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A Minmax Theorem for Concave-Convex Mappings with no Regularity Assumptions*

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We prove that zero-sum games with a concave-convex payoff mapping defined on a product of convex sets have a value as soon as the payoff mapping is bounded and one of the set is bounded and finite dimensional. In particular, no additional regularity assumption is required, such as lower or upper semicontinuity of the function or compactness of the sets. We provide several examples that show that our assumptions are minimal.

Introduction

Classical min-max theorems (see in particular von Neumann [3], Fan [1] and Sion [4] or see Sorin [5] and the first chapter of Mertens-Sorin-Zamir [2] for a survey of zero-sum games) require, in addition to convexity (or variants of it), regularity assumptions on the mapping, such as lower or upper semi-continuity.

We consider in this short note zero-sum games with a concave-convex mapping defined on a product of convex sets. We give simple assumptions leading to existence of a value without assuming any additional regularity of the mapping.

Theorem 1. *Let X and Y be two nonempty convex sets and $f : X \times Y \to \mathbb{R}$ be a concave-convex mapping, i.e., $f(\cdot, y)$ is concave and $f(x, \cdot)$ is convex for every $x \in X$ and $y \in Y$. Assume that*

- *X is finite dimensional,*
- *X is bounded,*
- *$f(x, \cdot)$ is lower bounded for some x in the relative interior of X.*

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Then the zero-sum game on $X \times Y$ with payoff f has a value, i.e.,

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\sup_{x\in X} f(x,y).$$

Remark 2. The assumptions of this theorem are satisfied in particular as soon as both X and Y are finite dimensional and bounded. Indeed, in that case $f(x,\cdot)$, being convex, is lower-bounded for any x

Proof. Without loss of generality, we can assume that X has a non-empty interior in $\mathbb{R}^n$ that contains 0 and that $f(0,\cdot)$ is nonnegative. For every $\varepsilon > 0$, we also define $X_\varepsilon = (1-\varepsilon)\bar{X}$, where $\bar{X}$ is the closure of X.

Since X_ε is included in the interior of the finite dimensional space X, every mapping $f(\cdot,y)$ being concave is continuous on X_ε. Hence, classic minimax theorems [1, Theorem 2] yield that

$$\sup_{x\in X_\varepsilon}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\sup_{x\in X_\varepsilon} f(x,y) := v_\varepsilon. \tag{1}$$

A first direct implication of Equation (1) is that

$$\limsup_{\varepsilon\to 0} v_\varepsilon \le \sup_{x\in X}\inf_{y\in Y} f(x,y). \tag{2}$$

The second implication is that, for every $\epsilon > 0$ there exists $y_\varepsilon \in Y$ such that $\sup_{x\in X_\varepsilon} f(x,y_\varepsilon) \le v_\varepsilon + \varepsilon$. Consider $x \in X$ that does not belong to X_ε, and let $x_\varepsilon = (1-\varepsilon)x \in X_\varepsilon$. Since $f(\cdot,y)$ is concave, one has that

$$\frac{f(x,y)-f(x_\varepsilon,y)}{\|x-x_\varepsilon\|} \le \frac{f(x_\varepsilon,y)-f(0,y)}{\|x_\varepsilon - 0\|} \le \frac{f(x_\varepsilon,y)}{\|x_\varepsilon\|}.$$

The definition of x_ε and the specific choice of $y = y_\varepsilon$ immediately gives that

$$f(x,y_\varepsilon) \le f(x_\varepsilon,y_\varepsilon)\left(1+\frac{\varepsilon}{1-\varepsilon}\right) \le (v_\varepsilon+\varepsilon)\left(1+\frac{\varepsilon}{1-\varepsilon}\right).$$

Taking the supremum in x yields

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X} f(x,y_\varepsilon) \le (v_\varepsilon+\varepsilon)\left(1+\frac{\varepsilon}{1-\varepsilon}\right).$$

Letting ε goes to 0, we obtain

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \liminf_{\varepsilon\to 0} v_\varepsilon \tag{3}$$

Equations (2) and (3) give the result. □

We now prove that our three assumptions cannot be dispensed with, by the mean of the following examples.

– Assume that $X = Y = \Delta([0,1))$ and that f is given by the bilinear extension of

$$f(i,j) = \begin{cases} 0 & \text{if } 0 = i < j \text{ or } 0 < j < i, \\ 1 & \text{if } 0 = j \leq i \text{ or } 0 < i \leq j, \end{cases} \qquad \forall i,j \in [0,1]. \tag{4}$$

Then f is bounded, bilinear and since $f(x,y) = (x \otimes y)\{0 = j \leq i\} \cup \{0 < i \leq j\}$ it is even 1-Lipschitz with respect to the total variation distance. The sets X and Y are bounded, and infinite dimensional. For any $x \in X$ and $\varepsilon > 0$, let $j > 0$ be such that x puts a probability less than ε on the interval $]0,j]$. Then $f(x,j) \leq \varepsilon$. Similarly, for any $y \in Y$, there exists $i > 0$ such that $f(i,y) \leq 1 - \varepsilon$. Hence,

$$\sup_{x\in X} \inf_{y\in Y} f(x,y) = 0 < \inf_{y\in Y} \sup_{x\in X} f(x,y) = 1.$$

– Assume that $X = Y = [1,+\infty)$ and define f by

$$f(x,y) = \begin{cases} \frac{x}{2y} & \text{if } x \leq y, \\ 1 - \frac{y}{2x} & \text{if } x \geq y. \end{cases} \tag{5}$$

Then X and Y are both finite dimensional but unbounded, f is concave-convex and bounded. Also note that on $X \times Y$,

$$0 \leq \frac{\partial f(x,y)}{\partial x} \leq \frac{1}{2y} \leq \frac{1}{2}$$

and similarly

$$0 \geq \frac{\partial f(x,y)}{\partial y} \geq -\frac{1}{2x} \geq -\frac{1}{2}$$

hence f is Lipschitz. But for any integer n, $f(x,nx) = \frac{1}{2n}$ while $f(ny,y) = 1 - \frac{1}{2n}$. Hence,

$$\sup_{x\in X} \inf_{y\in Y} f(x,y) = 0 < \inf_{y\in Y} \sup_{x\in X} f(x,y) = 1.$$

– Assume that $X = [0,1]$, $Y = [1,+\infty)$ and that

$$f(x,y) = \begin{cases} -xy & \text{if } x > 0, \\ -y & \text{if } x = 0. \end{cases} \tag{6}$$

Then X is bounded and finite dimensional, f is concave-linear and unbounded from below. But for any integer n, $f\left(x, \frac{n}{x}\right) = -n$ while $f\left(\frac{1}{ny}, y\right) = -\frac{1}{n}$. Hence,

$$\sup_{x\in X} \inf_{y\in Y} f(x,y) = -\infty < \inf_{y\in Y} \sup_{x\in X} f(x,y) = 0.$$

Remark 3. These examples also show that the first two assumptions of Theorem 2 cannot be dispensed with even when f is Lipschitz continuous.

Remark 4. Also recall that in the less demanding case of quasiconcave/convex mappings, regularity assumptions are necessary [4].

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