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On Subdifferentials of a Minimal Time Function in Hausdorff Topological Vector Spaces at Points Outside the Target Set

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The present paper is a continuation of the study of the minimal time functions in Hausdorff topological vector spaces, started by the author in [4]. We consider in this work the case of points outside the target set and we prove and extend various important properties on directional derivatives and subdifferentials of $T_{S,\Omega}$ at points $x \notin S$ in the convex and nonconvex cases. These results are used to prove various new characterizations of the convex tangent cone, Clarke tangent cone, Bouligand tangent cone, and Clarke normal cone to the enlargement $S(r)$ ($r := T_{S,\Omega}(\bar{x}) > 0$) of S at $\bar{x} \notin S$ in terms of the minimal time function at $\bar{x}$, in Hausdorff topological vector spaces. Our results extend various existing results, in convex and nonconvex cases, from Banach spaces and normed vector spaces to Hausdorff topological vector spaces [3, 6, 7, 16, 17]. Even in Banach spaces and in normed vector spaces our results are new and they extend various existing results on the distance function to closed sets by taking Ω to be the closed unit ball. Applications to normal and subdifferential regularities in normed vector spaces are also given in the last section of the paper.

Keywords: Minimal time function, Clarke tangent cone, Clarke normal cone, tangential regularity, directional regularity

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1. Introduction

Let E be a Hausdorff topological vector space and Ω be a nonempty subset of E. Let S be a nonempty closed subset of E. The so-called minimal time function associated with Ω to the set S is defined by

$$T_{S,\Omega}(x) := \inf\{t > 0 : S \cap (x + t\Omega) \neq \emptyset\}, \quad \forall x \in E. \tag{1}$$

This function plays a highly important role in variational analysis, optimization, control theory, Hamilton-Jacobi partial differential equations, approximation theory, etc.; the reader can find more discussions in [8, 13, 14, 15, 16, 23, 24, 31, 32] and the references therein. The value $T_{S,\Omega}(x)$ can be seen as the minimal

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time $t \geq 0$ associated with the differential inclusion $\dot{x}(\tau) \in \Omega$, almost everywhere on $[0, t]$, with $x(0) = x$ and $x(t) \in S$, in other words,

$$T_{S,\Omega}(x) = \begin{cases} \inf\{t > 0 : \exists x(\cdot) \text{ s.t. } \dot{x}(\tau) \in \Omega, \text{ a.e. } [0,t], \\ \qquad\qquad \text{with } x(0) = x, x(t) \in S\}, & \forall x \notin S; \\ 0, & \forall x \in S. \end{cases}$$

It is with high importance to notice that the minimal time function $T_{S,\Omega}$ covers three crucial functions in variational analysis: the indicator function, the distance function, and the Minkowski function, by taking some particular cases of S and Ω as follows:

1. If $S = \{0\}$, then $T_{S,\Omega}(-x) = T_{\{0\},\Omega}(-x) = \rho_\Omega(x)$ coincides with the Minkowski (gauge) function associated with Ω, i.e.,

$$\rho_\Omega(x) = \inf\{t > 0 : t^{-1}x \in \Omega\}.$$

2. If $\Omega = \{0\}$, then $T_{S,\Omega}(x) = T_{S,\{0\}}(x) = \psi_S(x)$, coincides with the indicator function associated with S. Here $\psi_S(x) = 0$, if $x \in S$ and $\psi_S(x) = +\infty$, otherwise.
3. If E is normed, $S = \{0\}$, and $\Omega = I\!B$ is the closed unit ball in E, then $T_{S,\Omega}(x) = T_{\{0\},\boldsymbol{B}}(x) = \|x\|$.
4. If E is normed and $\Omega = I\!B$, then $T_{S,\Omega}(x) = T_{S,\boldsymbol{B}}(x) = \inf_{s \in S} \|x - s\|$, coincides with the usual distance function associated with S.

From the above cases, we can see the importance of the study of the function $T_{S,\Omega}$ in normed vector spaces as well as in Hausdorff topological vector spaces. This type of study will unify the study of all the above functions. The study of the directional derivatives, subdifferentials, and regularity of $T_{S,\Omega}$ at points in S are given in [4]. In the present work, we continue the study of $T_{S,\Omega}$ at points outside the set S. We extend various existing results on directional derivatives and subdifferentials of $T_{S,\Omega}$ at $\bar{x} \notin S$ and their relationships to tangent and normal cones to the enlargement $S(r)$ with $r := T_{S,\Omega}(\bar{x}) > 0$ of the set S in Hausdorff topological vector space. Our results in Section 5 establish the relationships between the normal regularity of the enlargement $S(r)$ and the subdifferential regularity of its associated minimal time function. These results extend those proved in [3, 6] from the case of distance functions to the case of minimal time functions.

2. Notations and Preliminaries

Throughout the whole paper (unless otherwise specified), we assume that E is a Hausdorff topological vector space. We will denote by E^* the topological dual of E, and by $\langle \cdot, \cdot \rangle$ the pairing between the spaces E and E^*.

Let S be a nonempty closed subset of E and $\bar{x}$ be a point in S. Let us recall the following classical tangent cones $K(S; \bar{x})$ of Bouligand and $T^C(S; \bar{x})$ of Clarke.

- The Bouligand tangent cone (also called contingent cone) $K(S;\bar{x})$ to S at $\bar{x}$ is the set of all $h \in E$ such that for every neigbourhood H of h in E and for every $\epsilon > 0$, there exists $t \in (0,\epsilon)$ such that

$$(\bar{x} + tH) \cap S \neq \emptyset.$$

- The Clarke tangent cone $T^C(S;\bar{x})$ to S at $\bar{x}$ is the set of all $h \in E$ such that for every neigbourhood H of h in E there exist a neigbourhood X of $\bar{x}$ in E and a real number $\epsilon > 0$ such that

$$(x + tH) \cap S \neq \emptyset \quad \text{for all } x \in X \cap S \text{ and } t \in (0,\epsilon).$$

$T^C(S;\bar{x})$ is a closed convex cone (see [14]) and $K(S;\bar{x})$ is a closed cone (that may be nonconvex).

Obviously $T^C(S;\bar{x}) \subset K(S;\bar{x})$. When equality holds in this inclusion i.e. $T^C(S;\bar{x}) = K(S;\bar{x})$, one says that S is *tangentially regular* at $\bar{x}$.

For a closed convex set S and $\bar{x} \in S$, both tangent cones coincide and they are equal to the convex tangent cone defined by $T(S;\bar{x}) = \mathrm{cl}[I\!R_+(S-\bar{x})]$, where $I\!R_+$ denotes the set of all non negative real numbers. The convex normal cone to S at $\bar{x}$ is defined as the negative polar of $T(S;\bar{x})$, that is, $N(S;\bar{x}) := (T(S;\bar{x}))^-$ where $A^- := \{x^* \in E^* : \ \langle x^*, h\rangle \le 0, \ \forall h \in A\}$, for any subset $A \subset E$. Recall also ([29]) that $N(S;\bar{x})$ is also characterized by $N(S;\bar{x}) = \{x^* \in E^* : \langle x^*, x - \bar{x}\rangle \le 0, \forall x \in S\}$.

Let f be a function from E into $I\!R \cup \{-\infty,+\infty\}$ with $|f(\bar{x})| < +\infty$. The generalized directional derivative $f^{\uparrow}(\bar{x};.)$ is defined (see Rockafellar [27]) by

$$\begin{aligned} f^{\uparrow}(\bar{x};h) &= \limsup_{\substack{(x,\alpha)\downarrow_f \bar{x}\\ t\downarrow 0}} \inf_{h'\to h} t^{-1}\left[f(x+th') - \alpha\right] \\ &:= \sup_{H\in\mathcal{N}(h)} \left[\limsup_{\substack{(x,\alpha)\downarrow_f \bar{x}\\ t\downarrow 0}} \left(\inf_{h'\in H} t^{-1}\left[f(x+th')-\alpha\right]\right)\right], \end{aligned}$$

where $(x,\alpha) \downarrow_f \bar{x}$ means $(x,\alpha) \in \mathrm{epi}\, f := \{(z,\beta) \in E \times I\!R; f(z) \le \beta\}$ and $(x,\alpha) \longrightarrow (\bar{x}, f(\bar{x}))$ and $\mathcal{N}(h)$ denotes the filter of neighborhoods of h.

If f is lower semicontinuous (l.s.c.) at $\bar{x}$, the definition can be expressed in the following simpler form

$$f^{\uparrow}(\bar{x};h) = \limsup_{\substack{x\to^f \bar{x}\\ t\downarrow 0}} \inf_{h'\to h} t^{-1}\left[f(x+th') - f(x)\right],$$

where $x \to^f \bar{x}$ means $x \to \bar{x}$ and $f(x) \to f(\bar{x})$. Recall that f is l.s.c. at $\bar{x}$ if and only if $f(\bar{x}) \le \liminf_{x\to\bar{x}} f(x)$.

If f is Lipschitz around $\bar{x}$, then $f^{\uparrow}(\bar{x};h)$ coincides with the Clarke directional derivative $f^0(\bar{x};.)$ defined by

$$f^0(\bar{x};h) = \limsup_{\substack{x\to\bar{x}\\ t\downarrow 0}} t^{-1}\left[f(x+th) - f(x)\right].$$

Recall that (see [27]) f is said to be locally Lipschitz at $\bar{x}$ if there exist a closed balanced neighborhood V of zero and $X \in \mathcal{N}(\bar{x})$ such that

$$|f(x) - f(y)| \leq \rho_V(y - x), \quad \text{for all } x, y \in X.$$

Recall that (see [30]) a set A is said to be balanced provided that $[0,1]A = A$.

Even if f is not necessarily locally Lipschitz at $\bar{x}$ we will put

$$f^0(\bar{x}; h) := \limsup_{\substack{(x,\alpha)\downarrow_f \bar{x} \\ t \downarrow 0}} t^{-1} \left[f(x + th) - \alpha \right].$$

The lower Hadamard directional derivative of f at $\bar{x}$ is defined (see Penot [26]) by

$$f^H(\bar{x}; h) = \liminf_{\substack{h' \to h \\ t \downarrow 0}} t^{-1} \left[f(\bar{x} + th') - f(\bar{x}) \right],$$

and when f is Lipschitz around $\bar{x}$, it coincides with the lower Dini directional derivative defined by

$$f^-(\bar{x}; h) = \liminf_{t \downarrow 0} t^{-1} \left[f(\bar{x} + th) - f(\bar{x}) \right].$$

We recall the definition of directionally Lipschitz functions introduced in [27]. The function f is said to be directionally Lipschitz at $\bar{x}$ with respect to a vector h (see [27]) if

$$\limsup_{\substack{(x,\alpha)\downarrow_f \bar{x} \\ (t,h') \to (0^+, h)}} t^{-1} \left[f(x + th') - \alpha \right] < +\infty,$$

and this is reduced when f is l.s.c. at $\bar{x}$ to

$$\limsup_{\substack{(t,h') \to (0^+, h) \\ x \to_f \bar{x}}} t^{-1} \left[f(x + th') - f(x) \right] < +\infty.$$

If this relation holds for some h, one says that f is directionally Lipschitz at $\bar{x}$. Observe that f is Lipschitz around $\bar{x}$ if and only if it is directionally Lipschitz at $\bar{x}$ with respect to the vector zero (or equivalentely with respect to every vector in E). Recall also from [27] that if f is directionally Lipschitz at $\bar{x}$ in the direction $\bar{h}$, then $f^{\uparrow}(\bar{x}; \bar{h}) = f^0(\bar{x}; \bar{h})$.

We also recall that the Clarke (resp. Hadamard, Dini) subdifferential of f at $\bar{x}$ is defined by

$$\partial^C f(\bar{x}) = \{x^* \in E^* : \langle x^*, h \rangle \leq f^{\uparrow}(\bar{x}; h), \text{ for all } h \in E\}$$

(resp.

$$\partial^H f(\bar{x}) = \{x^* \in E^* : \langle x^*, h \rangle \leq f^H(\bar{x}; h), \text{ for all } h \in E\}$$
$$\partial^- f(\bar{x}) = \{x^* \in E^* : \langle x^*, h \rangle \leq f^-(\bar{x}; h), \text{ for all } h \in E\}).$$

Clearly, we always have $f^H(\bar{x}; h) \leq f^{\uparrow}(\bar{x}; h)$ for all $h \in E$ and we will say that f is *directionally regular* at $\bar{x}$ whenever

$$f^{\uparrow}(\bar{x}; h) = f^H(\bar{x}; h), \quad \text{for all } h \in E.$$

For a convex function $f : E \to I\!R \cup \{+\infty\}$ and $\bar{x} \in \operatorname{dom} f := \{x \in E : f(x) < \infty\}$, the directional derivative (resp. the convex subdifferential) of f is defined by

$$f'(\bar{x}; v) := \lim_{t \downarrow 0} t^{-1}[f(\bar{x} + tv) - f(\bar{x})], \quad \forall v \in E.$$

(resp.

$$\begin{aligned}\partial f(\bar{x}) &:= \{x^* \in E^* : \langle x^*, v\rangle \le f'(\bar{x}; v), \ \forall v \in E\}\\ &= \{x^* \in E^* : \langle x^*, x - \bar{x}\rangle \le f(x) - f(\bar{x}), \ \forall x \in E\}.)\end{aligned}$$

Note that for l.s.c. convex functions we have $f'(\bar{x}; h) = \sigma_{\partial f(\bar{x})}(h)$, whenever $\bar{x} \in \operatorname{core}(\operatorname{dom} f)$. Here σ_A denotes the support function associated with a closed set $A \subset E^*$ defined by $\sigma_A(h) := \sup\{\langle x^*, h\rangle : x^* \in A\}$ and $\operatorname{core}(S) := \{x \in E : \forall h \in E, \exists t_{x,h} > 0, \text{ such that } x + [0, t_{x,h}]h \subset S\}$ is called the algebraic interior of S.

3. The Convex case

Before starting the study of minimal time functions in the convex case, we need to recall some important results proved in [4]. We start with the following lemmas which are needed in all the proofs of our work.

Lemma 3.1. *Let E be Hausdorff topological vector space and let S be a nonempty closed subset of E. Assume that Ω is a bounded set in E with $0 \in \Omega$. Then the following assertions hold:*

(*1*) $T_{S,\Omega}(x)=0$ *if and only if* $x \in S$;

(*2*) *If, in addition Ω is convex, then $T_{S,\Omega}$ is convex on its domain if and only if S is convex.*

We point out that Part (*2*) in the previous lemma has been proved in Proposition 3.4 in [24] in Banach spaces.

Lemma 3.2. *Let E be Hausdorff topological vector space.*

(*a*) *Assume that Ω is a compact subset in E and S is a closed subset in E. Then $T_{S,\Omega}$ is lower semicontinuous at any $x \in \operatorname{dom} T_{S,\Omega}$;*

(*b*) *Assume that Ω is convex weakly compact and S is closed convex. Then $T_{S,\Omega}$ is lower semicontinuous at any $x \in \operatorname{dom} T_{S,\Omega}$;*

(*c*) *Assume that Ω is convex and $0 \in \operatorname{int} \Omega$. Then $T_{S,\Omega}$ is globally Lipschitz on E.*

Part (*b*) covers the result proved in Proposition 3.5 in [24] in Banach spaces. Indeed, they proved the lower semicontinuity of the minimal time function whenever Ω is closed bounded convex, S is closed convex, and the space E is either finite dimensional or reflexive Banach space.

Lemma 3.3. *Let E be Hausdorff topological vector space.*

(*1*) *Assume that S is a nonempty closed convex set in E, $\bar{x} \in S$, and Ω is a bounded convex set with $0 \in \Omega$. Then we have*

$$\partial T_{S,\Omega}(\bar{x}) = N(S;\bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) \leq 1\}.$$

(*2*) *Let S be a nonempty closed set in E with $\bar{x} \in S$ and let Ω be a bounded set in E with $0 \in \Omega$. Then for all $\bar{h} \in E$*

$$T^H_{S,\Omega}(\bar{x};\bar{h}) \leq T_{K(S,\bar{x}),\Omega}(\bar{h}).$$

Let S be a nonempty closed set in E, Ω be a bounded set in E with $0 \in \Omega$, and let $\bar{x} \notin S$. Put $r := T_{S,\Omega}(\bar{x})$. Clearly by Lemma 3.1 we have $r > 0$. Define now the enlargement set of S as follows

$$S(r) := \{x \in E : T_{S,\Omega}(x) \leq r\}. \tag{2}$$

Obviously, we have $\bar{x} \in S(r)$ and $S \subset S(r)$. Also, observe that by Lemma 3.1 the set $S(r)$ is convex whenever S is closed convex and Ω is convex. If, in addition, Ω is weakly compact, then $S(r)$ is closed convex. The following lemma is needed in our next proofs. It extends Proposition 3.1 in [24] (see also Lemma 4.1 in [23]) from Banach spaces to Hausdorff topological vector spaces.

Lemma 3.4. *Let E be a Hausdorff topological vector space. Assume that S is a nonempty closed set in E, $\bar{x} \notin S$, and Ω is a bounded convex set with $0 \in \Omega$. Then for any $x \in E$ we have*

$$T_{S,\Omega}(x) \leq T_{S(r),\Omega}(x) + r$$

and the for any $x \notin S(r)$ we have

$$T_{S,\Omega}(x) = T_{S(r),\Omega}(x) + r.$$

Proof. First we prove the inequality

$$T_{S,\Omega}(x) \leq T_{S(r),\Omega}(x) + r, \quad \text{for any } x \in E.$$

Assume that $x \in E$ with $T_{S(r),\Omega}(x) < \infty$. The case $T_{S(r),\Omega}(x) = \infty$ is obvious. Then we choose any $\delta > 0$ and $w \in \Omega$ such that $x + \delta w \in S(r)$, that is, $T_{S,\Omega}(x + \delta w) \leq r$. Then, for any $\epsilon > 0$ we can choose $\lambda > 0$ and $z \in \Omega$ such that

$$T_{S,\Omega}(x + \delta w) \leq \lambda < T_{S,\Omega}(x + \delta w) + \epsilon \quad \text{and} \quad x + \delta w + \lambda z \in S.$$

By convexity of Ω we have $\delta w + \lambda z \in (\delta + \lambda)\Omega$ and hence

$$x + \delta w + \lambda z \in S \cap [x + (\delta + \lambda)\Omega] \neq \emptyset.$$

Therefore,

$$T_{S,\Omega}(x) \leq \delta + \lambda < \delta + T_{S,\Omega}(x + \delta w) + \epsilon < \delta + r + \epsilon.$$

Since δ is taken arbitrarily satisfying $(x+\delta\Omega)\cap S(r)\neq\emptyset$, we obtain by taking the infimum over all δ the inequality

$$T_{S,\Omega}(x)\leq T_{S(r),\Omega}(x)+r+\epsilon,\quad \forall\epsilon>0,$$

and by taking $\epsilon\to 0$ we get the first desired inequality

$$T_{S,\Omega}(x)\leq T_{S(r),\Omega}(x)+r.$$

Assume now that $x\notin S(r)$ and let us prove the reverse inequality, i,e., $T_{S(r),\Omega}(x)+r\leq T_{S,\Omega}(x)$. The case $T_{S,\Omega}(x)=\infty$ is obvious so we suppose that $T_{S,\Omega}(x)<\infty$. Then we choose any $\delta>0$ and $w\in\Omega$ such that $x+\delta w\in S$. Clearly, $T_{S,\Omega}(x)\leq\delta$ and since $x\notin S(r)$ we have $\delta>r$. Therefore,

$$x+(\delta-r)w+rw\in S\cap[x+(\delta-r)w+r\Omega]\neq\emptyset,$$

and so

$$T_{S,\Omega}(x+(\delta-r)w)\leq r,\quad \text{that is, } x+(\delta-r)w\in S(r).$$

Thus

$$x+(\delta-r)w\in S(r)\cap[x+(\delta-r)\Omega]\neq\emptyset.$$

Then

$$T_{S(r),\Omega}(x)\leq\delta-r.$$

Since δ is taken arbitrarily we can take the infimum over all $\delta>0$ satisfying $S\cap[x+\delta\Omega]\neq\emptyset$ and hence we obtain

$$T_{S(r),\Omega}(x)+r\leq T_{S,\Omega}(x).$$

The proof is complete. □

The following theorem summarizes all our main results in the convex case. It extends Theorem 1 in [7] from the case of distance functions to minimal time function and from normed vector spaces to Hausdorff topological vector spaces.

Theorem 3.5. *Let E be a Hausdorff topological vector space. Assume that S is a closed convex set in E, $\bar{x}\notin S$, and Ω is a weakly compact convex set with $0\in\Omega$. Then we have*

(1) $$\partial T_{S,\Omega}(\bar{x})=N(S(r);\bar{x})\cap\{x^*\in E^*:\sigma_\Omega(-x^*)=1\};$$

(2) If, in addition, $\bar{x}\in\operatorname{core}(\operatorname{dom}T_{S,\Omega})$, then

$$T'_{S,\Omega}(\bar{x};h)\leq T_{T(S(r);\bar{x}),\Omega}(h),\quad \textit{for all } h\in E$$

(3) and

$$T(S(r);\bar{x})=\{h\in E:T'_{S,\Omega}(\bar{x};h)\leq 0\}.$$

Proof. (*1*) Let $x^* \in N(S(r); \bar{x})$ with $\sigma_\Omega(-x^*) = 1$. Then by definition of the convex normal cone we have

$$\langle x^*, x - \bar{x} \rangle \leq 0, \quad \text{for all } x \in S(r). \tag{3}$$

Fix any $x \notin S(r)$, i.e., $T_{S,\Omega}(x) > r$ and let $\epsilon > 0$. Assume that $T_{S,\Omega}(x) < \infty$, the other case is obvious. By definition of minimal time function there exist $w \in \Omega$, $s \in S$ and $\delta > 0$ such that

$$0 < T_{S,\Omega}(x) \leq \delta < T_{S,\Omega}(x) + \epsilon \quad \text{with } s = x + \delta w.$$

Take $z := x + (\delta - r)w$. Then $z + rw = x + \delta w \in S \cap (z + r\Omega) \neq \emptyset$ and so $T_{S,\Omega}(z) \leq r$, i.e., $z \in S(r)$. Therefore, the relations (3) and $\sigma_\Omega(-x^*) = 1$ imply

$$\langle x^*, x - \bar{x} \rangle = \langle x^*, z - \bar{x} \rangle + (\delta - r) \langle -x^*, w \rangle \leq \delta - r < T_{S,\Omega}(x) - T_{S,\Omega}(\bar{x}) + \epsilon.$$

By taking $\epsilon \to 0$, we obtain

$$\langle x^*, x - \bar{x} \rangle \leq T_{S,\Omega}(x) - T_{S,\Omega}(\bar{x}), \quad \text{for all } x \notin S(r). \tag{4}$$

Assume now that $x \in S(r)$, i.e., $T_{S,\Omega}(x) \leq r$. The case $T_{S,\Omega}(x) = r$ is obvious. Indeed, we have by (3)

$$\langle x^*, x - \bar{x} \rangle \leq 0 = T_{S,\Omega}(x) - T_{S,\Omega}(\bar{x}).$$

Assume now, that $T_{S,\Omega}(x) < r$. For any $\epsilon \in (0, r - T_{S,\Omega}(x))$, there exists $\delta > 0$ such that

$$0 \leq T_{S,\Omega}(x) \leq \delta < T_{S,\Omega}(x) + \epsilon < r \quad \text{with } S \cap (x + \delta\Omega) \neq \emptyset.$$

By the assumption $\sigma_\Omega(-x^*) = 1$, there exists some point $u \in \Omega$ such that $\langle x^*, u \rangle < \epsilon - 1$. Take $z := x - (r - \delta)u$. The convexity of Ω ensures that

$$x + \delta\Omega = z + (r - \delta)u + \delta\Omega \subset z + (r - \delta)\Omega + \delta\Omega \subset z + r\Omega$$

and so $\emptyset \neq S \cap (x + \delta\Omega) \subset S \cap (z + r\Omega)$ which yields $T_{S,\Omega}(z) \leq r$, i.e., $z \in S(r)$. Therefore, the relations (3) and $\sigma_\Omega(-x^*) = 1$ imply

$$\begin{aligned} \langle x^*, x - \bar{x} \rangle = &\ \langle x^*, z - \bar{x} \rangle + (r - \delta) \langle x^*, u \rangle \\ &< (r - \delta)(\epsilon - 1) = \delta - r + \epsilon(r - \delta) \\ &< T_{S,\Omega}(x) - T_{S,\Omega}(\bar{x}) + \epsilon(1 + r). \end{aligned}$$

Taking $\epsilon \to 0$ yields

$$\langle x^*, x - \bar{x} \rangle \leq T_{S,\Omega}(x) - T_{S,\Omega}(\bar{x}), \quad \text{for all } x \in S(r). \tag{5}$$

Consequently, from (4) and (5) we obtain

$$\langle x^*, x - \bar{x} \rangle \leq T_{S,\Omega}(x) - T_{S,\Omega}(\bar{x}), \quad \text{for all } x \in E,$$

which means that $x^* \in \partial T_{S,\Omega}(\bar{x})$.

Conversely, let $x^* \in \partial T_{S,\Omega}(\bar{x})$. Then by definition we have

$$\langle x^*, x - \bar{x}\rangle \leq T_{S,\Omega}(x) - T_{S,\Omega}(\bar{x}), \quad \text{for all } x \in E. \tag{6}$$

Using Lemma 3.1 and the definition of $S(r)$ we get

$$\langle x^*, x - \bar{x}\rangle \leq 0, \quad \text{for all } x \in S(r).$$

This means that $x^* \in N(S(r), \bar{x})$. We have to prove that $\sigma_\Omega(-x^*) = 1$. By Lemma 3.4 we have

$$T_{S,\Omega}(x) - T_{S,\Omega}(\bar{x}) \leq T_{S(r),\Omega}(x) - T_{S(r),\Omega}(\bar{x}), \quad \forall x \in E$$

which ensures together with (6) that

$$\langle x^*, x - \bar{x}\rangle \leq T_{S(r),\Omega}(x) - T_{S(r),\Omega}(x), \quad \text{for all } x \in E.$$

This means that $x^* \in \partial T_{S(r),\Omega}(\bar{x})$. Since $\bar{x} \in S(r)$, we have by Proposition 3.3, $\sigma_\Omega(-x^*) \leq 1$. It remains to prove the inequality $\sigma_\Omega(-x^*) \geq 1$. To do that, we fix $\epsilon \in (0, r)$. By the definition of the minimal time function, there exist $u \in \Omega$, $s \in S$, and $\delta \in (r, r + \epsilon^2)$ satisfying $s = \bar{x} + \delta u$. Therefore,

$$s = \bar{x} + \delta u = \bar{x} + \epsilon u + (\delta - \epsilon)u \in S \cap (\bar{x} + \epsilon u + (\delta - \epsilon)\Omega)$$

and hence $T_{S,\Omega}(\bar{x} + \epsilon u) \leq \delta - \epsilon$. Using (6) we obtain

$$\langle x^*, \epsilon u\rangle \leq T_{S,\Omega}(\bar{x} + \epsilon u) - T_{S,\Omega}(\bar{x}) \leq \delta - \epsilon - r$$

Thus

$$\langle -x^*, \epsilon u\rangle \geq r - \delta + \epsilon \geq \epsilon - \epsilon^2.$$

and so

$$\langle -x^*, u\rangle \geq 1 - \epsilon, \quad \forall \epsilon \in (0, r).$$

Taking $\epsilon \to 0$, we get $\sigma_\Omega(-x^*) \geq \langle -x^*, u\rangle \geq 1$ and hence the proof of (*1*) is complete.

(*2*) Let $h \in E$. Assume first that $h \in T(S(r); \bar{x})$. Then by Lemma 3.1 we have $T_{T(S(r),\bar{x}),\Omega}(h) = 0$. Also we have $\langle x^*, h\rangle \leq 0$, for all $x^* \in N(S(r), \bar{x})$. Hence by (*1*) we have $\langle x^*, h\rangle \leq 0$, for all $x^* \in \partial T_{S,\Omega}(\bar{x})$ and so $\sigma_{\partial T_{S,\Omega}(\bar{x})}(h) \leq 0$. Using now the assumption $\bar{x} \in \operatorname{core}(\operatorname{dom} T_{S,\Omega})$ to write $T'_{S,\Omega}(\bar{x}, h) = \sigma_{\partial T_{S,\Omega}(\bar{x})}(h)$ and consequently we obtain the needed inequality

$$T'_{S,\Omega}(\bar{x}, h) \leq 0 = T_{T(S(r),\bar{x}),\Omega}(h).$$

Assume now that $h \notin T(S(r); \bar{x})$. Without loss of generality we assume that $T_{T(S(r),\bar{x}),\Omega}(h) < \infty$. Put $\alpha := T_{T(S(r),\bar{x}),\Omega}(h)$. Let $\delta > 0$ such that

$$(h + \delta\Omega) \cap T(S(r), \bar{x}) \neq \emptyset.$$

There exists $w \in \Omega$ such that $h + \delta w \in T(S(r), \bar{x})$. Therefore, by (*1*) we get

$$\langle x^*, h\rangle = \langle x^*, h + \delta w\rangle + \delta \langle -x^*, w\rangle \le \delta\sigma_\Omega(-x^*) = \delta, \quad \text{for all } x^* \in \partial T_{S,\Omega}(\bar{x}).$$

This ensures that

$$\sigma_{\partial T_{S,\Omega}(\bar{x})}(h) \le \delta$$

and since we have

$$T'_{S,\Omega}(\bar{x}, h) = \sigma_{\partial T_{S,\Omega}(\bar{x})}(h)$$

we obtain

$$T'_{S,\Omega}(\bar{x}, h) \le \delta.$$

Since δ is taken arbitrarily satisfying $(h + \delta\Omega) \cap T(S(r), \bar{x}) \neq \emptyset$ we take the infimum over δ in the last inequality to get

$$T'_{S,\Omega}(\bar{x}, h) \le \inf\{\delta > 0 : \text{ such that } (h + \delta\Omega) \cap T(S(r), \bar{x}) \neq \emptyset\} = T_{T(S(r),\bar{x}),\Omega}(h).$$

Thus completing the proof of (*2*).

(*3*) The inclusion $T(S(r), \bar{x}) \subset \{h \in E : T'_{S,\Omega}(\bar{x}; h) \le 0\}$ follows from the assertion (*2*). Conversely, let $h \in E$ with $T'_{S,\Omega}(\bar{x}; h) \le 0$. Let (t_j) be a net of positive real numbers converging to zero. Let $j_0 \in J$ such that

$$T_{S,\Omega}(\bar{x} + t_j h) > 2t_j^2 > 0, \quad \text{for any } j \ge j_0. \tag{7}$$

This fact is ensured by the fact that $\bar{x} \notin S$ and the lower semicontinuity of $T_{S,\Omega}$ which is ensured by Part (*b*) in Lemma 3.2. Indeed, if we assume by contradiction that (7) is not true, that is, for any $j_0 \in J$, there exists some $j \ge j_0$ such that $T_{S,\Omega}(\bar{x} + t_j h) \le 2t_j^2$. This ensures the existence of a subnet $(t_{s(j)})$ of (t_j) such that $T_{S,\Omega}(\bar{x} + t_{s(j)} h) \le 2t_{s(j)}^2$, for all j. Therefore, the lower semicontinuity of $T_{S,\Omega}$ ensures that

$$0 \le T_{S,\Omega}(\bar{x}) \le \liminf_j T_{S,\Omega}(\bar{x} + t_{s(j)} h) \le \lim_j 2t_{s(j)}^2 = 0$$

and so by Lemma 3.1 we get $\bar{x} \in S$ which contradicts the assumption $\bar{x} \notin S$ and so (7) holds. The assumption $T'_{S,\Omega}(\bar{x}; h) \le 0$ ensures the existence of some $j_1 \ge j_0$ such that

$$T_{S,\Omega}(\bar{x} + t_j h) \le r + t_j^2, \quad \text{for any } j \ge j_1. \tag{8}$$

Thus, combining (7) and (8) we obtain

$$0 < 2t_j^2 < T_{S,\Omega}(\bar{x} + t_j h) \le r + t_j^2, \quad \text{for any } j \ge j_1. \tag{9}$$

Using now the definition of the minimal time function to choose for any $j \in J$ some $w_j \in \Omega$ and $\delta_j > 0$ such that

$$T_{S,\Omega}(\bar{x} + t_j h) \le \delta_j < T_{S,\Omega}(\bar{x} + t_j h) + t_j^2 \quad \text{and} \quad \bar{x} + t_j h + \delta_j w_j \in S.$$

Hence

$$0 < \delta_j - 2t_j^2 < r, \quad \forall j \ge j_1.$$

Let $z := \bar{x} + t_j h + 2t_j^2 w_j$ and $h_j := h + 2t_j w_j$. Then

$$z + (\delta_j - 2t_j^2)w_j = \bar{x} + t_j h + \delta_j w_j \in S \cap (z + (\delta_j - 2t_j^2)\Omega) \neq \emptyset.$$

So, $T_{S,\Omega}(z) \leq \delta_j - 2t_j^2 \leq r$, that is, $z \in S(r)$. Therefore,

$$\bar{x} + t_j h_j = \bar{x} + t_j h + 2t_j^2 w_j \in S(r).$$

Now, by the boundedness of the set Ω we get the convergence of $2t_j w_j$ to zero. This ensures the convergence of $h + 2t_j w_j \to h$ which means that $h \in T(S(r), \bar{x})$ and hence the proof is finished. □

Taking now the case Ω is the closed unit ball of a normed vector space E we get the following well known result (see for instance [7]).

Corollary 3.6. *Assume that E is a normed vector space, S is a closed convex set in E, and $\bar{x} \notin S$. Then we have*

(*1*) $$\partial d_S(\bar{x}) = N(S(r); \bar{x}) \cap \{x^* \in E^* : \|x^*\| = 1\};$$

(*2*) $$d'_S(\bar{x}; h) \leq d_{T(S(r);\bar{x})}(h), \quad \textit{for all } h \in E;$$

(*3*) $$T(S(r); \bar{x}) = \{h \in E : d'_S(\bar{x}; h) \leq 0\}.$$

Using Theorem 3.5 we deduce the following important result on the subdifferential of the Minkowski function at nonzero points in Hausdorff topological vector spaces. It extends Theorem 6.1 in [16] from Banach spaces to Hausdorff topological vector spaces and from the case of $0 \in \operatorname{int}\Omega$ to the case 0 not necessarily an interior point of Ω.

Theorem 3.7. *Let E be a Hausdorff topological vector space and let Ω be a weakly compact convex set in E with $0 \in \Omega$. Then for any $\bar{x} \neq 0$ we have*

$$\partial\rho_\Omega(\bar{x}) = \{x^* \in E^* : \sigma_\Omega(x^*) = 1 \textit{ and } \rho_\Omega(\bar{x}) = \langle x^*, \bar{x}\rangle\}. \tag{10}$$

Proof. Let $\bar{x} \neq 0$. Put $S = \{0\}$, $r := \rho_\Omega(\bar{x}) = T_{S,\Omega}(-\bar{x}) > 0$ and $\Omega_{\bar{x}} := \{x \in E : \rho_\Omega(x) \leq r\}$. Clearly $S(r) = -\Omega_{\bar{x}}$ and $N(S(r); -\bar{x}) = -N(\Omega_{\bar{x}}; \bar{x})$. Therefore, we have by Part (*1*) in Theorem 3.5

$$\begin{aligned} \partial\rho_\Omega(\bar{x}) = -\partial T_{S,\Omega}(-\bar{x}) &= -N(S(r); -\bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(x^*) = 1\} & (11)\\ &= N(\Omega_{\bar{x}}; \bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(x^*) = 1\}. & (12) \end{aligned}$$

Let $x^* \in \partial\rho_\Omega(\bar{x})$. Then $\sigma_\Omega(x^*) = 1$ and for any $x \in \Omega_{\bar{x}}$ we have

$$\langle x^*, x - \bar{x}\rangle \leq 0. \tag{13}$$

Since Ω is convex and $0 \in \Omega$ we have $\frac{\bar{x}}{r+\epsilon} \in \Omega$ for any $\epsilon > 0$. Then

$$\left\langle x^*, \frac{\bar{x}}{r+\epsilon}\right\rangle \leq \sigma_\Omega(x^*) = 1, \quad \text{that is } \langle x^*, \bar{x}\rangle \leq r + \epsilon, \quad \forall\epsilon > 0,$$

which ensures by taking $\epsilon \to 0$ that

$$\langle x^*, \bar{x} \rangle \leq r.$$

On the other hand, we have $rh \in \Omega_{\bar{x}}$ for any $h \in \Omega$ and hence by (13) we obtain

$$\langle x^*, rh \rangle \leq \langle x^*, \bar{x} \rangle .$$

Combining this inequality with the previous one we get

$$\langle x^*, h \rangle \leq \left\langle x^*, \frac{\bar{x}}{r} \right\rangle \leq 1, \quad \forall h \in \Omega.$$

This ensures that

$$\sigma_\Omega(x^*) \leq \left\langle x^*, \frac{\bar{x}}{r} \right\rangle \leq 1,$$

which ensures that $\left\langle x^*, \frac{\bar{x}}{r} \right\rangle = 1$ and so

$$\langle x^*, \bar{x} \rangle = r = \rho_\Omega(\bar{x}),$$

and hence the proof of the inclusion (10) is complete.

Conversely, we fix any $x^* \in E^*$ with $\sigma_\Omega(x^*) = 1$ and $\langle x^*, \bar{x} \rangle = \rho_\Omega(\bar{x})$. Observe that our assumptions on Ω ensure that $\{x \in E : \rho_\Omega(x) < 1\} \subset \Omega$. Then for any $x \in \Omega_{\bar{x}}$, we have $\rho_\Omega(x) \leq r < r + \epsilon$, for any $\epsilon > 0$ and so $\frac{x}{r+\epsilon} \in \Omega$ for any $\epsilon > 0$. Hence

$$\begin{aligned} \langle x^*, x \rangle &= (r + \epsilon) \left\langle x^*, \frac{x}{r + \epsilon} \right\rangle \leq (r + \epsilon)\sigma_\Omega(x^*) \\ &= (r + \epsilon) = \rho_\Omega(\bar{x}) + \epsilon = \langle x^*, \bar{x} \rangle + \epsilon, \end{aligned}$$

that is,

$$\langle x^*, x - \bar{x} \rangle \leq \epsilon, \quad \text{for any } \epsilon > 0.$$

Taking $\epsilon \to 0$ we obtain

$$\langle x^*, x - \bar{x} \rangle \leq 0, \quad \text{for any } x \in \Omega_{\bar{x}},$$

which ensures by definition that $x^* \in N(\Omega_{\bar{x}}, \bar{x})$. Therefore, we use (11) to get $x^* \in \partial\rho_\Omega(\bar{x})$ and hence the proof is complete. □

Based on Theorem 3.7 we deduce the following necessary condition for a point $\bar{u} \in S$ to be a generalized projection $\pi_{S,\Omega}(\bar{x})$ on a closed set S in Hausdorff topological vector spaces.

Theorem 3.8. *Let E be a Hausdorff topological vector space, S be a nonempty closed subset of E, Ω be a weakly compact convex set in E with $0 \in \Omega$ and let $\bar{x} \notin S$. If $\bar{u} \in \pi_{S,\Omega}(\bar{x})$, then there exists $x^* \in N^C(S; \bar{u})$ such that $\sigma_\Omega(-x^*) = 1$ and $\langle x^*, \bar{x} - \bar{u} \rangle = \rho_\Omega(\bar{u} - \bar{x})$.*

Proof. Let $\bar{u} \in \pi_{S,\Omega}(\bar{x})$, that is, $\rho_\Omega(\bar{u}-\bar{x}) = \inf_{s\in S} \rho_\Omega(s-\bar{x})$. Then $\bar{u}$ minimizes the function $\rho_\Omega(\cdot - \bar{x})$ over S, which ensures that $0 \in \partial\rho_\Omega(\cdot - \bar{x})(\bar{u}) + N^C(S;\bar{u})$ and hence there exists $x^* \in N^C(S;\bar{u})$ such that $x^* \in -\partial\rho_\Omega(\cdot - \bar{x})(\bar{u})$. Then using Theorem 3.7 we get $\sigma_\Omega(-x^*) = 1$ and $\rho_\Omega(\bar{u}-\bar{x}) = \langle x^*, \bar{x}-\bar{u}\rangle$. □

Another corollary of Theorem 3.7 establishes a characterization of generalized projection $\pi_{S,\Omega}$ on closed convex sets in Hausdorff topological vector spaces.

Theorem 3.9. *Let E be a Hausdorff topological vector space, S be a nonempty closed convex subset of E, Ω be a weakly compact convex set in E with $0 \in \Omega$ and let $\bar{x} \notin S$. Then $\bar{u} \in \pi_{S,\Omega}(\bar{x})$ if and only if there exists $x^* \in N(S;\bar{u})$ such that $\sigma_\Omega(-x^*) = 1$ and $\langle x^*, \bar{x}-\bar{u}\rangle = \rho_\Omega(\bar{u}-\bar{x})$.*

Proof. Let $\bar{u} \in S$. Then $\bar{u} \in \pi_{S,\Omega}(\bar{x})$ if and only if $\bar{u}$ minimizes the function $\rho_\Omega(\cdot - \bar{x})$ over S, which is equivalent to $0 \in \partial\rho_\Omega(\cdot - \bar{x})(\bar{u}) + N(S;\bar{u})$ (since S and ρ_Ω are convex). Then the conclusion follows from Theorem 3.7. □

4. The Nonconvex case

Throughout this section S is a nonempty closed subset (**not necessarily convex**) of the Hausdorff topological vector space E and $\bar{x} \notin S$.

Theorem 4.1. *Let E be a Hausdorff topological vector space, S be a nonempty closed subset of E, Ω be a bounded set in E with $0 \in \Omega$ and let $\bar{x} \notin S$. Assume that $T_{S,\Omega}$ is l.s.c. at $\bar{x}$. Then for all $\bar{h} \in E$ we have*

$$(1)\qquad T^H_{S,\Omega}(\bar{x};\bar{h}) \le T^H_{S(r),\Omega}(\bar{x};\bar{h}) \le T_{K(S(r);\bar{x}),\Omega)}(\bar{h});$$

$$(2)\qquad \begin{aligned} K(S(r);\bar{x}) &= \{h \in E : T^H_{S,\Omega}(\bar{x};\bar{h}) \le 0\} \\ &= \{h \in E : T^H_{S(r),\Omega}(\bar{x};\bar{h}) = 0\}. \end{aligned}$$

Proof. (*1*) Fix any $\bar{h} \in E$. Since $\bar{x} \in S(r)$ then the second inequality follows from Lemma 3.3 (see Theorem 4.8 in [4]). So, we only show the first inequality. By Lemma 3.4, we have for any $t > 0$ and any $h \in E$

$$T_{S,\Omega}(\bar{x}+th) - T_{S,\Omega}(\bar{x}) \le T_{S(r),\Omega}(\bar{x}+th).$$

This ensures

$$\liminf_{\substack{t\to 0^+\\ h\to\bar{h}}} t^{-1}[T_{S,\Omega}(\bar{x}+th) - T_{S,\Omega}(\bar{x})] \le \liminf_{\substack{t\to 0^+\\ h\to\bar{h}}} t^{-1}T_{S(r),\Omega}(\bar{x}+th),$$

and hence the desired inequality is proved.

(*2*) By the first part, we have the following inclusions

$$\begin{aligned} K(S(r);\bar{x}) &\subset \{h \in E : T^H_{S(r),\Omega}(\bar{x};\bar{h}) = 0\} \\ &\subset \{h \in E : T^H_{S,\Omega}(\bar{x};\bar{h}) \le 0\}. \end{aligned}$$

So, we will show the following one

$$\{h \in E : T^H_{S,\Omega}(\bar{x}; \bar{h}) \le 0\} \subset K(S(r); \bar{x}).$$

Let $h \in E$ with $T^H_{S,\Omega}(\bar{x}; h) \le 0$. Let (t_j, h_j) be a net in $(0, \infty) \times E$ converging to $(0, h)$ and satisfying the limit

$$T^H_{S,\Omega}(\bar{x}; h) = \lim_j t_j^{-1}[T_{S,\Omega}(\bar{x} + t_j h_j) - T_{S,\Omega}(\bar{x})].$$

Similarly to the proof of Theorem 3.5, we use the lower semicontinuity of $T_{S,\Omega}$ and the assumption $\bar{x} \notin S$ to find some $j_0 \in J$ such that

$$T_{S,\Omega}(\bar{x} + t_j h_j) > 2t_j^2 > 0, \quad \text{for any } j \ge j_0. \tag{14}$$

Using now the assumption $T^H_{S,\Omega}(\bar{x}; h) \le 0$ ensures the existence of some $j_1 \ge j_0$ such that

$$T_{S,\Omega}(\bar{x} + t_j h_j) \le r + t_j^2, \quad \text{for any } j \ge j_1. \tag{15}$$

Thus, combining (14) and (15) we obtain

$$0 < 2t_j^2 < T_{S,\Omega}(\bar{x} + t_j h_j) \le r + t_j^2, \quad \text{for any } j \ge j_1. \tag{16}$$

Using now the definition of the minimal time function to choose for any $j \ge j_1$ some $w_j \in \Omega$ and $\delta_j > 0$ such that

$$T_{S,\Omega}(\bar{x} + t_j h_j) \le \delta_j < T_{S,\Omega}(\bar{x} + t_j h_j) + t_j^2 \quad \text{and} \quad \bar{x} + t_j h_j + \delta_j w_j \in S.$$

Hence

$$0 \le \delta_j - 2t_j^2 < r, \quad \forall j \ge j_1.$$

Let $z := \bar{x} + t_j h_j + 2t_j^2 w_j$ and $h'_j := h_j + 2t_j w_j$. Then

$$z + (\delta_j - 2t_j^2) w_j = \bar{x} + t_j h_j + \delta_j w_j \in S \cap (z + (\delta_j - 2t_j^2)\Omega) \ne \emptyset.$$

So, $T_{S,\Omega}(z) \le \delta_j - 2t_j^2 \le r$, that is, $z \in S(r)$.
Therefore,

$$\bar{x} + t_j h'_j = \bar{x} + t_j h_j + 2t_j^2 w_j \in S(r).$$

Now, by the boundedness of the set Ω we get the convergence of $2t_j w_j$ to zero. This ensures the convergence of $h'_j = h_j + 2t_j w_j \to h$ which means that $h \in K(S(r), \bar{x})$ and hence the proof is finished. □

Taking E to be a normed vector space and Ω to be the closed unit ball of E, we get the following corollary proved in Corollary 4.4 in [3] (see also [5]).

Corollary 4.2. *Let E be a normed vector space, S be a nonempty closed subset of E, and let $\bar{x} \notin S$. Then for all $\bar{h} \in E$ we have*

(1) $$d^H_S(\bar{x}; \bar{h}) \le d^H_{S(r)}(\bar{x}; \bar{h}) \le d_{K(S(r);\bar{x})}(\bar{h}).$$

(2) $$\begin{aligned} K(S(r); \bar{x}) &= \{h \in E : d^H_S(\bar{x}; h) \le 0\} \\ &= \{h \in E : d^H_{S(r)}(\bar{x}; h) = 0\}. \end{aligned}$$

Now, we turn to establish similar results for the Clarke tangent cone to $S(r)$ and the generalized directional derivative of $T_{S,\Omega}$ at $\bar{x} \notin S$.

Theorem 4.3. *Let E be a Hausdorff topological vector space, S be a nonempty closed subset of E, Ω be a bounded set in E with $0 \in \Omega$ and let $\bar{x} \notin S$. Assume that $T_{S,\Omega}$ is l.s.c. at $\bar{x}$. Then*

(*a*)
$$\{h \in E : T^{\uparrow}_{S,\Omega}(\bar{x}; h) \leq 0\} \subset T^C(S(r); \bar{x}). \tag{17}$$

If, $T_{S,\Omega}$ is directionally regular at $\bar{x}$, then equality holds in (17) *and $S(r)$ is tangentially regular at $\bar{x}$.*

(*b*) *Assume that E is locally convex and $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$. Then*

$$\Big(\{h \in E : T^{\uparrow}_{S,\Omega}(\bar{x}; h) \leq 0\}\Big)^{-} = \mathrm{cl}_{w^*}\left(I\!R_+ \partial^C T_{S,\Omega}(\bar{x})\right).$$

Proof. (*a*) Let $\bar{x}, \bar{h} \in E$ with $T^{\uparrow}_{S,\Omega}(\bar{x}; \bar{h}) \leq 0$. Let

$$\beta := T^{\uparrow}_{S,\Omega}(\bar{x}; h) = \sup_{H \in \mathcal{N}(\bar{h})} \inf_{\substack{\delta > 0 \\ X \in \mathcal{N}(\bar{x})}} \sup_{\substack{t \in (0,\delta) \\ (x,\alpha) \in \operatorname{epi} T_{S,\Omega} \cap [r-\delta, r+\delta]}} \inf_{h \in H} t^{-1}\left[T_{S,\Omega}(x + th) - \alpha\right].$$

Consider any $H \in \mathcal{N}(\bar{h})$. Then

$$\inf_{\substack{\delta > 0 \\ X \in \mathcal{N}(\bar{x})}} \sup_{\substack{t \in (0,\delta) \\ (x,\alpha) \in \operatorname{epi} T_{S,\Omega} \cap (X \times [r-\delta, r+\delta])}} \inf_{h \in H} t^{-1}\left[T_{S,\Omega}(x + th) - \alpha\right] \leq \beta \leq 0.$$

There exist $\delta_0 > 0$ and $X_0 \in \mathcal{N}(\bar{x})$ such that

$$\inf_{h \in \bar{H}} t^{-1}\left[T_{S,\Omega}(x + th) - \alpha\right] \leq 0, \tag{18}$$

for any $t \in (0, \delta_0)$ and for any $(x, \alpha) \in \operatorname{epi} T_{S,\Omega} \cap (X_0 \times [r - \delta_0, r + \delta_0])$. Fix any $x \in X_0 \cap S(r)$ and any $t \in (0, \delta_0)$. Then $(x, r) \in \operatorname{epi} T_{S,\Omega} \cap (X_0 \times [r - \delta_0, r + \delta_0])$. So there exists by (18) some point $h_0 \in H$ such that

$$t^{-1}\left[T_{S,\Omega}(x + th_0) - r\right] \leq 0, \quad \text{i.e., } T_{S,\Omega}(x + th_0) \leq r,$$

which means that $x + th_0 \in S(r)$ and hence we obtain

$$(x + tH) \cap S(r) \neq \emptyset, \quad \forall x \in X_0 \cap S(r), \forall t \in (0, \delta_0).$$

Therefore, we conclude that for any $H \in \mathcal{N}(\bar{h})$, there exist $\delta_0 > 0$ and $X_0 \in \mathcal{N}(\bar{x})$ such that $(x + tH) \cap S(r) \neq \emptyset$, $\forall x \in X_0 \cap S(r), \forall t \in (0, \delta_0)$. This ensures by the definition of Clarke tangent cone that $\bar{h} \in T^C(S; \bar{x})$.

Assume now that $T_{S,\Omega}$ is directionally regular at $\bar{x}$. Then $T^{\uparrow}_{S,\Omega}(\bar{x}; h) = T^H_{S,\Omega}(\bar{x}; h)$, for all $h \in E$ and hence by Part (*2*) of the previous theorem and the inclusion (17) we obtain

$$K(S(r); \bar{x}) = \{h \in E : T^H_{S,\Omega}(\bar{x}; h) \leq 0\} = \{h \in E : T^{\uparrow}_{S,\Omega}(\bar{x}; h) \leq 0\} \subset T(S(r); \bar{x}).$$

Since the reverse inclusion $T(S(r); \bar{x}) \subset K(S(r); \bar{x})$ always holds we get the tangential regularity of $S(r)$ at $\bar{x}$ and the equality form in (17) holds.

(b) Assume now that E is locally convex and $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$. Denote

$$T := \{h \in E : T^{\uparrow}_{S,\Omega}(\bar{x}; h) \leq 0\}.$$

Let us prove that $\partial^C T_{S,\Omega}(\bar{x}) \subset T^-$. Since $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$, we take some $x^* \in E^*$ such that

$$\langle x^*, h\rangle \leq T^{\uparrow}_{S,\Omega}(\bar{x}; h), \quad \forall h \in E.$$

Hence, for any $h \in T$, we have

$$\langle x^*, h\rangle \leq T^{\uparrow}_{S,\Omega}(\bar{x}; h) \leq 0,$$

that is, $x^* \in T^-$. Since T^- is a closed convex cone we obtain

$$\mathrm{cl}_{w^*}(I\!R_+ \partial^C T_{S,\Omega}(\bar{x})) \subset T^- = \Big(\{h \in E : T^{\uparrow}_{S,\Omega}(\bar{x}; h) \leq 0\}\Big)^- .$$

Conversely, we have to prove the direct inclusion of (b), i.e.,

$$\Big(\{h \in E : T^{\uparrow}_{S,\Omega}(\bar{x}; h) \leq 0\}\Big)^- \subset \mathrm{cl}_{w^*}(I\!R_+ \partial^C T_{S,\Omega}(\bar{x})).$$

Since $T^{\uparrow}_{S,\Omega}(\bar{x}; \cdot)$ is l.s.c. and sublinear by Theorem 2 in Rockafellar [27], the set T is a closed convex cone and hence $(T^-)^- = T$. Therefore, we will prove that the negative polar of $\mathrm{cl}_{w^*}(I\!R_+ \partial^C T_{S,\Omega}(\bar{x}))$ is included in T, that is,

$$\big(\mathrm{cl}_{w^*}(I\!R_+ \partial^C T_{S,\Omega}(\bar{x}))\big)^- \subset \{h \in E : T^{\uparrow}_{S,\Omega}(\bar{x}; h) \leq 0\}.$$

Note first that

$$(\mathrm{cl}_{w^*}(I\!R_+ \partial^C T_{S,\Omega}(\bar{x})))^- = (\partial^C T_{S,\Omega}(\bar{x}))^-.$$

So consider any $h \in (\partial^C T_{S,\Omega}(\bar{x}))^-$. Then we have

$$\langle x^*, h\rangle \leq 0, \quad \forall x^* \in \partial^C T_{S,\Omega}(\bar{x}).$$

Hence (since $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$)

$$T^{\uparrow}_{S,\Omega}(\bar{x}; h) = \sigma_{\partial^C T_{S,\Omega}(\bar{x})}(h) = \sup_{x^* \in \partial^C T_{S,\Omega}(\bar{x})} \langle x^*, h\rangle \leq 0,$$

which ensures that $\bar{h} \in T$. Thus completing the proof. $\square$

The following corollary characterizes the Clarke tangent cone and the Clarke normal cone to the enlargement set $S(r)$ in terms of the generalized directional derivative of $T_{S,\Omega}$ at $\bar{x}$. It extends Corollary 4.4 in [3] (see also [5]).

Corollary 4.4. *Let E be a Hausdorff topological vector space, S be a nonempty closed subset of E, Ω be a bounded set in E with $0 \in \Omega$ and let $\bar{x} \notin S$. If $T_{S,\Omega}$ is l.s.c. and directionally regular at $\bar{x}$ with $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$, then we have*

$$T^C(S(r); \bar{x}) = \{h \in E : T^{\uparrow}_{S,\Omega}(\bar{x}; h) \leq 0\} = \{h \in E : T^{\uparrow}_{S(r),\Omega}(\bar{x}; h) = 0\}$$

and

$$N^C(S(r); \bar{x}) = \mathrm{cl}_{w^*}\left(I\!R_+ \partial^C T_{S,\Omega}(\bar{x})\right) = \mathrm{cl}_{w^*}\left(I\!R_+ \partial^C T_{S(r),\Omega}(\bar{x})\right).$$

The following corollary extends Corollary 1 in [9] from normed vector spaces to Hausdorff Topological vector spaces. It says that if we put the calmness of $T_{S,\Omega}$ instead of its l.s.c. and the assumption $0 \notin \partial^C T_{S,\Omega}(\bar{x})$ instead of the assumption $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$ in Corollary 4.4 we may remove the weak star closedness in the second equation of the second relation in Corollary 4.4. Recall that f is said to be calm at $\bar{x}$ if there exist a closed balanced neighborhood V of zero and $X \in \mathcal{N}(\bar{x})$ such that

$$|f(x) - f(\bar{x})| \leq \rho_V(\bar{x} - x), \quad \text{for all } x \in X.$$

Corollary 4.5. *Let E be a Hausdorff topological vector space, S be a nonempty closed subset of E, Ω be a bounded set in E with $0 \in \Omega$ and let $\bar{x} \notin S$. If $T_{S,\Omega}$ is calm and directionally regular at $\bar{x}$ and $0 \notin \partial^C T_{S,\Omega}(\bar{x})$, then we have*

$$N^C(S(r); \bar{x}) = I\!R_+ \partial^C T_{S,\Omega}(\bar{x}).$$

Proof. Using the calmness of $T_{S,\Omega}$ at $\bar{x}$ and Banach-Alaoglu theorem (see Theorem 3.5 in [30]), we get the weak star compactness of $\partial^C T_{S,\Omega}(\bar{x})$ in E^*. Therefore, the assumption $0 \notin \partial^C T_{S,\Omega}(\bar{x})$ with the weak star compactness of $\partial^C T_{S,\Omega}(\bar{x})$ ensures the weak star closedness of the cone generated by $\partial^C T_{S,\Omega}(\bar{x})$, that is, $\mathrm{cl}_{w^*}\left(I\!R_+ \partial^C T_{S,\Omega}(\bar{x})\right) = I\!R_+ \partial^C T_{S,\Omega}(\bar{x})$. Thus the conclusion follows from the previous corollary. □

The next results depend upon the non-emptiness of the minimum set for the minimal time function $T_{S,\Omega}$ defined as follows

$$\pi_{S,\Omega}(u) := \{x \in S : T_{S,\Omega}(u) = \rho_\Omega(x - u)\}.$$

We begin with the following lemma. It extends Proposition 3.4 in [24] from Banach spaces to Hausdorff topological vector spaces (see also Lemma 3.1 in [23] for the case $0 \in \operatorname{int} \Omega$ in Banach spaces). This lemma is needed in the proof of our next results.

Lemma 4.6. *Let E be a Hausdorff topological vector space, S be a nonempty closed subset of E, Ω be a convex set in E and let $\bar{x} \notin S$ with $T_{S,\Omega}(\bar{x}) < \infty$. For every $x \in \pi_{S,\Omega}(\bar{x})$, and for every $t \in (0,1]$ we have*

$$T_{S,\Omega}(tx + (1-t)\bar{x}) = (1-t) T_{S,\Omega}(\bar{x}). \tag{19}$$

This ensures that $x \in \pi_{S,\Omega}(tx + (1-t)\bar{x})$, for any $t \in (0,1]$.

Proof. Let $x \in \pi_{S,\Omega}(\bar{x})$. Then $x \in S$ with $T_{S,\Omega}(\bar{x}) = \rho_\Omega(x - \bar{x}) < \infty$. First we prove the inequality

$$T_{S,\Omega}(tx + (1-t)\bar{x}) \leq (1-t) T_{S,\Omega}(\bar{x}), \quad \forall t \in (0,1].$$

Put $\alpha := \rho_\Omega(x - \bar{x}) > 0$ and fix any $t \in (0,1]$. Clearly we can find for any $\epsilon > 0$ some $\alpha_0 > 0$ such that $x - \bar{x} \in \alpha_0 \Omega$ and $\alpha_0 \in [\alpha, \alpha + \epsilon)$ and so

$$x = tx + (1-t)\bar{x} + \alpha_0(1-t)\frac{(x - \bar{x})}{\alpha_0} \in S \cap [tx + (1-t)\bar{x} + \alpha_0(1-t)\Omega] \neq \emptyset,$$

which ensures that

$$T_{S,\Omega}(tx+(1-t)\bar{x}) \leq \alpha_0(1-t) < (1-t)[\rho_\Omega(x-\bar{x})+\epsilon] \leq (1-t)T_{S,\Omega}(\bar{x})+\epsilon, \ \ \forall\epsilon > 0.$$

Taking $\epsilon \to 0^+$ finishes the proof of the first inequality.

By the first part we have $T_{S,\Omega}(tx+(1-t)\bar{x}) < \infty$ whenever $T_{S,\Omega}(\bar{x}) < \infty$ and so for any $\epsilon > 0$ we can find some $w \in \Omega$, some $\delta > 0$, and some $\delta_0 > 0$ such that

$$T_{S,\Omega}(tx+(1-t)\bar{x}) \leq \delta < T_{S,\Omega}(tx+(1-t)\bar{x}) + \epsilon \quad \text{and} \quad tx+(1-t)\bar{x}+\delta w \in S,$$

and

$$x-\bar{x} \in \delta_0\Omega \ \text{ with } 0 < \rho_\Omega(x-\bar{x}) \leq \delta_0 < \rho_\Omega(x-\bar{x})+\epsilon.$$

Then by convexity of Ω we have $\delta_0 t\frac{(x-\bar{x})}{\delta_0} + \delta w \in (\delta_0 t+\delta)\Omega$ and so

$$\bar{x}+t(x-\bar{x})+\delta w = \bar{x}+\delta_0 t\frac{(x-\bar{x})}{\delta_0}+\delta w \in S \cap [\bar{x}+(\delta_0 t+\delta)\Omega] \neq \emptyset.$$

Hence

$$\begin{aligned} T_{S,\Omega}(\bar{x}) &\leq \delta_0 t+\delta \\ &< t(\rho_\Omega(x-\bar{x})+\epsilon)+T_{S,\Omega}(tx+(1-t)\bar{x})+\epsilon \\ &< tT_{S,\Omega}(\bar{x})+T_{S,\Omega}(tx+(1-t)\bar{x})+2\epsilon \end{aligned}$$

and so by taking $\epsilon \to 0^+$ we obtain

$$(1-t)T_{S,\Omega}(\bar{x}) \leq T_{S,\Omega}(tx+(1-t)\bar{x}).$$

This completes the proof of the lemma. □

We use this lemma to prove the following proposition on lower Dini directional derivatives and Dini subdifferentials of $T_{S,\Omega}$.

Proposition 4.7. *Assume that the assumptions of Lemma* 4.6 *are fulfilled and assume that* $0 \in \Omega$. *Then for every* $\bar{x} \notin S$ *and for every* $x \in \pi_{S,\Omega}(\bar{x})$ *we have*

$$T^{-}_{S,\Omega}(\bar{x}; x-\bar{x}) = -T_{S,\Omega}(\bar{x})$$

and

$$\partial^{-}T_{S,\Omega}(\bar{x}) \subset \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\}.$$

Proof. By Lemma 4.6, we have

$$T^{-}_{S,\Omega}(\bar{x}; x-\bar{x}) = \liminf_{t\to 0^+} t^{-1}\left[T_{S,\Omega}(\bar{x}+t(x-\bar{x})) - T_{S,\Omega}(\bar{x})\right] = -T_{S,\Omega}(\bar{x})$$

We turn now to show the inclusion

$$\partial^{-}T_{S,\Omega}(\bar{x}) \subset \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\}.$$

Fix any $x^* \in \partial^- T_{S,\Omega}(\bar{x})$. Then

$$\langle x^*, h\rangle \leq T^-_{S,\Omega}(\bar{x}; h), \quad \forall h \in E.$$

Then by the first part we have for any $x \in \pi_{S,\Omega}(\bar{x})$

$$\langle x^*, x - \bar{x}\rangle \leq T^-_{S,\Omega}(\bar{x}; x - \bar{x}) = -T_{S,\Omega}(\bar{x}) = -\rho_\Omega(x - \bar{x})$$

and so

$$\left\langle -x^*, \frac{x - \bar{x}}{\rho_\Omega(x - \bar{x})}\right\rangle \geq 1,$$

since $\rho_\Omega(x - \bar{x}) \neq 0$ because $\bar{x} \notin S$. Thus, for any $\epsilon > 0$ there exist $w \in \Omega$ and $\alpha_0 > 0$ such that

$$0 < \rho_\Omega(x - \bar{x}) \leq \alpha_0 < \rho_\Omega(x - \bar{x}) + \epsilon$$

and $x - \bar{x} \in \alpha_0 \Omega$. Thus

$$\begin{aligned}\sigma_\Omega(-x^*) &\geq \left\langle -x^*, \frac{x - \bar{x}}{\alpha_0}\right\rangle = \frac{\rho_\Omega(x - \bar{x})}{\alpha_0}\left\langle -x^*, \frac{x - \bar{x}}{\rho_\Omega(x - \bar{x})}\right\rangle \\ &\geq \frac{\rho_\Omega(x - \bar{x})}{\alpha_0} \geq \frac{\rho_\Omega(x - \bar{x})}{\rho_\Omega(x - \bar{x}) + \epsilon}\end{aligned}$$

and hence

$$\sigma_\Omega(-x^*) \geq 1 - \frac{\epsilon}{\rho_\Omega(x - \bar{x}) + \epsilon}, \quad \forall \epsilon > 0.$$

Taking $\epsilon \to 0^+$ yields

$$\sigma_\Omega(-x^*) \geq 1.$$

Let us prove the reverse inequality

$$\sigma_\Omega(-x^*) \leq 1.$$

Let $(t_j)_j$ be a net in $(0, \infty)$ converging to 0 and satisfying

$$T^-_{S,\Omega}(\bar{x}; h) = \lim_j t_j^{-1}[T_{S,\Omega}(\bar{x} + t_j h) - T_{S,\Omega}(\bar{x})].$$

By Lemma 3.4, we have

$$T_{S,\Omega}(x) - T_{S,\Omega}(\bar{x}) \leq T_{S(r),\Omega}(x) - T_{S(r),\Omega}(\bar{x}), \quad \forall x \in E,$$

which ensures together to the definition of the enlargement set $S(r)$ that

$$t_j^{-1}[T_{S,\Omega}(\bar{x} + t_j h) - T_{S,\Omega}(\bar{x})] \leq t_j^{-1} T_{S(r),\Omega}(\bar{x} + t_j h) \leq \rho_\Omega(-h),$$

and so

$$t_j^{-1}[T_{S,\Omega}(\bar{x} + t_j h) - T_{S,\Omega}(\bar{x})] \leq \rho_\Omega(-h).$$

Taking the limit on j we get

$$T^-_{S,\Omega}(\bar{x}; h) = \lim_j t_j^{-1}[T_{S,\Omega}(\bar{x} + t_j h) - T_{S,\Omega}(\bar{x})] \leq \rho_\Omega(-h).$$

Using the fact that $x^* \in \partial^- T_{S,\Omega}(\bar{x})$ we write

$$\langle -x^*, -h\rangle \le T^-_{S,\Omega}(\bar{x}; h) \le \rho_\Omega(-h), \quad h \in E.$$

Fix any $v \in \Omega$ and take $h := -v$ in the previous inequality we get

$$\langle -x^*, v\rangle = \langle -x^*, -h\rangle \le \rho_\Omega(v) \le 1.$$

This ensures that $\sigma_\Omega(-x^*) \le 1$ and hence the proof is finished. □

The following theorem establishes another relationship between the Clarke subdifferential of $T_{S,\Omega}$ and the Clarke normal cone of $S(r)$ at points outside the set S. It extends Theorem 3.2 in [16] from Banach spaces to Hausdorff topological vector spaces. Also the assumption $0 \in \operatorname{int}\Omega$ is not needed in our proof contrarily to the proofs in [16].

Theorem 4.8. *Let E be a Hausdorff topological vector space, S be a nonempty closed subset of E, Ω be a bounded convex set in E with $0 \in \Omega$ and let $\bar{x} \notin S$. Assume that $\pi_{S,\Omega}(\bar{x}) \neq \emptyset$, $T_{S,\Omega}$ is directionally regular at $\bar{x}$, and $T_{S,\Omega}$ is directionally Lipschitz at $\bar{x}$ in the direction $x - \bar{x}$ for some $x \in \pi_{S,\Omega}(\bar{x})$. Then we have*

$$\partial^C T_{S,\Omega}(\bar{x}) \subset N^C(S(r); \bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\}.$$

If, in addition, $T_{S,\Omega}$ is calm at $\bar{x}$, then we have

$$\partial^C T_{S,\Omega}(\bar{x}) = N^C(S(r); \bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\}.$$

Proof. The inclusion $\partial^C T_{S,\Omega}(\bar{x}) \subset N^C(S(r); \bar{x})$ follows directly from Corollary 4.4. Let us prove the inclusion $\partial^C T_{S,\Omega}(\bar{x}) \subset \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\}$. Fix $\bar{h} \in \Omega$ and any $V \in \mathcal{N}(0)$ and let $H := -\bar{h} + V \in \mathcal{N}(-\bar{h})$. Set $W := V \cap (-V) \in \mathcal{N}(0)$. Since Ω is bounded we may choose some $\alpha > 1$ such that $\Omega \subset \alpha W$. Take now $\tilde{h} := -\bar{h} + \frac{1}{\alpha}\bar{h}$. Clearly $\frac{\alpha}{1-\alpha}\tilde{h} = \bar{h} \in \Omega$. On the other hand, we have $\alpha^{-1}\bar{h} \in \alpha^{-1}\Omega \subset W$ and so

$$\tilde{h} = -\bar{h} + \alpha^{-1}\bar{h} \in -\bar{h} + W \subset -\bar{h} + V = H.$$

Fix now any $\epsilon > 0$ and take $t_0 \in (0, \epsilon)$. Hence we have (since $\frac{\alpha}{1-\alpha}\tilde{h} \in \Omega$)

$$\begin{aligned}\bar{x} = \bar{x} + t_0\tilde{h} - t_0\tilde{h} &= \bar{x} + t_0\tilde{h} + t_0\left(\frac{\alpha-1}{\alpha}\right)\left(\frac{\alpha}{1-\alpha}\right)\tilde{h} \\ &\in \left[\bar{x} + t_0\tilde{h} + t_0\left(\frac{\alpha-1}{\alpha}\right)\Omega\right] \cap S(r) \neq \emptyset,\end{aligned}$$

which ensures by the definition of the minimal time function that

$$T_{S(r),\Omega}(\bar{x} + t_0\tilde{h}) \le t_0\left(\frac{\alpha-1}{\alpha}\right) < t_0.$$

Therefore

$$\inf_{\substack{h \in -\bar{h}+V \\ t \in (0,\epsilon)}} t^{-1} T_{S(r),\Omega}(\bar{x}+th) \le t_0^{-1} T_{S(r),\Omega}(\bar{x}+t_0\tilde{h}) < 1, \text{ for all } \epsilon > 0, \bar{h} \in \Omega, V \in \mathcal{N}(0)$$

and so

$$\begin{aligned} T^{H}_{S(r),\Omega}(\bar{x};-\bar{h}) = &\liminf_{\substack{t\to 0+\\ h\to -\bar{h}}} t^{-1}T_{S(r),\Omega}(\bar{x}+th)\\ = &\sup_{\substack{\epsilon>0\\ V\in\mathcal{N}(0)}} \inf_{\substack{h\in -\bar{h}+V\\ t\in(0,\epsilon)}} t^{-1}T_{S(r),\Omega}(\bar{x}+th)\le 1, \end{aligned}$$

for all $\bar{h}\in\Omega$. Take now any $x^*\in\partial^C T_{S,\Omega}(\bar{x})$. Then by the previous inequality, the definition of Clarke subdifferential, the directional regularity of $T_{S,\Omega}$ at $\bar{x}$, and Part *(1)* in Theorem 4.1 we have

$$\left\langle -x^*,\bar{h}\right\rangle \le T^{\uparrow}_{S,\Omega}(\bar{x};-\bar{h}) = T^{H}_{S,\Omega}(\bar{x};-\bar{h}) \le T^{H}_{S(r),\Omega}(\bar{x};-\bar{h})\le 1, \quad \text{for any } \bar{h}\in\Omega.$$

This inequality yields

$$\sigma_\Omega(-x^*) = \sup_{\bar{h}\in\Omega}\left\langle -x^*,\bar{h}\right\rangle \le 1.$$

It remains to show the inequality $\sigma_\Omega(-x^*)\ge 1$. Using the assumption of the directional Lipschitzness of $T_{S,\Omega}$ at $\bar{x}$ in the direction $x-\bar{x}$ we have

$$T^{\uparrow}_{S,\Omega}(\bar{x};x-\bar{x}) = T^{0}_{S,\Omega}(\bar{x};x-\bar{x}).$$

This equality with the directional regularity of $T_{S,\Omega}$ at $\bar{x}$ ensure

$$\begin{aligned} T^{-}_{S,\Omega}(\bar{x};x-\bar{x}) \le T^{0}_{S,\Omega}(\bar{x};x-\bar{x}) &= T^{\uparrow}_{S,\Omega}(\bar{x};x-\bar{x})\\ &= T^{H}_{S,\Omega}(\bar{x};x-\bar{x}) \le T^{-}_{S,\Omega}(\bar{x};x-\bar{x}), \end{aligned}$$

that is, $T^{-}_{S,\Omega}(\bar{x};x-\bar{x}) = T^{\uparrow}_{S,\Omega}(\bar{x};x-\bar{x})$ and hence by Proposition 4.7 we obtain

$$\langle x^*,x-\bar{x}\rangle \le T^{\uparrow}_{S,\Omega}(\bar{x};x-\bar{x}) = T^{-}_{S,\Omega}(\bar{x};x-\bar{x}) = -T_{S,\Omega}(\bar{x}) = -\rho_\Omega(x-\bar{x}).$$

Following the same lines in the proof of Proposition 4.7 we conclude $\sigma_\Omega(-x^*)\ge 1$ and hence the proof of the first inclusion of the theorem is complete.

Let us prove the reverse inclusion. Fix any $x^*\in N^C(S(r);\bar{x})$ with $\sigma_\Omega(-x^*)=1$. First let us check that $0\notin\partial^C T_{S,\Omega}(\bar{x})$. Indeed, by the assumption $\pi_{S,\Omega}(\bar{x})\ne\emptyset$ and the above reasoning we have

$$T^{\uparrow}_{S,\Omega}(\bar{x};x-\bar{x}) = T^{-}_{S,\Omega}(\bar{x};x-\bar{x}) = -T_{S,\Omega}(\bar{x}) < 0, \quad \text{for some } x\in\pi_{S,\Omega}(\bar{x}),$$

which ensures that $0\notin\partial^C T_{S,\Omega}(\bar{x})$. Therefore, by Corollary 4.5, there exists some $\lambda>0$ and $y^*\in\partial^C T_{S,\Omega}(\bar{x})$, such that $x^*=\lambda y^*$. Using the first part of the proof we obtain $\sigma_\Omega(-y^*)=1$ and so by combining this equality with the assumption $\sigma_\Omega(-x^*)=1$ we get $\lambda=1$. Thus completing the proof. $\square$

5. Applications to Normal and Subdifferential regularity in normed vector spaces

In this section we are going to show the importance of our previous results on generalized directional derivatives and Clarke subdifferentials of $T_{S,\Omega}$ at points $\bar{x}$ outside the set S, in the study of the relationships between the subdifferential regularity of $T_{S,\Omega}$ and the normal regularity of the enlargement set $S(r)$ where $r := T_{S,\Omega}(\bar{x})$.

In all this section we assume that E is a normed vector space. We are going to study the relationship between the Fréchet (resp. Mordukhovich, proximal) normal regularity of the enlargement set $S(r)$ of a closed nonempty set S at $\bar{x} \notin S$, and the Fréchet (resp. Mordukhovich, proximal) subdifferential regularity of the minimal time function $T_{S,\Omega}$ associated with S at the same point $\bar{x}$. Our results will extend the results in [3, 6] from Banach spaces setting to normed vector spaces setting and from the case of distance function to the case of minimal time function by taking Ω to be the closed unit ball. To this end let us first recall some notations and definitions needed throughout this section. The various notions of subdifferentials and normal cones in the following definitions, and the relative facts provided in the following remarks are well-known (see for example [3, 6, 20, 21]).

Definition 5.1. Let $\bar{x} \in E$ with $f(\bar{x}) < \infty$.

(a) The ϵ-Fréchet subdifferential $\partial^F f(\bar{x})$ of f at $\bar{x}$ is defined by

$$\partial^F_\epsilon f(\bar{x}) := \left\{ x^* \in E^* \middle| \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq -\epsilon \right\}.$$

The Fréchet subdifferential $\partial^F f(\bar{x})$ of f at $\bar{x}$ is defined as the 0-subdifferential of f at the corresponding point, that is

$$\partial^F f(\bar{x}) := \left\{ x^* \in E^* \middle| \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

(b) The right-sided limiting Fréchet subdifferential (also called the right-sided Mordukhovich subdifferential) $\partial^M_{\geq} f(\bar{x})$ of f at $\bar{x}$ is defined by

$$\partial^M_{\geq} f(\bar{x}) := \limsup_{\substack{x \to^f \bar{x}, f(x) \geq f(\bar{x}) \\ \varepsilon \downarrow 0}} \partial^F_\varepsilon f(\bar{x}).$$

(c) The *proximal subdifferential* $\partial^P f(\bar{x})$ of f at $\bar{x}$ is defined by

$$\partial^P f(\bar{x}) := \left\{ x^* \in E^* \middle| \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|^2} > -\infty \right\}.$$

Remark 5.2. Let $\bar{x} \in E$ with $f(\bar{x}) < \infty$. Then the following inclusions are clear by definitions:

$$\partial^P f(\bar{x}) \subseteq \partial^F f(\bar{x}) \subseteq \partial^M_{\geq} f(\bar{x}) \subseteq \partial^C f(\bar{x}). \tag{20}$$

Definition 5.3. Let $\bar{x} \in S$.

(a) The set of Fréchet ϵ-normals to S at $\bar{x}$ is defined by

$$N_\epsilon^F(\bar{x}; S) := \left\{ x^* \in E^* : \limsup_{x \to^S \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \epsilon \right\}.$$

The set of Fréchet normals to S at $\bar{x}$ is defined as the Fréchet 0-normals to S at the same point, that is,

$$N^F(\bar{x}; S) := \left\{ x^* \in E^* : \limsup_{x \to^S \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

(b) The limiting Fréchet (also called Mordukhovich) normal cone to S at $\bar{x}$ is defined by

$$N^M(\bar{x}; S) := \limsup_{x \to^S \bar{x}, \varepsilon \downarrow 0} N_\varepsilon^F(\bar{x}; S).$$

(c) The proximal normal cone to S at $\bar{x}$ is defined by

$$N^P(\bar{x}; S) := \left\{ x^* \in E^* : \limsup_{x \to^S \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|^2} < +\infty \right\}.$$

Remark 5.4. Let $\bar{x} \in S$. By definition (see also [3, 9, 11, 21]), we have that

$$N^\Delta(\bar{x}; S) = \partial^\Delta \delta_S(\bar{x}) \text{ for any } \Delta \in \{F, P, C\}.$$

Then by Remark 5.2 we have that

$$N^P(\bar{x}; S) \subseteq N^F(\bar{x}; S) \subseteq N^M(\bar{x}; S) \subseteq N^C(\bar{x}; S). \tag{21}$$

The following concepts of regularity for sets and functions introduced and studied in [3, 6].

Definition 5.5. Let $\bar{x} \in E$, with $f(\bar{x}) < \infty$. We say that f is

(a) *Fréchet subdifferentially regular* at $\bar{x}$ if $\partial^F f(\bar{x}) = \partial^C f(\bar{x})$;
(b) *proximal subdifferentially regular* at $\bar{x}$ if $\partial^P f(\bar{x}) = \partial^C f(\bar{x})$;
(c) *Right-Sided Mordukhovich subdifferentially regular* at $\bar{x}$ if $\partial^F f(\bar{x}) = \partial^M_{\geq} f(\bar{x})$.

Definition 5.6. Let $\bar{x} \in S$. We say that S is

(a) *Fréchet normally regular* at $\bar{x}$ if $N^F(S; \bar{x}) = N^C(S; \bar{x})$;
(b) *proximal normally regular* at $\bar{x}$ if $N^P(S; \bar{x}) = N^C(S; \bar{x})$;
(c) *Mordukhovich normally regular* at $\bar{x}$ if $N^F(S; \bar{x}) = N^M(S; \bar{x})$.

Recall from [17] and [24] the following needed results.

Theorem 5.7. *Let S be a nonempty closed subset of E with $\bar{x} \notin S$ with $r := T_{S,\Omega}(\bar{x}) < \infty$ and let Ω be a bounded closed convex set in E with $0 \in \Omega$. Then*

(1) $\partial^F T_{S,\Omega}(\bar{x}) \subset N^F(S(r); \bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\}$ *(see [17]);*

(*2*) $\partial^P T_{S,\Omega}(\bar{x}) \subset N^P(S(r);\bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\}$ *(see* [17]*);*

(*3*) *If* $T_{S,\Omega}$ *is calm at* $\bar{x}$*, then we have equality form in* (1) *and* (2)*;*

(*4*) *If* E *is a Banach space, then*

$$\partial^M_{\geq} T_{S,\Omega}(\bar{x}) \subset N^M(S(r);\bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\},$$

and the equality $N^M(S(r);\bar{x}) = \mathbb{R}_+\partial^M_{\geq} T_{S,\Omega}(\bar{x})$ *holds provided that* $0 \in \operatorname{int}\Omega$. (*see* [24]).

We state our first result on the relationship between the Fréchet normal regularity of $S(r)$ at $\bar{x} \notin S$, and the Fréchet subdifferential regularity of the minimal time function $T_{S,\Omega}$ at the same point $\bar{x}$. This result extends Theorem 2.19 in [3](see also [6]) from reflexive Banach spaces setting to normed vector spaces setting and from the case of the distance function to the case of the minimal time function.

Theorem 5.8. *Let S be a nonempty closed subset in a normed vector space E with $\bar{x} \notin S$. Let Ω be a bounded closed convex set in E with $0 \in \Omega$. Assume that $T_{S,\Omega}$ is calm at $\bar{x}$ and $\pi_{S,\Omega}(\bar{x}) \neq \emptyset$.*

(*1*) *If $T_{S,\Omega}$ is directionally Lipschitz at $\bar{x}$ in the direction $x - \bar{x}$ for some $x \in \pi_{S,\Omega}(\bar{x})$, and $T_{S,\Omega}$ is is Fréchet subdifferentially regular at $\bar{x}$ with $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$, then $T_{S,\Omega}$ is directionally regular at $\bar{x}$ and $S(r)$ is Fréchet normally regular at $\bar{x}$;*

(*2*) *Conversely, if $S(r)$ is Fréchet normally regular at $\bar{x}$, $T_{S,\Omega}$ is directionally regular at $\bar{x}$, then $T_{S,\Omega}$ is Fréchet subdifferentially regular at $\bar{x}$.*

Proof. (*1*) Assume that $\pi_{S,\Omega}(\bar{x}) \neq \emptyset$ and let $x \in \pi_{S,\Omega}(\bar{x})$ for which $T_{S,\Omega}$ is directionally Lipschitz at $\bar{x}$ in the direction $x - \bar{x}$. Assume also that $T_{S,\Omega}$ is Fréchet subdifferentially regular at $\bar{x}$, that is, $\partial^F T_{S,\Omega}(\bar{x}) = \partial^C T_{S,\Omega}(\bar{x})$. Since $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$ and $T_{S,\Omega}$ is l.s.c. at $\bar{x}$ (because $T_{S,\Omega}$ is calm at $\bar{x}$) we have by Proposition 3.2 in [3] the directional regularity of $T_{S,\Omega}$ at $\bar{x}$. Therefore,

$$T^H_{S,\Omega}(\bar{x}, x - \bar{x}) = T^{\uparrow}_{S,\Omega}(\bar{x}, x - \bar{x}) = T^0_{S,\Omega}(\bar{x}, x - \bar{x}) \geq T^-_{S,\Omega}(\bar{x}, x - \bar{x})$$

and since we always have $T^H_{S,\Omega}(\bar{x}, x - \bar{x}) \leq T^-_{S,\Omega}(\bar{x}, x - \bar{x})$ then we get

$$T^{\uparrow}_{S,\Omega}(\bar{x}, x - \bar{x}) = T^-_{S,\Omega}(\bar{x}, x - \bar{x}).$$

Using now Proposition 4.7 to write

$$T^{\uparrow}_{S,\Omega}(\bar{x}, x - \bar{x}) = T^-_{S,\Omega}(\bar{x}, x - \bar{x}) = -T_{S,\Omega}(\bar{x}) < 0.$$

This ensures that $0 \notin \partial^C T_{S,\Omega}(\bar{x})$ and hence since $T_{S,\Omega}$ is calm at $\bar{x}$, we can write by using Corollary 4.5,

$$\begin{aligned} N^C(S(r);\bar{x}) &= \mathbb{R}_+\partial^C T_{S,\Omega}(\bar{x}) \\ &= \mathbb{R}_+\partial^F T_{S,\Omega}(\bar{x}) \\ &\subset N^F(S(r);\bar{x}). \end{aligned}$$

The last inclusion follows from Part (a) in Theorem 5.7 and hence we get the Fréchet normal regularity of $S(r)$ at $\bar{x}$.

(2) Conversely, assume that $S(r)$ is Fréchet normally regular at $\bar{x}$ (i.e., $N^F(S(r);\bar{x}) = N^C(S(r);\bar{x})$) as well as the other assumptions in *(2)* of the statement of the theorem. Thus, by Theorem 4.8 and Part *(3)* in Theorem 5.7 we have

$$\begin{aligned}\partial^C T_{S,\Omega}(\bar{x}) &\subset N^C(S(r);\bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\} \\ &= N^F(S(r);\bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\} \\ &= \partial^F T_{S,\Omega}(\bar{x}) \subset \partial^C T_{S,\Omega}(\bar{x}).\end{aligned}$$

This means that $T_{S,\Omega}$ is Fréchet subdifferentially regular at $\bar{x}$. □

Observe that the proof of the first part in the previous theorem needs the assumptions $\pi_{S,\Omega}(\bar{x}) \neq \emptyset$ and the directional Lipschitzness of $T_{S,\Omega}$ at $\bar{x}$ in the direction $x-\bar{x}$ for some $x \in \pi_{S,\Omega}(\bar{x})$ which is needed to guarantee that $0 \notin \partial^C T_{S,\Omega}(\bar{x})$. In order to remove these assumptions we will add the reflexivity assumption on the space E as the following proposition shows.

Proposition 5.9. *Let S be a nonempty closed subset in a reflexive Banach space E with $\bar{x} \notin S$. Let Ω be a bounded closed convex set in E with $0 \in \Omega$. Assume that $T_{S,\Omega}$ is calm at $\bar{x}$ with $r := T_{S,\Omega}(\bar{x}) < \infty$ and $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$. If the function $T_{S,\Omega}$ is Fréchet subdifferentially regular at $\bar{x}$, then $S(r)$ is Fréchet normally regular at $\bar{x}$;*

Proof. Assume that $T_{S,\Omega}$ is Fréchet subdifferentially regular at $\bar{x}$, that is, $\partial^F T_{S,\Omega}(\bar{x}) = \partial^C T_{S,\Omega}(\bar{x})$. As in the proof of the previous theorem we have the directional regularity of $T_{S,\Omega}$ at $\bar{x}$. Then by using Corollary 4.4 and Part *(3)* in Theorem 5.7, we have

$$\begin{aligned}N^C(S(r);\bar{x}) &= \mathrm{cl}_{w^*}\left(I\!R_+\partial^C T_{S,\Omega}(\bar{x})\right) \\ &= \mathrm{cl}_{w^*}\left(I\!R_+\partial^F T_{S,\Omega}(\bar{x})\right) \\ &= \mathrm{cl}_{w^*}\left(N^F(S(r);\bar{x})\right) \\ &= N^F(S(r);\bar{x}).\end{aligned}$$

The last equality follows from the fact that $N^F(S(r);\bar{x})$ is weakly star closed in reflexive Banach spaces. Indeed, it is well known that $N^F(S(r);\bar{x})$ is strongly closed (see for example Proposition 1.8 in [3]) and convex, which ensures its weak star closedness by the reflexivity of the space. Thus completing the proof. □

Now, we turn to the study of the equivalence between the Mordukhovich normal regularity of $S(r)$ at $\bar{x}$ and the Right-Sided Mordukhovich subdifferential regularity of $T_{S,\Omega}$ at the same point $\bar{x}$ in Banach spaces.

Theorem 5.10. *Let S be a nonempty closed subset in a Banach space E with $\bar{x} \notin S$. Let Ω be a bounded closed convex set in E with $0 \in \Omega$. Assume that $T_{S,\Omega}$ is calm at $\bar{x}$ and $S(r)$ is Mordukhovich normally regular at $\bar{x}$. Then $T_{S,\Omega}$ is Right-Sided Mordukhovich subdifferentially regular at $\bar{x}$. Conversely, if $0 \in \operatorname{int}\Omega$*

and $T_{S,\Omega}$ is Right-Sided Mordukhovich subdifferentially regular at $\bar{x}$, then $S(r)$ is Mordukhovich normally regular at $\bar{x}$.

Proof. (*1*) Assume that $T_{S,\Omega}$ is calm at $\bar{x}$ and $S(r)$ is Mordukhovich normally regular at $\bar{x}$. Then

$$N^F(S(r);\bar{x}) = N^M(S(r);\bar{x}).$$

We have by Part (*4*) in Theorem 5.7 the inclusion

$$\partial^M_{\geq} T_{S,\Omega}(\bar{x}) \subset N^M(S;\bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\}.$$

Since $T_{S,\Omega}$ is calm at $\bar{x}$, we can use Part (*3*) in Theorem 5.7 to write the inclusion

$$N^F(S;\bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\} = \partial^F T_{S,\Omega}(\bar{x})$$

Thus, combining the three above relations we get

$$\begin{aligned}\partial^M_{\geq} T_{S,\Omega}(\bar{x}) &\subset N^M(S;\bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\}\\ &= N^F(S;\bar{x}) \cap \{x^* \in E^* : \sigma_\Omega(-x^*) = 1\}\\ &= \partial^F T_{S,\Omega}(\bar{x}).\end{aligned}$$

This means that $T_{S,\Omega}$ is Right-Sided Mordukhovich subdifferentially regular at $\bar{x}$, since the reverse inclusion $\partial^F T_{S,\Omega}(\bar{x}) \subset \partial^M_{\geq} T_{S,\Omega}(\bar{x})$ always holds.

(2) Conversely, assume that $0 \in \operatorname{int}\Omega$ and $T_{S,\Omega}$ is Right-Sided Mordukhovich subdifferentially regular at $\bar{x}$. Then $\partial^F T_{S,\Omega}(\bar{x}) = \partial^M_{\geq} T_{S,\Omega}(\bar{x})$. Hence, by using Parts (*2*) and (*4*) in Theorem 5.7, we obtain

$$N^M(S(r);\bar{x}) = I\!R_+ \partial^M_{\geq} T_{S,\Omega}(\bar{x}) = I\!R_+ \partial^F T_{S,\Omega}(\bar{x}) = N^F(S(r);\bar{x}). \tag{22}$$

This means that $S(r)$ is Mordukhovich normally regular at $\bar{x}$. □

Following the same above ideas for the Fréchet case, we use Corollary 4.4 and Parts (*2*) and (*3*) in Theorem 5.7 to prove the relationship between the proximal normal regularity of $S(r)$ at $\bar{x} \notin S$, and the proximal subdifferential regularity of the minimal time function $T_{S,\Omega}$ at the same point $\bar{x}$. This result extends Theorem 2.21 in [3] (see also [6]) from the case of distance functions to the case of minimal time functions and from the case of reflexive Banach spaces to normed vector spaces setting.

Theorem 5.11. *Let S be a nonempty closed subset in a normed vector space E with $\bar{x} \notin S$. Let Ω be a bounded closed convex set in E with $0 \in \Omega$. Assume that $T_{S,\Omega}$ is calm at $\bar{x}$.*

(*1*) *If $\pi_{S,\Omega}(\bar{x}) \neq \emptyset$, $T_{S,\Omega}$ is directionally Lipschitz at $\bar{x}$ in the direction $x - \bar{x}$ for some $x \in \pi_{S,\Omega}(\bar{x})$, and $T_{S,\Omega}$ is proximal subdifferentially regular at $\bar{x}$ with $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$, then $T_{S,\Omega}$ is directionally regular at $\bar{x}$ and $S(r)$ is proximal normally regular at $\bar{x}$;*

(*2*) *If E is a reflexive Banach space, $\partial^C T_{S,\Omega}(\bar{x}) \neq \emptyset$, and the function $T_{S,\Omega}$ is proximal subdifferentially regular at $\bar{x}$, then $T_{S,\Omega}$ is directionally regular at $\bar{x}$ and $S(r)$ is proximal normally regular at $\bar{x}$;*

(*3*) *Conversely, if $S(r)$ is proximal normally regular at $\bar{x}$, $T_{S,\Omega}$ is directionally regular at $\bar{x}$, and $\pi_{S,\Omega}(\bar{x}) \neq \emptyset$, then $T_{S,\Omega}$ is proximal subdifferentially regular at $\bar{x}$.*

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References

[1] C. D. Aliprantis, K. C. Border: Infinite Dimensional Analysis, Springer, Berlin (1994).

[2] J. M. Borwein, M. Fabian: A note on regularity of sets and of distance functions in Banach space, J. Math. Anal. Appl. 182(2) (1994) 566–570.

[3] M. Bounkhel: Regularity Concepts in Nonsmooth Analysis, Theory and Applications, Springer Optimization and its Applications 59, Springer, Berlin (2011).

[4] M. Bounkhel: On subdifferentials of a minimal time function in topological vector spaces, Appl. Anal. 93(8) 1761–1791.

[5] M. Bounkhel: Scalarization of the normal Fréchet regularity of set-valued mappings, N. Z. J. Math. 33(2) (2004) 129–146.

[6] M. Bounkhel, L. Thibault: On various notions of regularity of sets in nonsmooth analysis, Nonlinear Anal. 48 (2002) 223–246.

[7] J. V. Burke, M. C. Ferris, M. Qian: On the Clarke subdifferential of the distance function of a closed set, J. Math. Anal. Appl. 166 (1992) 199–213.

[8] P. M. Cannarsa, C. Sinestri: Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control, Birkhäuser, Boston (2004).

[9] F. H. Clarke: Optimization and Nonsmooth Analysis, Wiley, New York (1983).

[10] F. H. Clarke: Optimal solutions to differential inclusions, J. Optim. Theory Appl. 19 (1976) 469–479.

[11] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, P. R. Wolenski: Nonsmooth Analysis and Control Theory, Springer, New York (1998).

[12] F. H. Clarke, R. J. Stern, P. R. Wolenski: Proximal smoothness and the lower-C^2 property, J. Convex Analysis 2 (1995) 117–144.

[13] G. Colombo, P. R. Wolenski: The subgradient formula for the minimal time function in the case of constant dynamics in Hilbert space, J. Glob. Optim. 28 (2004) 269–282.

[14] G. Colombo, P. R. Wolenski: Variational analysis for a class of minimal time functions in Hilbert spaces, J. Convex Analysis 11 (2004) 335–361.

[15] F. S. De Blasi, J. Myjak: On a generalized best approximation problem, J. Approx. Theory 94 (1998) 54–72.

[16] Y. He, K. F. Ng: Subdifferentials of a minimum time function in Banach spaces, J. Math. Anal. Appl. 321 (2006) 896–910.

[17] Y. Jiang, Y. He: Subdifferentials of a minimum time function in normed spaces, J. Math. Anal. Appl. 358 (2009) 410–418.

[18] J. B. Hiriart-Urruty: Tangent cones, generalized gradients and mathematical programing in Banach spaces, Math. Oper. Res. 4 (1979) 79–97.

[19] J. B. Hiriart-Urruty: New concepts in nondifferentiable programming, Bull. Soc. Math. Fr., Suppl., Mém. 60 (1979) 57–85.

[20] A. D. Ioffe: Proximal analysis and approximate subdifferentials, J. Lond. Math. Soc., II. Ser. 41 (1990) 175–192.

[21] B. S. Mordukhovich: Variational Analysis and Generalized Differentiation I: Basic Theory, Grundlehren der Mathematischen Wissenschaften 330, Springer, Berlin (2006).

[22] B. S. Mordukhovich: Variational Analysis and Generalized Differentiation II: Applications, Grundlehren der Mathematischen Wissenschaften 331, Springer, Berlin (2006).

[23] B. S. Mordukhovich, N. M. Nam: Limiting subgradients of minimal time functions in Banach spaces, J. Global Optim. 46(4) (2010) 615–633.

[24] B. S. Mordukhovich, N. M. Nam: Subgradients of minimal time functions under minimal assumptions, J. Convex Analysis 18 (2011) 915–947.

[25] N. M. Nam, M. C. Villalobos, N. T. An: Minimal time functions and the smallest intersecting ball problem with unbounded dynamics, J. Optim. Theory Appl. 154 (2012) 768–791.

[26] J. P. Penot: Calcul sous-différentiel et optimisation, J. Funct. Anal. 27 (1978) 248–276.

[27] R. T. Rockafellar: Generalized directional derivatives and subgradients of non-convex functions, Can. J. Math. 39 (1980) 257–280.

[28] R. T. Rockafellar: Lipschitzian properties of multifunctions, Nonlinear Anal. 9 (1985) 867–885.

[29] R. T. Rockafellar: Convex Analysis, Princeton Univ. Press, Princeton (1970).

[30] W. Rudin: Functional Analysis, McGraw-Hill, New York (1973).

[31] P. Soravia: Generalized motion of a front propagating along its normal direction: a differential games approach, Nonlinear Anal. 22 (1994) 1247–1262.

[32] P. R. Wolenski, Z. Yu: Proximal analysis and the minimal time function, SIAM J. Control Optim. 36 (1998) 1048–1072.