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Boundedness Criterions for the Hardy Operator in Weighted $L^{p(\cdot)}(0,l)$ Space*

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Equivalent conditions are proved for the Hardy type weighted inequality

$$\left\| W(\cdot)^{-1}\sigma(\cdot)^{\frac{1}{p(\cdot)}} \int_0^x f(t)dt \right\|_{L^{p(\cdot)}(0,l)} \leq C \left\| \omega(\cdot)^{\frac{1}{p(\cdot)}} f(\cdot) \right\|_{L^{p(\cdot)}(0,l)}, \quad f \geq 0$$

to be fulfilled in the norms of a Lebesgue space with variable exponent $L^{p(\cdot)}(0,l)$. It is assumed that the function $p(.)$ is a monotone function.

Keywords: Hardy operator, Hardy type inequality, variable exponent, weighted inequality, necessary and sufficient condition

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1. Introduction

The aim of this paper is to give a simple characterization of some weighted boundedness problem for the Hardy operator $Hf(x) = \int_0^x f(t)dt$. More precisely,

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we study Hardy's type general weighted inequality

$$\left\| W(\cdot)^{-1}\sigma(\cdot)^{\frac{1}{p(\cdot)}} Hf(\cdot)\right\|_{L^{p(\cdot)}(0,l)} \le C \left\|\omega(\cdot)^{\frac{1}{p(\cdot)}} f(\cdot)\right\|_{L^{p(\cdot)}(0,l)}, \quad f \ge 0 \tag{1}$$

in the norms of the variable exponent Lebesgue space $L^{p(\cdot)}(0,l)$, where the constant $C > 0$ does not depend on a measurable function f, $W(x) = \int_0^x \sigma(t)dt$; $\sigma(t) = \omega(t)^{-\frac{1}{p(t)-1}}$, $l > 0$ (see, e.g. [18] for this inequality in the case of constant exponent $p(.)$).

There are a lot of examples from the theory of differential equations and analysis, in which essential use is made on the boundedness of the Hardy operator in some space (Lebesgue space, weighted Lebesgue space, Lorentz space, Orlicz space, weak Lebesgue space, variable Lebesgue space and so on, see e.g. in [18, 19, 6, 31, 7, 3]).

We study the boundedness of the Hardy operator in the variable exponent Lebesgue space with general weights. The investigation of these spaces is actual in connection with the modeling of electroreological fluids which can change their constant physical parameters instantaneously (see, [29, 32]). Subsequent studies of this topic lead to the rapid development of a suitable existence, uniqueness, and regularity theories of nonlinear elliptic and parabolic equations with partial derivatives (see, the bibliography e.g. in [1, 13, 26]). Also, the required mathematical methods of analysis were elaborated to study the boundedness of principal integral operators (maximal operator, fractional operator, singular operator, commutators and so on) in the variable exponent Lebesgue spaces $L^{p(\cdot)}$ (see, the monographs [8, 10]).

As it seems to us, the first results of investigation of the variable exponent Hardy inequality were obtained in the works [30, 12, 16, 14, 25]; subsequently, in the works [11, 28, 15, 20, 23] were proved necessary and sufficient conditions for the power weighted case. We refer to the works [9, 22, 21, 24] concerning necessary and sufficient conditions in the case of general weights and [4, 5] for the case of monotone functions f, where was essentially used a logarithmic continuity condition for the exponent function.

In this paper, for the case of general weights, we prove an equivalent condition for the inequality (1) to hold. The paper is organized as follows: in Section 2 we give necessary preliminaries on variable exponent Lebesgue spaces and listed the main results; in Sections 3, 4 we prove our main statement for the Hardy operator. More precisely, in Theorem 2.1, we have proved that a regularity condition is not needed if the exponent p is decreasing at small neighborhood of the origin; in Theorem 2.2, we prove that the condition

$$\int_a^1 W(x)^{-\frac{1}{p'(x)}} \frac{\sigma(x)dx}{W(x)} \le CW(a)^{-\frac{1}{p'(a)}}, \quad 0 < a < l \tag{2}$$

and several other equivalent conditions are necessary and sufficient for the inequality (1) to hold if the function $p(.)$ is increasing on the interval $(0,l)$.

2. Main results and the notation

For the basic properties of spaces $L^{p(.)}$ we refer to the monographs [8], [10]. Throughout this paper, it is assumed that $p(x)$ is a measurable function in $(0,l)$, taking its values in the interval $[1,\infty)$ with $p^+ = \sup\{p(x) : x \in (0,l)\} < \infty$. The space of functions $L^{p(.)}(0,l)$ is introduced as the class of measurable functions $f(x)$ in $(0,l)$ which have a finite $I_{p(.)}(f) = \int_0^l |f(x)|^{p(x)}\, dx$ modular (put also $I_{p(.);E}(f) = \int_E |f(x)|^{p(x)} dx$). A norm in $L^{p(.)}(0,l)$ is given in the form

$$\|f\|_{L^{p(.)}(0,l)} = \inf\left\{\lambda > 0 : I_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

(Also by $\|f\|_{p(.);E}$ we denote the $p(.)$ norm over set E.)

For $1 < p^-, p^+ < \infty$ the space $L^{p(.)}(0,l)$ is a reflexive Banach space. The relation between the modular and the norm is expressed by the inequalities

$$\|f\|^{p^+}_{L^{p(.)}(0,l)} \leq I_p(f) \leq \|f\|^{p^-}_{L^{p(.)}(0,l)}, \quad \|f\|_{L^{p(.)}(0,l)} \leq 1 \tag{3}$$

$$\|f\|^{p^-}_{L^{p(.)}(0,l)} \leq I_p(f) \leq \|f\|^{p^+}_{L^{p(.)}(0,l)}, \quad \|f\|_{L^{p(.)}(0,l)} > 1. \tag{4}$$

These inequalities enable us to derive estimates of the norm in the modular form.

For the function $p(x)$ with $1 \leq p(x) < \infty$ the $p'(x)$ denotes the conjugate function of $p(x)$, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and $p' = \infty$ if $p = 1$. We denote by $C, C_1, C_2, ...$ various positive constants whose values may vary at each appearance. $B(x,r) = (x-r, x+r)$. We write $u \sim v$ if there are positive constants C_1, C_2 such that $C_1 u(x) \leq v(x) \leq C_1 u(x)$. The characteristic function of the set E is denoted by χ_E.

We say a function $u : (0,1) \to (-\infty, +\infty)$ is almost increasing (decreasing) if there exists a constant $C > 0$ such that $u(t_1) \leq Cu(t_2)$ $(u(t_2) \leq Cu(t_1))$ for $0 < t_1 \leq t_2 < l$.

Throughout the paper, we assume that $p : (0,l) \to [1,\infty)$ and $\omega : (0,l) \to (0,\infty)$ be a measurable function such that $\sigma = \omega^{-\frac{1}{p(x)-1}} \in L(0,a)$ for all $a \in (0,l)$ and $W(x) = \int_0^x \sigma(t)dt$.

The following main results are obtained in this paper. Below we present the main results obtained in this paper.

Theorem 2.1. *Let p be decreasing on some interval $(0,\varepsilon)$. Then it holds the inequality* (1) *for every positive measurable function f.*

Theorem 2.2. . *Let p be increasing on $(0,l)$. Then the following statements are equivalently:*

1) *There exists a constant $C > 0$ such that the inequality* (1) *holds for a function $f \geq 0$;*

2) *The condition* (2) *is satisfied;*

3) *There is an* $\varepsilon > 0$ *such that the function* $W(x)^{-\frac{1}{p'(x)}+\varepsilon}$ *is almost decreasing;*

4) *The condition*

$$\int_a^l \left[W(a)^{\frac{1}{p'(a)}} W(x)^{-\frac{1}{p'(x)}}\right]^{p(x)} \frac{\sigma(x)dx}{W(x)} \leq C, \quad 0 < a < l \tag{5}$$

is satisfied;

5) *The condition*

$$\|W(x)^{-1}\sigma(.)^{\frac{1}{p(.)}}\chi_{(0,l)}(.)\|_{L^{p(.)}(0,l)} \leq CW(a)^{-\frac{1}{p'(a)}}, \quad 0 < a < l \tag{6}$$

is satisfied.

Corollary 2.3. *Let* $p : (0, l) \to [0, \infty)$ *be a decreasing function on some interval* $(0, \varepsilon)$, $\varepsilon > 0$ *such that* $p(0) > 1$ *and the condition*

$$\int_x^l t^{-\frac{1}{p'(t)}} \frac{dt}{t} \leq Cx^{-\frac{1}{p'(x)}}, \quad 0 < x < l \tag{7}$$

is satisfied. Then it holds the inequality

$$\left\| x^{-1} \int_0^x f(t)dt \right\|_{L^{p(.)}(0,l)} \leq C \|f(.)\|_{L^{p(.)}(0,l)} \tag{8}$$

for a positive measurable function f.

Proof. Put $\omega(x) = 1$, $\sigma(x) = 1$, $W(x) = x$. Then the condition (2) is the condition (7) and Corollary 2.3 follows from accretion of Theorem 2.2.

3. Proof of Theorem 2.1

To prove the inequality (1) it suffices to consider the case when f is a positive measurable function such that $\|\omega(.)^{\frac{1}{p(.)}}f(.)\|_{L^{p(.)}(0,l)} \leq 1$ (see, [8], [10]). It follows from (3) that $I_{p(.)}\left(\omega^{\frac{1}{p(.)}}f\right) \leq 1$. In order to prove Theorem 2.1 we have to show

$$\|W(.)^{-1}\sigma(.)^{\frac{1}{p(.)}}Hf(.)\|_{L^{p(.)}(0,l)} \leq C_1. \tag{9}$$

To prove (9), we shall establish the estimate

$$I_{p(.)}\left(W(x)^{-1}\sigma(x)^{\frac{1}{p(x)}}Hf\right) \leq C_2.$$

Using the triangle inequality of $p(.)$-norms for $\delta \in (0, l)$ we have

$$\begin{aligned} &\|W(.)^{-1}\sigma(.)^{\frac{1}{p(.)}}Hf(.)\|_{L^{p(.)}(0,l)} \\ &\leq \|W(x)^{-1}\sigma(.)^{\frac{1}{p(.)}}Hf\|_{L^{p(.)}(0,\delta)} + \|W(.)^{-1}\sigma(.)^{\frac{1}{p(.)}}Hf(.)\|_{L^{p(.)}(\delta,l)} := i_1 + i_2 \end{aligned} \tag{10}$$

Define by $w(W)$ an inverse function for $W(w) = \int_0^w \sigma(u)du$. Replacing the variable $s = w(z)$, $s \to z$ in the expression of $Hf(x)$ one can easily see that

$$Hf(x) = \int_0^x f(s)ds = \int_0^{W(x)} \frac{f(w(z))}{\sigma(w(z))}dz.$$

The new change of variables $z = W(x)t$, $z \to t$ gives

$$Hf(x) = W(x)\int_0^1 \frac{f(w(tW(x)))}{\sigma(w(tW(x)))}dt.$$

Estimate of i_1. Using this and Minkowski's inequality for $L^{p(.)}$ norms (see, [8], [10]), we see that

$$\begin{aligned} i_1 = {} & \left\| W(.)^{-1}\sigma(.)^{\frac{1}{p(.)}} Hf \right\|_{L^{p(.)}(0,\delta)} = \left\| \int_0^1 \sigma(.)^{\frac{1}{p(.)}} \frac{f(w(tW(.)))}{\sigma(w(tW(.)))}dt \right\|_{L^{p(.)}(0,\epsilon)} \\ \leq {} & \int_0^1 \left\| \sigma(.)^{\frac{1}{p(.)}} \frac{f(w(tW(.)))}{\sigma(w(tW(.)))} \right\|_{L^{p(.)}(0,\delta)} dt. \end{aligned} \tag{11}$$

First, estimate

$$\left\| \sigma(.)^{\frac{1}{p(.)}} \frac{f(w(tW(.)))}{\sigma(w(tW(.)))} \right\|_{L^{p(.)}(0,\delta)}, \quad 0 < t < l.$$

Since $w\left(\frac{W(v)}{t}\right) > v$ and p is decreasing on $0, \delta$, we have $p\left(w\left(\frac{W(v)}{t}\right)\right) \leq p(v)$ for $v \in (0,\delta)$. Therefore,

$$\begin{aligned} & \int_0^\delta \left(\frac{\sigma(x)^{\frac{1}{p(x)}} f(w(tW(x)))}{t^{-\frac{1}{p(\delta)}} \sigma(w(tW(x)))} \right)^{p(x)} dx \\ = {} & \int_0^\delta \left(\frac{f(w(tW(x)))}{\sigma(w(tW(x)))} t^{\frac{1}{p(\delta)}} \right)^{p(x)} \sigma(x)dx \\ \leq {} & \int_0^\delta \left(\frac{f(w(tW(x)))}{\sigma(w(tW(x)))} \right)^{p(x)} t\sigma(x)dx \ \text{ since } \begin{cases} w(tW(x)) = v, \ tW(x) = W(v) \\ t\sigma(x)dx = dW(v) = \sigma(v)dv, \end{cases} \\ \leq {} & \int_0^{w(tW(\delta))} \left(\frac{f(v)}{\sigma(v)} \right)^{p\left(w\left(\frac{W(v)}{t}\right)\right)} \chi_{\left\{\frac{f(v)}{\sigma(v)} \geq 1\right\}}(v)\sigma(v)dv \\ & + \int_0^{w(tW(\delta))} \chi_{\left\{\frac{f(v)}{\sigma(v)} < 1\right\}}(v)\sigma(v)dv \int_0^\delta \left(\frac{f(v)}{\sigma(v)} \right)^{p(v)} \chi_{\left\{\frac{f(v)}{\sigma(v)} \geq 1\right\}}(v)\sigma(v)dv \\ & + \int_0^\delta \chi_{\left\{\frac{f(v)}{\sigma(v)} < 1\right\}}(v)\sigma(v)dv \leq 1 + W(\delta). \end{aligned} \tag{12}$$

This implies

$$\int_0^\delta \left(\frac{\sigma(x)^{\frac{1}{p(x)}}}{(1+W(\delta))^{\frac{1}{p(\delta)}}\, t^{-\frac{1}{p(\delta)}}} \cdot \frac{f\left(w(tW(x))\right)}{\sigma\left(w(tW(x))\right)} \right)^{p(x)} dx \le 1, \quad 0<t<l.$$

Therefore and using the definition of p-norms, we get

$$\left\| \sigma(.)^{\frac{1}{p(.)}} \frac{f\left(w(tW(.))\right)}{\sigma\left(w(tW(.))\right)} \right\|_{L^{p(.)}(0,\delta)} \le (1+W(\delta))^{\frac{1}{p(\delta)}}\, t^{-\frac{1}{p(\delta)}}, \quad 0<t<l. \tag{13}$$

Using (13) and (11) we get the estimate

$$i_1 \le (1+W(\delta))^{\frac{1}{p(\delta)}} \int_0^1 t^{-\frac{1}{p(\delta)}} dt \le (p(\delta))' \,(1+W(\delta))^{\frac{1}{p(\delta)}} \le C. \tag{14}$$

Estimate of i_2. Using Young's inequality, we get

$$\begin{aligned}
&\left(W(x)^{-1} \sigma(x)^{\frac{1}{p(x)}} Hf(x) \right)^{p(x)} \\
&= \sigma(x) W(x)^{-p(x)} \left(\int_0^x \frac{f(s)}{\sigma(s)} \sigma(s) ds \right)^{p(x)} \\
&\le \sigma(x)\,(1+W(l))^{\frac{p^+}{p^-}-1}\, W(\delta)^{-p^+} \int_0^x \frac{f(s)}{\sigma(s)} \sigma(s) ds \\
&\le C\sigma(x) \int_0^x \frac{1}{p(x)} \left(\frac{f(s)}{\sigma(s)} \right)^{p(x)} \sigma(s) ds + C\sigma(x) \int_0^x \frac{1}{p'(x)} \sigma(s) ds \\
&\le C\sigma(x), \quad \delta<x<l
\end{aligned}$$

since

$$\begin{aligned}
\int_0^x \left(\frac{f(s)}{\sigma(s)} \right) \sigma(s) ds &\le 2 \left\| \sigma(.)^{-\frac{1}{p'(.)}} f(.) \right\|_{L^{p(.)}(0,l)} \cdot \left\| \sigma(.)^{-\frac{1}{p'(.)}} f(.) \right\|_{L^{p'(.)}(0,l)} \\
&\le 2\,(1+W(l))^{\frac{1}{p^-}}
\end{aligned}$$

and

$$W(x) \ge W(\delta) \ \text{ for } x \in (\delta, l).$$

Therefore,

$$I_{p(.);(\delta,l)} \left(W(.)^{-1} \sigma(.)^{\frac{1}{p(.)}} Hf(.) \right) \le C_2 W(l).$$

Hence

$$i_2 = \left\| W(.)^{-1} \sigma(.)^{\frac{1}{p(.)}} Hf(.) \right\|_{L^{p(.)}(\delta,l)} \le C_3.$$

Using this estimate and (14) in (10), we complete the proof of Theorem 2.1.

4. Proof of Theorem 2.2

To prove Theorem 2.2 we need some lemmas. Throughout the section we assume $p : (0,l) \to [1,\infty)$ and $\omega : (0,l) \to (0,\infty)$ be measurable functions such that $\sigma(x) = \omega(x)^{-\frac{1}{p(x)-1}} \in L^1(0,a)$ for all $a \in (0,l)$ and $W(x) = \int_0^x \sigma(t)dt$.

Lemma 4.1. *Assume the inequality* (1) *be satisfied; then there exists a constant* $C > 0$ *such that the condition*

$$\left(\frac{1}{p(a)} - \frac{1}{p(a^1)}\right) \ln \frac{1}{W(a)} \le C, \quad 0 < W(a) < \frac{1}{2} \tag{15}$$

holds, where the $a^1 > a$ *is such that* $W(a^1) = 2W(a)$. *Moreover, the condition* (5) *is satisfied.*

Proof. Insert a test function

$$f_0(t) = \sigma(t)W(t)^{-\frac{1}{p(t)}}\chi_{(a_1,a)}(t), \quad t \in (0,l),\ 0 < W(a) < \frac{1}{2}$$

into the inequality (1), where $W(a_1) = \frac{1}{2}W(a)$. Then

$$I_{p(.)}\left(\omega f_0\right) = \int_{W(a_1)}^{W(a)} W(t)^{-1} dW(t) = \ln 2 \le 1,$$

hence $\left\|\omega^{\frac{1}{p(.)}} f_0\right\|_{L^{p(.)}(0,l)} \le 1.$

It follows from (1) that $\left\|\sigma(.)^{\frac{1}{p(.)}} W(.)^{-1} Hf_0(.)\right\|_{L^{p(.)}(0,1)} \le C$. This means $I_{p(.)}\left(\sigma(.)^{\frac{1}{p(.)}} W(.)^{-1} Hf_0\right) \le C_1$, therefore

$$\int_a^1 \sigma(x)W(x)^{-p(x)} \left[\int_0^x \sigma(t)W(t)^{-\frac{1}{p(t)}}\chi_{(a_1,a)}dt\right]^{p(x)} dx \le C_2. \tag{16}$$

or

$$\int_{W(a)}^{W(l)} \left[\int_{\frac{1}{2}W(a)}^{W(a)} \frac{dW(t)}{W(t)^{\frac{1}{p(t)}}}\right]^{p(x)} \frac{dW(x)}{W(x)^{p(x)}} \le C_2.$$

This implies that

$$\int_{W(a)}^{W(l)} \left[\int_{\frac{1}{2}W(a)}^{W(a)} W(x)^{-\frac{1}{p'(x)}} W(t)^{\frac{1}{p'(t)}} \frac{dW(t)}{W(t)}\right]^{p(x)} \frac{dW(x)}{W(x)} \le C_2. \tag{17}$$

Since $\frac{1}{p'(.)}$ is increasing, for $W(t) \in \left(\frac{1}{2}W(a), W(a)\right)$ we have

$$W(t)^{\frac{1}{p'(t)}} \ge \left(\frac{1}{2}W(a)\right)^{\frac{1}{p'(t)}} \ge CW(a)^{\frac{1}{p'(a)}},$$

where $C=\min\left\{\left(\frac{1}{2}\right)^{\frac{1}{p'(t)}},\ \left(\frac{1}{2}W(l)\right)^{\frac{1}{p'(0)}-\frac{1}{p'(l)}}\right\}$. Using this and (17) we have

$$C_3 \geq \int_{W(a)}^{W(l)} \left[W(x)^{\frac{1}{p'(x)}}\ W(a)^{\frac{1}{p'(a)}}\right]^{p(x)} \frac{dW(x)}{W(x)}.$$

This proves estimate (5). □

Further,

$$\begin{aligned} C_3 &\geq \int_{2W(a)}^{4W(a)} \left[W(x)^{\frac{1}{p'(x)}}\ W(a)^{\frac{1}{p'(a)}}\right]^{p(x)} \frac{dW(x)}{W(x)} \\ &\geq\ (4W(a))^{-\frac{1}{p'(a^1)}}\, W(a)^{\frac{1}{p'(a)}} \ln 2 \end{aligned}$$

since for $W(x)\in(2W(a),4W(a))$, we have $W(x)^{-\frac{1}{p'(x)}} \geq CW(a)^{-\frac{1}{p'(a^1)}}$, where $C=\min\left\{\left(\frac{1}{4}\right)^{\frac{1}{p'(t)}},\ \left(\frac{1}{4}W(l)\right)^{\frac{1}{p'(l)}-\frac{1}{p'(0)}}\right\}$. Hence

$$\left(\frac{1}{W(a)}\right)^{\frac{1}{p(a)}-\frac{1}{p(a^1)}} \leq C_4 \tag{18}$$

or

$$\left(\frac{1}{p(a)}-\frac{1}{p(a^1)}\right)\ln\frac{1}{W(a)} \leq C_5. \tag{19}$$

and

$$W(a)^{\frac{1}{p'(a^1)}} \sim W(a)^{\frac{1}{p'(a)}}. \tag{20}$$

We need also the following assertion to complete the proof of Lemma 4.1.

Claim 4.2. *For* $\frac{1}{2}W(x)<W(y)<2W(x)$ *the estimates*

$$C_1W(x)^{-\frac{1}{p'(x)}} \leq W(y)^{-\frac{1}{p'(y)}} \leq C_2W(x)^{-\frac{1}{p'(x)}} \tag{21}$$

hold.

Proof. Since $\frac{1}{p'(y)} \leq \frac{1}{p'(x_1)}$ using estimate (15) of Lemma 4.1, we have

$$\begin{aligned} W(y)^{-\frac{1}{p'(y)}} &\leq \left(\frac{W(x)}{2}\right)^{-\frac{1}{p'(y)}} \\ &\leq \left(\left(\frac{1}{W(x)}\right)^{\frac{1}{p'(x_1)}-\frac{1}{p'(x)}} 2^{\frac{1}{p'(l)}}+1\right) W(x)^{\frac{1}{p'(x)}} \\ &\leq C_1W(x)^{-\frac{1}{p'(x)}}. \end{aligned} \tag{22}$$

By the same way,

$$
\begin{aligned}
W(x)^{-\frac{1}{p'(x)}} &\le \left(\frac{W(y)}{2}\right)^{-\frac{1}{p'(y)}} \\
&\le \left(\left(\frac{1}{W(y)}\right)^{\frac{1}{p'(y_1)}-\frac{1}{p'(y)}} 2^{\frac{1}{p'(l)}}+1\right) W(y)^{\frac{1}{p'(y)}} \\
&\le C_1 W(y)^{-\frac{1}{p'(y)}}.
\end{aligned}
\tag{23}
$$

This proves the estimate (21). Now, applying the estimate (21) for the interior integral in (16), we get (5).

This completes the proof of Lemma 4.1. □

Lemma 4.3. *Let the inequality* (1) *be satisfied. Then the condition* (2) *holds. Moreover, there exists an* $\varepsilon > 0$ *such that the function* $W(x)^{-\frac{1}{p'(x)}+\varepsilon}$ *is almost decreasing.*

Proof. We proof that the function $W(x)^{-\frac{1}{p'(x)}}$ is almost decreasing, i.e.

$$
CW(t_1)^{-\frac{1}{p'(t_1)}} \ge W(t_2)^{-\frac{1}{p'(t_2)}} \quad \text{for } 0 < t_1 < t_2 \le l.
$$

Put $t_1 = a$ and $2^{n-1}W(a) < t_2 < 2^n W(a)$ for some $n \in \mathbb{N}$. Let the sequence of numbers $\{a_n\}$ defined as $W(a_n) = 2^n W(a)$, $n \in \mathbb{N}$. Let the $k_0 \in \mathbb{N}$ be such that $2^{k_0}W(a) < l < 2^{k_0+1}W(a)$.

Using Lemma 4.1 we get condition (15). According to (21) we see that

$$
\begin{aligned}
C &\ge \sum_{k=1}^{k_0} \int_{2^{k-1}W(a)}^{2^k W(a)} \left[W(a)^{\frac{1}{p'(a)}} W(x)^{-\frac{1}{p'(x)}}\right]^{p(x)} \frac{dW(x)}{W(x)} \\
&\ge \sum_{k=1, k\in\mathbb{N}'}^{k_0} \left[W(a)^{\frac{1}{p'(a)}} \left(2^k W(a)\right)^{-\frac{1}{p'(a_k)}}\right]^{p^+} \ln 2 \\
&\qquad + \sum_{k=1, k\in\mathbb{N}''}^{k_0} \left[W(a)^{\frac{1}{p'(a)}} \left(2^k W(a)\right)^{-\frac{1}{p'(a_k)}}\right]^{p^-} \ln 2
\end{aligned}
\tag{24}
$$

where the sum over $k \in \mathbb{N}'$ is extended to all $k \in \mathbb{N}$ such that

$$
W(a)^{\frac{1}{p'(a)}} \left(2^k W(a)\right)^{-\frac{1}{p'(a_k)}} \le 1 \tag{25}
$$

and a sum over $k \in \mathbb{N}''$ in opposite case. For any $k \in \mathbb{N}$ from (21) we get

$$
W(a)^{\frac{1}{p'(a)}} \left(2^k W(a)\right)^{-\frac{1}{p'(a_k)}} \le C_2,
$$

where the constant $C_2 > 1$ and does not depend on $k \in \mathbb{N}$. This inequality with $k = n$ and the condition (15) yields

$$W(t_2)^{-\frac{1}{p'(t_2)}} W(t_1)^{\frac{1}{p'(t_1)}} \le C_3,$$

i.e. the function $W(x)^{-\frac{1}{p'(x)}}$ is almost decreasing.

Now, since the function $W(x)^{-\frac{1}{p'(x)}}$ is almost decreasing we infer from (5):

$$\int_a^l C_3^{p^-} \left[\frac{1}{C_3} W(a)^{\frac{1}{p'(a)}} W(x)^{-\frac{1}{p'(x)}}\right]^{p^+} \frac{dW(x)}{W(x)} \le C_4, \quad 0 < a < l$$

or

$$\int_a^l W(x)^{-\frac{p^+}{p'(x)}} \frac{\sigma(x)dx}{W(x)} \le CW(a)^{-\frac{p^+}{p'(a)}}, \quad 0 < a < l \tag{26}$$

Since we had proved the function $W(x)^{-\frac{1}{p'(x)}}$ is almost decreasing it follows from Bari-Stechkin [2] theorem (see [17, 27] for general results) that there exists an $\varepsilon > 0$ such that

$$W(x)^{-\frac{p^+}{p'(x)}+\varepsilon}$$

is almost decreasing on $(0, l)$. This implies that the function

$$W(x)^{-\frac{1}{p'(x)}+\varepsilon_1}$$

is almost decreasing on $(0, l)$; $\varepsilon_1 = \frac{\varepsilon}{p^+}$. From this we easily deduce the condition (2):

$$\begin{aligned} &\int_a^1 W(x)^{-\frac{1}{p'(x)}} \frac{\sigma(x)dx}{W(x)} \\ &= \int_a^l W(x)^{-\frac{1}{p'(x)}+\varepsilon} \frac{\sigma(x)dx}{W(x)^{1+\varepsilon}} CW(a)^{-\frac{1}{p'(a)}+\varepsilon} \int_a^l \frac{\sigma(x)dx}{W(x)^{1+\varepsilon}} \le C_1 W(a)^{-\frac{1}{p'(a)}}. \end{aligned}$$

This completes the proof of Lemma 4.3 □

Lemma 4.4. *Let the condition* (2) *be satisfied. Then the condition* (15) *holds. Moreover, there exists* $\varepsilon > 0$ *such that the function* $W(x)^{-\frac{1}{p'(x)}+\varepsilon}$ *is almost decreasing.*

Proof. Use the argument of Lemma 4.1. It follows from (2) that

$$\int_{2W(a)}^{4W(a)} W(x)^{-\frac{1}{p'(x)}} \frac{dW(x)}{W(x)} \le CW(a)^{-\frac{1}{p'(a)}}, \quad 0 < a < l.$$

Therefore,

$$CW(a)^{-\frac{1}{p'(a)}} \ge \int_{2W(a)}^{4W(a)} W(x)^{-\frac{1}{p'(x)}} \frac{dW(x)}{W(x)} \ge (4W(a))^{-\frac{1}{p'(a^1)}} \ln 2 \tag{27}$$

or

$$\left(\frac{1}{W(a)}\right)^{\frac{1}{p(a^1)}-\frac{1}{p(a)}} \leq C_1.$$

This implies condition (15).

To prove the second assertion of Lemma 4.4, let us denote

$$g(x) = \int_x^l W(x)^{-\frac{1}{p'(x)}} \frac{\sigma(x)dx}{W(x)}.$$

Then $g'(x) = -W(x)^{-\frac{1}{p'(x)}} \frac{\sigma(x)}{W(x)}$ and by (2) we have

$$\sigma(x)g(x) \leq CW(x)g'(x).$$

Integrating this inequality in x over (t_1, t_2) we get

$$W(t_2)^{\frac{1}{C}} g(t_2) \leq W(t_1)^{\frac{1}{C}} g(t_1),$$

therefore,

$$W(t_2)^{\frac{1}{C}} g(t_2) \leq C_1 W(t_1)^{-\frac{1}{p'(t_1)}+\frac{1}{C}},$$

On other hand, using (2) and monotony of $\frac{1}{p'(x)}$, we get

$$g(t_2) \geq \int_{W(t_2)}^{2W(t_2)} W(t)^{-\frac{1}{p'(x)}} \frac{dW(x)}{W(x)} \geq C_2 W(t_2)^{-\frac{1}{p'(t_2)}} \ln 2.$$

Therefore,

$$W(t_2)^{-\frac{1}{p'(t_2)}+\frac{1}{C}} \leq C_3 W(t_1)^{-\frac{1}{p'(t_1)}+\frac{1}{C}}.$$

This completes the proof of Lemma 4.4. □

Lemma 4.5. *Let there exists an* $\varepsilon > 0$ *such that the function* $W^{-\frac{1}{p'(x)}+\varepsilon}$ *be almost decreasing. Then the condition* (2) *is satisfied. Moreover, it holds the inequality* (1) *for any measurable function* f.

Proof. Let us show the condition (2). Using almost decreasing of the function $W^{-\frac{1}{p'(x}+\varepsilon}$, we have

$$\begin{aligned} \int_a^l W(x)^{-\frac{1}{p'(x)}} \frac{\sigma(x)dx}{W(x)} &\leq \int_a^l W^{-\frac{1}{p'(x)}+\varepsilon} \frac{\sigma(x)dx}{W(x)^{1+\varepsilon}} \\ &\leq C \int_a^l W(x)^{-\frac{1}{p'(a)}+\varepsilon} \frac{\sigma(x)dx}{W(x)^{1+\varepsilon}} = \frac{C}{\varepsilon} W(a)^{-\frac{1}{p'(a)}} \end{aligned} \tag{28}$$

The condition (2) has been proved.

Let $f(x)$ be a measurable function such that

$$\left\|\omega(x)^{\frac{1}{p(.)}} f(.)\right\|_{L^{p(.)}(0,l)} \leq 1.$$

Then $I_{p(.)}\left(\omega(.)^{\frac{1}{p(.)}} f(.)\right) \leq 1$ In order to prove Lemma 4.5 we have to show

$$\left\|\sigma(x)^{\frac{1}{p(.)}} W(x)^{-1} H f(.)\right\|_{L^{p(.)}(0,l)} \leq C_1.$$

By the Minkowski inequality for $L^{p(.)}$ norms we have the inequalities

$$\begin{aligned} \left\|\sigma(.)^{\frac{1}{p(.)}} W(.)^{-1} H f(.)\right\|_{L^{p(.)}(0,l)} &\leq \left\|\sigma(.)^{\frac{1}{p(.)}} W(.)^{-1} \sum_{n=1}^{\infty} \int_{x_{n-1}}^{x_n} f(t) dt\right\|_{L^{p(.)}(0,l)} \\ &\leq \sum_{n=1}^{\infty} \left\|\sigma(.)^{\frac{1}{p(.)}} W(.)^{-1} \int_{x_{n-1}}^{x_n} f(t) dt\right\|_{L^{p(.)}(0,l)}, \end{aligned} \tag{29}$$

where the sequence x_n is defined as $W(x_n) = 2^{-n} W(x);\ n = 1, 2, ...$

Denote

$$p_{x,n}^{-} = \inf\{p(t) : t \in (x_{n-1}, x_n]\}$$

By assumption,

$$W(t)^{\frac{1}{p'(t)} - \varepsilon} \leq C\ W(x)^{\frac{1}{p'(x)} - \varepsilon},$$

where t is a point in $(x_{n-1}, x_n],\ 0 < x < l$ and the constant C does not depend on n.

Since the condition (2) is satisfied, it follows from Lemma 4.3 that the condition (15) holds. By using (15) and $x_{n-1} < t \leq x_n$, we have the estimates

$$W(t)^{\frac{1}{p'(t)}} = W(t)^{\varepsilon} W(t)^{\frac{1}{p'(t)} - \varepsilon} \leq C\ W(x)^{\varepsilon} W(x)^{\frac{1}{p'(x)} - \varepsilon} \leq C\ 2^{-n\varepsilon} W(x)^{\frac{1}{p'(x)}}$$

or

$$W(x)^{-\frac{1}{p'(x)}} \leq C\ 2^{-n\varepsilon} W(t)^{-\frac{1}{p'(t)}},$$

with $t \in (x_{n-1}, x_n]$ is any fixed point.

Therefore and due to Holder's inequality for $x \in (0, l)$ we get

$$\begin{aligned} \int_{x_{n-1}}^{x_n} f(t) dt \leq &\left(\int_{x_{n-1}}^{x_n} \left(\frac{f(t)}{\sigma(t)}\right)^{p_{x,n}^{-}} \sigma(t) dt\right)^{\frac{1}{p_{x,n}^{-}}} \left(\int_{x_{n-1}}^{x_n} \sigma(t) dt\right)^{\frac{1}{(p_{x,n}^{-})'}} \\ &\left(\int_{x_{n-1}}^{x_n} \left(\frac{f(t)}{\sigma(t)}\right)^{p_{x,n}^{-}} \sigma(t) dt\right)^{\frac{1}{p_{x,n}^{-}}} \left(2^{-n} W(x)\right)^{\frac{1}{(p_{x,n}^{-})'}} \end{aligned} \tag{30}$$

By using (15), Corollary 4.2 and $x_{n-1} < t < x_n$ it follows that

$$\begin{aligned} W(x)^{-\frac{1}{p'(x)}} \left(2^{-n} W(x)\right)^{\frac{1}{(p^-_{x,n})'}} &\le C_1\, W(x)^{-\frac{1}{p'(x)}} W(t)^{\frac{1}{(p^-_{x,n})'}} \\ &\le W(x)^{-\frac{1}{p'(x)}} W(t)^{\frac{1}{p'(t)}} \le C\, 2^{-n\varepsilon} \end{aligned} \tag{31}$$

Combining (30) and (31) we get

$$\begin{aligned} &W(x)^{-1} \int_{x_{n-1}}^{x_n} f(t)dt \\ &\le C_2\, 2^{-n\varepsilon} W(x)^{-\frac{1}{p(x)}} \left(\int_{x_{n-1}}^{x_n} \left(\frac{f(t)}{\sigma(t)}\right)^{p^-_{x,n}} \sigma(t)dt \right)^{\frac{1}{p^-_{x,n}}}, \end{aligned} \tag{32}$$

where $0 < x < l$, $n = 1, 2, ...$ and the constant C_2 does not depend on x, n. On other hand,

$$\begin{aligned} &\int_{x_{n-1}}^{x_n} \left(\frac{f(t)}{\sigma(t)}\right)^{p^-_{x,n}} \sigma(t)dt \\ \le &\int_{x_{n-1}}^{x_n} \left(\frac{f(t)}{\sigma(t)}\right)^{p(t)} \chi_{\{\frac{f(t)}{\sigma(t)} \ge 1\}} \sigma(t)dt + \int_{x_{n-1}}^{x_n} \sigma(t)dt \le 1 + 2^{-n} W(l) \le C_3 \end{aligned} \tag{33}$$

Using the last inequality, it follows from (32) that

$$\begin{aligned} &I_{p(.)}\left(\sigma(x)^{\frac{1}{p(x)}} W(x)^{-1} Hf(x)\right) \\ &\le C_4 2^{-n\varepsilon p^-} \int_0^l W(x)^{-1} \left(\int_{x_{n-1}}^{x_n} \left(\frac{f(t)}{\sigma(t)}\right)^{p^-_{x,n}} \sigma(t)dt \right)^{\frac{p(x)}{p^-_{x,n}}} \sigma(x)dx \\ &\le C_4 C_3^{\frac{p^+}{p^-}-1} 2^{-n\varepsilon p^-} \int_0^l W(x)^{-1} \left[\int_{x_{n-1}}^{x_n} \left(\frac{f(t)}{\sigma(t)}\right)^{p(t)} \sigma(t)dt \right] \sigma(x)dx \end{aligned}$$

Due to Fubini's theorem this is exceeded by

$$\begin{aligned} &C_4 C_3^{\frac{p^+}{p^-}-1} 2^{-n\varepsilon p^-} \int_0^{2^{-n}W(l)} \left(\int_{2^{n-1}W(t)}^{2^n W(t)} \frac{dW(x)}{W(x)} \right) \left(\left(\frac{f(t)}{\sigma(t)}\right)^{p(t)} + 1 \right) dW(t) \\ &= C_5 \ln 2 \cdot\, 2^{-n\varepsilon p^-} \int_0^{2^{-n}W(l)} \left(\left(\frac{f(t)}{\sigma(t)}\right)^{p(t)} + 1 \right) dW(t) \le C_6 2^{-n\varepsilon p^-}. \end{aligned}$$

Therefore,

$$\left\| \sigma(.)^{\frac{1}{p(.)}} W(.)^{-1} \int_{x_{n-1}}^{x_n} f(t)dt \right\|_{L^{p(.)}(0,l)} \le C 2^{-\frac{n\varepsilon p^-}{p^+}} \tag{34}$$

By (34) and (29) we get

$$\left\|\sigma(x)^{\frac{1}{p(x)}} W(x)^{-1} Hf(x)\right\|_{L^{p(\cdot)}(0,l)} \le C \sum_{n=1}^{\infty} 2^{-\frac{n\varepsilon p^-}{p^+}} \le C_1$$

This completes the proof of Lemma 4.5. □

Lemma 4.6. *Let the condition* (5) *be satisfied. Then the condition* (15) *also holds. Moreover, it is satisfied the condition* (2) *too.*

Proof. We need to derive only the condition (15) to obtain the condition (2) by the way of Lemma 4.3. Let us derive the estimate (15) from the condition (5).

From (5) it follows that

$$\int_{2W(a)}^{4W(a)} \left(W(x)^{-\frac{1}{p'(x)}} W(a)^{\frac{1}{p'(a)}}\right)^{p(x)} \frac{dW(x)}{W(x)} \le C.$$

Since $\frac{1}{p'(x)}$ is increasing, we have

$$\int_{2W(a)}^{4W(a)} \left((4W(a))^{-\frac{1}{p'(a^1)}} W(a)^{\frac{1}{p'(a)}}\right)^{p(x)} \frac{dW(x)}{W(x)} \le C, \tag{35}$$

where $W(a^1) = 2W(a)$. Suppose $(4W(a))^{-\frac{1}{p'(a^1)}} W(a)^{\frac{1}{p'(a)}}$ is greater than 1. Then it follows from (35) that

$$\left((4W(a))^{-\frac{1}{p'(a^1)}} W(a)^{\frac{1}{p'(a)}}\right)^{p^-} \ln 2 \le C$$

Whence,

$$\left(\frac{1}{W(a)}\right)^{\frac{1}{p'(a^1)} - \frac{1}{p'(a)}} \le 1 + C_2$$

or

$$\left(\frac{1}{p'(a^1)} - \frac{1}{p'(a)}\right) \ln \frac{1}{W(a)} \le \ln(1 + C_2)$$

This completes the proof of estimate (15)

It remains to apply the proof of Lemma 4.3 to derive the estimate (15) form (5) □

Now, we are ready to prove Theorem 2.2

4.1. Proof of Theorem 2.2

Proof. The implication *4)* $\iff$ *5)* follows from the definition of $p(.)$ norm directly. The implication *1)* $\implies$ *4)* has been proved in Lemma 4.1. The implication *4)* $\implies$ *2)* was proved in Lemma 4.6. The implication *2)* $\implies$ *1)* follows from Lemma 4.5. The implication *1)* $\implies$ *2)* follows from Lemma 4.3. The implication *2)* $\implies$ *3)* follows from Lemma 4.4. The implication *3)* $\implies$ *1)* follows from Lemma 4.5.

This completes the proof of Theorem 2.2.

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