $a + z \notin A + \operatorname{int} M$. □

Lemma 5.5. *Let E be a Banach space, $M \in \mathcal{Q} \cap \mathcal{P} \cap \mathcal{F}2\mathcal{C}$, $A \in \mathcal{WCS}(M)$ be a closed set, and $x_0 \in U_M(A)$. Then $P_M(x_0, A) \neq \emptyset$.*

Proof. According to (21), we have $M \in \mathcal{SC} \cap \mathcal{K}$. By Lemma 4.5 the space E is reflexive. Lemma 5.4 and (14), (26) imply that $x_0 \in S_M(A)$. It follows from Proposition 5.2 that there exists a sequence $\{x_k\} \subset T_M(A) \cap S_M(A)$ such that $\lim_{k\to\infty}\|x_k - x_0\| = 0$. According to Remark 2.4, for every $k \in \mathbb{N}$ one can find $a_k \in P_M(x_k, A)$. For any $k \in \mathbb{N} \cup \{0\}$ we denote $\varrho_k = \varrho_M(x_k, A)$. Since $x_k \in S_M(A)$, it follows that $\varrho_k > 0$ for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$ put $z_k = \frac{x_k - a_k}{\varrho_k}$. Lemma 5.1(*ii*) implies that

$$a_k + z_k \notin A + \operatorname{int} M, \quad \forall k \in \mathbb{N}. \tag{27}$$

Fix a number $\varrho' \in (\varrho_0, 1)$. Since $\lim_{k\to\infty}\mu_M(x_0 - a_k) = \lim_{k\to\infty}\mu_M(x_k - a_k) = \lim_{k\to\infty}\varrho_k = \varrho_0 < \varrho'$, there exists $k_1 \in \mathbb{N}$ such that $\mu_M(x_0 - a_k) < \varrho'$ for all $k \geq k_1$. Consequently, by (27) we get $a_k \in (x_0 - \varrho' M) \setminus (a_1 + z_1 - \operatorname{int} M)$ for all $k \geq k_1$. By Proposition 4.1, we deduce that the sequence $\{a_k\}$ is bounded. Using (27), we have $\mu_M(a_k + z_k - a_n) \geq 1$ for all $n, k \in \mathbb{N}$. Bearing in mind that $a_k + z_k = x_k + (1-\varrho_k)z_k$ and $a_n = x_n - \varrho_n z_n$, we get $\mu_M(x_k + (1-\varrho_k)z_k - x_n + \varrho_n z_n) \geq 1$ for all $n, k \in \mathbb{N}$. Hence, subadditivity of the Minkowski functional implies that $\mu_M((1-\varrho_k)z_k + \varrho_n z_n) + \mu_M(x_k - x_n) \geq 1$ for all $n, k \in \mathbb{N}$. Using the convergence of the sequence $\{x_k\}$ and Remark 2.1 we deduce that $\lim_{n,k\to\infty}\mu_M(x_k - x_n) = 0$. Therefore, $\liminf_{n,k\to\infty}\mu_M((1-\varrho_k)z_k + \varrho_n z_n) \geq 1$. Lemma 4.7(*i*) implies that $\{z_k\}$ is a Cauchy sequence and, consequently, converges. Therefore, the sequence $\{a_k\}$ converges to some $a_0 \in E$. Since A is closed, it follows that $a_0 \in A$. Passing to the limit in $\varrho_M(x_k, A) = \mu_M(x_k - a_k)$, we deduce that $\varrho_M(x_0, A) = \mu_M(x_0 - a_0)$. Consequently, $a_0 \in P_M(x_0, A)$. □

Lemma 5.6. *Let E be a Banach space, $M \in \mathcal{Q} \cap \mathcal{P} \cap \mathcal{F}2\mathcal{C}$. Let $A \in \mathcal{WCS}(M)$ be a closed set. Then for any $r \in (0,1)$ the set $A + rM$ is closed.*

Proof. Suppose the sequence $\{x_k\} \subset A + rM$ converges to $x_0 \in E$. It suffices to prove that $x_0 \in A + rM$. Since $\varrho_M(x_k, A) \leq r$ for all $k \in \mathbb{N}$, it follows that $\varrho_M(x_0, A) \leq r$. If $\varrho_M(x_0, A) < r$, then, according to Remark 2.2, we get $x_0 \in A + rM$. Now assume that $\varrho_M(x_0, A) = r$. Lemma 5.5 implies that there exists $a_0 \in P_M(x_0, A)$. Therefore, $\mu_M(x_0 - a_0) = \varrho_M(x_0, A) = r$ and, consequently, we have $x_0 \in a_0 + rM \subset A + rM$. □

6. Continuity of the M-projection implies the weak convexity

Lemma 6.1. *Let $M \in \mathcal{Q}$, $A \subset E$. Then*

(*i*) $\varrho_M(x_1, A) - \varrho_M(x_2, A) \leq \mu_M(x_1 - x_2)$, $\forall x_1, x_2 \in E$;
(*ii*) *the function $\varrho_M(\cdot, A)$ is Lipschitz continuous on E with the constant $\frac{1}{\varrho(0,\partial M)}$.*

Proof. Using (9) and the subadditivity of the Minkowski functional we deduce (*i*). Remark 2.1 and the assertion (*i*) imply the assertion (*ii*). □

Lemma 6.2. *Let $M \in \mathcal{Q}$, $A \subset E$, $x_0 \in E \setminus A$, $a_0 \in P_M(x_0, A)$, $x \in [a_0, x_0]$. Then $a_0 \in P_M(x, A)$.*

Proof. Since $a_0 \in P_M(x_0, A)$, it follows that $\mu_M(x_0 - a_0) = \varrho_M(x_0, A)$. By Lemma 6.1(*i*) we have $\varrho_M(x_0, A) \leq \varrho_M(x, A) + \mu_M(x_0 - x)$. Bearing in mind that $a_0 \in A$, we deduce that $\varrho_M(x, A) \le \mu_M(x - a_0)$. Using $x \in [a_0, x_0]$, we get $\mu_M(x - a_0) + \mu_M(x_0 - x) = \mu_M(x_0 - a_0)$. Thus,

$$\begin{aligned}\mu_M(x_0 - a_0) = \varrho_M(x_0, A) &\le \varrho_M(x, A) + \mu_M(x_0 - x)\\ &\le \mu_M(x - a_0) + \mu_M(x_0 - x) = \mu_M(x_0 - a_0).\end{aligned}$$

Consequently, every inequality in this chain is an equality. Therefore, $\varrho_M(x, A) = \mu_M(x - a_0)$. Hence, by (11) we have $a_0 \in P_M(x, A)$. □

Lemma 6.3. *Let $M \in \mathcal{Q}$, $A \subset E$, $x_0 \in E \setminus A$, $\delta > 0$, $v \in E$, $\tau_1 > 0$. Assume that*

$$\mu_M(x_0 - \pi(x_0) - \tau_1 v) < \varrho_M(x_0, A),$$

the M-projection $P_M(x, A)$ is the singleton $\{\pi(x)\}$ for every $x \in \mathfrak{B}_\delta(x_0)$, and the function $\pi : \mathfrak{B}_\delta(x_0) \to A$ is continuous at x_0 in the direction v, i.e., $\lim_{t\to+0} \pi(x_0 + tv) = \pi(x_0)$. Then there exists $\tau_0 > 0$ such that for any $t \in (0, \tau_0)$ we have

$$\varrho_M(x_0 + tv, A) > \varrho_M(x_0, A). \tag{28}$$

Proof. Put $\varrho_0 = \varrho_M(x_0, A)$. The assumptions imply that

$$\lim_{t\to+0} \mu_M(x_0 - \pi(x_0 + tv) - \tau_1 v) = \mu_M(x_0 - \pi(x_0) - \tau_1 v) < \varrho_M(x_0, A) = \varrho_0.$$

Consequently, there exists $\tau_0 > 0$ such that

$$\mu_M(x_0 - \pi(x_0 + tv) - \tau_1 v) < \varrho_0, \quad \forall t \in (0, \tau_0). \tag{29}$$

Let us fix any $t \in (0, \tau_0)$ and prove (28). We denote $\pi_t = \pi(x_0+tv)$. Since $\pi_t \in A$, it follows that $\varrho_0 \le \mu_M(x_0 - \pi_t)$. Hence, using (29) and the subadditivity of the Minkowski functional, we get

$$\begin{aligned}\varrho_0 \le \mu_M(x_0 - \pi_t) &\le \frac{\tau_1}{\tau_1 + t}\mu_M(x_0 - \pi_t + tv) + \frac{t}{\tau_1 + t}\mu_M(x_0 - \pi_t - \tau_1 v)\\ &< \frac{\tau_1}{\tau_1 + t}\mu_M(x_0 - \pi_t + tv) + \frac{t}{\tau_1 + t}\varrho_0.\end{aligned}$$

Consequently, $\varrho_M(x_0 + tv, A) = \mu_M(x_0 - \pi_t + tv) > \varrho_0 = \varrho_M(x_0, A)$. □

For a functional $p \in E^*$ we use $s(p, M)$ to denote the value of the *support function* of the set $M \subset E$:

$$s(p, M) = \sup_{x\in M} \langle p, x\rangle. \tag{30}$$

Lemma 6.4. *Let $M \in \mathcal{Q} \cap \mathcal{SC}$, $A \subset E$, $x_0 \in E \setminus A$, $\delta > 0$. Assume that for any $x \in \mathfrak{B}_\delta(x_0)$ the M-projection $P_M(x, A)$ is the singleton $\{\pi(x)\}$ and the function $\pi : \mathfrak{B}_\delta(x_0) \to A$ is directionally continuous at x_0 (i.e., $\pi(\cdot)$ is continuous at x_0 in any direction $v \in E$). Suppose that*

$$D = \{x \in E \mid \varrho_M(x, A) \ge \varrho_M(x_0, A)\},$$

$y_0 \in E$, $0 < \varrho_{-M}(y_0, D) < \varrho_M(x_0, A)$, $x_0 \in P_{-M}(y_0, D)$. Then $y_0 \in [\pi(x_0), x_0]$.

Proof. Denote

$$\varrho_0 = \varrho_M(x_0, A), \qquad \varrho_1 = \varrho_{-M}(y_0, D), \qquad a_0 = \pi(x_0),$$
$$w_0 = \frac{x_0 - a_0}{\varrho_0}, \qquad w_1 = \frac{x_0 - y_0}{\varrho_1}.$$

Hence,

$$w_0 \in \partial M, \qquad w_1 \in \partial M. \tag{31}$$

Assume that

$$w_0 \neq w_1. \tag{32}$$

If $(w_0 - \operatorname{int} M) \cap (\operatorname{int} M - w_1) = \emptyset$, then by the separation theorem there exists a functional $p \in E^* \setminus \{0\}$ such that

$$\langle p, x - w_1\rangle \leq \langle p, w_0 - x'\rangle, \quad \forall x, x' \in M.$$

Using (31), we get $\langle p, w_1\rangle = s(p, M) = \langle p, w_0\rangle$. This contradicts (32), bearing in mind that M is strictly convex. Thus, $(w_0 - \operatorname{int} M) \cap (\operatorname{int} M - w_1) \neq \emptyset$ and, consequently, there exists

$$v \in (w_0 - \operatorname{int} M) \cap (\operatorname{int} M - w_1). \tag{33}$$

Hence, $w_1 + v \in \operatorname{int} M$ and due to convexity of M, we get

$$w_1 + tv \in \operatorname{int} M, \quad \forall t \in (0, 1]. \tag{34}$$

According to (33), we have $w_0 - v \in \operatorname{int} M$, that is $\mu_M(w_0 - v) < 1$ and, therefore, $\mu_M(x_0 - a_0 - \varrho_0 v) < \varrho_0$. By Lemma 6.3 there exists $\tau_0 > 0$ such that

$$\varrho_M(x_0 + tv, A) > \varrho_M(x_0, A), \quad \forall t \in (0, \tau_0).$$

Consequently,

$$x_0 + tv \in D, \quad \forall t \in (0, \tau_0). \tag{35}$$

Since $x_0 \in P_{-M}(y_0, D)$ and $\varrho_1 = \varrho_{-M}(y_0, D)$, Lemma 5.1($i$) implies that $(y_0 + \varrho_1 \operatorname{int} M) \cap D = \emptyset$. Thus, by (35) we have $x_0 + tv \notin y_0 + \varrho_1 \operatorname{int} M$ for all $t \in (0, \tau_0)$. Hence,

$$w_1 + \frac{t}{\varrho_1} v = \frac{x_0 - y_0}{\varrho_1} + \frac{t}{\varrho_1} v \notin \operatorname{int} M, \quad \forall t \in (0, \tau_0).$$

This contradicts (34). Therefore, the assumption (32) does not hold, that is $w_0 = w_1$. Thus, $\frac{x_0 - a_0}{\varrho_0} = \frac{x_0 - y_0}{\varrho_1}$, $0 < \varrho_1 < \varrho_0$. Consequently, $y_0 \in [a_0, x_0] = [\pi(x_0), x_0]$. □

Lemma 6.5. *Let E be a reflexive Banach space, $M \in \mathcal{Q} \cap \mathcal{P} \cap \mathcal{SC} \cap \mathcal{K}$. Let $A \subset E$ be a closed set. Suppose that the M-projection mapping $x \mapsto P_M(x, A)$ is single-valued and continuous on $U_M(A)$. Then $A \in \mathcal{WCS}(M)$.*

Proof. Let $a_0 \in A$ and $v \in N^1_M(a_0, A)$. For all $t \geq 0$ we put

$$x(t) = a_0 + tv. \tag{36}$$

According to (13), it suffices to prove that

$$a_0 \in P_M(x(1), A). \tag{37}$$

Denote

$$t_0 = \sup\{t > 0 \mid a_0 \in P_M(x(t), A)\}. \tag{38}$$

Using (12), we get $t_0 > 0$. By Lemma 6.2 we have

$$a_0 \in P_M(x(t), A), \quad \forall t \in (0, t_0).$$

According to (11), for any $t \in (0, t_0)$ we get $\varrho_M(x(t), A) = \mu_M(x(t) - a_0) = t$. Passing to the limit as $t \to t_0 - 0$, we deduce that

$$\varrho_M(x(t), A) = \mu_M(x(t) - a_0) = t, \quad \forall t \in (0, t_0]. \tag{39}$$

Hence, by (11) we have

$$a_0 \in P_M(x(t), A), \quad \forall t \in (0, t_0]. \tag{40}$$

If $t_0 \geq 1$, then (40) implies (37). Now assume that $t_0 \in (0, 1)$. According to (14), we get $x(t_0) \in U_M(A)$. Put $x_0 = x(t_0)$. Note that $\mu_M(x_0 - a_0 - t_0 v) = 0 < t_0 = \varrho_M(x_0, A)$. Lemma 6.3 implies that there exists $\tau_0 > 0$ such that

$$\varrho_M(x(t_0 + t), A) = \varrho_M(x_0 + tv, A) > \varrho_M(x_0, A) = t_0, \quad \forall t \in (0, \tau_0).$$

Consequently, one can find $\tau \in (t_0, 2t_0)$ such that $t_1 := \varrho_M(x(\tau), A) \in (t_0, 1)$. Consider the nonempty closed set

$$D = \{x \in E \mid \varrho_M(x, A) \geq t_1\}. \tag{41}$$

According to (39) and Lemma 6.1(i), we get for all $d \in D$

$$t_1 \leq \varrho_M(d, A) \leq \varrho_M(x(t_0), A) + \mu_M(d - x(t_0)) = t_0 + \mu_M(d - x(t_0)).$$

Therefore,

$$\varrho_{-M}(x(t_0), D) = \inf_{d \in D} \mu_{-M}(x(t_0) - d) = \inf_{d \in D} \mu_M(d - x(t_0)) \geq t_1 - t_0 > 0.$$

Since $x(\tau) \in D$, it follows that

$$\varrho_{-M}(x(t_0), D) \leq \mu_M(x(\tau) - x(t_0)) = \mu_M((\tau - t_0)v) = \tau - t_0 < t_0 < t_1. \tag{42}$$

Consequently,

$$0 < \varrho_{-M}(x(t_0), D) < t_1 \leq \varrho_{-M}(a_0, D).$$

Using (26), we deduce that

$$x(t_0) \in S_{-M}(D).$$

According to Proposition 5.2, the set $T_{-M}(D) \cap S_{-M}(D)$ is dense in $S_{-M}(D)$. Hence, there exists a sequence $\{z_k\} \subset T_{-M}(D) \cap S_{-M}(D)$ such that

$$\lim_{k\to\infty} \|z_k - x(t_0)\| = 0. \tag{43}$$

Bearing in mind Remark 2.4, for every $k \in \mathbb{N}$ one can find $d_k \in P_{-M}(z_k, D)$. Thus, $d_k \in \partial D$ and, consequently, $\varrho_M(d_k, A) = t_1 \in (0, 1)$ and $d_k \in U_M(A)$. Therefore, for any $k \in \mathbb{N}$ there exists $a_k \in P_M(d_k, A)$. Since $D = \{x \in E \mid \varrho_M(x, A) \geq \varrho_M(d_k, A)\}$, by Lemma 6.4 it follows that

$$z_k \in [a_k, d_k], \quad \forall k \in \mathbb{N}. \tag{44}$$

Using Lemma 6.2, we get

$$a_k \in P_M(z_k, A), \quad \forall k \in \mathbb{N}.$$

According to (40), we have $a_0 \in P_M(x(t_0), A)$. Bearing in mind (43), single-valuedness and continuity of the M-projection on $U_M(A)$, we deduce that

$$\lim_{k\to\infty} \|a_k - a_0\| = 0. \tag{45}$$

Using (44), we get

$$\begin{aligned} d_k &= a_k + \frac{\mu_M(d_k - a_k)}{\mu_M(z_k - a_k)}(z_k - a_k) \\ &= a_k + \frac{\mu_M(d_k - a_k)}{\mu_M(d_k - a_k) - \mu_M(d_k - z_k)}(z_k - a_k), \quad \forall k \in \mathbb{N}. \end{aligned} \tag{46}$$

According to (43), (42), we have $\mu_M(d_k - z_k) = \mu_{-M}(z_k - d_k) = \varrho_{-M}(z_k, D) \to \varrho_{-M}(x(t_0), D) < t_1$ as $k \to \infty$. Thus, by (43), (45), (46) we get

$$d_k \to d_0 := a_0 + \frac{t_1}{t_1 - \varrho_{-M}(x(t_0), D)}(x(t_0) - a_0) \quad (k \to \infty). \tag{47}$$

Denoting $t_2 = \frac{t_1 t_0}{t_1 - \varrho_{-M}(x(t_0), D)}$ and using (36), we have $d_0 = a_0 + t_2 v$. Passing to the limit in $\mu_M(d_k - a_k) = \varrho_M(d_k, A) = t_1$ and bearing in mind (45), (47), we deduce that $t_2 = \mu_M(d_0 - a_0) = \varrho_M(d_0, A) = t_1$. Thus, $d_0 = x(t_1)$, $a_0 \in P_M(x(t_1), A)$, $t_1 > t_0$. This contradicts (38). $\square$

7. Differentiability of the M-distance function

Lemma 7.1. *Let $M \in \mathcal{Q}$, $A \in \mathcal{WCS}(M)$, $x_0 \in U_M(A)$, $a_0 \in P_M(x_0, A)$. Assume that $p \in E^*$ is the Fréchet derivative of the Minkowski functional $\mu_M(\cdot)$ at $(x_0 - a_0)$. Then p is the Fréchet derivative of the function $\varrho_M(\cdot, A)$ at x_0.*

Proof. Denote $\varrho_0 = \varrho_M(x_0, A)$. Since $a_0 \in P_M(x_0, A)$, it follows that $\varrho_0 = \mu_M(x_0 - a_0)$. Denoting $y = a_0 + \frac{x_0 - a_0}{\varrho_0}$ and using Lemma 5.1(*ii*), we get $y \notin A + \operatorname{int} M$ and, therefore, $\varrho_M(y, A) \geq 1$. By Lemma 6.1(*i*) we have

$$1 \leq \varrho_M(y, A) \leq \varrho_M(x, A) + \mu_M(y - x), \quad \forall x \in E. \tag{48}$$

Since $y - x_0 = \frac{1-\varrho_0}{\varrho_0}(x_0 - a_0)$, it follows that $\mu_M(y - x_0) = \frac{1-\varrho_0}{\varrho_0}\mu_M(x_0 - a_0) = 1 - \varrho_0$. Hence, using (48), we deduce that

$$\mu_M(y - x_0) - \mu_M(y - x) \leq \varrho_M(x, A) - \varrho_0 = \varrho_M(x, A) - \varrho_M(x_0, A), \quad \forall x \in E. \tag{49}$$

Bearing in mind that $y - x_0 = \frac{1-\varrho_0}{\varrho_0}(x_0 - a_0)$, the function $\mu_M(\cdot)$ is positive homogeneous and p is the Fréchet derivative of the function $\mu_M(\cdot)$ at $(x_0 - a_0)$, we deduce that p is the Fréchet derivative of the same function at $(y - x_0)$. According to (15), we have

$$\lim_{x \to x_0} \frac{\mu_M(x - a_0) - \mu_M(x_0 - a_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} = 0, \tag{50}$$

$$\lim_{x \to x_0} \frac{\mu_M(y - x) - \mu_M(y - x_0) + \langle p, x - x_0 \rangle}{\|x - x_0\|} = 0. \tag{51}$$

Using (49), (51), we get

$$\liminf_{x \to x_0} \frac{\varrho_M(x, A) - \varrho_M(x_0, A) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \geq 0. \tag{52}$$

Since $a_0 \in A$, it follows that $\varrho_M(x, A) \leq \mu_M(x - a_0)$ for all $x \in E$. Consequently,

$$\varrho_M(x, A) - \varrho_M(x_0, A) \leq \mu_M(x - a_0) - \mu_M(x_0 - a_0), \quad \forall x \in E.$$

Using (50), we deduce that

$$\limsup_{x \to x_0} \frac{\varrho_M(x, A) - \varrho_M(x_0, A) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \leq 0.$$

Recalling (52), we complete the proof. $\square$

Lemma 7.2. *Let E be a reflexive Banach space, $M \in \mathcal{Q} \cap \mathcal{SC} \cap \mathcal{K}$. Assume that a functional $p \in E^* \setminus \{0\}$ and a bounded sequence $\{z_k\} \subset \partial M$ are such that $\lim_{k\to\infty}\langle p, z_k \rangle = s(p, M)$ (see (30)). Then $\{z_k\}$ converges.*

Proof. First we assume that $\{z_k\}$ weakly converges to some $z_0 \in E$. Since M is convex closed, it follows that $z_0 \in M$. Thus, $\langle p, z_0\rangle = \lim_{k\to\infty}\langle p, z_k\rangle = s(p, M)$. Consequently, $z_0 \in \partial M$. Using the Kadec property, we deduce that $\{z_k\}$ converges to z_0. Thus, the proof is completed provided that $\{z_k\}$ weakly converges.

Now assume that $\{z_k\}$ does not weakly converge. Using reflexiveness of E, one can find two subsequences of $\{z_k\}$, which converge to different points z_0 and z_0'. As showed above, we have $z_0, z_0' \in \partial M$, $\langle p, z_0\rangle = \langle p, z_0'\rangle = s(p, M)$. This contradicts the strict convexity of M. □

Lemma 7.3. *Let E be a reflexive Banach space, $M \in \mathcal{Q} \cap \mathcal{P} \cap \mathcal{SC} \cap \mathcal{K}$. Let $A \subset E$ be a closed set. Assume that $x_0 \in S_M(A)$ (see* (26)*) and $p \in \partial_F \varrho_M(x_0, A)$, where $\partial_F \varrho_M(x_0, A)$ is the Fréchet subdifferential of the function $\varrho_M(\cdot, A)$ at x_0 (see* (16)*). Then $x_0 \in T_M(A)$.*

Proof. Let $\{a_k\}$ be a minimizing sequence of the problem (8). Denote

$$\varrho_0 = \varrho_M(x_0, A), \qquad \varrho_k = \mu_M(x_0 - a_k). \tag{53}$$

Note that

$$\lim_{k\to\infty} \varrho_k = \varrho_0 > 0. \tag{54}$$

Since $x_0 \in S_M(A)$, it follows that $\varrho_M(x_0, A) > 0$ and one can find $\hat{\varrho} \in \mathbb{R}$ and $\hat{x} \in E$ such that $\varrho_M(x_0, A) < \hat{\varrho} < \varrho_M(\hat{x}, A)$. By (54) there exists $k_0 \in \mathbb{N}$ such that $\varrho_k < \hat{\varrho}$ for all $k \geq k_0$. Consequently, $a_k \in x_0 - \hat{\varrho} M$ for all $k \geq k_0$. Using $a_k \in A$, by Lemma 5.1(i) we have $a_k \notin \hat{x} - \varrho_M(\hat{x}, A)\,\mathrm{int}\, M$. According to Proposition 4.1 and using $\hat{\varrho} < \varrho_M(\hat{x}, A)$, we deduce that the sequence $\{a_k\}$ is bounded.

Without loss of generality assume that $\varrho_k > 0$ for all $k \in \mathbb{N}$. Thus, $\inf_{k\in\mathbb{N}} \varrho_k > 0$. Consequently, $\inf_{k\in\mathbb{N}} \|a_k - x_0\| > 0$. Bearing in mind that the sequence of

$$t_k = \frac{2^{-k} + \sqrt{\varrho_k - \varrho_0}}{\|a_k - x_0\|}$$

is infinitesimal, we assume that $t_k < 1$ for all $k \in \mathbb{N}$. Due to the boundedness of $\{a_k\}$, we have that the sequence

$$x_k = x_0 + t_k(a_k - x_0)$$

converges to x_0. Since $p \in \partial_F \varrho_M(x_0, A)$, it follows that

$$\liminf_{k\to\infty} \frac{\varrho_M(x_k, A) - \varrho_M(x_0, A) - \langle p, x_k - x_0\rangle}{\|x_k - x_0\|} \geq 0.$$

Bearing in mind that $\varrho_M(x_k, A) \leq \mu_M(x_k - a_k) = (1 - t_k)\mu_M(x_0 - a_k) = (1 - t_k)\varrho_k$, we get

$$\liminf_{k\to\infty} \frac{(1 - t_k)\varrho_k - \varrho_0 - t_k\langle p, a_k - x_0\rangle}{t_k\|a_k - x_0\|} \geq 0.$$

Taking into account that $\frac{\varrho_k-\varrho_0}{t_k\|a_k-x_0\|} = \frac{\varrho_k-\varrho_0}{2^{-k}+\sqrt{\varrho_k-\varrho_0}} \to 0$ as $k\to\infty$, we deduce that

$$\liminf_{k\to\infty} \frac{\langle p, x_0-a_k\rangle - \varrho_0}{\|a_k-x_0\|} \geq 0.$$

Using the boundedness of $\{a_k\}$, we have

$$\liminf_{k\to\infty} \langle p, x_0-a_k\rangle \geq \varrho_0. \tag{55}$$

Thus, $p\neq 0$. For all $k\in\mathbb{N}$ we denote

$$z_k = \frac{x_0-a_k}{\varrho_k}. \tag{56}$$

By (54), (55) we get

$$\liminf_{k\to\infty} \langle p, z_k\rangle \geq 1. \tag{57}$$

Since $p\in\partial_F\varrho_M(x_0,A)$, it follows that $\liminf_{t\to+0}\frac{\varrho_M(x_0+tx,A)-\varrho_M(x_0,A)-t\langle p,x\rangle}{t}\geq 0$ for all $x\in E$. On the other hand, Lemma 6.1(i) implies that $\varrho_M(x_0+tx,A)-\varrho_M(x_0,A)\leq \mu_M(tx)=t\mu_M(x)$ for all $x\in E$, $t>0$. Hence,

$$\langle p, x\rangle \leq \mu_M(x), \quad \forall x\in E$$

and, consequently,

$$s(p,M)\leq 1. \tag{58}$$

Using (53), (56), we get $z_k\in\partial M$. Taking into account (57), (58), we have

$$s(p,M)\leq 1\leq \liminf_{k\to\infty}\langle p,z_k\rangle \leq \limsup_{k\to\infty}\langle p,z_k\rangle \leq s(p,M).$$

Thus,

$$\lim_{k\to\infty}\langle p,z_k\rangle = s(p,M) = 1. \tag{59}$$

The boundedness of $\{a_k\}$ together with (54), (56) imply that $\{z_k\}$ is bounded. According to Lemma 7.2, we deduce that $\{z_k\}$ converges. Using (56), we come to convergence of the minimizing sequence $\{a_k\}$. Consequently, $x_0\in T_M(A)$. □

8. Proof of Theorem 2.5

Suppose that the assumptions of Theorem 2.5 hold. Using (21), we have $M\in\mathcal{Q}\cap\mathcal{P}\cap\mathcal{LUC}\cap\mathcal{F}2\mathcal{C}$. Lemma 4.5 implies that E is reflexive.

(i) $\Rightarrow$ (ii). Suppose that the assertion (i) of Theorem 2.5 holds, i.e. $A\in\mathcal{WCS}(M)$. Fix any $x_0\in U_M(A)$. By Lemma 5.5 we have $P_M(x_0,A)\neq\emptyset$. Hence, according to Lemma 5.3 we get $x_0\in T_M(A)$. Thus, (ii) holds.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. According to Remark 2.4, the M-projection of any $x\in U_M(A)$ onto A is a singleton: $P_M(x,A)=\{\pi(x)\}$. It suffices to prove that the function $\pi(\cdot)$ is continuous on $U_M(A)$. Indeed, suppose that a sequence $\{x_k\} \subset U_M(A)$ converges to $x_0 \in U_M(A)$. Lemma 6.1(ii) implies that $\lim_{k\to\infty} \varrho_M(x_k, A) = \varrho_M(x_0, A)$. Consequently, $\lim_{k\to\infty} \mu_M(x_k - \pi(x_k)) = \varrho_M(x_0, A)$. So, $\{\pi(x_k)\}$ is a minimizing sequence of the problem (8). Since the problem (8) is well posed, it follows that $\{\pi(x_k)\}$ converges to some $\pi_0 \in E$. Thus, $\pi_0 \in P_M(x_0, A)$ and, therefore, $\pi_0 = \pi(x_0)$, $\lim_{k\to\infty} \|\pi(x_k) - \pi(x_0)\| = 0$.

(iii) $\Rightarrow$ (i) follows from Lemma 6.5.

Now we suppose that the Minkowski functional of M is Fréchet differentiable on $E \setminus \{0\}$.

(i) $\Rightarrow$ (iv). Suppose that (i) holds. Lemma 5.4 and the assumption $U_M(A) \neq \emptyset$ imply that $A + \operatorname{int} M \neq E$. Hence, using ($iii$) and Lemma 7.1, we get (iv).

(iv) $\Rightarrow$ (v) holds trivially.

(v) $\Rightarrow$ (ii). Suppose that (v) holds. Fix any $x_0 \in U_M(A)$. Since $A + \operatorname{int} M \neq E$, it follows that $\sup_{x\in E} \varrho_M(x, A) \geq 1 > \varrho_M(x_0, A)$. Hence, $x_0 \in S_M(A)$. By Lemma 7.3, we get $x_0 \in T_M(A)$. Thus, (ii) holds.

Now we suppose that the Minkowski functional of M is continuously differentiable on $E \setminus \{0\}$.

(i) $\Rightarrow$ (vi). Suppose that (i) holds. As shown above, we have $A + \operatorname{int} M \neq E$. Using Lemma 7.1 and the continuity of the M-projection (assertion (iii)), we deduce (vi).

(vi) $\Rightarrow$ (iv) holds trivially. □

9. Proofs of Lemmas 3.1, 3.2 and Theorems 3.3, 3.4

Proof of Lemma 3.1. Since $u \in \pi_\gamma f(x)$, by (18) there exists $r > 0$ such that $(f \boxplus \gamma_r)(x + ru) = f(x) + \gamma_r(ru)$. Hence, according to (6), (17) we have

$$f(x+v) - f(x) \geq \gamma_r(ru) - \gamma_r(ru - v) = r\left(\gamma(u) - \gamma\left(u - \frac{v}{r}\right)\right), \quad \forall v \in E.$$

Therefore, by (15) we deduce that

$$\begin{aligned} &\liminf_{v\to 0} \frac{f(x+v) - f(x) - \langle \gamma'(u), v\rangle}{\|v\|} \\ &\geq r \liminf_{v\to 0} \frac{\gamma(u) - \gamma\left(u - \frac{v}{r}\right) - \left\langle \gamma'(u), \frac{v}{r}\right\rangle}{\|v\|} = 0. \end{aligned}$$

Using (16), we get $\gamma'(u) \in \partial_F f(x)$. □

Lemma 9.1. *Let $\gamma : E \to \mathbb{R}$ be a convex continuous coercive function. Assume a function $f : E \to \mathbb{R} \cup \{+\infty\}$ to be such that* $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$*,* $x_0 \in \operatorname{dom} f$*. Then*

$$\begin{aligned} \pi_\gamma f(x_0) = \{u \in E \mid \exists r > 0,\ \exists \delta > 0 :& \\ \forall v \in \mathfrak{B}_\delta(0)\ f(x_0) + \gamma_r(ru) &\leq f(x_0 + v) + \gamma_r(ru - v)\}. \end{aligned}$$

Proof. If $u \in \pi_\gamma f(x_0)$, then (6) and (18) imply that

$$\exists r > 0 : \ \forall v \in E \ \ f(x_0) + \gamma_r(ru) = (f \boxplus \gamma_r)(x_0 + ru) \le f(x_0 + v) + \gamma_r(ru - v).$$

Now we suppose that the equality to be proved is wrong. Then there exist $u \in E$, $r > 0$, and $\delta \in (0, 1)$ such that $u \notin \pi_\gamma f(x_0)$ and

$$f(x_0) + \gamma_r(ru) \le f(x_0 + v) + \gamma_r(ru - v), \quad \forall v \in \mathfrak{B}_\delta(0). \tag{60}$$

Convexity of γ and (17) imply that

$$\gamma_r(ru) - \gamma_r(ru - v) \ge \gamma_\varepsilon(\varepsilon u) - \gamma_\varepsilon(\varepsilon u - v), \quad \forall \varepsilon \in (0, r), \quad \forall v \in E.$$

Hence, using (60), we have

$$f(x_0) + \gamma_\varepsilon(\varepsilon u) \le f(x_0 + v) + \gamma_\varepsilon(\varepsilon u - v), \quad \forall v \in \mathfrak{B}_\delta(0), \ \forall \varepsilon \in (0, r). \tag{61}$$

Let $\{\varepsilon_k\}$ be an infinitesimal sequence of positive numbers, $\varepsilon_k < r$ for all $k \in \mathbb{N}$. Since $u \notin \pi_\gamma f(x_0)$, using (6), (18), we get

$$(f \boxplus \gamma_{\varepsilon_k})(x_0 + \varepsilon_k u) < f(x_0) + \gamma_{\varepsilon_k}(\varepsilon_k u), \quad \forall k \in \mathbb{N}.$$

Therefore, by (6) for any $k \in \mathbb{N}$ one can find $v_k \in E$ such that

$$f(x_0 + v_k) + \gamma_{\varepsilon_k}(\varepsilon_k u - v_k) < f(x_0) + \gamma_{\varepsilon_k}(\varepsilon_k u). \tag{62}$$

By comparing (61) with (62) we deduce that $\|v_k\| > \delta$ for all $k \in \mathbb{N}$.

Fix any $x_1 \in \operatorname{dom}(f \boxplus \gamma)$ and put $y_1 = (f \boxplus \gamma)(x_1)$. Hence, due to (6) we have $y_1 \le f(x_0 + v_k) + \gamma(x_1 - x_0 - v_k)$ for all $k \in \mathbb{N}$. Thus, (62) implies that

$$y_1 - \gamma(x_1 - x_0 - v_k) + \gamma_{\varepsilon_k}(\varepsilon_k u - v_k) < f(x_0) + \gamma_{\varepsilon_k}(\varepsilon_k u), \quad \forall k \in \mathbb{N}.$$

Denote $x_2 = x_1 - x_0$, $y_2 = f(x_0) - y_1$. Then, using (17), we get

$$\varepsilon_k \gamma\left(u - \frac{v_k}{\varepsilon_k}\right) - \gamma(x_2 - v_k) < y_2 + \varepsilon_k \gamma(u), \quad \forall k \in \mathbb{N}. \tag{63}$$

Since function γ is continuous at the point x_2, there exists $\sigma \in \left(0, \frac{\delta}{2}\right)$ such that

$$\gamma(x) < \gamma(x_2) + 1, \quad \forall x \in \mathfrak{B}_{3\sigma}(x_2). \tag{64}$$

For every $k \in \mathbb{N}$ denote

$$\lambda_k = \left(1 - \frac{\sigma}{\|v_k\|}\right) \varepsilon_k. \tag{65}$$

Since $\lim_{k\to\infty} \varepsilon_k = 0$ and $\|v_k\| > \delta > 0$, it follows that $\lim_{k\to\infty} \lambda_k = 0$. Therefore, one can find $k_0 \in \mathbb{N}$ such that

$$2\lambda_k(2\sigma + \|u\| + \|x_2\|) < \sigma, \quad \forall k \ge k_0. \tag{66}$$

Consequently, $\lambda_k \in \left(0, \frac{1}{2}\right)$ for all $k \geq k_0$. For every $k \in \mathbb{N}$ we consider

$$z_k = x_2 - \lambda_k u - \frac{\sigma}{\|v_k\|} v_k.$$

Using (66), we have

$$\begin{aligned}\left\| \frac{z_k}{1-\lambda_k} - x_2 \right\| &\leq \frac{\lambda_k}{1-\lambda_k} \|z_k\| + \|z_k - x_2\| \\ &\leq 2\lambda_k(\|x_2\| + 2\sigma) + 2\sigma \leq 3\sigma, \quad \forall k \geq k_0.\end{aligned}$$

Therefore, (64) implies that

$$\gamma\left(\frac{z_k}{1-\lambda_k}\right) < \gamma(x_2) + 1, \quad \forall k \geq k_0. \tag{67}$$

Since

$$x_2 - v_k = \lambda_k \left(u - \frac{v_k}{\varepsilon_k}\right) + (1-\lambda_k)\frac{z_k}{1-\lambda_k},$$

thanks to convexity of γ, it follows that

$$\gamma(x_2 - v_k) \leq \lambda_k \gamma\left(u - \frac{v_k}{\varepsilon_k}\right) + (1-\lambda_k)\gamma\left(\frac{z_k}{1-\lambda_k}\right), \quad \forall k \geq k_0.$$

Combining this inequality with (63) and (67), we get

$$\varepsilon_k \gamma\left(u - \frac{v_k}{\varepsilon_k}\right) < \lambda_k \gamma\left(u - \frac{v_k}{\varepsilon_k}\right) + (1-\lambda_k)(\gamma(x_2)+1) + y_2 + \varepsilon_k \gamma(u), \quad \forall k \geq k_0.$$

Consequently, using (65), we have

$$\sup_{k \in \mathbb{N}} \frac{\varepsilon_k}{\|v_k\|} \gamma\left(u - \frac{v_k}{\varepsilon_k}\right) < +\infty. \tag{68}$$

Since $\lim_{k\to\infty} \frac{\|v_k\|}{\varepsilon_k} = +\infty$, the inequality (68) contradicts the coercivity of γ. $\square$

Lemma 9.2. *Let $\gamma : E \to \mathbb{R}$ be defined by $\gamma(x) = \sigma\|x\|^2$ for some $\sigma > 0$. Assume a function $f : E \to \mathbb{R} \cup \{+\infty\}$ to be such that* $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$, *$x_0 \in \operatorname{dom} f$, and $u \in E$. Then the following statements are equivalent:*

(i) $u \in 2\sigma\pi_\gamma f(x_0)$;

(ii) there exists $\alpha > 0$ such that for all $v \in E$ we have

$$f(x_0 + 2\alpha v) - f(x_0) \geq \alpha(\|u\|^2 - \|u - v\|^2); \tag{69}$$

(iii) there exist $\alpha > 0$ and $\delta > 0$ such that for all $v \in \mathfrak{B}_\delta(0)$ the inequality (69) *is valid.*

Proof. Using (17), we have

$$\gamma_r(x) = \frac{\sigma}{r}\|x\|^2, \quad \forall r > 0,\ \forall x \in E. \tag{70}$$

$(i) \Rightarrow (ii)$. Suppose that (i) holds. Denote $u_0 = \frac{1}{2\sigma}u$, $\alpha = \frac{r}{4\sigma}$. According to (18), there exists $r > 0$ such that

$$(f \boxplus \gamma_r)(x_0 + ru_0) = f(x_0) + \gamma_r(ru_0).$$

Hence, using (6) and (70), we get

$$\begin{aligned} f(x_0 + 2\alpha v) - f(x_0) &\geq \gamma_r(ru_0) - \gamma_r(ru_0 - 2\alpha v) \\ &= \alpha(\|u\|^2 - \|u - v\|^2), \quad \forall v \in E. \end{aligned}$$

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$. Suppose that (iii) holds. Denote $u_0 = \frac{1}{2\sigma}u$, $r = 4\sigma\alpha$. Then

$$f(x_0 + 2\alpha v) - f(x_0) \geq \gamma_r(ru_0) - \gamma_r(ru_0 - 2\alpha v), \quad \forall v \in \mathfrak{B}_\delta(0).$$

Lemma 9.1 implies that $u_0 \in \pi_\gamma f(x_0)$. $\square$

Proof of Lemma 3.2. Fix any $u \in 2\sigma\pi_\gamma f(x_0)$. According to Lemma 9.2, there exists $\alpha > 0$ such that

$$f(x_0 + 2\alpha v) - f(x_0) \geq \alpha(\|u\|^2 - \|u - v\|^2), \quad \forall v \in E.$$

Consequently,

$$4\alpha(y - f(x_0)) \geq 4\alpha^2\|u\|^2 - \|2\alpha u + x_0 - x\|^2, \quad \forall x \in \operatorname{dom} f,\ \forall y \geq f(x).$$

Thus,

$$\begin{aligned} &(y - f(x_0) + 2\alpha)^2 + \|2\alpha u + x_0 - x\|^2 \\ &\geq 4\alpha^2(1 + \|u\|^2), \quad \forall x \in \operatorname{dom} f,\ \forall y \geq f(x). \end{aligned} \tag{71}$$

Denote

$$\overline{x}_0 = (x_0, f(x_0)), \qquad \overline{u} = (u, -1). \tag{72}$$

By (4) and (71) we have

$$\|\overline{x} - \overline{x}_0 - 2\alpha\overline{u}\| \geq 2\alpha\sqrt{1 + \|u\|^2} = 2\alpha\|\overline{u}\|, \quad \forall \overline{x} \in \operatorname{epi} f.$$

Hence,

$$\overline{x}_0 \in P\left(\overline{x}_0 + 2\alpha\overline{u}, \operatorname{epi} f\right).$$

Therefore, by (2) we get $\overline{u} \in N^P(\overline{x}_0, \operatorname{epi} f)$, that is $(u,-1) \in N^P((x_0, f(x_0)), \operatorname{epi} f)$. Using (3), we have $u \in \partial^P f(x_0)$. So, we have proved the inclusion $2\sigma\pi_\gamma f(x_0) \subset \partial^P f(x_0)$. Now it suffices to prove the inverse one:

$$\partial^P f(x_0) \subset 2\sigma\pi_\gamma f(x_0). \tag{73}$$

Take any $u \in \partial^P f(x_0)$ and denote $\overline{x}_0$, $\overline{u}$ as in (72). According to (2) and (3), there exists $\beta > 0$ such that

$$\overline{x}_0 \in P\left(\overline{x}_0 + \beta\overline{u}, \operatorname{epi} f\right).$$

Hence,

$$(y - f(x_0) + \beta)^2 + \|x_0 + \beta u - x\|^2 \geq \beta^2(1 + \|u\|^2), \quad \forall x \in \operatorname{dom} f,\ \forall y \geq f(x). \tag{74}$$

Denote

$$\eta = \sqrt{1 + \|u\|^2} - \|u\|.$$

Then

$$\|x_0 + \beta u - x\| < \beta\sqrt{1 + \|u\|^2}, \quad \forall x \in \operatorname{int} \mathfrak{B}_{\eta\beta}(x_0).$$

Using (74), we have

$$f(x) - f(x_0) + \beta \geq \sqrt{\beta^2(1 + \|u\|^2) - \|x_0 + \beta u - x\|^2} > 0, \quad \forall x \in \operatorname{int} \mathfrak{B}_{\eta\beta}(x_0).$$

Denote

$$\alpha = \frac{\beta}{4(1 + \|u\|^2)}, \qquad \varepsilon = \frac{\eta\beta}{2\alpha}.$$

Thus,

$$\begin{aligned} &f(x_0 + 2\alpha v) - f(x_0) \\ &\geq \sqrt{\beta^2(1 + \|u\|^2) - \|\beta u - 2\alpha v\|^2} - \beta, \quad \forall v \in \operatorname{int} \mathfrak{B}_\varepsilon(0). \end{aligned} \tag{75}$$

Consider the function $\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}$:

$$\varphi(t) = \alpha(\|u\| + t)^2 + \sqrt{\beta^2(1 + \|u\|^2) - (\beta\|u\| + 2\alpha t)^2}, \quad t \in (-\varepsilon, \varepsilon).$$

Since $\varphi'(0) = 0$ and $\varphi''(0) = \alpha > 0$ there exists $\delta \in (0, \varepsilon)$ such that

$$\varphi(t) \geq \varphi(0), \quad \forall t \in [-\delta, \delta]. \tag{76}$$

Fix any $v \in \mathfrak{B}_\delta(0)$ and denote $t = \|u - v\| - \|u\|$. Then $t \in [-\delta, \delta]$ and

$$\|\beta u - 2\alpha v\| \leq 2\alpha\|u - v\| + (\beta - 2\alpha)\|u\| = \beta\|u\| + 2\alpha t.$$

Therefore, using (76), we get

$$\begin{aligned} &\alpha\|u - v\|^2 + \sqrt{\beta^2(1 + \|u\|^2) - \|\beta u - 2\alpha v\|^2} \\ &\geq \varphi(t) \geq \varphi(0) = \alpha\|u\|^2 + \beta, \quad \forall v \in \mathfrak{B}_\delta(0). \end{aligned} \tag{77}$$

Combining (75) and (77), we get

$$f(x_0 + 2\alpha v) - f(x_0) \geq \alpha(\|u\|^2 - \|u - v\|^2), \quad \forall v \in \mathfrak{B}_\delta(0).$$

Hence, Lemma 9.2 implies that $u \in 2\sigma\pi_\gamma f(x_0)$. So, the inclusion (73) is proved. □

Proof of Theorem 3.3. Fix any $x \in \operatorname{dom} f$, $u \in \partial^P f(x)$, $t \geq 2\sigma$, $v \in E$, $p \in J(u - tv)$. Take $q \in J(u - 2\sigma v)$. Since the normalized duality mapping is the subdifferential of the convex function $\frac{1}{2}\|\cdot\|^2$, it follows that

$$\frac{\|u\|^2}{2} - \frac{\|u - 2\sigma v\|^2}{2} \geq \langle q, 2\sigma v\rangle.$$

Denoting $u_0 = \frac{u}{2\sigma}$, we get

$$\gamma(u_0) - \gamma(u_0 - v) = \sigma(\|u_0\|^2 - \|u_0 - v\|^2) = \frac{1}{4\sigma}(\|u\|^2 - \|u - 2\sigma v\|^2) \geq \langle q, v\rangle.$$

Monotonicity of the subdifferential and the inequality $t \geq 2\sigma$ imply that $\langle q - p, v\rangle \geq 0$. Thus,

$$\gamma(u_0) - \gamma(u_0 - v) \geq \langle p, v\rangle. \tag{78}$$

According to Lemma 3.2, we have $u_0 \in \pi_\gamma f(x)$. Hence, using (19), (78), we deduce that

$$f(x + v) - f(x) \geq \gamma(u_0) - \gamma(u_0 - v) \geq \langle p, v\rangle.$$ □

Proof of Theorem 3.4. Since

$$\begin{aligned}\langle 2\sigma u, x - x_0\rangle - \sigma\|x - x_0\|^2 &= \sigma\|u\|^2 - \sigma\|u + x_0 - x\|^2 \\ &= \gamma(u) - \gamma(u + x_0 - x), \quad \forall x, x_0, u \in H,\end{aligned}$$

the statement (ii) is equivalent to

$$f(x) - f(x_0) \geq \gamma(u) - \gamma(u + x_0 - x), \quad \forall x_0 \in \operatorname{dom} f,\ \forall x \in H,\ \forall u \in \frac{1}{2\sigma}\partial^P f(x_0).$$

Hence, using (6), we have

$$(ii) \quad \Leftrightarrow \quad \left((f \boxplus \gamma)(x_0 + u) = f(x_0) + \gamma(u), \quad \forall x_0 \in \operatorname{dom} f,\ \forall u \in \frac{1}{2\sigma}\partial^P f(x_0)\right). \tag{79}$$

Since f is lower semicontinuous, by Lau's theorem [17] there exists $x_0 \in \operatorname{dom} f$ such that $\partial^P f(x_0) \neq \emptyset$. Using (79), we deduce that the statement (ii) implies that $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$. Thus, according to (79) and Lemma 3.2, we get

$$(ii) \quad \Leftrightarrow \quad \begin{cases} \operatorname{dom}(f \boxplus \gamma) \neq \emptyset, \\ (f \boxplus \gamma)(x_0 + u) = f(x_0) + \gamma(u), \quad \forall x_0 \in \operatorname{dom} f,\ \forall u \in \pi_\gamma f(x_0). \end{cases}$$

So, by (19), we get $(i) \Leftrightarrow (ii)$. The equivalence $(ii) \Leftrightarrow (iii)$ is proved in Theorem 5.1 in [7]. □

10. Proofs of Theorems 3.5–3.7

Lemma 10.1. *Assume the function $\gamma : E \to \mathbb{R}$ to be uniformly convex on the ball $\mathfrak{B}_R(0)$ for all $R > 0$. Then the set* epi γ *is boundedly uniformly convex.*

Proof. Denote $M = \operatorname{epi}\gamma$. Consider bounded sequences $\{z_k\} \subset M$ and $\{z'_k\} \subset M$ such that $\lim_{k\to\infty} \varrho\left(\frac{z_k+z'_k}{2}, \partial M\right) = 0$. It suffices to prove that

$$\lim_{k\to\infty} \|z_k - z'_k\| = 0. \tag{80}$$

Pick a sequence $\{w_k\} \subset \partial M$ such that $\lim_{k\to\infty} \|z_k + z'_k - 2w_k\| = 0$. Thus, for any $k \in \mathbb{N}$ we have $z_k = (x_k, y_k)$, $z'_k = (x'_k, y'_k)$, $w_k = (u_k, \gamma(u_k))$, where $x_k, x'_k, u_k \in E$, $y_k \geq \gamma(x_k)$, $y'_k \geq \gamma(x'_k)$,

$$\lim_{k\to\infty} \|x_k + x'_k - 2u_k\| = 0, \tag{81}$$

$$\lim_{k\to\infty} |y_k + y'_k - 2\gamma(u_k)| = 0. \tag{82}$$

Since $\{z_k\}$ and $\{z'_k\}$ are bounded sequences, it follows that there exists $R > 0$ such that $\{x_k\} \subset \mathfrak{B}_R(0)$ and $\{x'_k\} \subset \mathfrak{B}_R(0)$. Using the convexity and the boundedness of the function γ on the ball $\mathfrak{B}_{3R}(0)$, we deduce that the function γ is Lipschitz continuous on the ball $\mathfrak{B}_{2R}(0)$. Taking into account (81), we get $\lim_{k\to\infty}\left(\gamma\left(\frac{x_k+x'_k}{2}\right) - \gamma(u_k)\right) = 0$. Hence, denoting $\varepsilon_k = y_k + y'_k - 2\gamma\left(\frac{x_k+x'_k}{2}\right)$ and using (82), we have $\lim_{k\to\infty} \varepsilon_k = 0$. Thus, taking into account convexity of γ, we get

$$2\gamma\left(\frac{x_k + x'_k}{2}\right) \leq \gamma(x_k) + \gamma(x'_k) \leq y_k + y'_k = 2\gamma\left(\frac{x_k + x'_k}{2}\right) + \varepsilon_k.$$

Consequently,

$$\lim_{k\to\infty} \left(\gamma(x_k) + \gamma(x'_k) - 2\gamma\left(\frac{x_k + x'_k}{2}\right)\right) = 0, \tag{83}$$

$$\lim_{k\to\infty} (\gamma(x_k) - y_k) = 0, \qquad \lim_{k\to\infty} (\gamma(x'_k) - y'_k) = 0. \tag{84}$$

Bearing in mind that the function γ is uniformly convex on $\mathfrak{B}_R(0)$, by (83) we get

$$\lim_{k\to\infty} \|x_k - x'_k\| = 0. \tag{85}$$

Hence, thanks to the Lipschitz continuity of γ on $\mathfrak{B}_R(0)$, we deduce that $\lim_{k\to\infty}(\gamma(x_k) - \gamma(x'_k)) = 0$. Therefore, by (84) we get $\lim_{k\to\infty}(y_k - y'_k) = 0$. This and (85) imply (80). □

Lemma 10.2. *Let $\gamma : E \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function, $\gamma(0) < 0$, and $0 \in \operatorname{int} \operatorname{dom} \gamma$. Then for a function $f : E \to \mathbb{R} \cup \{+\infty\}$, points $x_0 \in \operatorname{dom} f$, $u_0 \in \operatorname{dom} \gamma$ and a number $r > 0$ the following statements are equivalent:*

(*i*) $(f \boxplus \gamma_r)(x_0 + ru_0) = f(x_0) + r\gamma(u_0)$;

(*ii*) $(x_0, f(x_0)) \in P_{\operatorname{epi} \gamma}\left((x_0 + ru_0, f(x_0) + r\gamma(u_0)), \operatorname{epi} f\right)$.

Proof. Denote $M = \operatorname{epi} \gamma$, $A = \operatorname{epi} f$, $\overline{x}_0 = (x_0, f(x_0))$, $\overline{u}_0 = (u_0, \gamma(u_0))$. Note that

$$\overline{u}_0 \in \partial M, \qquad \overline{x}_0 \in \partial A. \tag{86}$$

The statement (*ii*) can be equivalently rewritten in the form $\overline{x}_0 \in P_M(\overline{x}_0 + r\overline{u}_0, A)$. Using (11) and (86), we deduce that (*ii*) is equivalent to $\varrho_M(\overline{x}_0 + r\overline{u}_0, A) = r$. According to (86) and Lemma 5.1(*ii*), we have

$$(ii) \quad \Leftrightarrow \quad \overline{x}_0 + r\overline{u}_0 \notin A + r \operatorname{int} M.$$

Hence,

$$(ii) \quad \Leftrightarrow \quad \left(\overline{x}_0 + r\overline{u}_0 - \overline{x} \notin r \operatorname{int} M, \quad \forall \overline{x} \in A\right).$$

Consequently,

$$\begin{aligned}(ii) \quad \Leftrightarrow \quad &\left((x_0 + ru_0 - x, f(x_0) + r\gamma(u_0) - y) \notin \operatorname{int} \operatorname{epi} \gamma_r,\right.\\ &\left.\forall x \in \operatorname{dom} f,\ \forall y \geq f(x)\right).\end{aligned}$$

Thus,

$$(ii) \quad \Leftrightarrow \quad \left(\gamma_r(x_0 + ru_0 - x) \geq f(x_0) + r\gamma(u_0) - f(x), \quad \forall x \in \operatorname{dom} f\right).$$

Using (6), we deduce that (*ii*) is equivalent to $(f \boxplus \gamma_r)(x_0 + ru_0) \geq f(x_0) + r\gamma(u_0)$. Inverse inequality is due to (6) and (17). $\square$

Proof of Theorem 3.5. Denote $M = \operatorname{epi} \gamma$, $A = \operatorname{epi} f$. Given $\overline{x} \in A$, $\overline{u} \in \partial M$, $x \in \operatorname{dom} f$, $u \in \operatorname{dom} \gamma$, and $r > 0$ we consider the statements

$$\begin{aligned}\mathfrak{S}[\overline{x}, \overline{u}, r]: &\quad \overline{x} \in P_M(\overline{x} + r\overline{u}, A);\\ \mathfrak{F}[x, u, r]: &\quad (f \boxplus \gamma_r)(x + ru) = f(x) + r\gamma(u).\end{aligned}$$

Lemma 10.2 implies that

$$\begin{aligned}&\mathfrak{F}[x, u, r]\\ \Leftrightarrow \quad &\mathfrak{S}\left[(x, f(x)), (u, \gamma(u)), r\right], \quad \forall x \in \operatorname{dom} f,\ \forall u \in \operatorname{dom} \gamma,\ \forall r > 0.\end{aligned} \tag{87}$$

According to (12), (18) for all $\overline{x} \in A$, $\overline{u} \in \partial M$, $x \in \operatorname{dom} f$, $u \in \operatorname{dom} \gamma$ we have

$$\begin{aligned}\overline{u} \in N_M^1(\overline{x}, A) \quad &\Leftrightarrow \quad (\exists r > 0 :\ \mathfrak{S}[\overline{x}, \overline{u}, r]),\\ u \in \pi_\gamma f(x) \quad &\Leftrightarrow \quad (\exists r > 0 :\ \mathfrak{F}[x, u, r]).\end{aligned}$$

Thus, (13) and (19) imply that

$$\begin{aligned} & A \in \mathcal{WCS}(M) \\ \Leftrightarrow \quad & ((\exists r > 0: \ \mathfrak{S}[\overline{x}, \overline{u}, r]) \Rightarrow \mathfrak{S}[\overline{x}, \overline{u}, 1], \quad \forall \overline{x} \in A, \ \forall \overline{u} \in \partial M), \end{aligned} \tag{88}$$

$$\begin{aligned} & f \in \mathcal{WCF}(\gamma) \\ \Leftrightarrow \quad & ((\exists r > 0: \ \mathfrak{F}[x, u, r]) \Rightarrow \mathfrak{F}[x, u, 1], \quad \forall x \in \operatorname{dom} f, \ \forall u \in \operatorname{dom} \gamma). \end{aligned} \tag{89}$$

If for some $\overline{x} \in A$, $\overline{u} \in \partial M$, $r > 0$ the statement $\mathfrak{S}[\overline{x}, \overline{u}, r]$ holds, then $\overline{x} = (x, f(x))$, $\overline{u} = (u, \gamma(u))$, where $x \in \operatorname{dom} f$, $u \in \operatorname{dom} \gamma$. Hence, (88) implies that the statement (ii) of Theorem 3.5 is equivalent to

$$\begin{aligned} & (\exists r > 0: \ \mathfrak{S}\left[(x, f(x)), (u, \gamma(u)), r\right]) \\ \Rightarrow\ & \mathfrak{S}\left[(x, f(x)), (u, \gamma(u)), 1\right], \quad \forall x \in \operatorname{dom} f, \ \forall u \in \operatorname{dom} \gamma. \end{aligned}$$

Using (87), we have

$$(ii) \quad \Leftrightarrow \quad ((\exists r > 0: \ \mathfrak{F}[x, u, r]) \Rightarrow \mathfrak{F}[x, u, 1], \quad \forall x \in \operatorname{dom} f, \ \forall u \in \operatorname{dom} \gamma).$$

This and (89) complete the proof. □

Proof of Theorem 3.6. Consider the function $\tilde{\gamma}(x) = \gamma(x) - \gamma(0) - 1$. Note that $\tilde{\gamma}(0) = -1$ and all the assumptions and the assertions of Theorem 3.6 preserve the same sense if we replace γ by $\tilde{\gamma}$. Hence, without loss of generality we assume that $\gamma(0) < 0$. Thus, $\operatorname{epi} \gamma$ is a quasiball. Denote $M = \operatorname{epi} \gamma$, $A = \operatorname{epi} f$. By Theorem 3.5, we have $A \in \mathcal{WCS}(M)$. Since the function f is lower semicontinuous, it follows that the set A is closed. According to Lemma 2.5 in [15], the coercivity of γ implies that $M \in \mathcal{P}$. By Lemma 10.1, we get $M \in \mathcal{BUC}$.

(i). Since $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$, it follows that $\operatorname{dom} f \neq \emptyset$ and there exist points $x_1 \in \operatorname{dom} f$, $x_2 \in \operatorname{dom}(f \boxplus \gamma)$. Due to (6), for any $x \in E$ we have

$$(f \boxplus \gamma_r)(x) \leq f(x_1) + \gamma_r(x - x_1),$$

$$(f \boxplus \gamma)(x_2) = (f \boxplus \gamma_r \boxplus \gamma_{1-r})(x_2) \leq (f \boxplus \gamma_r)(x) + \gamma_{1-r}(x_2 - x).$$

Consequently,

$$(f \boxplus \gamma_r)(x) \in \mathbb{R}, \quad \forall x \in E.$$

According to Lemma 5.6, the set $A + rM = \operatorname{epi} f + \operatorname{epi} \gamma_r$ is closed. Using (7), we deduce that the set $\operatorname{epi}(f \boxplus \gamma_r)$ is closed. Hence, the function $f \boxplus \gamma_r$ is lower semicontinuous. Let us prove that this function is upper semicontinuous. Fix any $x_0 \in E$, sequence $\{x_k\} \subset E$, which converges to x_0 and a number $y_0 > (f \boxplus \gamma_r)(x_0)$. According to (6), there exists $u_0 \in E$ such that $f(u_0) + \gamma_r(x_0 - u_0) < y_0$. Using continuity of γ_r, we have

$$\limsup_{k \to \infty} (f \boxplus \gamma_r)(x_k) \leq \limsup_{k \to \infty} (f(u_0) + \gamma_r(x_k - u_0)) = f(u_0) + \gamma_r(x_0 - u_0) < y_0.$$

Passing to the limit as $y_0 \to (f \boxplus \gamma_r)(x_0)$, we get

$$\limsup_{k\to\infty}(f \boxplus \gamma_r)(x_k) \le (f \boxplus \gamma_r)(x_0).$$

Consequently, the function $f \boxplus \gamma_r$ is upper semicontinuous. Thus, this function is continuous.

(ii). Fix any $x_0 \in E$ and a minimizing sequence $\{u_k\}$ of the problem (20). Thus,

$$\lim_{k\to\infty}(f(u_k) + \gamma_r(x_0 - u_k)) = (f \boxplus \gamma_r)(x_0). \tag{90}$$

Denote

$$\overline{x}_0 = (x_0, (f \boxplus \gamma_r)(x_0)), \qquad \overline{a}_k = (u_k, f(u_k)),$$
$$\overline{w}_k = (x_0 - u_k, \gamma_r(x_0 - u_k)), \quad k \in \mathbb{N}.$$

Since (90), it follows that $\lim_{k\to\infty} \|\overline{x}_0 - \overline{a}_k - \overline{w}_k\| = 0$. According to Remark 2.1, we get $\lim_{k\to\infty} \mu_M(\overline{x}_0 - \overline{a}_k - \overline{w}_k) = 0$. Using $\overline{w}_k \in \operatorname{epi}\gamma_r = rM$, we have $\mu_M(\overline{w}_k) \le r$. Consequently,

$$\limsup_{k\to\infty} \mu_M(\overline{x}_0 - \overline{a}_k) \le \limsup_{k\to\infty}(\mu_M(\overline{x}_0 - \overline{a}_k - \overline{w}_k) + \mu_M(\overline{w}_k)) \le r.$$

Since $\overline{x}_0 \in \partial \operatorname{epi}(f \boxplus \gamma_r) = \partial(A + rM)$, it follows that $\varrho_M(\overline{x}_0, A) = r$. Thus, $\{\overline{a}_k\} \subset A$ is a minimizing sequence of the problem

$$\min_{\overline{a}\in A} \mu_M(\overline{x}_0 - \overline{a}). \tag{91}$$

Theorem 2.5 implies that $\overline{x}_0 \in T_M(A)$. Hence, the sequence $\{\overline{a}_k\}$ converges. Therefore, the sequence $\{u_k\}$ also converges.

(iii). Fix any $x_0 \in E$. Let $\{u_k\}$ be a minimizing sequence of the problem (20), that is (90) holds. As shown above, $\{u_k\}$ converges to some $u_0 \in E$. Passing to limit in (90) and bearing in mind the lower semicontinuity of the functions f and γ_r, we deduce that $f(u_0) + \gamma_r(x_0 - u_0) \le (f \boxplus \gamma_r)(x_0)$. Thus, $u_0 = u_{\min}(x_0)$ is the minimizer of the problem (20). Suppose a sequence $\{x_k\} \subset E$ converges to x_0. For any $k \in \mathbb{N} \cup \{0\}$ we denote

$$u_k = u_{\min}(x_k), \qquad \overline{a}_k = (u_k, f(u_k)), \qquad \overline{x}_k = (x_k, (f \boxplus \gamma_r)(x_k)).$$

Lemma 10.2 implies that $\overline{a}_k \in P_M(\overline{x}_k, A)$ for all $k \in \mathbb{N} \cup \{0\}$. By above proved assertion (i) we have $\lim_{k\to\infty} \|\overline{x}_k - \overline{x}_0\| = 0$. Theorem 2.5 implies that $\lim_{k\to\infty} \|\overline{a}_k - \overline{a}_0\| = 0$. Consequently, $u_{\min}(x_0) = \lim_{k\to\infty} u_{\min}(x_k)$. $\square$

Proof of Theorem 3.7. As in the proof of Theorem 3.6 without loss of generality we assume that $\gamma(0) < 0$. Denote $M = \operatorname{epi}\gamma$, $A = \operatorname{epi} f$, $U_M(A) = \{\overline{x} \in E \times \mathbb{R} \mid 0 < \varrho_M(\overline{x}, A) < 1\}$. Fix a point $\overline{x}_0 \in U_M(A)$. Put $r = \varrho_M(\overline{x}_0, A)$. Hence,

$r \in (0,1)$. We represent the point $\overline{x}_0$ as $\overline{x}_0 = (x_0, y_0)$, where $x_0 \in E$, $y_0 \in \mathbb{R}$. Lemma 5.1(*ii*) implies that $\overline{x}_0 \notin A + r \operatorname{int} M$. Consequently, for any $u \in \operatorname{dom} f$ we have $(x_0 - u, y_0 - f(u)) \notin r \operatorname{int} M = \operatorname{int} \operatorname{epi} \gamma_r$. Thus, $\gamma_r(x_0 - u) \geq y_0 - f(u)$. Hence,

$$y_0 \leq \inf_{u \in \operatorname{dom} f} (f(u) + \gamma_r(x_0 - u)) = (f \boxplus \gamma_r)(x_0). \tag{92}$$

Let $\{\overline{a}_k\}$ be a minimizing sequence of the problem (91). For any $k \in \mathbb{N}$ we denote $r_k = \mu_M(\overline{x}_0 - \overline{a}_k)$. Thus,

$$\lim_{k \to \infty} r_k = r \tag{93}$$

and

$$\overline{x}_0 - \overline{a}_k \in r_k M, \quad \forall k \in \mathbb{N}. \tag{94}$$

Since $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$, it follows that $A + M \neq E \times \mathbb{R}$. Hence, there exists a point $\overline{b} \in (E \times \mathbb{R}) \setminus (A + M)$. Using (94), we have

$$\overline{a}_k \in (\overline{x}_0 - r_k M) \setminus (\overline{b} - M), \quad \forall k \in \mathbb{N}. \tag{95}$$

According to Lemma 2.5 in [15], the coercivity of γ implies that $M \in \mathcal{P}$. Using (93), (95), $r \in (0,1)$ and Proposition 4.1, we deduce that the sequence $\{\overline{a}_k\}$ is bounded. For any $k \in \mathbb{N}$ we represent $\overline{a}_k$ in the form $\overline{a}_k = (u_k, v_k)$, where $u_k \in E$, $v_k \in \mathbb{R}$. Since $\overline{a}_k \in A$, it follows that $v_k \geq f(u_k)$ for all $k \in \mathbb{N}$. Using (94), we get

$$\gamma_{r_k}(x_0 - u_k) \leq y_0 - v_k \leq y_0 - f(u_k), \quad \forall k \in \mathbb{N}. \tag{96}$$

Since the function γ is convex and bounded on any bounded set, it is Lipschitz continuous on any bounded set. Bearing in mind that the sequence $\{\overline{a}_k\}$ is bounded and, consequently, the sequence $\{u_k\}$ is bounded too, by (93) we have

$$\lim_{k \to \infty} \left(r_k \gamma \left(\frac{x_0 - u_k}{r_k} \right) - r \gamma \left(\frac{x_0 - u_k}{r} \right) \right) = 0.$$

Thus, according to (17),

$$\lim_{k \to \infty} (\gamma_{r_k}(x_0 - u_k) - \gamma_r(x_0 - u_k)) = 0. \tag{97}$$

Using (92), (96), we get

$$\limsup_{k \to \infty} (f(u_k) + \gamma_r(x_0 - u_k)) \leq y_0 \leq (f \boxplus \gamma_r)(x_0). \tag{98}$$

Therefore, $\{u_k\}$ is the minimizing sequence of the problem (20). According to the assumptions, the sequence $\{u_k\}$ converges to some $u_0 \in E$. Bearing in mind that f is lower semicontinuous and γ is continuous, by (98), we have

$$\begin{aligned} f(u_0) + \gamma_r(x_0 - u_0) &\leq \liminf_{k \to \infty} (f(u_k) + \gamma_r(x_0 - u_k)) \\ &\leq \limsup_{k \to \infty} (f(u_k) + \gamma_r(x_0 - u_k)) \\ &\leq y_0 \leq (f \boxplus \gamma_r)(x_0) \leq f(u_0) + \gamma_r(x_0 - u_0). \end{aligned}$$

Hence, (97) implies that

$$\lim_{k\to\infty}(y_0 - f(u_k)) = \gamma_r(x_0 - u_0) = \lim_{k\to\infty}\gamma_{r_k}(x_0 - u_k).$$

Using (96), we deduce that the sequence $\{v_k\}$ converges. So, the sequence $\{\overline{a}_k\} = \{(u_k, v_k)\}$ converges as well. Thus, $\overline{x}_0 \in T_M(A)$. Since $\overline{x}_0$ is an arbitrary point of $U_M(A)$, we get $U_M(A) \subset T_M(A)$. Lemma 10.1 implies that $M \in \mathcal{BUC}$. According to Theorem 2.5, we have $A \in \mathcal{WCS}(M)$. Hence, by Theorem 3.5, we get $f \in \mathcal{WCF}(\gamma)$. □

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