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On p-Convex, Proximally Smooth, Quasi-Convex, Strictly Quasi-Convex and Approximately Convex Sets

G. Chavent
Ceremade, Université Paris-Dauphine, 75775 Paris Cedex 16, France;
and: Inria-Rocquencourt, BP 105, 78153 Le Chesnay Cedex, France
guy.chavent@free.fr

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We compare four generalizations of convex sets which ensure good properties of the projection: p-convexity $\simeq$ ρ-prox regularity, proximal smoothness, quasi-convexity and strict quasi-convexity, and one generalization based on the distance function: approximate convexity. We prove that i) p-convex sets essentially coincide with quasi-convex sets, ii) strictly quasi-convex sets are a subclass of proximally smooth sets and iii) p-convex or quasi-convex sets are approximately convex, but the converse is false. The definition and main properties of these approaches are recalled without demonstration but with unified notations. We compare the Lipschitz properties of the projection on the different families of sets, and show that strict quasi-convexity ensures moreover the unimodality of the distance to a point over the set, and hence the computability of the projection by local optimization algorithms. Sufficient size$\times$curvature conditions for strict quasi-convexity are also recalled.

Keywords: Generalized set convexity, p-convexity, prox-regularity, approximate convexity, proximal smoothness, quasi-convexity, strict quasi-convexity, geodesics, single-valued projection, closest points, Hilbert space

1. Introduction

We compare in this note five generalizations of convex sets C of a Hilbert space H. The desirable properties we are interested in on some neighborhood ϑ of such a "generalized convex" D are:

$$\text{every } z \in \vartheta \text{ has at most one projection on } D, \tag{1}$$

$$\text{the projection on } D \text{ is locally Lipschitz-continuous on } \vartheta, \tag{2}$$

$$\forall z \in \vartheta, \text{ the "distance from } z \text{ to } D\text{" is unimodal on } D, \tag{3}$$

where unimodal refers to a function that has no stationary point other than its global minimum.

The last property is usually not included in the classical generalizations, but it is important for the numerical determination of the projection: the absence of

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parasitic stationary point of the "distance to z" function guarantees that any local minimization algorithm will converge to the global minimum whatever the initial guess in D.

The mainstream approach to this problem starts with H. Federer [18](1959), who introduced the notion of *positively reached* subsets of $I\!R^n$, which admitted a *cylindrical* neighborhood where (1)(2) hold. The extension to the case of Hilbert spaces was made by J.-P. Vial [26](1983), A. Shapiro [22](1994) and F. H. Clarke, R. J. Stern, and P. R. Wolenski [10](1995) under the more descriptive name of *proximally smooth* sets. In 1983, M. Degiovani, A. Marino, M. Tosques [16] introduced the (p, q)-convex *functions*, which led in 1988 A. Canino to define in [3] the class of *p-convex* sets, which possess a *non necessarily cylindrical* neighborhoods ϑ where (1)(2) hold, provided that D is *locally closed* (Definition 2.4 below). Hence p-convex sets can be seen as generalization of *locally closed convex sets*.

Motivated by a local version of (1)(2), R. A. Poliquin, R. T. Rockafellar and L. Thibault [21](2000) defined the local notion of *prox-regularity* at a point x, which ensures regularity properties of the projection on $D \cap \{y \in D | |y - x| \leq \epsilon\}$ for some $\epsilon > 0$. Then in 2010, G. Colombo and L. Thibault have shown [13, Proposition 23] that *prox-regularity at all points* is equivalent for a (locally) closed set to a global *ρ-prox-regularity* property, which they defined in the same way as p-convexity, but with a different normal cone. Here again, (locally) closed ρ-prox-regular sets D possess a *non necessarily cylindrical* neighborhoods ϑ where (1)(2) hold - we refer to [13] for an extensive study of ρ-prox-regularity.

But it turns out that, for locally closed sets, p-convexity coincide with ρ-prox-regularity (see [13, Proposition 7 (b)] and Remark 2.2 below), and that for closed sets either notion contains proximally smooth and positively reached sets (see Remark 2.7). Hence p-convex sets will represent the mainstream approach in our comparison.

All these approaches use a notion of *normal cone* at a point x of D to evaluate how far one can move away from D on this cone and still retain the properties (1)(2). Here the Lipschitz stability (2) is obtained for the *distance in H* of the projections.

Other points of view, which are not based on the satisfaction of (1)(2), have also been used for the generalization of convex sets. We mention here the class of *approximately convex* sets introduced by G. Colombo and V. Goncharov [12, Section 5] under the name of "property ω", D. Aussel, A. Daniilidis and L. Thibault [1] under the name of "subsmoothness", and studied further by H. Van Ngai and J. P. Penot in [25]. It is based on the *approximate convexity* of the distance function $z \rightsquigarrow d(z, D)$ on a neighborhood of $x \in D$ in H (Definition 3.1 below). H. Van Ngai and J. P. Penot have also introduced in [25] the weaker notion of *intrinsically approximately convex* sets (Definition 3.2 below), where the approximate convexity of the distance function is required to hold only for the points of the neighborhood which belong to D. These notions are weaker than thoses based on (1)(2): [13, Proposition 9 (ii)]) shows that closed ρ-prox-regular sets are intrinsically approximately convex.

Let us now come to the challengers. Inverse problems are often formulated as *non-linear least square* problems with a *convex* admissible parameter set C:

$$\hat{x} \text{ minimizes } J(x) = \frac{1}{2} | \varphi(x) - z |^2 \text{ over } C, \tag{4}$$

where $x \in C$ is the unknown parameter, $z \in H$ is the data and φ the map to be inverted. The solution of (4) implies the projection of z on $D = \varphi(C)$. The *quasi-convex* subsets of H, which also possess a *non necessarily cylindrical* regular neighborhood ϑ where (1)(2) hold, have been initially developped as a first attempt for the analysis of this situation by the author in [6](1991) and [9](2010). In this approach, the set D is supposed to be - at least locally - arcwise connected by finite curvature paths γ, and the theory of quasi-convex sets consists in adapting the classical proofs for convex sets to the case where the segments are replaced by the paths γ (when $D = \varphi(C)$, the images by φ of segments of C are natural candidates for the paths γ). With this approach, the local Lipschitz continuity of the projection required by (2) is satisfied for the stronger *arc-length distance in* D. This property is important for inverse problems, as it gives a chance of obtaining stability for the solution $\hat{x}$ of (4) under suitable conditions on the derivative φ'. By construction, quasi-convex sets contain *all convex sets*.

But quasi-convex sets are only a first step for the analysis of (4): one wants to find conditions which ensure that the non-linear inverse problem (4) can be solved numerically by *local minimization algorithms* as soon as the noise or error level $d(z, D)$ on the data is *smaller than some* $\delta > 0$. This was the motivation for the introduction of the subclass of *strictly quasi-convex* (*s.q.c.* in short) sets in [7, 5, 9], which satisfy the additional *unimodality* property (3) on a *cylindrical* neighborhood $\vartheta = \{z \in H | d(z, D) < \delta\}$ of D. Sufficient conditions for strict quasi-convexity have been developed based on the curvature and the deflection along the finite curvature curves of D.

Now one hint on what the paper is *not* about: other notions of "generalized convex sets" have been introduced for the sake of calculus of variation, see for example B. Dacorogna and A. M. Ribeiro [14](2006) and B. Dacorogna [15](2007), which have no relation to the ones considered in this paper, though one of them bears the same name (quasiconvex).

The paper is organized as follows: we recall first in Sections 2, 3, 4, 5, without demonstration but with unified notations, the definition and main properties of p-convex sets, which "coincide" with ρ-prox-regular sets (Remark 2.2) and contain proximally smooth sets (Remark 2.7), of approximately convex and intrinsically approximately convex sets, of quasi-convex and of s.q.c. sets. We highlight in particular the different Lipschitz properties of the projection, and show that by construction the unimodality property (3) is satisfied for s.q.c.

sets. Then we give in Section 6 the proof of our sole and main result:

Theorem 1.1. *Let H be a Hilbert space, and $D \subset H$ be given. Then:*

1. *D quasi-convex $\Longrightarrow$ D p-convex,*
2. *D locally closed and p-convex $\Longrightarrow$ D quasi-convex,*
3. *D closed and s.q.c. $\Longrightarrow$ D proximally smooth,*
4. *D locally closed and p-convex $\Longrightarrow$ D approximately convex, but the converse implication is false.*

The proof of part 1 is easy, but that of part 2 requires to equip D with finite curvature paths, so the work of A. Cannino [3](1988) and [4](1991) on geodesics of p-convex sets is key to the proof, then part 3 follow easily. The proof of part 4 is based on [13, Proposition 9 (ii)], plus a simple counterexample.

In all the sequel, H is a Hilbert space, $|\cdot|$ denotes the norm in H, and $B_o(x, \epsilon)$ (resp. $B(x, \epsilon)$) denotes the open (resp. closed) ball of H centered at x with radius ϵ, and π denotes the projection on D, when it exists.

2. P-convex sets

We follow in this section the presentation of A. Canino [3, 4]. We denote by I_D the characteristic function of D:

$$I_D(x) = \begin{cases} 0 & \text{if } x \in D, \\ +\infty & \text{if } x \in H \setminus D. \end{cases}$$

We will call (outward) normal cone to D at $x \in D$ the set $\partial^- I_D(x)$, where ∂^- denotes the Fréchet-subdifferential. $\partial^- I_D(x)$ is a closed convex cone for all $x \in D$.

Definition 2.1. A subset $D \in H$ is p-convex if there exists a continuous function $p : D \rightsquigarrow I\!R^+$ such that:

$$(n, y - x) \leq p(x)|n||y - x|^2 \quad \forall x, y \in D \text{ and } \forall n \in \partial^- I_D(x), \tag{5}$$

or equivalently:

$$\begin{cases} B(x + n, |n|) \cap D = \{x\} \\ \text{whenever } x \in D \cap \partial D,\ n \in \partial^- I_D(x) \setminus \{0\}, \text{ and } 2p(x)|n| < 1. \end{cases} \tag{6}$$

Convex sets satisfy Definition 2.1 with $p(x) = 0$, so they are p-convex. Notice also that if x is an interior point of D, then $\partial^- I_D(x) = \{0\}$, so that (5) is satisfied independently of the value of $p(x)$, which is why (6) needs to be checked only at all points x of D which are on its boundary ∂D.

Hence conditions (5) or (6) will impose smoothness only on the part of ∂D which is in D. In particular, all open subsets of H are trivially p-convex!

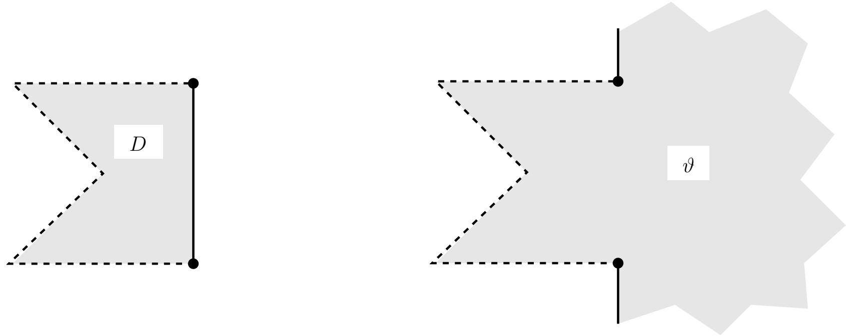

Figure 2.1: A p-convex set D and the associated set $\vartheta \supset D$ (interior points of a set: grey area, boundary points in the set: thick lines and points, boundary points not in the set: dashed lines). Here D is not locally closed, and ϑ is not open.

Remark 2.2. The family of ρ-prox-regular sets defined in [13, Definition 1] satisfies also Definition 2.1, but with ∂^- being the proximal-subdifferential and with $1/(2\rho(x))$ instead of $p(x)$. The Fréchet normal cone contains the proximal normal cone, so for general sets p-convexity implies ρ-prox regularity. But in the case of interest of locally closed sets, ρ-prox-regular sets satisfy the normal regularity property [13, Proposition 7 - (b)], so the two cones coincide, and ρ-prox-regularity coincides with p-convexity. $\square$

Then one defines:

$$\vartheta = \left\{ z \in H \mid \limsup_{\substack{|z-x| \to d(z,D), \\ x \in D}} 2p(x)|z-x| < 1, \text{ and} \right. \tag{7}$$

$$\left. \exists r \geq 0: \quad D \cap B(z,r) \text{ is closed and not empty} \right\} \supset D,$$

which has the nice following properties:

Property 2.3. Let $D \subset H$ be p-convex, and ϑ defined by (7). Then:

$$\vartheta = \{ z \in H \mid z \text{ has a projection } X \text{ on } D \text{ and } 2p(X)|z-X| < 1 \},$$

and the projection π on D is uniquely defined and locally Lipschitz continuous:

$$\begin{cases} \forall z_0, z_1 \in \vartheta : \\ |X_0 - X_1| \leq \left((1 - p(X_0)|z_0 - X_0| - p(X_1)|z_1 - X_1|\right)^{-1} |z_0 - z_1| \,. \end{cases} \tag{8}$$

where we have used the notations $X_0 = \pi(z_0)$ etc...

Hence ϑ satisties properties (1) and (2)... up to the fact that it is not necessarily open, as one can see on the simple example of Figure 2.1. The following additional property will be required:

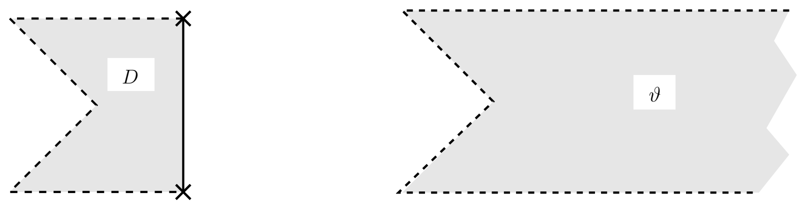

Figure 2.2: A locally closed p-convex set D and the associated open p-convex neighborhood $\vartheta \supset D$ (interior points of a set: grey area, boundary points in the set: thick line, boundary points not in the set: dashed lines and thick crosses).

Definition 2.4. The set $D \subset H$ is *locally closed* if $\forall x \in D$ there exists $r > 0$ such that $D \cap B(x, r)$ is closed in H.

In other words, D is locally closed set as soon as $D \cap \partial D$ is an open subset of ∂D for the topology induced by H. Then one has:

Property 2.5. Let $D \subset H$ be locally closed and p-convex. Then ϑ defined by (7) is open. We call ϑ a *(regular) p-convex neighborhood* of D.

Figure 2.2 shows a simple locally closed p-convex set, with the associated set ϑ, which is open as expected from Property 2.5.

With the requirement that p is a continuous function, the existence of a smallest function p which makes D p-convex is not guaranteed, as one can see on very simple examples. Hence a *largest p-convex neighborhood* does not necessarily exists.

Remark 2.6. The Lipschitz stability result (8) for p-convex sets is *global* on ϑ in the sense that it holds for any two points of ϑ, with a constant always ≥ 1 (and equal to 1 when D is convex). On the opposite, as we shall see in Remark 4.3 next section, the Lipschitz stability for quasi-convex sets will hold only *locally*, but with a constant which can be ≥ 1 but also < 1. □

Remark 2.7. The class of *δ-proximally smooth sets* is defined in [10] as the family of closed sets D which possess an open cylindrical neighborhood ϑ of thickness δ where the "distance to D" function is continuously differentiable. Though the motivation there is somewhat different from (1)(2), it turns out, using property (d) of Theorem 4.1 in [10], that this class is made of all closed p-convex sets which satisfy:

$$\forall x \in \partial D, \ p(x) = (2\delta)^{-1} = \text{constant},$$

with the desired neighborhood ϑ being given by (7), which simplifies to:

$$\vartheta = \{z \in H \text{ such that } d(z, D) < \delta\}.$$

Proximally smooth sets have been used in the study of "sweeping process" [11, 23], and for the analysis of crowd motion [19, 24, 20]. □

The following tools are available to recognize p-convex sets other than by their definition:

Property 2.8. Let $D \subset H$ be locally closed. Then D is p-convex if and only if, for any $x \in D$, there exists $r > 0$ and $p \geq 0$ such that:

$$\forall y_0, y_1 \in B(x,r) \cap D,\ \exists y \in B(x,r) \cap D \text{ s.t.: } |y - \frac{y_0 + y_1}{2}| \leq p|y_0 - y_1|^2.$$

Property 2.9. The image of a convex by a $\mathcal{C}^{1,1}_{\text{loc}}$-diffeomorphism is p-convex.

We summarize now two results from A. Canino [4] on the geodesics of p-convex sets which are needed for the comparison with quasi-convex sets.

Definition 2.10. A curve $\gamma : [0,1] \rightsquigarrow D$ is said to be a geodesic on D if $\gamma \in W^{2,1}(0,1;H)$ and $\gamma''(t) \in \partial^- I_D(\gamma(t))$ a.e. in $]0,1[$. A geodesic γ satisfies automatically:

$$\gamma \in W^{2,\infty}(0,1;H) \quad \text{and} \quad |\gamma'(t)| = cst \stackrel{\text{def}}{=} \ell(\gamma) \text{ a.e. on }]0,1[,$$

where $\ell(\gamma)$ denotes the arc length of γ.

Property 2.11. Let $D \subset H$ be p-convex, and γ a geodesic of D. Then:

$$|\gamma''(t)| \leq 2p(\gamma(t))\ |\gamma'|^2 \ \text{ a.e. on } [0,1].$$

This shows that the curvature of a geodesic γ of a p-convex set at $\gamma(t)$ is smaller than $2p(\gamma(t))$.

Property 2.12. Let $D \subset H$ be p-convex and locally closed. Then $\forall x \in D$, $\exists\, \bar{r} > 0$ such that:

$$\begin{gathered} 0 < r \leq \bar{r}, \quad x_0, x_1 \in D \cap B_o(x,r) \\ \Downarrow \\ \exists \text{ a unique geodesic curve } \gamma \text{ of } D \text{ connecting } x_0 \text{ and } x_1, \\ \text{and: } \ \gamma(t) \in B_o(x,r) \quad \forall t \in [0,1]. \end{gathered}$$

We conclude this section with a local property of the "distance to D" which generalizes slightly [13, Proposition 9-(i)], as it will be useful in the comparison with *approximately convex* sets:

Property 2.13. Let $D \subset H$ be p-convex and locally closed, $x \in D$ and $\bar{p} > p(x)$ be given. Then $\exists\, \bar{\delta} > 0$ such that

$$\begin{gathered} z_0, z_1 \in B(x, \bar{\delta}) \text{ and } t \in [0,1] \\ \Downarrow \\ d(z_t, D) \leq (1-t)d(z_0, D) + td(z_1, D) + \bar{p}\ t(1-t)|z_0 - z_1|^2 \end{gathered}$$

where $z_t = (1-t)z_0 + tz_1$.

One sees that for locally closed p-convex sets, the "convexity default" term $d(z_t, D) - (1-t)d(z_0, D) - td(z_1, D)$ goes to zero as $|z_0 - z_1|^2$. The proof will be given at the end of Section 6.4.

3. Approximately convex sets

We recall for sake of completeness their definition (we follow here [25]):

Definition 3.1. The set $D \subset H$ is approximately convex if $\forall x \in D$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that:

$$\begin{gathered} z_0, z_1 \in B(x, \delta) \text{ and } t \in [0,1] \\ \Downarrow \\ d(z_t, D) \leq (1-t)d(z_0, D) + td(z_1, D) + \epsilon\, t(1-t)|z_0 - z_1| \end{gathered} \tag{9}$$

where $z_t = (1-t)z_0 + tz_1$.

Notice that a still weaker notion, *intrinsic approximate convexity*, has also been introduced in [25]:

Definition 3.2. The set $D \subset H$ is intrinsically approximately convex if $\forall x \in D$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that:

$$\begin{gathered} z_0, z_1 \in D,\ z_0, z_1 \in B(x, \delta) \text{ and } t \in [0,1] \\ \Downarrow \\ d(z_t, D) \leq \epsilon\, t(1-t)|z_0 - z_1| \end{gathered}$$

where $z_t = (1-t)z_0 + tz_1$.

In these two definitions, the "convexity default" term $d(z_t, D) - (1-t)d(z_0, D) - td(z_1, D)$ is only required to go to zero faster than $|z_0 - z_1|$ (compare with Property 2.13).

We limit ourselves in this paper to the comparison of approximately convex and p-convex sets, as stated in point 4 of Theorem 1.1 and proved in Section 6.4, and refer to references [12, Section 5] [1] [25] for further study of these families of set.

4. Quasi-convex sets

We follow the presentation of [9], Chapter 6, with a slight generalization of the definition for harmonization purpose with p-convex sets: we shall require only in Definition 4.2 the existence of paths which connect any two η-projections.

Definition 4.1. $\mathcal{P} \subset W^{2,\infty}([0,1]; H)$ is a *family of finite curvature paths* of D if:

$$\forall \gamma \in \mathcal{P},\ \forall t \in [0,1],\ \gamma(t) \in D \ \text{ and } \ |\gamma'(t)| = \text{cst} \stackrel{\text{def}}{=} \ell(\gamma), \tag{10}$$

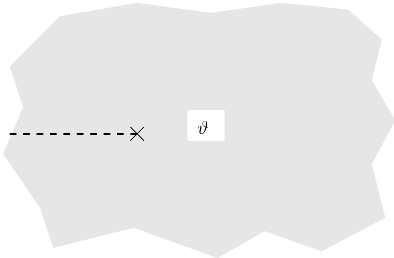

Figure 4.1: The largest open regular quasi-convex neighborhood $\vartheta \supset D$ associated to the quasi-convex sets of Figures 2.1 and 2.2 (interior points of a set: grey area, boundary points not in the set: dashed line and thick cross).

and if it is stable by restriction:

$$\begin{gathered} \gamma \in \mathcal{P},\ 0 \leq t' < t'' \leq 1 \\ \Downarrow \\ \tilde{\gamma} : t \in [0,1] \rightsquigarrow \gamma(t' + t(t'' - t')) \text{ belongs to } \mathcal{P}, \end{gathered} \tag{11}$$

where $\ell(\gamma)$ is the (arc-)length of the curve γ.

Definition 4.2. A subset $D \subset H$ is *quasi-convex* if there exists a family $\mathcal{P}$ of finite curvature paths of D, a neighborhood ϑ of D in F, and a lower semi-continuous (l.s.c.) function $\eta_{\max} : \vartheta \rightsquigarrow]0, +\infty]$ such that $\forall z \in \vartheta$, $\forall\, 0 < \eta < \eta_{\max}(z)$, $\exists \alpha > 0$ such that:

$$\text{any two } \eta\text{-projections of } z \text{ can be connected by a path } \gamma \in \mathcal{P} \tag{12}$$

and:

$$d^2_{z,\gamma} \text{ is uniformly } \alpha\text{-convex} \Leftrightarrow \frac{1}{2}\frac{d^2}{dt^2}(d^2_{z,\gamma})(t) \geq \alpha|\gamma'|^2 \quad \forall t \in [0,1], \tag{13}$$

where an η-projection of z is any $x \in D$ which satisfies $|z - x| \leq d(z, D) + \eta$, and $d_{z,\gamma} : t \rightsquigarrow |z - \gamma(t)|$ denotes the "distance to z" function along γ.

We call such a neighborhood ϑ a *(regular) quasi-convex neighborhood* of D.

When D is convex, then $\mathcal{P} = \{\text{segments of } D\}$, $\vartheta = H$ and $\eta_{max}(z) = +\infty$ satisfy Definition 4.2, and D is quasi-convex with $\alpha = 1$.

Remark 4.3. The case where the best (i.e. the largest) α satisfies $\alpha < 1$ corresponds to z being outside of a "non-convex side" of D.

But notice that the best α can also be strictly greater than 1, if z happens to be outside of a "convex side" of D *and* if the finite curvature paths γ "follow the shape of the boundary" of D near this boundary. $\square$

Property 4.4. Let $D \subset H$ be quasi-convex for some family $\mathcal{P}$ of finite curvature paths. Then there exists a *largest open regular quasi-convex neighborhood* ϑ of D, and a *largest l.s.c. function* $\eta_{\max} : \vartheta \to]0, +\infty]$ such that $\mathcal{P}, \vartheta, \eta_{max}$ satisfy Definition 4.2.

The sets D of Figures 2.1 and 2.2 are quasi-convex. The associated largest quasi-convex neighborhood ϑ is shown in Figure 4.1.

Definition 4.5. Let $D \subset H$ be quasi-convex, and $x_0, x_1 \in D$ be two η-projections of some $z \in \vartheta$ for some $0 < \eta < \eta_{max}$. The *arc-length distance* $d_{\mathcal{P}}$ of x_0 and x_1 in D is then defined by:

$$d_{\mathcal{P}}(x_0, x_1) = \sup_{\gamma \in \mathcal{P}\,,\, \gamma(j)=x_j\,,\, j=0,1} \ell(\gamma),$$

where $\ell(\gamma)$ is the arc-length of γ defined in (10).

We recall now the properties of quasi-convex sets with respect to projection:

Property 4.6. Let $D \subset H$ be quasi-convex, and $\mathcal{P}, \vartheta, \eta_{\max}$ be chosen according to Definition 4.2. Then, for any $z \in \vartheta$:

1. there exists at most one projection $\widehat{X}$ of z on D
2. any minimizing sequence $X_n \in D$ of the "distance to z" function over D is a Cauchy sequence for both $|X - Y|$ and $d_{\mathcal{P}}(X, Y)$. Hence X_n converges in H to the (unique) projection $\widehat{X}$ of z on the closure $\overline{D}$ of D.
3. If $z_0, z_1, \in \vartheta$ admit projections X_0, X_1 on D, the following implication holds:

$$|z_0 - z_1| + \max_{j=0,1} d(z_j, D) \le d < \min_{j=0,1} \{d(z_j, D) + \eta_{\max}(z_j)\} \text{ for some } d > 0, \quad (14)$$

implies:

$$\begin{cases} |X_0 - X_1| \le d_{\mathcal{P}}(X_0, X_1) \le \alpha^{-1}\,|z_0 - z_1| \\ \alpha = \dfrac{1}{2}(\alpha_0 + \alpha_1) > 0, \end{cases} \quad (15)$$

where $\alpha_j > 0, j = 0, 1$ are defined by (12)(13), with $z = z_j$ and $\eta = \eta_j$ given by:

$$0 \le \eta_j = d - d(z_j, D) < \eta_{\max}(z_j), \quad (16)$$

Notice that, because of the lower semi-continuity of η_{max}, condition (14) is necessarily satisfied when z_1 is close enough to z_0: the Lipschitz stability result for quasi-convex sets is *local*. According to the situations mentionned in Remark 4.3, the local Lipschitz constant α^{-1} can be ≥ 1 or < 1.

When D is convex and $\mathcal{P}$ is chosen to be made of the segments of D, then $\eta_{max} = +\infty$, $d_{\mathcal{P}}(x, y) \equiv |x - y|$ and $\alpha = 1$, and Property 4.6 reduces to the usual properties of the projection on convex sets.

5. Strictly quasi-convex (s.q.c.) sets

Definition 4.2 of quasi-convex sets is quite technical, and not easy to check in applications. Moreover, quasi-convex sets satisfy only properties (1) and (2), but miss to satisfy the computationally important unimodality property (3). So we recall now the notion of *strictly quasi-convex (s.q.c.)* sets [7, 5, 9], which not only satisfies (1) (2) *and* (3) on a *cylindrical* neighborhood, but also for which convenient sufficient conditions are available.

We follow the presentation of [9], but for sake of conciseness we skip the original definition, which is also quite technical, and use as definition the more intuitive characterization of s.q.c. sets by the global radius of curvature.

For the rest of this section, D is a subset of the Hilbert space H, equipped with a family $\mathcal{P}$ of finite curvature paths. In order to ensure a *global* unimodality property, we shall require throughout the section that the paths of $\mathcal{P}$ connect *any two points* of D.

Definition 5.1. A function $f \in \mathcal{C}^1([0,1])$ has a *stationary point* at $t \in [0,1]$ (for the minimization of f) if:

$$f'(t)\delta t \;\geq\; 0 \quad \forall \delta t \in I\!R, \quad t+\delta t \in [0,1].$$

Definition 5.2. Let $\gamma \in \mathcal{P}$ be given. Then the *affine normal cone* $N(\gamma, t)$ to γ at some $t \in [0,1]$ is defined by:

$$N(\gamma,t) = \big\{ z \in H \mid d^2_{z,\gamma} \text{ has a stationary point at } t \big\}$$

where $d^2_{z,\gamma}(t) = |z - \gamma(t)|^2$ as in Definition 4.2.

Definition 5.3. Let $\gamma \in \mathcal{P}$ be given. Then, for any $t, t' \in [0,1]$ with $t \neq t'$, the *global radius of curvature* of γ at t seen from t' is:

$$\rho_G(\gamma, t, t') = d\left(\gamma(t), N(\gamma,t) \cap N(\gamma,t')\right) \in [0, +\infty],$$

with the natural convention that $\rho_G(\gamma,t,t') = +\infty$ if $N(\gamma,t)$ and $N(\gamma,t')$ don't intersect. It is related to the *usual radius of curvature*

$$\rho(\gamma,t) = |\gamma'|^2 / |\gamma''(t)| \in]0, \infty] \tag{17}$$

by the formula:

$$\begin{cases} \text{for almost every } t \in [0,1], \\ \rho_G(\gamma,t,t') \to \rho(\gamma,t) \text{ and } \rho_G(\gamma,t',t) \to \rho(\gamma,t) \text{ when } t' \to t. \end{cases}$$

Consideration of the worst case for all $t, t' \in [0,1]$, then for all γ of $\mathcal{P}$ leads to the

Definition 5.4. Let $(D,\mathcal{P})$ and $\gamma \in \mathcal{P}$ be given. The *global radius of curvature of* γ is defined by:

$$R_G(\gamma) \overset{\text{def}}{=} \inf_{t,t'\in[0,1],\ t\neq t'} \rho_G(\gamma,t,t'),$$

and that of the set D equipped with the family $\mathcal{P}$ by:

$$R_G(D,\mathcal{P}) \overset{\text{def}}{=} \inf_{\gamma\in\mathcal{P}} R_G(\gamma).$$

We define now our new family of sets:

Definition 5.5. A subset $D \subset H$ is *strictly quasi-convex (s.q.c. in short)* if there exists a family $\mathcal{P}$ of finite curvature paths such that:

$$\begin{cases} D \text{ is fully connected by } \mathcal{P}: \\ \forall x_0, x_1 \in D,\ \exists \gamma \in \mathcal{P} \text{ such that } \gamma(0) = x_0, \gamma(1) = x_1, \end{cases}$$

and:

$$R_G(D,\mathcal{P}) > 0.$$

The cylindrical neighborhood:

$$\vartheta = \{z \in F \mid d(z,D) < R_G(D,\mathcal{P})\}, \tag{18}$$

is then called a *(regular) s.q.c. neighborhood* of D.

By construction, the unimodality property (3) holds for s.q.c. sets:

Property 5.6. Let D equipped with $\mathcal{P}$ be s.q.c., and ϑ be defined by (18). Then if $z \in \vartheta$ admits a projection X on D, the "distance to z" function has no parasitic stationary point on D other that X.

And, though it is not obvious from the definition, s.q.c. sets are indeed quasi-convex:

Property 5.7. Let D equipped with $\mathcal{P}$ be s.q.c.. Then D, ϑ defined by (18) and $\eta_{\max}$ defined by:

$$\eta_{\max}(z) \ = \ R_G(D,\mathcal{P}) - d(z,D).$$

satisfy Definition 4.2 of quasi-convex sets. Hence Property 4.6 holds on ϑ, and formula (14) (15) (16) for the local Lipschitz stability reduce to:

$$|z_0 - z_1| + \max_{j=0,1} d(z_j,D) \leq d < R_G(D,\mathcal{P}) \ \text{ for some } d > 0$$

implies:

$$|X_0 - X_1| \leq \ell(\gamma) \leq (1 - d\ /R(\gamma))^{-1}\ |z_0 - z_1|,$$

for any $\gamma \in \mathcal{P}$ connecting X_0 to X_1, where $\ell(\gamma)$ is the arc length of γ and $R(\gamma) \in]0,\infty]$ is its radius of curvature defined by - see (17):

$$R(\gamma) = \inf_{t\in[0,1]} \rho(\gamma,t).$$

The quasi-convex sets D of Figures 2.1 and 2.2 cannot be made s.q.c. for any family $\mathcal{P}$ of finite curvature paths, as the largest regular quasi-convex neighborhood ϑ of Figure 4.1 does not contain any cylindrical neighborhood of D.

We conclude this section with a sufficient condition for strict quasi-convexity based on lower/upper bounds R, L, Θ to the radius of curvature, arc length and deflection of the paths $\gamma \in \mathcal{P}$:

$$0 < R \leq \inf_{\gamma \in \mathcal{P}} \inf_{t \in [0,1]} |\rho(\gamma, t)|, \tag{19}$$

$$L \geq \sup_{\gamma \in \mathcal{P}} \ell(\gamma), \tag{20}$$

$$\Theta \geq \sup_{\gamma \in \mathcal{P}} \sup_{t,t' \in [0,1]} Arg \cos < \frac{\gamma'(t)}{|\gamma'|}, \frac{\gamma'(t')}{|\gamma'|} > \in [0, \pi]. \tag{21}$$

One checks easily that L/R is also an upper bound to the deflection, so one can always suppose that the available upper bound Θ is better than L/R, i.e.:

$$\Theta \leq L/R. \tag{22}$$

Property 5.8. Let R, L, Θ satisfy (19) (20) (21) (22). Then the global radius of curvature R_G of D equipped with the family of paths $\mathcal{P}$ satisfies:

$$R_G(D, \mathcal{P}) \geq \begin{cases} R, & 0 \leq \Theta \leq \pi/2, \\ R \sin \Theta + (L - R\Theta) \cos \Theta, & \pi/2 \leq \Theta \leq \pi. \end{cases}$$

This gives immediately sufficient conditions for D to be s.q.c.: for example if D is fully connected by $\mathcal{P}$ and if $\Theta \leq \pi/2$, then $R_G(D, \mathcal{P}) = R > 0$ and D is s.q.c. with a neighborhood ϑ of thickness R.

These conditions are particularly convenient in the case where D is the image by a smooth application φ of a convex C. Let us denote by $\widetilde{\mathcal{P}}$ the family of curves of D obtained as image by φ of the segments of C:

$$\widetilde{\mathcal{P}} = \{\tilde{\gamma} : t \in [0,1] \rightsquigarrow \varphi((1-t)x_0 + tx_1),\ x_0, x_1 \in C\}.$$

Here t is not any more the arc-length along $\tilde{\gamma}$, and even the $\mathcal{C}^2$ regularity of φ is not enough to ensure that the curves $\tilde{\gamma}$ have a finite curvature. However the following property holds:

Property 5.9. Let C be a convex set of a Banach space E, and $\varphi : C \rightsquigarrow H$ a mapping such that:

$$\widetilde{\mathcal{P}} \subset W^{2,\infty}([0,1]; H)$$

and:

$$\exists R > 0 \text{ s.t. } \forall \tilde{\gamma} \in \widetilde{\mathcal{P}} \text{ one has } |\tilde{\gamma}''(t)| \leq \frac{1}{R} |\tilde{\gamma}'(t)|^2 \text{ for a.e. } t \in [0,1].$$

Then the lower/upper bounds R, L, Θ can be estimated as follows:

i) the family of curves $\widetilde{\mathcal{P}}$ is, once reparameterized by arc length, a family of finite curvature paths which connects all points of $D = \varphi(C)$, and R is a lower bound to its radii of curvature in the sense of (19).

ii) If moreover:

$$\exists\, k \geq 0 \text{ s.t. } \forall \tilde{\gamma} \in \widetilde{\mathcal{P}} \text{ one has } |\tilde{\gamma}'(t)| \leq k\,|x_1 - x_0|_E \text{ for a.e. } t \in [0,1],$$

then any number $L > 0$ which satisfies:

$$\forall \tilde{\gamma} \in \widetilde{\mathcal{P}},\ \int_0^1 |\tilde{\gamma}'(t)| dt \leq L \leq k \operatorname{diam}(C)$$

is an upper bound to arc length in the sense of (20).

iii) And finally, any number $\Theta \geq 0$ which satisfies:

$$\forall \tilde{\gamma} \in \widetilde{\mathcal{P}},\ \int_0^1 \theta(t) dt \leq \Theta \leq L/R \text{ with } |\tilde{\gamma}''(t)| \leq \theta(t)|\tilde{\gamma}'(t)| \text{ for a.e. } t \in [0,1]$$

is an upper bound to the deflection in the sense of (21).

An example of application of these results to a source estimation problem in a two-dimensional non-linear elliptic equation is given in [8, 9].

6. Proof of Theorem 1.1

6.1. Quasi-convex sets are p-convex

Let $D \subset H$ be quasi-convex and denote by ϑ an associated q.c. neighborhood. Without loss of generality, we can suppose that ϑ is open. Hence the function

$$r_{\max}(x) = d(x, H \setminus \vartheta) \quad \forall x \in H \tag{23}$$

satisfies:

$$r_{\max} \text{ is continuous on } D, \quad r_{\max}(x) > 0 \ \ \forall x \in \vartheta.$$

Now define $p : D \rightsquigarrow I\!R^+$ by:

$$p(x) = (2r_{\max}(x))^{-1}, \tag{24}$$

which is also continuous. We check that it satisfies (6): let $x \in \partial D$, $n \in \partial^- I_D(x) \setminus 0$ be given such that $2p(x)|n| < 1$. By definition of p, the normal n satisfies:

$$|n| < r_{\max}(x). \tag{25}$$

Suppose for a while that $x+n \notin \vartheta$. Then $x+n \in H \setminus \vartheta$, so that:

$$|n| = |x-(x+n)| \geq d(x, H \setminus \vartheta) = r_{\max}(x),$$

which contradicts (25). Hence $x+n \in \vartheta$. But ϑ is a regular quasi-convex neighborhood of D, so $x+n$ admits at most one projection on D, which is necessarily x by construction.

Hence $B(x+n, |n|) \cap D = \{x\}$, so condition (6) of Definition 2.1 is satisfied, and D is p-convex.

6.2. Locally closed p-convex sets are quasi-convex

We start with a lemma:

Lemma 6.1. *Let $D \subset H$ be p-convex and locally closed. Then $\forall x \in D$, $\forall \bar{p} > p(x)$, $\exists r(x) > 0$ such that:*

$$\begin{array}{c} z \in B_o(x, r(x)),\ 0 < \eta < 2\,(r(x) - |x-z|) \\ \Downarrow \\ \alpha = \eta\bar{p} > 0 \textit{ satisfies } (12)\ (13). \end{array} \tag{26}$$

Proof: Let $x \in D$ and $\bar{p} > p(x)$ be given. Because of the continuity of $x \rightsquigarrow p(x)$, one can always suppose that $\bar{r}$ associated to x by Property 2.12 is chosen small enough so that it satisfies moreover:

$$p(x') \leq \bar{p} \quad \forall x' \in B(x, \bar{r}). \tag{27}$$

We check now that the function:

$$r(x) = \min\{\bar{r}/2,\ (6\bar{p})^{-1}\}. \tag{28}$$

has the required properties. Let $z \in H$ and $\eta > 0$ be chosen as in (26), and $x_0, x_1 \in D$ be two η-projections of z on D:

$$|z - x_j| \leq d(z, D) + \eta \leq |z - x| + \eta, \quad j = 0, 1.$$

Then for $j = 0, 1$ one has:

$$|x - x_j| \leq |x - z| + |z - x_j| \leq 2|z - x| + \eta < 2r(x) \leq \bar{r}.$$

Hence $x_0, x_1 \in D \cap B_o(x, 2r(x))$ where $2r(x) \leq \bar{r}$, and Property 2.12 shows the existence of a geodesic γ of D which connects x_0 and x_1, so that condition (12) is satisfied. This proves the first part of the lemma.

But Property 2.12 says also that γ stays inside $B_o(x, 2r(x))$. Hence (27) shows that:

$$|p(\gamma(t))| \leq \bar{p} \quad \forall t \in [0, 1].$$

We check now that the function $d^2_{z,\gamma} : t \rightsquigarrow |z-\gamma(t)|^2$ is uniformly α-convex:

$$\frac{1}{2}\frac{d^2}{dt^2}(d^2_{z,\gamma})(t) = |\gamma'|^2 - (z-\gamma(t), \gamma''(t)),$$

where:

$$\begin{aligned}(z-\gamma(t),\gamma''(t)) &\le |z-\gamma(t)|\,|\gamma''(t)| \\ &\le |z-\gamma(t)|\,2\bar{p}\,|\gamma'|^2 \qquad \text{(Property 2.11 and (29))} \\ &\le \Big(\underbrace{|z-x|}_{<r(x)-\eta/2} + \underbrace{|x-\gamma(t)|}_{<2r(x)}\Big)\,2\bar{p}\,|\gamma'|^2 \\ &\le (6\bar{p}\,r(x) - \eta\bar{p})\,|\gamma'|^2 \\ &\le (1-\eta\bar{p})|\gamma'|^2 \qquad \text{(equation (28))}\end{aligned}$$

Hence:

$$\frac{1}{2}\frac{d^2}{dt^2}(d^2_{z,\gamma})(t) \ge \eta\bar{p}\,|\gamma'|^2,$$

which shows that (13) is satisfied and ends the proof of the lemma. □

We use now this lemma to construct the family of paths $\mathcal{P}$, the neighborhood ϑ and the l.s.c. function η_{max} which satisfy Definition 4.2. We set:

$$\mathcal{P} = \{\text{geodesics } \gamma \text{ whose existence is asserted by Property 2.12}\}.$$

Definition 2.10 shows that geodesic curves are in $W^{2,\infty}(0,1;H)$, and condition (11) is automatically satisfied as any part of a geodesic is a geodesic. Hence $\mathcal{P}$ *is a family of finite curvature paths* in the sense of Definition 4.1.

Then we define an *open neighborhood* ϑ by:

$$\vartheta = \cup_{x\in D}\, B_o(x, r(x)),$$

where $r(x)$ is associated to $x \in D$ by Lemma 6.1 with $\bar{p} = p(x)+1$.

In order to define $\eta_{max} : \vartheta \rightsquigarrow I\!R^+$, we associate first to any $Z \in \vartheta$ one point $x_Z \in D$ such that $Z \in B_o(x_Z, r(x_Z))$ (the existence of x_Z follows from the definition of ϑ). So we can define a "local" function $\eta_Z : \vartheta \rightsquigarrow I\!R^+$ by:

$$\forall z \in \vartheta, \eta_Z(z) = \begin{cases} 2\,(r(x_Z) - |x_Z - Z|) & \text{if } |z-Z| < (r(x_Z) - |x_Z - Z|), \\ 0 & \text{if } |z-Z| \ge (r(x_Z) - |x_Z-Z|). \end{cases} \tag{29}$$

The function η_Z is obviously l.s.c., so we define the l.s.c. function η_{max} by:

$$\eta_{max} = \sup_{Z\in\vartheta} \eta_Z. \tag{30}$$

We check now that $\mathcal{P}$, ϑ and η_{max} defined as above satisfy the Definition 4.2 of quasi-convex sets: let $z \in \vartheta$ and $0 < \eta < \eta_{max}(z)$ be given.

By definition of η_{max}, there exists $Z \in \vartheta$ such that:

$$0 < \eta < \eta_Z(z) \leq \eta_{max}(z).$$

Hence, using (29):

$$\begin{aligned} |z - Z| &< (r(x_Z) - |x_Z - Z|), \\ \eta &< 2(r(x_Z) - |x_Z - Z|). \end{aligned} \tag{31}$$

But (31) gives:

$$\begin{aligned} |z - x_Z| &\leq |z - Z| + |Z - x_Z| \\ &< (r(x_Z) - |x_Z - Z|) + |Z - x_Z| \\ &< r(x_Z). \end{aligned} \tag{32}$$

So Lemma 6.1 applied to $x = x_Z$ and $\bar{p} = p(x_Z) + 1$ shows that conditions (12) and (13) hold with $\alpha = \eta\,(p(x_Z) + 1)$ for any η-projections x_0, x_1 of z on D, hence D satisfies the Definition 4.2 of quasi-convex sets.

6.3. Closed s.q.c. sets are proximally smooth

Let $D \subset H$ be closed and s.q.c. for some family of finite curvature path $\mathcal{P}$, and denote by $R_G > 0$ its global radius of curvature. Then D is quasi-convex (Property 5.7) for the same family $\mathcal{P}$ and the cylindrical neighborhood ϑ given by (18).

We check now that the function $x \in D \rightsquigarrow p(x) = (2R_G)^{-1} =$ constant satisfies the condition (6) of the Definition 2.1 of p-convex sets: let $x \in \partial D$ and $n \in \partial^- I_D(x) \setminus \{0\}$ be given such that $2p(x)|n| < 1$. Then $|n| < R_G$, hence $d(x + n, D) < R_G$, so $x + n \in \vartheta$. This shows that x is the unique projection of $x + n$ on D, and (6) is satisfied.

Hence D is a closed p-convex set with a function $p(x) = (2R_G)^{-1}$ =constant, which amounts to say that D is δ-proximally smooth for $\delta = R_G > 0$ using Remark 2.7

Remark 6.2. Property (c) of Theorem 4.1 in [10] shows that, for a closed R_G-proximally smooth set one has:

$$\forall z \in \vartheta \setminus D : d(z, D) + d(z, H \setminus \vartheta) = R_G.$$

Hence the function $p(x)$ defined by equations (23) (24) satisfies also $p(x) = (2R_G)^{-1}$ =constant on ∂D. $\square$

6.4. Locally closed p-convex sets are approximately convex, but the converse is false.

We prove first the implication in point 4 of Theorem 1.1 Let $D \subset H$ be p-convex and locally closed. In order to prove that it is approximately convex, let $x \in H$ and $\epsilon > 0$ be given. According to Definition 3.1 one has to find $\delta > 0$ such that (9) holds. Choose then $\bar{p} > p(x)$, for example $\bar{p} = p(x) + 1$. Property 2.13 of p-convex sets shows the existence of $\bar{\delta} > 0$ such that $z_0, z_1 \in B(x, \delta)$ implies:

$$\forall t \in [0,1],\ d(z_t, D) \leq (1-t)d(z_0, D) + td(z_1, D) + \bar{p}\, t(1-t)|z_0 - z_1|^2 \qquad (33)$$

as soon as $\delta \leq \bar{\delta}$. Then (9) holds for $\delta = \mathrm{Min}\{\bar{\delta}, \epsilon/(2\bar{p})\}$, and D is approximately convex.

We give now a counter example which proves that the converse implication is false. The closed set

$$D = \left\{(\eta, \zeta) \in \mathbb{R}^2 \middle| -1 \leq \eta \leq 1,\ \zeta = |\eta|^\alpha\right\}$$

is not quasi-convex - and hence not p-convex - for $1 < \alpha < 2$, as there is no finite curvature path connecting a point with negative abscissa to one with positive abscissa. But it is approximately convex (and hence also intrinsically approximately convex), as one can prove by tedious but quite natural calculations that:

$$d(z_t, D) \leq (1-t)d(z_0, D) + td(z_1, D) + \alpha\delta^{\alpha-1}(1 + \alpha\delta^{\alpha-1})\, t(1-t)|z_0 - z_1|$$

for any couple of points $z_0, z_1 \in B(0, \delta/2)$.

This ends the proof of Theorem 1.1.

Proof of Property 2.13. Let $x \in D$ and $\bar{p} > p(0)$ be given. We want to determine $\bar{\delta} > 0$ such that (33) is satisfied as soon as:

$$z_0, z_1 \in B(x, \bar{\delta}),\ 0 \leq t \leq 1,\ z_t = (1-t)z_0 + tz_1.$$

To that purpose, let us choose $0 < \tau < 1$ such that:

$$(1-\tau)^2 \bar{p} > p(x). \qquad (34)$$

By construction,

$$\vartheta_\tau = \{z \in \vartheta \mid 2p(X)|z - X| < \tau\ \} \quad (\text{where } X = \pi(z))$$

is an open subset of the set ϑ defined in (7). So for $\bar{\delta}$ small enough, one has $B(x, 4\bar{\delta}) \subset \vartheta_\tau$, and $z_j \in B(x, \bar{\delta})$ has a unique projection $X_j \in B(x, 2\bar{\delta}) \subset \vartheta_\tau$ for $j = 0, 1$, which satisfy $|X_0 - X_1| \leq (1-\tau)^{-1}|z_0 - z_1|$, cf (8). Define then:

$$\forall t \in [0,1]: \quad X_t = (1-t)X_0 + tX_1 \in B(x, 2\bar{\delta}) \subset \vartheta_\tau,$$

which has a unique projection $\pi(X_t) \in B(x, 4\bar{\delta})$. The p-convexity property (5) is satisfied at $\pi(X_t)$ for all $t \in [0,1]]$, so the proof of [13, Proposition 9 (ii)] applies. Hence:

$$\begin{aligned} d(z_t, D) &\leq |z_t - X_t| + d(X_t, D) \\ &\leq (1-t)|z_0 - X_0| + t|z_1 - X_1| + p(\pi(X_t))\, t(1-t)\, |X_0 - X_1|^2 \\ &\leq (1-t)d(z_0, D) + td(z_1, D) + \sup_{y \in B(x, 4\bar{\delta})} p(y)\, t(1-t)\, (1-\tau)^{-2} |z_0 - z_1|^2 \end{aligned}$$

But p is a continuous function, so for $\bar{\delta}$ small enough one has (cf. (34)):

$$p(x) \leq \sup_{y \in B(x, 4\bar{\delta})} p(y) \leq (1-\tau)^2 \bar{p},$$

which proves that (33) is satisfied for $\bar{\delta}$ small enough. □

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