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Antidistance and Antiprojection in the Hilbert Space*

M. V. Balashov
Department of Higher Mathematics, Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudny, Moscow region, Russia 141700
balashov73@mail.ru

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In the present paper we characterize such convex closed sets in the real Hilbert space, that for each of these sets the operator of metric antiprojection on the set (which gives for a given point of the space the subset of points of the set, which are most farthest from the given point of the space) is singleton and Lipschitz continuous on the complementary to some neighborhood of the given set. We obtain new estimates of geometric properties of such a set as function of the size of the neighborhood of the set and the Lipschitz constant for the antiprojection operator. Properties of the metric antiprojector operator are investigated. We prove the stability of the antiprojection on a strongly convex set with respect to the point and to the set. We also consider the question: which points of the set are antiprojections of some points from the space?

Keywords: Hilbert space, strongly convex set of radius R, R-exposed points, distance and antidistance functions, antiprojection, weak convexity

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1. Introduction and main notations

Let $\mathcal{H}$ be a real *Hilbert* space. Denote by (p, x) the *inner product* for vectors $p, x \in \mathcal{H}$. Let $B_r(a) = \{x \in \mathcal{H} \mid \|x - a\| \leq r\}$.

For a subset $A \subset \mathcal{H}$ we denote by ∂A, $\operatorname{int} A$, $\operatorname{cl} A$ *the boundary, the interior and the closure* of the set A, respectively.

The *supporting function* for the subset $A \subset \mathcal{H}$ is the function of the form

$$s(p, A) = \sup_{a \in A}(p, a), \quad p \in \mathcal{H}.$$

The *normal cone* to the set $A \subset \mathcal{H}$ at the point $a \in A$ is defined by

$$N(A, a) = \{p \in \mathcal{H} \mid (p, a) \geq (p, x),\ \forall x \in A\}.$$

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The *convex hull* of the subset $A \subset \mathcal{H}$ is the smallest in the sense of the inclusion convex set, which contains A. We shall define it by $\operatorname{co} A$.

The *diameter* of the set A is the value $\operatorname{diam} A = \sup_{x,y\in A} \|x-y\|$.

The *distance function* from a point x to the set A will be denoted by

$$\varrho_A(x) = \inf_{a\in A} \|x-a\|.$$

We define by $P_A x$ the *metric projection* of the point $x \in \mathcal{H}$ on the set $A \subset \mathcal{H}$:

$$P_A x = \{a \in A \mid \|x-a\| = \varrho_A(x)\}.$$

Recall the definition of the *Minkowskii sum* for subsets $A, B \subset \mathcal{H}$

$$A + B = \bigcup_{b\in B} (A+b) = \{a+b \mid a \in A, b \in B\},$$

and the *geometric (or Minkowskii-Pontryagin)* difference

$$A \overset{*}{-} B = \{x \in \mathcal{H} \mid x + B \subset A\} = \bigcap_{b\in B} (A-b).$$

The *Hausdorff distance* between subsets $A, B \subset \mathcal{H}$ is defined as follows

$$\begin{aligned} h(A,B) = &\ \inf\{r > 0 \mid A \subset B + B_r(0),\ B \subset A + B_r(0)\} \\ = &\ \max\left\{\sup_{a\in A} \varrho_B(a),\ \sup_{b\in B} \varrho_A(b)\right\}. \end{aligned}$$

Definition 1.1 ([3]). Let $A \subset \mathcal{H}$, $x \in \mathcal{H}$. We shall call

$$f_A(x) = \sup_{a\in A} \|x-a\|$$

the *antidistance (function)* from the point $x \in \mathcal{H}$ to the set A.

The *antineighborhood* of radius $r > 0$ for the set A [3, 6] is defined as follows

$$T_A(r) = \{x \in \mathcal{H} \mid f_A(x) > r\}.$$

Definition 1.2 ([3]). Let $A \subset \mathcal{H}$, $x \in \mathcal{H}$. We shall call the set

$$AP_A x = \{a \in A \mid \|x-a\| = f_A(x)\}$$

the *antiprojection* of the point x on the set A. The mapping $\mathcal{H} \ni x \to AP_A x$ is called the *antiprojection operator.* We shall say that a point $a(x) \in AP_A x$ is an *element of the antiprojection* of the point $x \in \mathcal{H}$ on the set A. The antiprojection $AP_A x$ is unique if $AP_A x = \{a(x)\}$.

A subset $A \subset \mathcal{H}$ is called *strongly convex of radius* $R > 0$, if it can be represented as an intersection of closed balls of radius R.

Proposition 1.3 ([8, Theorems 4.1.3, 4.2.7]). *For a closed convex subset $A \subset \mathcal{H}$ the next properties are equvivalent:*

(i) $A = \bigcap_{x \in X} B_R(x)$ *for some* $X \subset \mathcal{H}$*;*

(ii) *For any* $p \in \partial B_1(0)$ *and* $a_p = \arg\max_{a \in A}(p, a)$ *we have the supporting principle*

$$A \subset B_R(a_p - Rp).$$

The present paper enhances the results from [3, 6].

In the paper [3] the next propositions were proved.

Proposition 1.4. *Let $A \subset \mathcal{H}$ be a strongly convex set of radius R, $r > R$. Then, for any points $x_0, x_1 \in T_A(r)$, one has the inequality*

$$\|a(x_0) - a(x_1)\| \leq \frac{R}{r - R}\|x_0 - x_1\|. \tag{1}$$

Note that under conditions of Proposition 1.4 the antiprojection $AP_A x$ is singleton for any point $x \in T_A(r)$.

Proposition 1.5. *Let $A \subset \mathcal{H}$ be a closed convex bounded subset, $r > 0$. Suppose that for any point $x \in T_A(r)$ we have $AP_A x = \{a(x)\}$ and $T_A(r) \ni x \to a(x)$ is Lipschitz continuous with the Lipschitz constant $C > 0$. Then the set A is strongly convex of radius r.*

In the first part of the present paper we shall refine the results of Propositions 1.4 and 1.5. We shall define the precise relationship between the Lipschitz constant C from Proposition 1.5 and the constant of strong convexity R for the set A.

In the second part we shall consider the properties of the antiprojection operator on the strongly convex set, in particular we shall prove the stability of the antiprojection on a strongly convex set with respect to the point and to the set in the Hausdorff metric. We also shall consider the multivalued antiprojection, which will be defined analogously with the multivalued metric projection on a closed convex set.

It is well known that a point $a \in A$ of the closed convex subset $A \subset \mathcal{H}$ is the metric projection of some point $x \in \mathcal{H} \backslash A$, if and only if $N(A, a) \neq \{0\}$. Moreover $P_A(a + N(A, a)) = \{a\}$. In the third part of the paper we consider the question: which points of an arbitrary closed convex set can be elements of antiprojections of some points from the space $\mathcal{H}$ and which exactly points?

Note that for any set $A \subset \mathcal{H}$ the set $T_A(r)$ can be expressed as

$$T_A(r) = (\mathcal{H} \backslash B_r(0)) + A,$$

see [2] for details.

2. The relationship between the constant of strong convexity of the set and the Lipschitz constant for the antiprojection operator on the set in the Hilbert space

It turns out that under conditions of Proposition 1.5 we can assert that the set A is strongly convex of radius $R = \frac{Cr}{C+1}$. This refines Proposition 1.5 and together with Proposition 1.4 gives the best possible relationship between the constant of strong convexity of the set A and the Lipschitz constant for the antiprojection operator.

Lemma 2.1 ([3, Lemma 1]). *Let $A \subset \mathcal{H}$ be a strongly convex set of radius R, $p_i \in \mathcal{H}$, $\|p_i\| \geq 1$ and $a_i = \arg\max_{a\in A}(p_i, a)$, $i = 0, 1$. Then*

$$\|a_0 - a_1\|^2 \leq R(p_0 - p_1, a_0 - a_1).$$

Theorem 2.2. *Suppose that conditions of Proposition* 1.5 *hold. Then the set A is strongly convex of radius $\frac{Cr}{C+1}$.*

Proof. From Proposition 1.5 the set A is strongly convex of radius r. Let R be the minimal radius of strong convexity for the set A. It is easy to see [8, Corollary 3.1.4, Theorem 4.3.2], that such radius exists.

Let us prove that any unit vector $p \in \partial B_1(0) \subset \mathcal{H}$ can be represented in the form

$$p = \frac{a(x) - x}{r} \tag{2}$$

for some point $x \in \partial T_A(r) = \{z \in \mathcal{H} \mid f_A(z) = r\}$.

Fix $p \in \partial B_1(0)$. Suppose that the point $a \in \partial A$ is such that $(p, a) = s(p, A)$. From Proposition 1.3

$$A \subset B_r(a - rp),$$

hence choosing $x := a - rp$, we see that $a(x) = a$.

With the help of the limit procedure it is easy to see that for any points $x_0, x_1 \in \operatorname{cl} T_A(r)$ we have

$$\|a(x_0) - a(x_1)\| \leq C\|x_0 - x_1\|.$$

Fix $x_0, x_1 \in \partial T_A(r)$. Define $a_i = a(x_i)$ and $p_i = \frac{x_i - a_i}{r} \in \partial B_1(0)$, $i = 0, 1$. We have

$$\|a_0 - a_1\| \leq C\|x_0 - x_1\| = Cr\left\|p_0 - p_1 + \frac{a_0 - a_1}{r}\right\|.$$

By Lemma 2.1 $(p_0 - p_1, a_0 - a_1) \leq -\frac{1}{R}\|a_0 - a_1\|^2$, thus

$$\|a_0 - a_1\|^2 \leq C^2r^2\|p_0 - p_1\|^2 + C^2\|a_0 - a_1\|^2 - \frac{2C^2r}{R}\|a_0 - a_1\|^2,$$

i.e.

$$\|a_0 - a_1\| \leq \frac{Cr}{\sqrt{1 + \frac{2r}{R}C^2 - C^2}} \cdot \|p_0 - p_1\|. \tag{3}$$

Points $x_0, x_1 \in \partial T_A(r)$ are arbitrary. From the property (2) for any unit vectors $p_0, p_1 \in \mathcal{H}$ with $(-p_i, a_i) = s(-p_i, A)$, $i = 0, 1$, we get (3). By [8, Theorem 4.3.2] the last assertion is equivalent to the strong convexity of the set A with radius

$$\frac{Cr}{\sqrt{1 + \frac{2r}{R} C^2 - C^2}}.$$

From the minimality of radius R we obtain that

$$R \leq \frac{Cr}{\sqrt{1 + \frac{2r}{R} C^2 - C^2}}. \tag{4}$$

Put $R_1 = \frac{Cr}{C-1}$, $R_2 = \frac{Cr}{C+1}$.

Suppose that $C \in (0, 1)$. The inequality (4) is equivalent to $R \in [R_1, R_2]$, hence $R \leq R_2$.

If $C = 1$, then $R \leq \frac{r}{2} = \frac{Cr}{C+1}$.

Under condition $C > 1$ the inequality (4) is equivalent to $R \in (-\infty, R_2] \cup [R_1, +\infty)$. Note that if $C > 1$ then $R_1 > r$, but we have $R \leq r$ by Proposition 1.5. So in this case we also get $R \leq R_2$. $\square$

Corollary 2.3. *Let C be the minimal Lipschitz constant in Proposition* 1.5. *Then $R = \frac{Cr}{C+1}$ is the minimal radius of strong convexity for the set A.*

Let a subset $A \subset \mathcal{H}$ be strongly convex with the minimal radius of strong convexity $R > 0$. Then the Lipschitz constant $C = \frac{R}{r-R}$ for the antiprojection $AP_A x = \{a(x)\}$, $x \in T_A(r)$, is the smallest possible (note that $r > R$).

Proof. The proof follows from Proposition 1.4, Theorem 2.2 and condition $C = \frac{R}{r-R} \Leftrightarrow R = \frac{Cr}{C+1}$. $\square$

3. Metric antiprojection and ε-antiprojection

Lemma 3.1. *Let $A, B \subset \mathcal{H}$ be closed convex bounded subsets, $x, y \in \mathcal{H}$. Then*

i) $|f_A(x) - f_B(x)| \leq h(A, B)$,
ii) $|f_A(x) - f_A(y)| \leq \|x - y\|$.

Proof. Denote $h = h(A, B)$.

Without loss of generality we may assume that $x = 0$.

Fix $\varepsilon > 0$. Suppose $b_\varepsilon \in B$ is such a point that $\|b_\varepsilon\| + \varepsilon > f_B(0)$. Suppose that a point $a_\varepsilon \in A$ satisfies the condition $\|a_\varepsilon - b_\varepsilon\| \leq h + \varepsilon$. Then

$$f_B(0) < \|b_\varepsilon\| + \varepsilon \leq \|a_\varepsilon\| + \|a_\varepsilon - b_\varepsilon\| + \varepsilon \leq f_A(0) + h + 2\varepsilon.$$

In the same way

$$f_A(0) \leq f_B(0) + h + 2\varepsilon,$$

i.e. $|f_A(0) - f_B(0)| \le h + 2\varepsilon$ for any $\varepsilon > 0$. Taking the limit $\varepsilon \to +0$, we get item *1*).

Item *2*) follows from the Lipschitz condition with the constant 1 for the norm $\|\cdot\|$. $\square$

Lemma 3.2. *Let $A, B \subset \mathcal{H}$ be closed bounded subsets. Suppose that the set B is strongly convex of radius R, $h = h(A, B)$. Let a point $x \in \mathcal{H}$ be such that $R < f_B(x)$. Suppose that $AP_B x = \{b(x)\}$ and $a(x) \in AP_A x$ exists. Then*

$$\|a(x) - b(x)\| \le 2\sqrt{\frac{f_B(x)}{f_B(x) - R}}\sqrt{Rh + h^2} + h. \tag{5}$$

Proof. Denote $a = a(x)$, $b = b(x) = AP_B x$. The point $b(x)$ exists (and it is unique) by Proposition 1.4. Without loss of generality we may assume that $x = 0$.

By Lemma 3.1 we have two possible cases. Case 1) $R < f_B(0) \le f_A(0) \le f_B(0) + h$, and case 2) $f_A(0) \le f_B(0) \le f_A(0) + h$, $R < f_B(0)$.

Case 1). Define $z = b - R\frac{b}{\|b\|}$. By Proposition 1.3 $B \subset B_R(z)$ and from the condition of the theorem for any $t > h$ we have the inclusion $A \subset B_{R+t}(z)$. Consider the triangle $a0b$ and the segment $[a, z]$, $a \in B_{R+t}(z) \setminus \operatorname{int} B_{f_B(0)}(0)$. We have the inequality $d = \|a - z\| \le R + t$.

With $\varphi = \angle a0b$, we have

$$\cos\varphi = \frac{f_A^2(0) + (f_B(0) - R)^2 - d^2}{2f_A(0)\,(f_B(0) - R)} \ge \frac{f_A^2(0) + (f_B(0) - R)^2 - (R + t)^2}{2f_A(0)\,(f_B(0) - R)},$$

and hence

$$\|a - b\|^2 = f_A^2(0) + f_B^2(0) - 2f_A(0)f_B(0)\cos\varphi \le \frac{f_B(0)}{f_B(0) - R}\left(2Rt + t^2\right).$$

Taking the limit $t \to h + 0$, we get

$$\|a - b\| \le \sqrt{\frac{f_B(0)}{f_B(0) - R}}\,\sqrt{2Rh + h^2}. \tag{6}$$

Case 2). The set $C = A + B_h(0)$ satisfies the conditions $f_C(0) = f_A(0) + h$, $c = a + h\frac{a}{\|a\|} \in AP_C 0$. Moreover $R < f_B(0) \le f_C(0) \le f_B(0) + h$. Thus we have the case 1) for the sets B and C with particular changes ($h(B, C) \le 2h$). Hence

$$\|c - b\| = \left\|a + h\frac{a}{\|a\|} - b\right\| \le \sqrt{\frac{f_B(0)}{f_B(0) - R}}\,\sqrt{2R(2h) + (2h)^2},$$

and

$$\|a - b\| \le 2\sqrt{\frac{f_B(0)}{f_B(0) - R}}\,\sqrt{Rh + h^2} + h. \tag{7}$$

The cases 1) and 2) give the estimates (6) and (7), and we get the conclusion of the lemma. □

Remark 3.3. The condition of strong convexity of radius R for the set B in Lemma 3.2 can be replaced by the inclusion

$$b(x) \in AP_B\left(b(x) - R\frac{b(x)-x}{\|b(x)-x\|}\right).$$

Then Formula (5) of Lemma 3.2 remains the same.

Theorem 3.4. *Let $A, B \subset \mathcal{H}$ be closed bounded sets and let $x, y \in \mathcal{H}$ and $h = h(A,B)$. Suppose that the set B is strongly convex of radius R and the condition $\min\{f_B(x), f_B(y)\} \geq r > R$ is satisfied. Then for $\{b\} = AP_B y$, $a \in AP_A x$, one has*

$$\|a-b\| \leq 2\sqrt{\frac{r}{r-R}}\sqrt{Rh+h^2} + h + \frac{R}{r-R}\|x-y\|. \tag{8}$$

Proof. Note that the antiprojection $AP_B y$ is singleton by Proposition 1.4. Let $c = AP_B x$ (the point c exists and it is unique by Proposition 1.4 again).

$$\|a-b\| \leq \|a-c\| + \|b-c\|.$$

From Lemma 3.2

$$\|a-c\| \leq 2\sqrt{\frac{f_B(x)}{f_B(x)-R}}\sqrt{Rh+h^2} + h \leq 2\sqrt{\frac{r}{r-R}}\sqrt{Rh+h^2} + h,$$

and from Proposition 1.4

$$\|b-c\| \leq \frac{R}{r-R}\|x-y\|.$$

Hence we obtain Formula (8). □

Theorem 3.4 refines Lemma 16 [6] for the case of the Hilbert space.

Lemma 3.5. *Let $B_1, B_2 \subset \mathcal{H}$ be closed convex bounded subsets with nonempty interior, $C_i = \mathcal{H}\setminus \operatorname{int} B_i$, $i = 1, 2$. Then*

$$h(C_1, C_2) \leq h(B_1, B_2).$$

Proof. Fix a point $z \in C_1$ with $z \notin C_2$. This means that $z \notin \operatorname{int} B_1$, $z \in \operatorname{int} B_2$. Let ϱ be the distance from the point z to the set C_2. By the separation theorem there exists a unit vector $p \in \mathcal{H}$ such that $(p, z-b) \geq 0$ for all $b \in B_1$. From the inclusion $B_\varrho(z) \subset B_2$ we have $z + \varrho p \in B_2$, hence

$$\begin{aligned} h(B_1, B_2) &\geq \varrho_{B_1}(z+\varrho p) = \inf_{b\in B_1}\|z+\varrho p - b\| \\ &\geq \inf_{b\in B_1}(p, z+\varrho p - b) \geq \varrho. \end{aligned}$$

Thus, $\sup_{z\in C_1}\varrho_{C_2}(z) \leq h(B_1,B_2)$. In the same way $\sup_{z\in C_2}\varrho_{C_1}(z) \leq h(B_1,B_2)$. Hence, $h(C_1,C_2) \leq h(B_1,B_2)$. □

Definition 3.6. Let $\varepsilon > 0$. Define

$$AP_A^{\varepsilon}(x) = A \setminus \operatorname{int} B_{f_A(x)-\varepsilon}(x).$$

We shall call the set $AP_A^{\varepsilon}(x)$ the ε-*antiprojection* of the point x on the set A.

Theorem 3.7. *Let $A_1, A_2 \subset \mathcal{H}$ be strongly convex subsets of radius $R_i > 0$, $x_1, x_2 \in \mathcal{H}$, $r_i = f_{A_i}(x_i) > R_i$, $\varepsilon_i \in (0, r_i - R_i)$, $i = 1, 2$. Then*

$$\begin{aligned} &h(AP_{A_1}^{\varepsilon_1}(x_1), AP_{A_2}^{\varepsilon_2}(x_2)) \\ &\le 16 \max\left\{\frac{R_1}{\varepsilon_1}, \frac{R_2}{\varepsilon_2}\right\} (2\|x_1 - x_2\| + 2h(A_1, A_2) + |\varepsilon_1 - \varepsilon_2|)\,. \end{aligned} \tag{9}$$

Proof. Consider the sets $C_i = \mathcal{H} \setminus \operatorname{int} B_{r_i - \varepsilon_i}(x_i)$, $i = 1, 2$. These sets are weakly convex in the Hilbert space [7] with the constant of weak convexity $r_i - \varepsilon_i > R_i$, $i = 1, 2$. Hereafter i means 1 or 2.

Denote $a_i = AP_{A_i} x_i$. We have $B_{\varepsilon_i}(a_i) \subset C_i$ and $AP_{A_i}^{\varepsilon_i}(x_i) = A_i \cap C_i$.

By Theorem 3.2.1 [7] we have

$$h\left(A_1 \cap C_1, A_2 \cap C_2\right) \le 16 \max\left\{\frac{R_1}{\varepsilon_1}, \frac{R_2}{\varepsilon_2}\right\} (h(A_1, A_2) + h(C_1, C_2))\,. \tag{10}$$

Using Lemmas 3.5 and 3.1 we get

$$\begin{aligned} h(C_1, C_2) &\le h\left(B_{r_1-\varepsilon_1}(x_1), B_{r_2-\varepsilon_2}(x_2)\right) \\ &\le |r_1 - \varepsilon_1 - r_2 + \varepsilon_2| + \|x_1 - x_2\| \\ &\le |r_1 - r_2| + |\varepsilon_1 - \varepsilon_2| + \|x_1 - x_2\| \\ &= |f_{A_1}(x_1) - f_{A_2}(x_2)| + |\varepsilon_1 - \varepsilon_2| + \|x_1 - x_2\| \\ &\le \|x_1 - x_2\| + h(A_1, A_2) + |\varepsilon_1 - \varepsilon_2| + \|x_1 - x_2\| \\ &= 2\|x_1 - x_2\| + h(A_1, A_2) + |\varepsilon_1 - \varepsilon_2|. \end{aligned}$$

Substituting the last estimate to Formula (10), we obtain (9). □

4. At which points the antiprojection is realized?

In the present section we shall consider the next question: which points of the set A can be elements of the antiprojections, for which points of the space and which antidistance is realized at these points?

The *strongly convex R-hull* of the set $A \subset \mathcal{H}$, $\operatorname{int}\left(B_R(0) \overset{*}{-} A\right) \neq \emptyset$ (condition of nonempty interior) is the intersection of all closed balls of radius R, each of which contains the set A. We shall denote the strongly convex R-hull of the set A by $\operatorname{strco}_R A$. In [8, Lemma 4.4.1] it was obtained that in the above case

$$\operatorname{strco}_R A = B_R(0) \overset{*}{-} \left(B_R(0) \overset{*}{-} A\right), \tag{11}$$

$$s\left(p, \operatorname{strco}_R A\right) = R\|p\| - \operatorname{co}\left(R\|p\| - s(p, A)\right). \tag{12}$$

Recall (see [1]) that for the set $A \subset \mathcal{H}$ the point $a \in A$ is called *R-exposed* $(R > 0)$, if there exists a ball $B_R(z) \subset \mathcal{H}$ with $A \subset B_R(z)$ and $A \cap \partial B_R(z) = \{a\}$. We shall denote the set of R-exposed points for the set A by $\exp_R A$. If the subset $A \subset \mathcal{H}$ is closed and is contained in the ball of radius $r \in (0, R)$, then $\exp_R A \neq \emptyset$ and, moreover, $A \subset \operatorname{strco}_R \exp_R A$ [1].

Let $A \subset \mathcal{H}$. Then all singleton antiprojections for the points from the set $\partial T_A(R)$ are exactly the points of the set $\exp_R A$. The last set coincides with the set $\exp_R \operatorname{strco}_R A$ in the case when the set A is (norm) compact.

Theorem 4.1. *Let a compact subset $A \subset \mathcal{H}$ be such that* $\operatorname{int}\left(B_R(0) \stackrel{*}{-} A\right) \neq \emptyset$. *Then* $\exp_R A = \exp_R \operatorname{strco}_R A$.

Proof. Fix $a \in \exp_R \operatorname{strco}_R A$, i.e. for some ball $B_R(z)$ we have $A \subset \operatorname{strco}_R A \subset B_R(z)$ and $\operatorname{strco}_R A \cap \partial B_R(z) = \{a\}$. Suppose that, $a \notin A$ (otherwise it is obvious that $a \in \exp_R A$). Then $A \subset \operatorname{int} B_R(z)$. By the theorem about topological separation of compact set and closed set with empty intersection there exists a number $\varepsilon > 0$ such that $A + B_\varepsilon(0) \subset B_R(z)$. Put $p = \frac{a-z}{R} \in \partial B_1(0)$. Then $A \subset B_R\left(z - \frac{\varepsilon}{2}p\right)$, and hence, by the definition, $\operatorname{strco}_R A \subset B_R\left(z - \frac{\varepsilon}{2}p\right)$. But from the other hand $a \notin B_R\left(z - \frac{\varepsilon}{2}p\right)$, and thus $a \notin \operatorname{strco}_R A$. This contradiction shows that the assumption $a \notin A$ is not true.

Vise versa, let $a \in \exp_R A$. Then for some ball $B_R(z)$ we have $A \subset B_R(z)$ and $A \cap \partial B_R(z) = \{a\}$. By the definition of strongly convex R-hull $\operatorname{strco}_R A \subset B_R(z)$ and thus $a \in \operatorname{strco}_R A \cap \partial B_R(z)$. Suppose that there exists a point $b \in (\operatorname{strco}_R A \cap \partial B_R(z)) \setminus A$.

Put $p = \frac{a-b}{\|a-b\|}$, $\varepsilon = \frac{\|a-b\|}{4} > 0$, $A_1 = \operatorname{cl}(A \setminus B_\varepsilon(a))$. The set A_1 is compact (maybe empty) and $A_1 \subset \operatorname{int} B_R(z)$. Hence there exists $\delta > 0$ such that $A_1 + B_\delta(0) \subset \operatorname{int} B_R(z)$ (if the set A_1 is empty, then put $\delta = \varepsilon$). Let $\gamma = \min\{\varepsilon, \delta\} > 0$. Then $A \subset B_R(z + \gamma p)$, and we have $\operatorname{strco}_R A \subset B_R(z + \gamma p)$. But $b \notin B_R(z + \gamma p)$. This contradiction shows, that $\operatorname{strco}_R A \cap \partial B_R(z) = \{a\}$. Hence $a \in \exp_R \operatorname{strco}_R A$. □

So, (for the set A) the set $\exp_R A$ is the set of singleton (unique) antiprojections, which realize the antidistance R. If $a \in \exp_R A$ and the ball $B_R(z)$ is such that $A \subset B_R(z)$, $A \cap \partial B_R(z) = \{a\}$, then $\{a\} = AP_A x$ for any point $x \in \left\{z + t\frac{z-a}{\|z-a\|} \mid t \geq 0\right\}$.

Let us make one more remark. Let $a \in AP_A x$ for some point $x \in \mathcal{H}$ and $f_A(x) = R$. Then $a \in AP_{\operatorname{strco}_R A} x$ and from Proposition 1.3 for any unit vector $p \in N(\operatorname{strco}_R A, a)$ and number $r \geq R$ we have $a \in AP_A(a - rp)$. In the case $a \in \operatorname{int} \operatorname{strco}_R A$ the point a is not an element of the antiprojection for any point $x \in \mathcal{H}$ with $f_A(x) = R$.

Consider a set of the type $A = \{x_k\}_{k=1}^N \subset \mathbb{R}^n$. Let us calculate the normal cone $N(\operatorname{strco}_R A, a)$ for any point $a \in A$. Further we shall assume that $x_i \neq x_j$ when $i \neq j$.

Recall that the *polar set* for the subset $A \subset \mathcal{H}$ is the set $A^\circ = \{p \in \mathcal{H} \mid s(p, A) \leq 1\}$. For a cone $K \subset \mathcal{H}$ we have $K^\circ = \{p \mid (p, x) \leq 0, \ \forall x \in K\}$.

For the set $A \subset \mathbb{R}^n$ and the point $a \in A$ the set

$$T(A, a) = \{p \in \mathbb{R}^n \mid a + \lambda p + o_p(\lambda) \in A, \ \lambda > 0\}$$

is called the *tangent cone* to the set A at the point a. For a convex closed subset $A \subset \mathcal{H}$ and $a \in A$

$$N(A, a) = T^\circ(A, a),$$

see [8, Proposition 1.4.6, Lemma 1.12.7].

Lemma 4.2. *Let $R > 0$, $x_1, x_2 \in \mathbb{R}^n$, $\|x_1 - x_2\| \leq 2R$. Then*

$$\begin{aligned} T &= T(\mathrm{strco}_R\{x_1, x_2\}, x_1) \\ &= \left\{ p \;\middle|\; (p, x_2 - x_1) \geq \|x_1 - x_2\| \|p\| \sqrt{1 - \frac{\|x_1 - x_2\|^2}{4R^2}} \right\}, \\ N &= N(\mathrm{strco}_R\{x_1, x_2\}, x_1) = \left\{ p \;\middle|\; (p, x_1 - x_2) \geq \|p\| \frac{\|x_1 - x_2\|^2}{2R} \right\}. \end{aligned}$$

Proof. The set $\mathrm{strco}_R\{x_1, x_2\}$ is spindle-like: the boundary of this set is union of arcs of the circles of radius R with the length less or equal πR. Fix any vector $p \in T$. Then the angle between p and $x_2 - x_1$ is less or equal the angle φ between the segment $[x_1, x_2]$ and the arc (of the circle of radius R with the length $\leq \pi R$ and the endpoints x_1, x_2) at the point x_1. By the Pythagoras theorem we get

$$\sin \varphi = \frac{\|x_1 - x_2\|}{2R}, \qquad \cos \varphi = \sqrt{1 - \frac{\|x_1 - x_2\|^2}{4R^2}},$$

and

$$(p, x_2 - x_1) \geq \|x_1 - x_2\| \|p\| \cos \varphi.$$

The first formula is proved. The second formula follows from the equality $N = T^\circ$. $\square$

Theorem 4.3. *Let $\{x_k\}_{k=1}^N \subset \mathbb{R}^n$, $R > 0$. Let $y \in \mathrm{strco}_R\{x_k\}_{k=1}^N$. Then*

$$N = N(\mathrm{strco}_R\{x_k\}_{k=1}^N, y) = \bigcap_{k=1}^N N(\mathrm{strco}_R\{x_k, y\}, y).$$

Proof. Put $N_k = N(\mathrm{strco}_R\{x_k, y\}, y)$ for any $1 \leq k \leq N$. By the inclusion $\mathrm{strco}_R\{x_i, y\} \subset \mathrm{strco}_R\{x_k\}_{k=1}^N$ we get $N \subset N_i$ for all i. Hence $N \subset \cap_k N_k$.

Suppose that there exists some vector $p \in \cap_k N_k$ ($\|p\| = 1$ without loss of generality) such that $p \notin N$. By Proposition 1.3 $\mathrm{strco}_R\{x_k, y\} \subset B_R(y - Rp)$, i.e.

$$\mathrm{strco}_R \left(\bigcup_{k=1}^N \mathrm{strco}_R\{x_k, y\} \right) \subset B_R(y - Rp). \tag{13}$$

From the inclusion $p \notin N$ by Proposition 1.3 we have

$$\mathrm{strco}_R\{x_k\}_{k=1}^N \backslash B_R(y - Rp) \neq \emptyset.$$

But it is easy to check that

$$\mathrm{strco}_R\{x_k\}_{k=1}^N = \mathrm{strco}_R\left(\bigcup_{k=1}^N \mathrm{strco}_R\{x_k, y\}\right).$$

This contradicts Formula (13). □

Theorem 4.4. *$N = N\left(\mathrm{strco}_R\{x_k\}_{k=1}^N, y\right) \neq \{0\}$ if and only if there exists a vector $p \in S$ with $\|p\| \leq 1$, where*

$$S = \left\{ x \;\middle|\; \left(x, \frac{y - x_k}{\|y - x_k\|}\right) \geq \frac{\|y - x_k\|}{2R},\ 1 \leq k \leq N \right\}.$$

Proof. Put $p_k = \frac{y - x_k}{\|y - x_k\|}$ and $c_k = \frac{\|y - x_k\|}{2R}$, $1 \leq k \leq N$. If $y = x_j$ for some $1 \leq j \leq N$ then $k \neq j$.

For all k we have $\|p_k\| = 1$ and $c_k > 0$.

Suppose that $N \neq \{0\}$. Then there exists a unit vector $p \in N$. By Theorem 4.3 $(p, p_k) \geq \|p\| c_k = c_k$ for all k, i.e. $p \in S$.

Suppose that there exists $p \in S$ with $\|p\| \leq 1$. Then for all k we have $(p_k, p) \geq c_k$. Obviously $p \neq 0$, hence

$$\left(p_k, \frac{p}{\|p\|}\right) \geq \frac{c_k}{\|p\|} \geq c_k.$$

So for all k and $\lambda \geq 0$ we have $(p_k, \lambda p) \geq \lambda \|p\| c_k$, i.e. $p \in N$. □

Theorem 4.4 permits to check the inclusion $y \in \partial A$. We should solve the problem of quadratic programming (notations from the proof of Theorem 4.4)

$$\min \|p\|^2, \qquad (p_k, p) \geq c_k,\ 1 \leq k \leq N.$$

(Note that if $y = x_j$ then $k \neq j$).

If we have a solution $p \in \mathbb{R}^n$, $\|p\| \leq 1$, then $N\left(\mathrm{strco}_R\{x_k\}_{k=1}^N, y\right) \neq \{0\}$ and $y \in \partial A$. Moreover, $p \in N\left(\mathrm{strco}_R\{x_k\}_{k=1}^N, y\right)$. Otherwise $y \in \mathrm{int}\, A$.

So, we can consider the set $\mathrm{strco}_R A$ without loss of generality. The last set $\mathrm{strco}_R A$, in comparison with the set A, is intersection of closed balls of radius R, and the class of such sets is well studied [8, Chapters 3, 4].

Unfortunately, as we can see from Formulae (11), (12) and Theorems 4.3, 4.4, the construction of the set $\mathrm{strco}_R A$ is the problem of high algorithmic complexity.

Thereby below we shall obtain few more simple local conditions, which will be allocate the point of the set A, which is an element of the antiprojection for some points of the space.

For any closed convex subset $A \subset \mathcal{H}$ and vector $p \in \mathcal{H}\backslash\{0\}$ define

$$A(p) = \{x \in A \mid (p, x) = s(p, A)\}.$$

We shall call $A(p)$ the set of *supporting elements of the set A in the direction of the vector p.*

Definition 4.5. Let a subset $A \subset \mathcal{H}$ be closed convex and bounded, $a \in \partial A$, $N(A, a) \neq \{0\}$. We shall say that the set A has the *Lipschitz supporting elements with respect to the point a and the vector $p \in N(A, a)$, $\|p\| = 1$, with the constant $L > 0$,* if for any unit vector $q \in \mathcal{H}$ and any point $z \in A(q)$ one has

$$\|a - z\| \leq L\|p - q\|.$$

Note, that Definition 4.5 does not mean that the set A is strictly convex. For example, the set from $\mathbb{R}^2$

$$A = \text{co}\{(1, 1), (-1, 1), (-1, -1), (1, -1)\}$$

satisfies Definition 4.5 for $a = (1, 1)$ and $p = \frac{1}{\sqrt{2}}(1, 1)$ with the constant $L = \frac{2}{\sqrt{2-\sqrt{2}}}$.

Theorem 4.6. *Let $A \subset \mathcal{H}$ be a closed convex bounded subset, $a \in \partial A$, $p \in N(A, a)$, $\|p\| = 1$. Suppose that the set A has the Lipschitz supporting elements with respect to the point a and the vector p with the constant $L > 0$. Then we have $a \in AP_A(a - Lp)$.*

Proof. Assume the contrary: $A \not\subset B_R(a - Lp)$. Put $w = a - Lp$ and

$$R_0 = \inf\{r > 0 \mid A \subset B_r(w)\},$$

$R_0 < +\infty$. The definition of R_0 implies that $A \subset w + R_0 B_1(0)$ and $R_0 > L$.

Choose a sequence $\{x_k\}_{k=1}^{\infty} \subset A$ such that the numbers $R_k = \|x_k - z\|$ satisfy the conditions $L < R_k$ and $\lim_{k\to\infty} R_k = R_0$.

Let $q_k = \frac{x_k - w}{R_k}$ and $H_k^+ = \{x \in \mathcal{H} \mid (q_k, x) \geq (q_k, x_k)\}$. Put $r_k = \sqrt{R_0^2 - R_k^2}$. By the definition $\lim_{k\to\infty} r_k = 0$.

Let $x \in B_{R_0}(w) \cap H_k^+$. Then

$$\|x - x_k\|^2 = \|x - w - R_k q_k\|^2 = \|x - w\|^2 + R_k^2 + 2R_k(w - x, q_k).$$

By the inclusion $x \in H_k^+$ we get

$$(w - x, q_k) \leq (w - x_k, q_k) = -R_k$$

and

$$\|x - x_k\|^2 \leq \|x - w\|^2 - R_k^2 \leq R_0^2 - R_k^2 = r_k^2.$$

Thus $A(q_k) \subset B_{R_0}(z) \cap H_k^+ \subset x_k + r_k B_1(0)$. For any natural k we obtain the following equality for any $z_k \in A(q_k)$:

$$z_k = w + R_k q_k + r_k e_k(z_k), \quad e_k(z_k) \in B_1(0).$$

Put

$$\alpha_k(z_k) = r_k^2 + 2r_k(e_k(z_k), R_k q_k - Lp),$$

we note that $\lim_{k\to\infty} \alpha_k(z_k) = 0$ uniformly on $z_k \in A(q_k)$ and that for any $z_k \in A(q_k)$

$$\begin{aligned}
&\|a - z_k\|^2 \\
&= \|R_k q_k + r_k e_k(z_k) - Lp\|^2 \\
&= \alpha_k(z_k) + (R_k - L)^2 + L^2\|q_k - p\|^2 + 2L(R_k - L)(q_k, q_k - p) \\
&\geq \alpha_k(z_k) + (R_k - L)^2 + L^2\|q_k - p\|^2.
\end{aligned}$$

By the last formula the Lipschitz condition with the Lipschitz constant L is not true for sufficiently large numbers k. □

Note, that for the ball $B_L(0)$, any point $a \in \partial B_L(0)$ and for the unit vector $p \in N(B_L(0), a)$ the result of Theorem 4.6 is precise.

Remark 4.7. Suppose that Definition 4.5 takes place for such unit vectors q, that $\|p - q\| \leq t_0$, $t_0 \in (0, 2)$. Then it takes place for all unit vectors q with the constant

$$L_1 = \max\left\{L, \frac{\operatorname{diam} A}{t_0}\right\}.$$

Indeed, if $\|p - q\| > t_0$ and $z \in A(q)$, then

$$\|a - z\| \leq \operatorname{diam} A \leq \frac{\|p - q\|}{t_0} \operatorname{diam} A.$$

Theorem 4.8. *Let $A \subset \mathcal{H}$ be a closed convex bounded subset, $a \in \partial A$, $p \in N(A, a)$, $\|p\| = 1$. Suppose that for the number $r > 0$ we have the inclusion $a \in AP_A(a - rp)$. Then the set A has the Lipschitz supporting elements with respect to the point a and the vector $p \in N(A, a)$, $\|p\| = 1$, with the constant $L = 2r$.*

Proof. From the condition of the theorem

$$A \subset B_r(a - rp), \qquad a \in A \bigcap \partial B_r(a - rp).$$

Let $\|q\| = 1$, $z_1 \in \partial B_r(a - rp)$: $(q, z_1) = s(q, B_r(a - rp))$. Any point $z \in A(q)$ belongs to the set ("segment")

$$S = \{x \in \mathcal{H} \mid (q, x) \geq (q, a)\} \bigcap B_r(a - rp).$$

It is easy to see that the diameter of the "segment" S does not exceed $\operatorname{diam} S \leq 2\|z_1 - a\| = 2r\|p - q\|$. Hence

$$\|a - z\| \leq \operatorname{diam} S \leq 2r\|p - q\|.$$

□

Consider the triangle $A = \mathrm{co}\{a, b, c\}$ in the plane $\mathbb{R}^2$ with $\|a - b\| = \|b - c\| = 1$, $\angle(bac) = \angle(bca) = \varphi$. Let $w = \frac{a+b}{2}$, $p = \frac{b-w}{\|b-w\|}$. Then the set A has the Lipschitz supporting elements with respect to the point b and the vector p ($p \in N(A, b)$, $\|p\| = 1$) with the constant $L(\varphi) = \frac{1}{2\sin(\varphi/2)}$. The radius of the circumcircle for the triangle A equals $R(\varphi) = \frac{1}{2\sin\varphi}$ and it center is $b - R(\varphi)p$. We have

$$\lim_{\varphi\to+0} \frac{L(\varphi)}{2R(\varphi)} = 1$$

Hence the result of Theorem 4.8 is precise.

Lemma 4.9. *Let $A \subset \mathcal{H}$ be a closed bounded subset. Suppose that $r > 0$ and for a point $x \in \partial T_A(r)$ the antiprojection is singleton: $AP_A x = \{a(x)\}$. The letter is equivalent to the fact that for any point $z \in A\backslash\{a(x)\}$ the inequality*

$$\|a(x) - z\| < \sqrt{2r}\sqrt{(p, a(x) - z)} \tag{14}$$

holds, where $p = \frac{a(x)-x}{r} \in \partial B_1(0)$.

Proof. We have $r = \|x - a(x)\|$. The condition $\|x - a(x)\| = f_A(x) > \|x - z\|$, $\forall z \in A\backslash\{a(x)\}$, is equivalent to the condition

$$\|a(x) - rp - z\| < r, \quad \forall z \in A\backslash\{a(x)\}.$$

Squared the last inequality, we obtain (14). □

Recall that the intersection of all convex cones each of which contains the convex set $A \subset \mathcal{H}$ is called the *convex conic hull* of A.

Theorem 4.10. *Let $A \subset \mathcal{H}$ be a closed bounded subset, $\beta \in (0, 1)$, $p \in \partial B_1(0)$ and $B_\beta(p) \subset N(A, a)$. Then $\{a\} = AP_A(a - rp)$ for all $r > \frac{f_A(a)}{2\beta}$.*

Remark 4.11. One of the most important classes of convex compact sets in $\mathbb{R}^n$ is the class of convex polyhedra. It is easy to see that only vertices of the convex polyhedron can be antiprojections of some points from $\mathbb{R}^n$. Theorem 4.10 permits to estimate the antidistance function to the vertex of polyhedron.

Proof. Put $A_0 = \mathrm{cl}\,\mathrm{co}\,A$.

Consider the polar set $T = (\mathrm{cone}\,(B_\beta(p)))^\circ$.

From the inclusion $\mathrm{cone}\,(B_\beta(p)) \subset N(A, a) = N(A_0, a)$ we have $T \supset N^\circ(A_0, a) = \mathrm{cl}\,\mathrm{cone}\,(A_0 - a)$ [8, Lemma 1.12.7, Proposition 1.4.6]. Thus, $A_0 - a \subset T$.

Let φ be the angle between the line $\{\lambda p \mid \lambda \in \mathbb{R}\}$ and (any) generatrix of the cone $\mathrm{cone}\,(B_\beta(p))$, $\sin\varphi = \beta$.

The cone T is defined by the formula

$$T = \{z \in \mathcal{H} \mid (q, z) \le 0,\ \forall q \in \mathrm{cone}\,(B_\beta(p))\},$$

hence T is the cone of revolution (as the polar set to the cone of revolution), the line $\{\lambda p \mid \lambda \in \mathbb{R}\}$ is the axis of symmetry for T. The angle between the axis of symmetry and any generatrix equals $\frac{\pi}{2} - \varphi$.

Fix any point $z \in A_0 \subset T + a$. The angle between p and $a - z$ is less than $\frac{\pi}{2} - \varphi$, hence, using Lemma 4.9, we get, that the condition

$$\frac{\|a-z\|^2}{2r} < (p, a-z)$$

follows from the condition

$$\frac{\|a-z\|^2}{2r} < \|a-z\| \cos\left(\frac{\pi}{2} - \varphi\right),$$

i.e.

$$\frac{\|a-z\|}{2r} < \sin\varphi = \beta.$$

The last condition follows from the inequality

$$\frac{f_{A_0}(a)}{2r} < \beta,$$

or, using $f_A(a) = f_{A_0}(a)$, from the inequality

$$\frac{f_A(a)}{2r} < \beta.$$

Thus for any number $r > \frac{f_A(a)}{2\beta}$ the point a is the unique antiprojection of the point $a - rp$. $\square$

Definition 4.12. Let $A \subset \mathcal{H}$ be a closed convex bounded subset, $a \in A$. We shall call the value

$$\delta_A(a,t) = \sup\left\{\delta \geq 0 \,\middle|\, \frac{a+z}{2} + B_\delta(0) \subset A, \quad z \in A, \ \|a-z\| = t\right\}$$

the *modulus of convexity of the set A at the point a.*

We shall say that the set A is *locally uniformly convex at the point $a \in A$,* if $\delta_A(a,t) > 0$ for any permissible values $t > 0$.

A closed convex subset A is locally uniformly convex at any point $a \in \operatorname{int} A$. Note also, that a locally uniformly convex set at the point $a \in A$ is not necessary strictly convex.

If the set A is uniformly convex with the modulus $\delta_A(t)$ [5, 4], then for any point $a \in A$ and permissible value $t > 0$ we have $\delta_A(a,t) \geq \delta_A(t)$.

Theorem 4.13. *Let $A \subset \mathcal{H}$ be a closed convex bounded subset, $a \in \partial A$, $C > 0$. Suppose that the modulus of convexity of the set A at the point a satisfies the condition $\delta_A(a,t) \geq Ct^2$ for any permissible values $t > 0$. Then $a \in \exp_R A$ for all $R > \frac{1}{4C}$.*

Proof. Fix $p \in N(A,a)$, $\|p\| = 1$. Define

$$R_0 = \sup_{z \in A} \frac{\|z-a\|^2}{2(p, a-z)}.$$

Then for any $R > R_0$ by Lemma 4.9 we have the inclusion $a \in \exp_R A$.

From the inclusion $\frac{a+z}{2} + B_{\delta_A(a,\|a-z\|)}(0) \subset A$ for all $z \in A$ we get $(p, a-z) \geq 2\delta_A(a, \|a-z\|)$. Hence

$$R_0 \leq \sup_{z \in A,\ \|a-z\|=t} \frac{t^2}{4\delta_A(a,t)} \leq \sup_{t \in (0, \operatorname{diam} A)} \frac{t^2}{4Ct^2} = \frac{1}{4C}. \qquad \square$$

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